ALGEBRAIC KAN EXTENSIONS IN DOUBLE CATEGORIES

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ABSTRACT. We study Kan extensions in three weakenings of the Eilenberg-Moore double category associated to a double monad, that was introduced by Grandis and Paré. To be precise, given a normal oplax double monad T on a double category $\mathcal{K}$, we consider the double categories consisting of pseudo T-algebras, 'weak' vertical T-morphisms, horizontal T-morphisms and T-cells, where 'weak' means either 'lax', 'colax' or 'pseudo'. Denoting these double categories by $\mathsf{Alg}_{\mathrm{w}}(T)$, where w = l, c or ps accordingly, our main result gives, in each of these cases, conditions ensuring that (pointwise) Kan extensions can be lifted along the forgetful double functor $\mathsf{Alg}_{\mathrm{w}}(T) \to \mathcal{K}$. As an application we recover and generalise a result by Getzler, on the lifting of pointwise left Kan extensions along symmetric monoidal enriched functors. As an application of Getzler's result we prove, in suitable symmetric monoidal categories, the existence of bicommutative Hopf monoids that are freely generated by cocommutative comonoids.

1. Introduction

When given a symmetric monoidal functor $j \colon A \to B$ one, instead of considering 'ordinary' Kan extensions along j, often considers 'symmetric monoidal' Kan extensions along j. Precisely, while we can consider Kan extensions along j in the 2-category Cat of categories, functors and transformations, it is often more useful to consider such extensions in the 2-category $\mathsf{sMonCat}$ of symmetric monoidal categories, symmetric monoidal functors and monoidal transformations.

For example consider algebras of 'PROPs': informally, a PROP $\mathbb{P}$ is a certain kind of symmetric monoidal category that describes a type of algebraic structure, on the objects A of any symmetric monoidal category, that involves operations of the form $A^{\otimes m} \to A^{\otimes n}$. There is, for instance, a PROP that describes monoids, and one that describes Hopf monoids. An algebra A of $\mathbb{P}$, in a symmetric monoidal category M, is then an object A in M that is equipped with the algebraic structure described by $\mathbb{P}$—formally, A is simply a symmetric monoidal functor $A \colon \mathbb{P} \to M$. Presenting algebraic structures as symmetric monoidal functors like this has the advantage that, often, freely generated such structures can be constructed as left Kan extensions in $\mathsf{sMonCat}$. For example, the PROP $\mathbb{C}$ of

cocommutative comonoids embeds into the PROP $\mathbb{H}$ of bicommutative Hopf monoids, and the bicommutative Hopf monoids in M that are freely generated by cocommutative comonoids correspond precisely to left Kan extensions along the embedding $\mathbb{C} \hookrightarrow \mathbb{H}$. We will show in §6 that, as a consequence of our main result, all such Kan extensions exist under reasonable conditions on M.

As a second example, Getzler shows in [Get09] that 'operads' in M—which describe algebraic structures involving operations of the form $A^{\otimes n} \to A$—can also be regarded as symmetric monoidal functors $\mathsf{T} \to M$, where T is a certain symmetric monoidal category that describes the algebraic structure of operads. Several generalisations of operads, such as 'cyclic operads' and 'modular operads' can be similarly presented. The main result of [Get09] gives conditions ensuring that left Kan extensions can be 'lifted' along the forgetful 2-functor $\mathsf{sMonCat} \to \mathsf{Cat}^1$ which, as a consequence, gives a coherent way of freely generating many types of generalised operad. To describe this 'lifting' of Kan extensions more precisely, let us consider symmetric monoidal functors $j \colon A \to B$ and $d \colon A \to M$: we say that the ordinary left Kan extension $l \colon B \to M$ of d along j, in Cat, can be lifted to $\mathsf{sMonCat}$ whenever l admits a 'canonical' symmetric monoidal structure that makes it into the left Kan extension of d along j in $\mathsf{sMonCat}$.

Using the language of double categories, the horizontal dual of the main result of this paper, which is stated below, can be thought of as generalising Getzler's result of 'lifting symmetric monoidal Kan extensions' to the broader idea of 'lifting algebraic Kan extensions'. In the remainder of this introduction we will informally explain some of the details of the main result and at the same time describe the contents of this paper. We will however not explain the condition (p) except for remarking that, to recover Getzler's result, we will apply the main result to the double monad whose algebras are symmetric monoidal categories and in that case (p) holds as soon as the tensor product of M preserves colimits in each variable.

1.1. Theorem. [Horizontal dual of Theorem 5.7.] *Let T be a normal pseudo double monad on a double category $\mathcal{K}$, and assume that T is pointwise left exact. Let 'weak' mean either 'colax', 'lax' or 'pseudo'. Given pseudo T-algebras A, B and M, consider the following conditions on a horizontal T-morphism $J \colon A \nrightarrow B$ and a weak vertical T-morphism $d \colon A \to M$:*

(p) *the algebraic structure of M preserves the pointwise left Kan extension of d along J;*

(e) *the structure cell of J is pointwise left d-exact;*

(l) *the forgetful double functor $\mathsf{Alg}_{\mathrm{w}}(T) \to \mathcal{K}$ lifts the pointwise left Kan extension of d along J.*

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[1]In fact it gives conditions ensuring that left Kan extensions can be lifted along the forgetful 2-functor $\mathcal{V}\text{-}\mathsf{sMonCat} \to \mathcal{V}\text{-}\mathsf{Cat}$, from the 2-category of symmetric monoidal categories enriched in a suitable closed symmetric monoidal category $\mathcal{V}$, to the 2-category of $\mathcal{V}$-categories.

The following hold:

(a) *if 'weak' means 'colax' then (e) implies (l);*

(b) *if 'weak' means 'lax' then (p) implies (l);*

(c) *if 'weak' means 'pseudo' then any two of (p), (e) and (l) imply the third.*

We start in §2 by recalling the relevant terminology on double categories. Briefly, the features of double categories that distinguish them from bicategories are that, besides objects, they consist of two types of morphisms, 'vertical' ones denoted $f\colon A \to B$ and 'horizontal' ones denoted $J\colon A \nrightarrow B$, as well as cells that are shaped like squares, which can be composed both vertically and horizontally. Amongst others, we will recall the notions of 'restriction' and 'extension' of a horizontal morphism along vertical morphisms, as well as the notion of 'tabulation' of a horizontal morphism—the latter can be thought of as generalising the notion of comma object in 2-categories. Following this we recall the notion of 'left Kan extension' in any double category $\mathcal{K}$ which, as can be deduced from the way the main result is stated, defines the extension of a vertical morphism $d\colon A \to M$ along a horizontal morphism $J\colon A \nrightarrow B$; the resulting extension, if it exists, is a vertical morphism of the form $l\colon B \to M$ which, like in the case of 2-categories, is defined by a cell satisfying a universal property. In fact, the notion of Kan extension in $\mathcal{K}$ generalises that of Kan extension in the 2-category $V(\mathcal{K})$, which is the 'vertical part' of $\mathcal{K}$, as soon as $\mathcal{K}$ has all restrictions. Double categories that have all restrictions are called 'equipments'.

We continue by recalling the stronger notion of 'pointwise' Kan extension and, to get some feeling for it, study in detail such extensions in the equipment $\mathcal{V}$-Prof consisting of categories enriched in a suitable monoidal category $\mathcal{V}$, $\mathcal{V}$-functors, '$\mathcal{V}$-profunctors' and their transformations. In doing so we are naturally lead to an extension of the notions of 'weighted limit' and 'Kan extension along $\mathcal{V}$-functors' to the setting in which $\mathcal{V}$ is just a monoidal category, where classically (see e.g. [Kel82]) $\mathcal{V}$ is assumed to be closed symmetric monoidal. At the end of §2 we recall the notion of 'pointwise left exactness', which is crucial in the statement of the main result: a cell of a double category is called pointwise left exact if, whenever it is vertically postcomposed with a cell defining a pointwise left Kan extension, the resulting composite defines again a such an extension. For example any $\mathcal{V}$-natural transformation of $\mathcal{V}$-functors that satisfies the 'left Beck-Chevalley condition', in the sense of e.g. [Gui80], gives rise to a pointwise left exact cell in $\mathcal{V}$-Prof.

In §3 we recall the description of the 2-category consisting of double categories, so-called 'normal pseudo' double functors, and double transformations; by a normal pseudo double monad $T = (T, \mu, \eta)$ on a double category $\mathcal{K}$, as in the main result, we simply mean a monad on $\mathcal{K}$ in this 2-category. We remark that, like all double transformations, the multiplication μ and the unit η consist of a natural family of vertical morphisms, one for each object in $\mathcal{K}$, as well as a natural family of cells, one for each horizontal morphism in $\mathcal{K}$. We call the double monad T pointwise left exact, as is required in the main result, whenever each of these cells is pointwise left exact.

Any normal pseudo double monad T on $\mathcal{K}$ induces a strict 2-monad on the vertical part $V(\mathcal{K})$ of $\mathcal{K}$, so that we can consider pseudo $V(T)$-algebras in $V(\mathcal{K})$, in the usual 2-categorical sense of e.g. [Str74], as well as 'weak' $V(T)$-morphisms between them, where we take 'weak' to mean either 'lax', 'colax' or 'pseudo'. Moreover, we shall consider a notion of 'horizontal T-morphism' between pseudo $V(T)$-algebras, which generalises slightly a notion introduced in [GP04], and show that, for each choice of 'weak', pseudo $V(T)$-algebras, weak $V(T)$-morphisms, horizontal T-morphisms, together with an appropriate notion of 'T-cell', form a double category $\mathsf{Alg}_{\mathrm{w}}(T)$, where the subscript $\mathrm{w} \in \{\mathrm{c}, \mathrm{l}, \mathrm{ps}\}$ according to the choice of weakness. We remark that, without going into details, a horizontal T-morphism $A \nrightarrow B$ is a horizontal morphism $J \colon A \nrightarrow B$ that is equipped with a cell $\bar{J}$ defining its algebraic structure. If the pointwise left Kan extension of a map $d \colon A \to M$ along J exists in $\mathcal{K}$ then its defining cell can be vertically postcomposed with $\bar{J}$ and, in these terms, condition (e) of the main result means that the resulting composite defines again a pointwise left Kan extension.

In §4 we consider Kan extensions in $\mathsf{Alg}_{\mathrm{w}}(T)$. We show that some restrictions (of horizontal morphisms) can be lifted along the double functor $\mathsf{Alg}_{\mathrm{w}}(T) \to \mathcal{K}$, that forgets the algebraic structure, and that all tabulations can be lifted. We conclude that, if $\mathcal{K}$ is an equipment that has so-called 'opcartesian' tabulations, then pointwise Kan extensions in $\mathsf{Alg}_{\mathrm{w}}(T)$ can be defined in terms of ordinary Kan extensions, in a way that is analogous to Street's definition of pointwise Kan extension in 2-categories, that was introduced in [Str74].

Finally, §5 is devoted to stating and proving the main result. We show that Getzler's result can be recovered and, in §6, that the existence of freely generated bicommutative Hopf monoids can be obtained as an application, as promised.

2. Kan extensions in double categories

Throughout this paper the terminology and notation of [Kou14] are used, which we recall here.

2.1. DOUBLE CATEGORIES. By a double category we mean a weakly internal category in the 2-category Cat of categories, functors and natural transformations, as follows.

2.2. DEFINITION. A *double category* $\mathcal{K}$ consists of a diagram of functors

$$\mathcal{K}_{\mathrm{c}}\ _{R}\times_{L}\mathcal{K}_{\mathrm{c}} \underset{\pi_2}{\overset{\pi_1}{\underset{\odot}{\rightrightarrows}}} \mathcal{K}_{\mathrm{c}} \underset{R}{\overset{L}{\underset{1}{\leftrightarrows}}} \mathcal{K}_{\mathrm{v}}$$

(where $\mathcal{K}_{\mathrm{c}}\ _{R}\times_{L}\mathcal{K}_{\mathrm{c}}$ is the pullback of R and L, with projections π_1 and π_2), such that

$$L \circ \odot = L \circ \pi_1, \qquad R \circ \odot = R \circ \pi_2 \qquad \text{and} \qquad L \circ 1 = \mathrm{id} = R \circ 1,$$

together with natural isomorphisms

$$\mathfrak{a}\colon (J \odot H) \odot K \cong J \odot (H \odot K), \quad \mathfrak{l}\colon 1_A \odot M \cong M \quad \text{and} \quad \mathfrak{r}\colon M \odot 1_B \cong M,$$

for all $(J, H, K) \in \mathcal{K}_\text{c } {}_R\times_L \mathcal{K}_\text{c } {}_R\times_L \mathcal{K}_\text{c}$ and $M \in \mathcal{K}_\text{v}$ with $LM = A$ and $RM = B$. The natural isomorphisms $\mathfrak{a}$, $\mathfrak{l}$ and $\mathfrak{r}$ are required to satisfy the usual coherence axioms for a monoidal category or bicategory (see e.g. Section VII.1 of [ML98]), while their images under both R and L must be identities.

The category $\mathcal{K}_\text{v}$ consists of the objects and *vertical morphisms* of $\mathcal{K}$, while the objects and morphisms of $\mathcal{K}_\text{c}$ form the *horizontal morphisms* and *cells* of $\mathcal{K}$. We denote a horizontal morphism $J \in \mathcal{K}_\text{c}$ with $LJ = A$ and $RJ = B$ as a barred arrow $J\colon A \nrightarrow B$, while a cell $\phi\colon J \to K \in \mathcal{K}_\text{c}$ with $L\phi = f\colon A \to C$ and $R\phi = g\colon B \to D$ will be depicted as

$$
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
{\scriptstyle f}\downarrow & \Downarrow{\scriptstyle \phi} & \downarrow{\scriptstyle g} \\
C & \xrightarrow[K]{} & D
\end{array}
\tag{1}
$$

and denoted by $\phi\colon J \Rightarrow K$; we call J and K the *horizontal source* and *target* of ϕ, while f and g are called its *vertical source* and *target*. A cell whose vertical source and target are identities is called *horizontal*.

The compositions of $\mathcal{K}_\text{v}$ and $\mathcal{K}_\text{c}$, which are associative and unital, define vertical compositions for the vertical morphisms and cells of $\mathcal{K}$, both of which we denote by $\circ$, and whose identities are denoted $\text{id}_A\colon A \to A$ and $\text{id}_J\colon J \Rightarrow J$ (which is a horizontal cell). The functors $\odot\colon \mathcal{K}_\text{c } {}_R\times_L \mathcal{K}_\text{c} \to \mathcal{K}_\text{c}$ and $1\colon \mathcal{K}_\text{v} \to \mathcal{K}_\text{c}$ define horizontal compositions for the horizontal morphisms and cells of $\mathcal{K}$, which are associative up to invertible horizontal cells $\mathfrak{a}\colon (J \odot H) \odot K \cong J \odot (H \odot K)$, that are called *associators*, and unital up to invertible horizontal cells $\mathfrak{l}\colon 1_A \odot M \cong M$ and $\mathfrak{r}\colon M \odot 1_B \cong M$, called *unitors*. A cell ϕ as in (1) that has units $J = 1_A$ and $K = 1_C$ as horizontal source and target is called *vertical*; we will often denote it by the more descriptive $\phi\colon f \Rightarrow g$.

To make our drawings of cells more readable we will depict both vertical identities and horizontal units simply as $A{=\!=}A$. Likewise when writing down, or depicting, compositions of cells we will often leave out the associators and unitors.

Every double category $\mathcal{K}$ contains both a *vertical 2-category* $V(\mathcal{K})$, consisting of its objects, vertical morphisms and vertical cells, as well as a *horizontal bicategory* $H(\mathcal{K})$, consisting of its objects, horizontal morphisms and horizontal cells. For details see Definition 1.8 of [Kou14]. Like 2-categories, any double category $\mathcal{K}$ has both a *vertical dual* $\mathcal{K}^\text{op}$, that is given by taking $(\mathcal{K}^\text{op})_\text{v} = (\mathcal{K}_\text{v})^\text{op}$ and $(\mathcal{K}^\text{op})_\text{c} = (\mathcal{K}_\text{c})^\text{op}$, and a *horizontal dual* $\mathcal{K}^\text{co}$, that is obtained by swapping the functors L and $R\colon \mathcal{K}_\text{c} \to \mathcal{K}_\text{v}$; for details see Definition 1.7 of [Kou14].

2.3. EXAMPLE. The archetypal double category is that of profunctors. Denoted Prof, it has small categories as objects and functors as vertical morphisms, while its horizontal morphisms $J\colon A \nrightarrow B$ are *profunctors*, that is functors of the form $J\colon A^\text{op} \times B \to \mathsf{Set}$. A cell ϕ in Prof, of the form (1), is a natural transformation $\phi\colon J \Rightarrow K(f, g)$ where $K(f, g) = K \circ (f^\text{op} \times g)$. The horizontal composite $J \odot H$ of profunctors $J\colon A \nrightarrow B$ and

$H\colon B \nrightarrow E$ is given by choosing a coequaliser $(J \odot H)(x,z)$ of each pair of functions

$$\coprod_{v\colon y_1 \to y_2 \in B} J(x, y_1) \times H(y_2, z) \rightrightarrows \coprod_{y \in B} J(x, y) \times H(y, z), \tag{2}$$

that are induced by postcomposing the maps in $J(x, y_1)$ with $v\colon y_1 \to y_2$ and precomposing the maps in $H(y_2, z)$ with v. The unit profunctor $1_A\colon A \nrightarrow A$ is given by the hom-sets $1_A(x_1, x_2) = A(x_1, x_2)$. We shall describe a $\mathcal{V}$-enriched variant of Prof, where $\mathcal{V}$ is a suitable monoidal category, in detail in Example 2.15.

2.4. EXAMPLE. A *span* $J\colon A \nrightarrow B$ in a category $\mathcal{E}$ is a diagram $A \xleftarrow{d_0} J \xrightarrow{d_1} B$ in $\mathcal{E}$. Spans can be composed as soon as $\mathcal{E}$ has pullbacks; with this composition objects and morphisms of $\mathcal{E}$, together with spans in $\mathcal{E}$ and their morphisms, form a double category $\mathsf{Span}(\mathcal{E})$. Details can be found in Example 1.5 of [Kou14].

2.5. EXAMPLE. Given a category $\mathcal{V}$ and sets A and B, a $\mathcal{V}$-*matrix* $J\colon A \nrightarrow B$ is simply a family of $\mathcal{V}$-objects $J(x, y)$, one for each pair of objects $x \in A$ and $y \in B$. If $\mathcal{V}$ is equipped with a monoidal structure $(\otimes, 1)$, and has coproducts that are preserved by $\otimes$ on both sides, then sets, functions between sets and $\mathcal{V}$-matrices form a double category $\mathcal{V}\text{-}\mathsf{Mat}$ as follows. A cell ϕ in $\mathcal{V}\text{-}\mathsf{Mat}$, of the form (1), is given by a family of $\mathcal{V}$-maps $\phi_{x,y}\colon J(x, y) \to K(fx, gy)$, while the horizontal composite $J \odot H$ of $J\colon A \nrightarrow B$ and $H\colon B \nrightarrow E$ is given by 'matrix multiplication':

$$(J \odot H)(x, z) = \coprod_{y \in B} J(x, y) \otimes H(y, z).$$

The unit matrix $1_A\colon A \nrightarrow A$ is given by $1_A(x, x) = 1$ and $1_A(x_1, x_2) = \emptyset$, the initial object of $\mathcal{V}$, whenever $x_1 \neq x_2$.

2.6. EQUIPMENTS. Important to the theory of double categories are the notions of cartesian and opcartesian cells. A cartesian cell defines the restriction of a horizontal morphism along a pair of vertical morphisms and, dually, opcartesian cells define extensions, as follows.

2.7. DEFINITION. The cell ϕ on the left below is called *cartesian* if any cell ψ, as in the middle, factors uniquely through ϕ as shown.

$$
\begin{array}{ccc}
\begin{array}{c}
A \xrightarrow{J} B \\
f \downarrow \quad \Downarrow\phi \quad \downarrow g \\
C \xrightarrow{K} D
\end{array}
&
\begin{array}{c}
X \xrightarrow{H} Y \\
h \downarrow \qquad \downarrow k \\
A \quad \Downarrow\psi \quad B \\
f \downarrow \qquad \downarrow g \\
C \xrightarrow{K} D
\end{array}
=
\begin{array}{c}
X \xrightarrow{H} Y \\
h \downarrow \quad \Downarrow\psi' \quad \downarrow k \\
A \xrightarrow{J} B \\
f \downarrow \quad \Downarrow\phi \quad \downarrow g \\
C \xrightarrow{K} D
\end{array}
&
\begin{array}{c}
A \xrightarrow{J} B \\
f \downarrow \qquad \downarrow g \\
C \quad \Downarrow\chi \quad D \\
h \downarrow \qquad \downarrow k \\
X \xrightarrow{L} Y
\end{array}
=
\begin{array}{c}
A \xrightarrow{J} B \\
f \downarrow \quad \Downarrow\phi \quad \downarrow g \\
C \xrightarrow{K} D \\
h \downarrow \quad \Downarrow\chi' \quad \downarrow k \\
X \xrightarrow{L} Y
\end{array}
\end{array}
$$

Vertically dual, the cell ϕ is called *opcartesian* if any cell χ as on the right factors uniquely through ϕ as shown.

If a cartesian cell like ϕ exists then we call J the *restriction of K along f and g*, and write $K(f,g) = J$; if $K = 1_C$ then we write $C(f,g) = 1_C(f,g)$. By their universal property any two cartesian cells defining the same restriction factor through each other as invertible horizontal cells. Moreover, since the vertical composite of two cartesian cells is again cartesian, and since vertical identities id_K are cartesian, it follows that restrictions are pseudofunctorial, in the sense that $K(f,g)(h,k) \cong K(f \circ h, g \circ k)$ and $K(\mathrm{id}, \mathrm{id}) \cong K$. Dually, if an opcartesian cell like ϕ exists then we call K the *extension of J along f and g*; like restrictions, extensions are unique up to isomorphism and pseudofunctorial. We shall usually not name cartesian and opcartesian cells, but simply depict them like the two cells below.

For each vertical morphism $f \colon A \to C$ the restriction $f_* = C(f, \mathrm{id}) \colon A \nrightarrow C$, if it exists, is called the *companion* of f; it is defined by a cartesian cell as on left below. Dually the extension of 1_A along f and id_A, if it exists, is called the *conjoint* of f; it is denoted by f^* and defined by an opcartesian cell as on the right.

$$
\begin{array}{ccc}
A & \xrightarrow{f_*} & C \\
f\downarrow & \text{cart} & \| \\
C & = & C
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & = & A \\
f\downarrow & \text{opcart} & \| \\
C & \xrightarrow{f^*} & A
\end{array}
$$

2.8. EXAMPLE. In Example 2.3 we have already used the notation $K(f,g)$ to denote the profunctor $K \circ (f^{\mathrm{op}} \times g) \colon A \nrightarrow B$. It is readily seen that the cell $\varepsilon \colon K(f,g) \Rightarrow K$ given by the identity transformation on $K \circ (f^{\mathrm{op}} \times g)$ is cartesian in the double category Prof, so that $K \circ (f^{\mathrm{op}} \times g)$ is indeed the restriction of K along f and g.

We can take the conjoint f^* of a functor $f \colon A \to C$ to be the restriction $C(\mathrm{id}, f)$: the opcartesian cell defining it is given by the actions $f \colon A(x_1, x_2) \to C(fx_1, fx_2)$ of f on the hom-sets. That $f^* \cong C(\mathrm{id}, f)$ is no coincidence is explained below.

Generalising the situation above, in every double category the conjoint f^* of a vertical morphism $f \colon A \to C$ can be equivalently defined as the restriction $C(\mathrm{id}, f)$. This is because the vertical identity cell 1_f factors uniquely through the opcartesian cell defining f^* as a cartesian cell that defines $C(\mathrm{id}, f)$, and conversely. Horizontally dual, the same relation exists between the companion f_* and the extension of 1_A along id_A and f.

Thus companions and conjoints are defined by cartesian, or equivalently opcartesian, cells. Conversely the existence of all companions and conjoints implies the existence of all restrictions and extensions: for any $K \colon C \nrightarrow D$ the composite on the left below is cartesian while, for any $J \colon B \nrightarrow A$, the composite on the right is opcartesian. For details see Theorem 4.1 of [Shu08] or Lemma 2.8 of [Kou14].

$$
\begin{array}{ccccccc}
A & \xrightarrow{f_*} & C & \xrightarrow{K} & D & \xrightarrow{g^*} & B \\
f\downarrow & \text{cart} & \| & & \| & \text{cart} & \downarrow g \\
C & = & C & \xrightarrow{K} & D & = & D
\end{array}
\qquad
\begin{array}{ccccccc}
B & = & B & \xrightarrow{J} & A & = & A \\
g\downarrow & \text{opcart} & \| & & \| & \text{opcart} & \downarrow f \\
D & \xrightarrow{g^*} & B & \xrightarrow{J} & A & \xrightarrow{f_*} & C
\end{array}
\qquad (3)
$$

In summary, the following conditions on a double category $\mathcal{K}$ are equivalent: $\mathcal{K}$ has all companions and conjoints; $\mathcal{K}$ has all restrictions; $\mathcal{K}$ has all extensions.

2.9. DEFINITION. An *equipment* is a double category that satisfies the conditions above.

2.10. EXAMPLE. Since the double category Prof has all restrictions, as we saw in Example 2.8, it is an equipment. The double categories $\mathsf{Span}(\mathcal{E})$ (Example 2.4) and $\mathcal{V}$-Mat (Example 2.5) are equipments as well. Indeed the extension of a span $A \xleftarrow{d_0} J \xrightarrow{d_1} B$ in $\mathcal{E}$, along morphisms $f \colon A \to C$ and $g \colon B \to D$, is given by the span $C \xleftarrow{f \circ d_0} J \xrightarrow{g \circ d_1} D$, while the restriction $K(f, g)$ of a $\mathcal{V}$-matrix $K \colon C \nrightarrow D$ is given by the family of $\mathcal{V}$-objects $K(f, g)(x, y) = K(fx, gy)$.

2.11. MONOIDS AND BIMODULES. Next we recall the notions of monoid and bimodule in double categories, following Section 11 of [Shu08]. These notions are useful: for example, as we recall below, monoids and bimodules in $\mathsf{Span}(\mathcal{E})$ are categories and profunctors internal in $\mathcal{E}$ while monoids and bimodules in $\mathcal{V}$-Mat are $\mathcal{V}$-enriched categories and $\mathcal{V}$-profunctors.

2.12. DEFINITION. Let $\mathcal{K}$ be a double category.

- A *monoid* in $\mathcal{K}$ consists of a quadruple $A = (A_0, A, \mu, \eta)$ where $A \colon A_0 \nrightarrow A_0$ is a horizontal morphism in $\mathcal{K}$ and $\mu \colon A \odot A \Rightarrow A$ and $\eta \colon 1_{A_0} \Rightarrow A$ are horizontal cells satisfying the usual coherence axioms for monoids.

- Given monoids A and C, a *morphism of monoids* $f \colon A \to C$ consists of a vertical morphism $f_0 \colon A_0 \to C_0$ and a cell f, as on the left below, such that $\mu_C \circ (f \odot f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_C \circ 1_{f_0}$.

$$
\begin{array}{ccc}
\begin{array}{ccc}
A_0 & \xrightarrow{A} & A_0 \\
{\scriptstyle f_0}\downarrow & \Downarrow f & \downarrow{\scriptstyle f_0} \\
C_0 & \xrightarrow[C]{} & C_0
\end{array}
&
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
{\scriptstyle f}\downarrow & \Downarrow \phi & \downarrow{\scriptstyle g} \\
C & \xrightarrow[K]{} & D
\end{array}
&
\begin{array}{ccc}
A_0 & \xrightarrow{J} & B_0 \\
{\scriptstyle f_0}\downarrow & \Downarrow \phi & \downarrow{\scriptstyle g_0} \\
C_0 & \xrightarrow[K]{} & D_0
\end{array}
\end{array}
$$

- Given monoids A and B, an *(A, B)-bimodule* $J \colon A \nrightarrow B$ consists of a horizontal morphism $J \colon A_0 \nrightarrow B_0$ that is equipped with horizontal cells $\lambda \colon A \odot J \Rightarrow J$ and $\rho \colon J \odot B \Rightarrow J$ defining the *actions* of A and B on J, which satisfy the usual coherence axioms for bimodules.

- Given morphisms of monoids $f \colon A \to C$ and $g \colon B \to D$, and bimodules $J \colon A \nrightarrow B$ and $K \colon C \nrightarrow D$, a *cell* ϕ as in the middle above is a cell in $\mathcal{K}$ as on the right, such that $\lambda_K \circ (f \odot \phi) = \phi \circ \lambda_J$ and $\rho_K \circ (\phi \odot g) = \phi \circ \rho_J$.

- The *horizontal composite* $J \odot_B H$ of bimodules $J \colon A \nrightarrow B$ and $H \colon B \nrightarrow E$ is the following reflexive coequaliser in $H(\mathcal{K})(A, E)$, if it exists:

$$
J \odot B \odot H \overset{\rho_J \odot \mathrm{id}}{\underset{\mathrm{id} \odot \lambda_H}{\rightrightarrows}} J \odot H \implies J \odot_B H.
$$

The following is Proposition 11.10 of [Shu08]. Recall that each double category $\mathcal{K}$ contains a bicategory $H(\mathcal{K})$ consisting of horizontal morphisms and horizontal cells. We say that $\mathcal{K}$ has *local reflexive coequalisers* if the categories $H(\mathcal{K})(A, B)$ have reflexive coequalisers that are preserved by horizontal composition on both sides.

2.13. PROPOSITION. [Shulman] *If an equipment $\mathcal{K}$ has local reflexive coequalisers then monoids and bimodules in $\mathcal{K}$, together with their morphisms and cells, again form an equipment $\mathsf{Mod}(\mathcal{K})$ that has local reflexive coequalisers, whose horizontal composition is given as above.*

2.14. EXAMPLE. Let $\mathcal{E}$ be a category with pullbacks. Monoids in $\mathsf{Span}(\mathcal{E})$ are *internal categories* in $\mathcal{E}$ and morphisms of such monoids are internal functors. Bimodules and their cells in $\mathsf{Span}(\mathcal{E})$ are *internal profunctors* in $\mathcal{E}$ and their transformations. Internal categories and internal profunctors are described in some detail in Example 1.6 of [Kou14]. If $\mathcal{E}$ has reflexive coequalisers preserved by pullback then $\mathsf{Span}(\mathcal{E})$ has local reflexive coequalisers, so that internal categories, functors, profunctors and transformations in $\mathcal{E}$ form an equipment $\mathsf{Mod}(\mathsf{Span}(\mathcal{E}))$ which is denoted $\mathsf{Prof}(\mathcal{E})$.

For a suitable monoidal category $\mathcal{V}$ the equipment of bimodules in $\mathcal{V}$-Mat (Example 2.5) is that of $\mathcal{V}$-enriched profunctors, which we shall now describe; it will be used as an example throughout. We remark that, in the case that $\mathcal{V}$ is closed symmetric monoidal—hence enriched over itself—, the classical meaning of a $\mathcal{V}$-profunctor $A \nrightarrow B$, where A and B are $\mathcal{V}$-categories, is that of a $\mathcal{V}$-functor of the form $A^{\mathrm{op}} \otimes B \to \mathcal{V}$. The advantage of defining $\mathcal{V}$-profunctors as bimodules in $\mathcal{V}$-Mat is that, while this extends the classical definition, in this way it is not necessary for $\mathcal{V}$ to be closed symmetric monoidal—$\mathcal{V}$ being a monoidal category suffices. This allows us, in the next subsection, to extend the classical notions of 'weighted limit' and 'enriched Kan extension' to settings in which the enriching category $\mathcal{V}$ is not closed symmetric monoidal.

2.15. EXAMPLE. Let $\mathcal{V} = (\mathcal{V}, \otimes, 1)$ be a monoidal category that has coproducts which are preserved by $\otimes$ on both sides. A monoid in $\mathcal{V}$-Mat is a category *enriched* in $\mathcal{V}$, while a morphism of such monoids is a $\mathcal{V}$-functor, both in the usual sense; see e.g. Section 1.2 of [Kel82]. A bimodule $J\colon A \nrightarrow B$ in $\mathcal{V}$-Mat is a $\mathcal{V}$-*profunctor* in the sense of Section 7 of [DS97]: it consists of a family $J(x, y)$ of $\mathcal{V}$-objects, indexed by pairs of objects $x \in A$ and $y \in B$, that is equipped with associative and unital actions

$$\lambda\colon A(x_1, x_2) \otimes J(x_2, y) \to J(x_1, y) \quad \text{and} \quad \rho\colon J(x, y_1) \otimes B(y_1, y_2) \to J(x, y_2)$$

satisfying the usual compatibility axiom for bimodules. Given a map[2] $f\colon x_1 \to x_2$ in A we write λ_f for the composite

$$J(x_2, y) \xrightarrow{f \otimes \mathrm{id}} A(x_1, x_2) \otimes J(x_2, y) \xrightarrow{\lambda} J(x_1, y);$$

likewise for $g\colon y_1 \to y_2$ in B we write $\rho_g = \rho \circ (\mathrm{id} \otimes g)\colon J(x, y_1) \to J(x, y_2)$.

---

[2]As is the custom, by a map $f\colon x_1 \to x_2$ in a $\mathcal{V}$-category A we mean a $\mathcal{V}$-map $f\colon 1 \to A(x_1, x_2)$.

If $\mathcal{V}$ is closed symmetric monoidal then $\mathcal{V}$-profunctors $J\colon A \nrightarrow B$ can be identified with $\mathcal{V}$-functors of the form $J\colon A^{\mathrm{op}} \otimes B \to \mathcal{V}$. Indeed the actions of J correspond, under the adjunctions $-\otimes J(x,y) \dashv [J(x,y),-]$ that are part of the closed structure on $\mathcal{V}$, to families of maps

$$A(x_1, x_2) \to [J(x_2, y), J(x_1, y)] \qquad \text{and} \qquad B(y_1, y_2) \to [J(x, y_1), J(x, y_2)]$$

respectively, which combine to form families of partial $\mathcal{V}$-functors $J(-,y)\colon A^{\mathrm{op}} \to \mathcal{V}$ and $J(x,-)\colon B \to \mathcal{V}$. The compatibility axiom for bimodules ensures that the latter correspond to a single $\mathcal{V}$-functor $J\colon A^{\mathrm{op}} \otimes B \to \mathcal{V}$; for details see Section 1.4 of [Kel82].

A cell of $\mathcal{V}$-profunctors

$$
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
{\scriptstyle f}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle g} \\
C & \xrightarrow[K]{} & D,
\end{array}
$$

called a *transformation*, consists of a family of $\mathcal{V}$-maps $\phi_{(x,y)}\colon J(x,y) \to K(fx, gy)$ (often simply denoted ϕ) that are compatible with the actions, in the sense that the identities

$$\lambda \circ (f \otimes \phi) = \phi \circ \lambda\colon A(x_1, x_2) \otimes J(x_2, y) \to K(fx_1, gy)$$

and
$$\rho \circ (\phi \otimes g) = \phi \circ \rho\colon J(x, y_1) \otimes B(y_1, y_2) \to K(fx, gy_2)$$

are satisfied, where f and g denote the actions of the $\mathcal{V}$-functors f and g on the hom-objects of A and B respectively. If $\mathcal{V}$ is closed symmetric monoidal then transformations in the above sense can be identified with the usual $\mathcal{V}$-natural transformations between the $\mathcal{V}$-functors $J\colon A^{\mathrm{op}} \otimes B \to \mathcal{V}$ and $K(f, g) = K \circ (f^{\mathrm{op}} \otimes g)$.

If $\mathcal{V}$ has reflexive coequalisers preserved by $\otimes$ on both sides then $\mathcal{V}$-Mat has local reflexive coequalisers, so that $\mathcal{V}$-categories, $\mathcal{V}$-functors, $\mathcal{V}$-profunctors and their transformations form a double category $\mathsf{Mod}(\mathcal{V}\text{-}\mathsf{Mat})$ which is denoted $\mathcal{V}$-Prof. Its horizontal composite $J \odot H$, of $\mathcal{V}$-profunctors $J\colon A \nrightarrow B$ and $H\colon B \nrightarrow E$, is obtained by choosing coequalisers $(J \odot H)(x, z)$ for the pairs of $\mathcal{V}$-maps

$$\coprod_{y_1, y_2 \in B} J(x, y_1) \otimes B(y_1, y_2) \otimes H(y_2, z) \rightrightarrows \coprod_{y \in B} J(x, y) \otimes H(y, z), \tag{4}$$

that are induced by letting $B(y_1, y_2)$ act on $J(x, y_1)$ and $H(y_2, z)$ respectively; compare the unenriched situation (2). The horizontal composite $\phi \odot \chi$ of the transformations on the left below is given by the family of unique factorisations shown on the right, where the $\mathcal{V}$-maps drawn horizontally are the coequalisers defining $(J \odot H)(x, z)$ and $(K \odot L)(fx, hz)$, and the $\mathcal{V}$-map on the left is induced by the tensorproducts $\phi_{(x,y)} \otimes \chi_{(y,z)}$.

$$
\begin{array}{ccccc}
A & \xrightarrow{J} & B & \xrightarrow{H} & E \\
{\scriptstyle f}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle g} & \Downarrow\chi & \downarrow{\scriptstyle h} \\
C & \xrightarrow[K]{} & D & \xrightarrow[L]{} & F
\end{array}
\qquad
\begin{array}{ccc}
\coprod\limits_{y \in B} J(x, y) \otimes H(y, z) & \longrightarrow & (J \odot H)(x, z) \\
\downarrow & & \vdots\,{\scriptstyle (\phi \odot \chi)_{(x,z)}} \\
\coprod\limits_{v \in D} K(fx, v) \otimes L(v, hz) & \longrightarrow & (K \odot L)(fx, hz)
\end{array}
$$

The unit $\mathcal{V}$-profunctor $1_A \colon A \nrightarrow A$, for a $\mathcal{V}$-category A, is given by the hom-objects $1_A(x_1, x_2) = A(x_1, x_2)$; its actions are given by the composition of A. Finally, $\mathcal{V}$-Prof is an equipment in which the restriction $K(f, g)$ of a $\mathcal{V}$-profunctor $K \colon C \nrightarrow D$ along $\mathcal{V}$-functors $f \colon A \to C$ and $g \colon B \to D$ is given by the family of $\mathcal{V}$-objects $K(f, g)(x, y) = K(fx, gy)$, that is equipped with actions induced by those of K.

We remark that, for $\mathcal{V}$-functors f and $g \colon A \to C$, the vertical cells $f \Rightarrow g$ in $\mathcal{V}$-Prof can be identified with $\mathcal{V}$-natural transformations $f \Rightarrow g$, in the classical sense; see Section 1.2 of [Kel82]. Indeed any vertical cell $\phi \colon f \Rightarrow g$ is given by a family of $\mathcal{V}$-maps $\phi_{(x_1, x_2)} \colon A(x_1, x_2) \to C(fx_1, gx_2)$ which is, because of its compatibility with the actions of A and C, completely determined by the composites

$$\phi_x = \left[1 \xrightarrow{\eta_x} A(x, x) \xrightarrow{\phi_{(x,x)}} C(fx, gx) \right],$$

which make the diagram below commute; that is, they form a $\mathcal{V}$-natural transformation of $\mathcal{V}$-functors $f \Rightarrow g$.

$$
\begin{array}{ccc}
A(x_1, x_2) & \xrightarrow{\ f \otimes \phi_{x_2}\ } & C(fx_1, fx_2) \otimes C(fx_2, gx_2) \\
{\scriptstyle \phi_{x_1} \otimes g} \downarrow & & \downarrow {\scriptstyle \mu} \\
C(fx_1, gx_1) \otimes C(gx_1, gx_2) & \xrightarrow[\ \mu\ ]{} & C(fx_1, gx_2)
\end{array}
$$

This identification induces an isomorphism

$$V(\mathcal{V}\text{-Prof}) \cong \mathcal{V}\text{-Cat} \tag{5}$$

between the vertical 2-category contained in $\mathcal{V}$-Prof and the 2-category $\mathcal{V}$-Cat of $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations.

2.16. KAN EXTENSIONS IN DOUBLE CATEGORIES. Analogous to that in 2-categories there is a notion of Kan extension in double categories, that was introduced in [GP08] and which is recalled below. The stronger notion of pointwise Kan extension, which we also consider, was introduced in [Kou14]. To get some feeling for the latter we shall consider pointwise Kan extensions in the double category $\mathcal{V}$-Prof in the next subsection.

2.17. DEFINITION. Let $d \colon B \to M$ and $J \colon A \nrightarrow B$ be morphisms in a double category. The cell ε in the right-hand side below is said to define r as the *right Kan extension of d along J* if every cell ϕ below factors uniquely through ε as shown.

$$
\begin{array}{ccccccc}
A & \xrightarrow{J} & B & & A & = A & \xrightarrow{J} & B \\
{\scriptstyle s}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle d} & = & {\scriptstyle s}\downarrow & \Downarrow\phi' \ \ \downarrow{\scriptstyle r} \ \Downarrow\varepsilon & & \downarrow{\scriptstyle d} \\
M & = & M & & M & = M & = & M
\end{array}
$$

We call the right Kan extension r above *pointwise* if cells ϕ of the more general form below also factor uniquely through ε, as shown.

$$
\begin{array}{ccc}
C \xrightarrow{H} A \xrightarrow{J} B & \quad & C \xrightarrow{H} A \xrightarrow{J} B \\
s\downarrow \quad \Downarrow\phi \quad \downarrow d & = & s\downarrow \quad \Downarrow\phi' \;\; \downarrow r \;\; \Downarrow\varepsilon \;\; \downarrow d \\
M =\!\!=\!\!=\!\!= M & & M =\!\!= M =\!\!= M
\end{array}
$$

The following, which is a simple consequence of the universal property of opcartesian cells, shows that the notion of Kan extension in double categories generalises that of Kan extension in 2-categories (for a definition see e.g. Section 3 of [Kou14]). Remember that any double category $\mathcal{K}$ contains a 2-category $V(\mathcal{K})$ of objects, vertical morphisms and vertical cells of $\mathcal{K}$.

2.18. PROPOSITION. *Let $d\colon B \to M$, $j\colon B \to A$ and $r\colon A \to M$ be morphisms in a double category $\mathcal{K}$, and assume that the conjoint $j^*\colon A \nrightarrow B$ of j exists. Consider a vertical cell ε as on the left below, as well as its factorisation through the opcartesian cell defining j^*, as shown.*

$$
\begin{array}{ccc}
B =\!\!= B & \quad & B =\!\!= B \\
j\downarrow \qquad \downarrow & & j\downarrow \text{opcart} \,\| \\
A \quad \Downarrow\varepsilon \quad \downarrow d & = & A \xrightarrow{\;\;j^*\;\;} B \\
r\downarrow \qquad \downarrow & & r\downarrow \quad \Downarrow\varepsilon' \;\; \downarrow d \\
M =\!\!= M & & M =\!\!= M
\end{array}
$$

The vertical cell ε defines r as the right Kan extension of d along j in $V(\mathcal{K})$ precisely if its factorisation ε' defines r as the right Kan extension of d along j^* in $\mathcal{K}$.

The following result for iterated pointwise Kan extensions is a straightforward consequence of Definition 2.17.

2.19. PROPOSITION. *Consider horizontally composable cells*

$$
\begin{array}{c}
A \xrightarrow{J} B \xrightarrow{H} C \\
s\downarrow \quad \Downarrow\gamma \;\; \downarrow r \;\; \Downarrow\varepsilon \;\; \downarrow d \\
M =\!\!= M =\!\!= M
\end{array}
$$

in a double category, and suppose that ε defines r as the pointwise right Kan extension of d along H. Then γ defines s as the pointwise right Kan extension of r along J precisely if $\gamma \odot \varepsilon$ defines s as the pointwise right Kan extension of d along $J \odot H$.

For completeness we record the definition of pointwise left Kan extension in a double category $\mathcal{K}$, which coincides with that of pointwise right Kan extension in its horizontal dual $\mathcal{K}^{\mathrm{co}}$.

2.20. DEFINITION. Let $d \colon A \to M$ and $J \colon A \nrightarrow B$ be morphisms in a double category. The cell η in the right-hand side below defines l as the *pointwise left Kan extension of d along J* if every cell ϕ below factors uniquely through η as shown.

$$
\begin{array}{ccc}
A \xrightarrow{J} B \xrightarrow{H} C & & A \xrightarrow{J} B \xrightarrow{H} C \\
d\downarrow \quad \Downarrow\phi \quad \downarrow k & = & d\downarrow \quad \Downarrow\eta \;\downarrow l \; \Downarrow\phi' \;\downarrow k \\
M =\!\!=\!\!=\!\!=\!\!= M & & M =\!\!= M =\!\!= M
\end{array}
$$

2.21. POINTWISE KAN EXTENSIONS IN $\mathcal{V}$-Prof. Here we study pointwise Kan extensions in the double category $\mathcal{V}$-Prof and show that they can be described in terms of '$\mathcal{V}$-weighted limits'. While such limits are classically defined in the case that $\mathcal{V}$ is a closed symmetric monoidal category, we shall consider a more general definition for which a monoidal structure on $\mathcal{V}$ alone suffices; this enables us to treat all cases of $\mathcal{V}$-Prof.

For the rest of this subsection we take $\mathcal{V}$ to be a cocomplete monoidal category whose tensor product preserves colimits on both sides, so that $\mathcal{V}$-categories, $\mathcal{V}$-functors, $\mathcal{V}$-profunctors and their transformations form a double category $\mathcal{V}$-Prof; see Example 2.15. We write 1 for the unit $\mathcal{V}$-category, that has a single object $*$ and hom-object $1(*, *) = 1$. We identify $\mathcal{V}$-functors $1 \to A$ with objects in A and $\mathcal{V}$-profunctors $1 \nrightarrow 1$ with $\mathcal{V}$-objects; transformations of such profunctors are identified with $\mathcal{V}$-maps.

By a $\mathcal{V}$-*weight* J on a $\mathcal{V}$-category B we mean a $\mathcal{V}$-profunctor $J \colon 1 \nrightarrow B$.

2.22. DEFINITION. Let $d \colon B \to M$ be a $\mathcal{V}$-functor, J a $\mathcal{V}$-weight on B and r an object of M. A transformation ε in $\mathcal{V}$-Prof, as in the right-hand side below, is said to define r as the *J-weighted limit of d* if every transformation ϕ below factors uniquely through ε as shown.

$$
\begin{array}{ccc}
1 \xrightarrow{H} 1 \xrightarrow{J} B & & 1 \xrightarrow{H} 1 \xrightarrow{J} B \\
s\downarrow \quad \Downarrow\phi \quad \downarrow d & = & s\downarrow \quad \Downarrow\phi' \;\downarrow r \; \Downarrow\varepsilon \;\downarrow d \\
M =\!\!=\!\!=\!\!=\!\!= M & & M =\!\!= M =\!\!= M
\end{array}
$$

The definition above extends the classical definition of weighted limit as follows. Note that if $\mathcal{V}$ is closed symmetric monoidal then $\mathcal{V}$-weights on B can be identified with $\mathcal{V}$-functors $J \colon B \to \mathcal{V}$, using the isomorphism $1^{\mathrm{op}} \otimes B = 1 \otimes B \cong B$; see Example 2.15. This recovers the classical definition of $\mathcal{V}$-weight, see e.g. Section 3.1 of [Kel82] where such $\mathcal{V}$-functors are called 'indexing types'. Using this identification, the transformation ε above can be regarded as a $\mathcal{V}$-natural transformation between the $\mathcal{V}$-functors $J \colon B \to \mathcal{V}$ and $M(r, d) \colon B \to \mathcal{V}$.

2.23. PROPOSITION. *Let d, J, r and ε be as above. If $\mathcal{V}$ is complete and closed symmetric monoidal then the transformation ε defines r as the J-weighted limit of d, in the above sense, precisely if the pair (r, ε), where $\varepsilon \colon J \Rightarrow M(r, d)$ is regarded as a $\mathcal{V}$-natural transformation of $\mathcal{V}$-functors $B \to \mathcal{V}$, forms the 'limit of d indexed by J' in the sense of Section 3.1 of [Kel82].*

PROOF. In terms of $\mathcal{V}$-natural transformations between $\mathcal{V}$-functors $B \to \mathcal{V}$, the unique factorisations through ε above can be restated as the following universal property for the composite $\mathcal{V}$-natural transformations

$$\varepsilon_s = \left[M(s,r) \otimes J \xRightarrow{\mathrm{id} \otimes \varepsilon} M(s,r) \otimes M(r,d) \xRightarrow{\mu} M(s,d) \right],$$

where $s \in M$: for any $\mathcal{V}$-natural transformation $\phi \colon H \odot J \Rightarrow M(s,d)$, where $H \in \mathcal{V}$, there exists a unique $\mathcal{V}$-map $\phi' \colon H \to M(s,r)$ such that

$$\phi = \left[H \odot J \xRightarrow{\phi' \odot \mathrm{id}} M(s,r) \odot J \xRightarrow{\varepsilon_s} M(s,r) \right].$$

Now the adjunctions $- \otimes Y \dashv [Y, -]$, that define the closed structure on $\mathcal{V}$, induce a bijective correspondence between $\mathcal{V}$-natural transformations $H \odot J \Rightarrow M(s,d)$, of $\mathcal{V}$-functors $B \to \mathcal{V}$, and $\mathcal{V}$-natural transformations $H \Rightarrow [J, M(s,d)]$, from the $\mathcal{V}$-object H to the $\mathcal{V}$-functor $[J, M(s,d)] \colon B^{\mathrm{op}} \otimes B \to \mathcal{V}$ given on objects by $(y_1, y_2) \mapsto [Jy_1, M(s, dy_2)]$; the latter in the sense of Section 2.1 of [Kel82]. Since this correspondence is natural in H the universality of the transformations $\varepsilon_s \colon M(s,r) \otimes J \Rightarrow M(s,d)$ above translates into the universality of the corresponding transformations $M(s,r) \Rightarrow [J, M(s,d)]$; in other words they define $M(s,r)$ as the end $\int_{y \in B} [Jy, M(s, dy)]$, for each $s \in M$. The latter is equivalent to the meaning of the definition of '(r, ε) is the limit of d indexed by J' that is given in Section 3.1 of [Kel82]; see the sentence therein containing formula (3.3). ∎

Next we describe the pointwise Kan extensions of $\mathcal{V}$-Prof in terms of weighted limits and, as a consequence, obtain an 'enriched variant' of Proposition 2.18. Recall that vertical cells in $\mathcal{V}$-Prof can be identified with $\mathcal{V}$-natural transformations of $\mathcal{V}$-functors, under the isomorphism (5).

2.24. PROPOSITION. *Consider $\mathcal{V}$-functors $r \colon A \to M$ and $d \colon B \to M$ as well as a $\mathcal{V}$-profunctor $J \colon A \nrightarrow B$.*

For a transformation ε in $\mathcal{V}$-Prof as on the left above, the following are equivalent:

(a) *ε defines r as the pointwise right Kan extension of d along J;*

(b) *for each $x \in A$ the composite in the middle above defines rx as the $J(x, \mathrm{id})$-weighted limit of d.*

In particular, pointwise right Kan extensions along a $\mathcal{V}$-weight $J\colon 1 \nrightarrow B$ coincide with J-weighted limits.

Finally assume that $\mathcal{V}$ is complete and closed symmetric monoidal. If $J = j^*$, for a $\mathcal{V}$-functor $j\colon B \to A$, and ε is the factorisation of a $\mathcal{V}$-natural transformation $\psi\colon rj \Rightarrow d$ through the opcartesian cell defining j^*, as on the right above, then condition (b) coincides with the meaning of the definition of 'ψ exhibits r as the right Kan extension of d along j' that is given in Section 4.1 of [Kel82].

PROOF. For each $x \in A$ we denote by ε_x the composite in the middle above.

(a) $\Rightarrow$ (b). By Theorem 2.29 below condition (a) implies that ε_x defines rx as the pointwise right Kan extension of d along $J(x, \mathrm{id})$, for each $x \in A$. By restricting the universal property of Definition 2.17 for ε_x to cells ϕ with $H\colon 1 \nrightarrow 1$, we obtain the universal property of Definition 2.22 for ε_x, so that (b) follows.

(b) $\Rightarrow$ (a). Consider a transformation ϕ in $\mathcal{V}$-Prof, as in the middle composite below; we have to show that it factors uniquely as $\phi = \phi' \odot \varepsilon$.

$$
\phi_{(z,x)} =
\begin{array}{c}
1 \xrightarrow{H(z,x)} 1 \xrightarrow{J(x,\mathrm{id})} B \\
z\downarrow \quad \mathrm{cart} \quad \downarrow x\;\mathrm{cart} \quad \| \\
C \xrightarrow{H} A \xrightarrow{J} B \\
s\downarrow \qquad \Downarrow\phi \qquad \downarrow d \\
M =\!=\!=\!=\!=\!= M
\end{array}
\quad = \quad
\begin{array}{c}
1 \xrightarrow{H(z,x)} 1 \xrightarrow{J(x,\mathrm{id})} B \\
sz\downarrow \;\; \Downarrow\phi'_{(z,x)} \;\; \downarrow rx \;\; \Downarrow\varepsilon_x \;\; \downarrow d \\
M =\!=\!= M =\!=\!= M
\end{array}
\tag{6}
$$

Because the cells ε_x define weighted limits, the restrictions $\phi_{(z,x)}$ of ϕ above, for each pair $z \in C$ and $x \in A$, factor uniquely through ε_x as cells $\phi'_{(z,x)}$ as shown. These factorisations are simply $\mathcal{V}$-maps $\phi'_{(z,x)}\colon H(z,x) \to M(sz, rx)$, and it remains to show that, as a family, they combine to form a transformation $\phi'\colon H \Rightarrow M(s,r)$. Indeed, the unique factorisations above then imply that ϕ factors uniquely as $\phi = \phi' \odot \varepsilon$, as needed.

Thus we have to show that the $\mathcal{V}$-maps $\phi'_{(z,x)}$ are compatible with the left and right actions of C, A and M. In terms of cells in $\mathcal{V}$-Prof, the compatibility with the left actions means that the identities

$$
\begin{array}{c}
1 \xrightarrow{C(z_1,z_2)} 1 \xrightarrow{H(z_2,x)} 1 \\
\| \qquad \Downarrow\lambda \qquad \| \\
1 \xrightarrow{H(z_1,x)} 1 \\
sz_1\downarrow \quad \Downarrow\phi'_{(z_1,x)} \quad \downarrow rx \\
M =\!=\!=\!=\!=\!= M
\end{array}
\quad = \quad
\begin{array}{c}
1 \xrightarrow{C(z_1,z_2)} 1 \xrightarrow{H(z_2,x)} 1 \\
sz_1\downarrow \;\; s\Downarrow \;\; sz_2\downarrow \;\; \Downarrow\phi'_{(z_2,x)} \;\; \downarrow rx \\
M =\!=\!= M =\!=\!= M
\end{array}
$$

hold, where λ is the $\mathcal{V}$-map $C(z_1, z_2) \otimes H(z_2, x) \to H(z_1, x)$ given by the left action of C on H and s is the $\mathcal{V}$-map $C(z_1, z_2) \to M(sz_1, sz_2)$ given by the action of the $\mathcal{V}$-functor s on hom-objects. Because factorisations through ε_x are unique we may equivalently show

that this identity holds after composing both sides on the right with ε_x. That it does is shown below where, to save space, only the non-identity cells are depicted while objects and morphisms are left out. The first and last identities here follow from the factorisations (6) while the second identity follows from the compatibility of ϕ with the action of C.

$$
\begin{array}{|c|c|}\hline \lambda & \\\hline \phi'_{(z_1,x)} & \varepsilon_x \\\hline\end{array}
\;=\;
\begin{array}{|c|}\hline \lambda \\\hline \phi_{(z_1,x)} \\\hline\end{array}
\;=\;
\begin{array}{|c|c|}\hline s & \phi_{(z_2,x)} \\\hline\end{array}
\;=\;
\begin{array}{|c|c|c|}\hline s & \phi'_{(z_2,x)} & \varepsilon_x \\\hline\end{array}
$$

That the family of $\mathcal{V}$-maps $\phi'_{(z,x)}$ is compatible with the right actions as well follows similarly, from the following equation of transformations (which are of the form $H(z,x_1) \odot A(x_1,x_2) \odot J(x_2,\mathrm{id}) \Rightarrow 1_M$) and the fact that factorisations through ε_{x_2} are unique.

$$
\begin{array}{|c|c|}\hline \rho & \\\hline \phi'_{(z,x_2)} & \varepsilon_{x_2} \\\hline\end{array}
\;=\;
\begin{array}{|c|}\hline \rho \\\hline \phi_{(z,x_2)} \\\hline\end{array}
\;=\;
\begin{array}{|c|}\hline \lambda \\\hline \phi_{(z,x_1)} \\\hline\end{array}
\;=\;
\begin{array}{|c|c|}\hline \lambda & \\\hline \phi'_{(z,x_1)} & \varepsilon_{x_1} \\\hline\end{array}
\;=\;
\begin{array}{|c|c|c|}\hline \phi'_{(z,x_1)} & r & \varepsilon_{x_2} \\\hline\end{array}
$$

The identities here follow from the factorisations (6); the fact that the domain $H \odot_A J$ of ϕ coequalises the actions of A on H and on J, see (4); the factorisations (6) again; the compatibility of ε with the left actions.

To prove the final assertion we assume that $J = j^*$ for a $\mathcal{V}$-functor $j\colon B \to A$, and that ε is the unique factorisation of a $\mathcal{V}$-natural transformation $\psi\colon rj \Rightarrow d$ through the opcartesian cell defining j^*. As soon as we consider ψ as a vertical cell of $\mathcal{V}$-Prof, under the isomorphism (5), it is easy to check that ε must be given by the $\mathcal{V}$-maps

$$
A(x,jy) \xrightarrow{r} M(rx,rjy) \xrightarrow{\rho_{\psi_y}} M(rx,dy),
$$

for pairs $x \in A$ and $y \in B$. It then follows from Proposition 2.23 that, if $\mathcal{V}$ is complete and closed symmetric monoidal, then condition (b) coincides with condition (ii) of Theorem 4.6 of [Kel82], which lists equivalent meanings of the definition of 'ψ exhibits r as the right Kan extension of d along j'. ∎

2.25. Pointwise Kan extensions in terms of Kan extensions. The main goal of [Kou14] was to give a condition on equipments $\mathcal{K}$ ensuring that an analogue of Proposition 2.18 holds for pointwise Kan extensions, where by 'pointwise Kan extension in the 2-category $V(\mathcal{K})$', we mean the classical notion given by Street in [Str74]. This condition is given in terms of the notion of tabulation, as follows.

2.26. Definition. Given a horizontal morphism $J\colon A \nrightarrow B$ in a double category $\mathcal{K}$, the *tabulation* $\langle J \rangle$ of J consists of an object $\langle J \rangle$ equipped with a cell π as on the left below, satisfying the following 1-dimensional and 2-dimensional universal properties.

$$
\begin{array}{ccc}
\langle J \rangle & == & \langle J \rangle \\
\pi_A \downarrow & \Downarrow \pi & \downarrow \pi_B \\
A & \xrightarrow{J} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & == & X \\
\phi_A \downarrow & \Downarrow \phi & \downarrow \phi_B \\
A & \xrightarrow{J} & B
\end{array}
$$

Given any other cell ϕ in $\mathcal{K}$ as on the right above, the 1-dimensional property states that there exists a unique vertical morphism $\phi' \colon X \to \langle J \rangle$ such that $\pi \circ 1_{\phi'} = \phi$.

The 2-dimensional property is the following. Suppose we are given a further cell ψ as in the identity below, which factors through π as $\psi' \colon Y \to \langle J \rangle$, like ϕ factors as ϕ'. Then for any pair of cells ξ_A and ξ_B as below, so that the identity holds, there exists a unique cell ξ' as on the right below such that $1_{\pi_A} \circ \xi' = \xi_A$ and $1_{\pi_B} \circ \xi' = \xi_B$.

$$
\begin{array}{ccc}
\begin{array}{c}
X \xrightarrow{H} Y == Y \\
\phi_A \downarrow \quad \Downarrow \xi_A \, |\psi_A \Downarrow \psi \, |\psi_B \\
A == A \xrightarrow[J]{} B
\end{array}
&
=
\quad
\begin{array}{c}
X == X \xrightarrow{H} Y \\
\phi_A \downarrow \quad \Downarrow \phi \, |\phi_B \Downarrow \xi_B \, |\psi_B \\
A \xrightarrow[J]{} B == B
\end{array}
&
\quad
\begin{array}{c}
X \xrightarrow{H} Y \\
\phi' \downarrow \quad \Downarrow \xi' \, |\psi' \\
\langle J \rangle == \langle J \rangle
\end{array}
\end{array}
$$

A tabulation is called *opcartesian* whenever its defining cell π is opcartesian.

2.27. EXAMPLE. The tabulation $\langle J \rangle$ of a profunctor $J \colon A^{\mathrm{op}} \times B \to \mathsf{Set}$ is the category that has triples (x, u, y) as objects, where $(x, y) \in A \times B$ and $u \colon x \to y$ in $J(x, y)$, while a map $(x, u, y) \to (x', u', y')$ is a pair $(p, q) \colon (x, y) \to (x', y')$ in $A \times B$ making the diagram

$$
\begin{array}{ccc}
x & \xrightarrow{u} & y \\
p \downarrow & & \downarrow q \\
x' & \xrightarrow[u']{} & y'
\end{array}
$$

commute in J. The functors π_A and π_B are the projections and the natural transformation $\pi \colon 1_{\langle J \rangle} \Rightarrow J$ maps the pair (p, q) to the diagonal $u' \circ p = q \circ u$. It is easy to check that π satisfies both the 1-dimensional and 2-dimensional universal property above, and that it is opcartesian.

2.28. EXAMPLE. Generalising the previous example, it was shown in Proposition 5.15 of [Kou14] that the double category $\mathsf{Prof}(\mathcal{E})$, of profunctors internal to $\mathcal{E}$, admits all opcartesian tabulations.

The main result, Theorem 5.11, of [Kou14] is the following.

2.29. THEOREM. *In an equipment $\mathcal{K}$ consider the cell ε on the left below.*

$$
\begin{array}{cc}
\begin{array}{c}
A \xrightarrow{J} B \\
r \downarrow \quad \Downarrow \varepsilon \quad \downarrow d \\
M == M
\end{array}
&
\qquad
\begin{array}{c}
C \xrightarrow{J(f, \mathrm{id})} B \\
f \downarrow \quad \mathrm{cart} \quad \| \\
A \xrightarrow[J]{} B \\
r \downarrow \quad \Downarrow \varepsilon \quad \downarrow d \\
M == M
\end{array}
\end{array}
$$

For the following conditions the implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) hold, while (c) $\Rightarrow$ (a) holds as soon as $\mathcal{K}$ has opcartesian tabulations.

(a) *The cell ε defines r as the pointwise right Kan extension of d along J;*

(b) *for all $f\colon C \to A$ the composite on the right above defines $r \circ f$ as the pointwise right Kan extension of d along $J(f, \mathrm{id})$;*

(c) *for all $f\colon C \to A$ the composite on the right above defines $r \circ f$ as the right Kan extension of d along $J(f, \mathrm{id})$.*

As a consequence the analogue of Proposition 2.18 for pointwise Kan extensions below follows, which is Proposition 5.12 of [Kou14].

2.30. PROPOSITION. *Let $d\colon B \to M$, $j\colon B \to A$ and $r\colon A \to M$ be morphisms in an equipment $\mathcal{K}$ that has opcartesian tabulations. Consider a vertical cell ε as on the left below, as well as its factorisation through the opcartesian cell defining j^*, as shown.*

$$
\begin{array}{ccc}
B = \!\!= B & & B = \!\!= B \\
j\downarrow \quad \quad \downarrow & & j\downarrow \ \mathrm{opcart} \ \| \\
A \quad \Downarrow\varepsilon \quad \Big| d \ = & & A \xrightarrow{\ j^*\ } B \\
r\downarrow \quad \quad \downarrow & & r\downarrow \ \Downarrow\varepsilon' \ \downarrow d \\
M = \!\!= M & & M = \!\!= M
\end{array}
$$

The vertical cell ε defines r as the pointwise right Kan extension of d along j in $V(\mathcal{K})$, in the sense of [Str74], precisely if its factorisation ε' defines r as the pointwise right Kan extension of d along j^* in $\mathcal{K}$.

2.31. EXACT CELLS. The final notions that we need to recall are those of 'exact' and 'initial' cell.

2.32. DEFINITION. In a double category consider a cell ϕ, as on the left below, and a vertical morphism $d\colon D \to M$. We call ϕ *(pointwise) right d-exact* if for any cell ε, as on the right, that defines r as the (pointwise) right Kan extension of d along K, the composite $\varepsilon \circ \phi$ defines $r \circ f$ as the (pointwise) right Kan extension of $d \circ g$ along J. If the converse holds as well then we call ϕ *(pointwise) d-initial*.

$$
\begin{array}{ccc}
A \xrightarrow{\ J\ } B & \quad\quad & C \xrightarrow{\ K\ } D \\
f\downarrow \ \Downarrow\phi \ \downarrow g & & r\downarrow \ \Downarrow\varepsilon \ \downarrow d \\
C \xrightarrow[\ K\ ]{} D & & M = \!\!= M
\end{array}
$$

If ϕ is (pointwise) right d-exact for all vertical morphisms $d\colon D \to M$, where M varies, then it is called *(pointwise) right exact*. Likewise ϕ is called *(pointwise) initial* whenever it is (pointwise) d-initial for all $d\colon D \to M$. (Pointwise) left exact and (pointwise) final cells are defined likewise.

The uniqueness of Kan extensions implies that the notions of right d-exactness and d-initialness do not depend on the choice of the cell ε that defines the Kan extension along d.

2.33. EXAMPLE. In Example 4.3 of [Kou14] it was shown that every 'initial' functor $f\colon A \to C$ induces an initial cell in Prof, while in Example 4.6 of the same paper any natural transformation $\phi\colon fj \Rightarrow kg$ between composites of functors, that satisfies the 'right Beck-Chevalley condition', was shown to give rise to a pointwise right exact cell. We shall return to this condition later, at the end of §5.

The following, which combines Proposition 4.2 and Corollary 4.5 of [Kou14], describes classes of initial and right exact cells.

2.34. PROPOSITION. *In a double category consider a cell ϕ as on the left below, and assume that the companions $f_*\colon A \nrightarrow C$ and $g_*\colon B \nrightarrow D$ of f and g exists. It follows that the opcartesian and cartesian cell in the composite on the right exist, see* (3), *and we write ϕ_* for the unique factorisation of ϕ through these cells, as shown.*

$$
\begin{array}{c}
A \xrightarrow{\;J\;} B \\
f \Big\downarrow \quad \Downarrow\phi \quad \Big\downarrow g \\
C \xrightarrow[K]{} D
\end{array}
\;=\;
\begin{array}{c}
A \xrightarrow{\;J\;} B \\
\| \quad \text{opcart} \quad \Big\downarrow g \\
A \xrightarrow[J]{} B \xrightarrow[g_*]{} D \\
\| \quad \Downarrow\phi_* \quad \| \\
A \xrightarrow[K(f,\,\mathrm{id})]{} D \\
f \Big\downarrow \quad \text{cart} \quad \| \\
C \xrightarrow[K]{} D
\end{array}
$$

The following hold:

(a) *if $f = \mathrm{id}_A$ and ϕ is opcartesian then ϕ is both initial and pointwise initial;*

(b) *ϕ is pointwise right exact if the following equivalent conditions hold: $\phi_* \circ \mathrm{opcart}$ is opcartesian; $\mathrm{cart} \circ \phi_*$ is cartesian; ϕ_* is invertible.*

3. Eilenberg-Moore double categories

In this section we recall the notion of a 'normal oplax double monad' T on a double category and consider several weakenings of the 'Eilenberg-Moore double category' associated to T, that was introduced in Section 7.1 of [GP04].

More precisely, by a 'normal oplax double monad' we shall mean a monad T in the 2-category $\mathsf{Dbl}_{\mathrm{no}}$ of double categories, 'normal oplax double functors' and their transformations. Since the assignment $\mathcal{K} \mapsto V(\mathcal{K})$, that maps a double category $\mathcal{K}$ to its vertical 2-category $V(\mathcal{K})$, extends to a 2-functor $\mathsf{Dbl}_{\mathrm{no}} \to 2\text{-}\mathsf{Cat}$, any normal oplax double monad T on $\mathcal{K}$ induces a strict 2-monad $V(T)$ on $V(\mathcal{K})$. After generalising slightly the notions of 'horizontal T-morphism' and 'T-cell', that were introduced in [GP04], we will, for any choice of 'weak' $\in \{\mathrm{colax}, \mathrm{lax}, \mathrm{pseudo}\}$, show that lax $V(T)$-algebras, weak $V(T)$-morphisms, horizontal T-morphisms and T-cells form a double category $\mathsf{Alg}_{\mathrm{w}}(T)$.

3.1. DOUBLE FUNCTORS AND THEIR TRANSFORMATIONS. We start by recalling the notions of double functor and double transformation; references include Sections 7.2 and 7.3 of [GP99] and Section 6 of [Shu08].

3.2. DEFINITION. Let $\mathcal{K}$ and $\mathcal{L}$ be double categories. A *lax double functor* $F\colon \mathcal{K} \to \mathcal{L}$ consists of a pair of functors $F_{\mathrm{v}}\colon \mathcal{K}_{\mathrm{v}} \to \mathcal{L}_{\mathrm{v}}$ and $F_{\mathrm{c}}\colon \mathcal{K}_{\mathrm{c}} \to \mathcal{L}_{\mathrm{c}}$ (both will be denoted F), such that $LF_{\mathrm{c}} = F_{\mathrm{v}}L$ and $RF_{\mathrm{c}} = F_{\mathrm{v}}R$, together with natural transformations

$$
\begin{array}{ccccc}
\mathcal{K}_{\mathrm{c}} \times_{\mathcal{K}_{\mathrm{v}}} \mathcal{K}_{\mathrm{c}} & \xrightarrow{\odot_{\mathcal{K}}} & \mathcal{K}_{\mathrm{c}} & \xleftarrow{\;1\;} & \mathcal{K}_{\mathrm{v}} \\
{\scriptstyle F_{\mathrm{c}} \times_{F_{\mathrm{v}}} F_{\mathrm{c}}} \downarrow & {\scriptstyle F_{\odot}}\nearrow & \downarrow{\scriptstyle F_{\mathrm{c}}} & \swarrow{\scriptstyle F_1} & \downarrow{\scriptstyle F_{\mathrm{v}}} \\
\mathcal{L}_{\mathrm{c}} \times_{\mathcal{L}_{\mathrm{v}}} \mathcal{L}_{\mathrm{c}} & \xrightarrow[\odot_{\mathcal{L}}]{} & \mathcal{L}_{\mathrm{c}} & \xleftarrow[1]{} & \mathcal{L}_{\mathrm{v}},
\end{array}
$$

whose components are horizontal cells, that satisfy the usual associativity and unit axioms for monoidal functors, see e.g. Section XI.2 of [ML98]. The transformations $F_{\odot}$ and F_1 are called the *compositor* and *unitor* of F.

Vertically dual, in the definition of an *oplax double functor* $F\colon \mathcal{K} \to \mathcal{L}$ the directions of the compositor $F_{\odot}$ and unitor F_1 are reversed. A *pseudo double functor* is a lax (or, equivalently, an oplax) double functor whose compositor and unitor are invertible; a lax, oplax or pseudo double functor whose unitor F_1 is the identity is called *normal*.

We shall mostly be interested in normal oplax double functors. Unpacking the above, such a double functor $F\colon \mathcal{K} \to \mathcal{L}$ maps the objects, vertical and horizontal morphisms, as well as cells of $\mathcal{K}$ to those of $\mathcal{L}$, in a way that preserves horizontal and vertical sources and targets. Moreover, vertical composition of morphisms and cells is preserved, as are the horizontal units: $F1_A = 1_{FA}$ and $F1_f = 1_{Ff}$ for each A and $f\colon A \to C$ in $\mathcal{K}$, while horizontal composition is preserved only up to natural coherent cells

$$
\begin{array}{ccc}
FA & \xrightarrow{F(J \odot H)} & FE \\
\| & \Downarrow F_{\odot} & \| \\
FA & \xrightarrow[FJ]{} FB \xrightarrow[FH]{} & FE
\end{array}
$$

for each composable pair $J\colon A \nrightarrow B$ and $H\colon B \nrightarrow E$ in $\mathcal{K}$. We will often use the unit axioms for F: they state that, for any horizontal morphism $J\colon A \nrightarrow B$, the composite

$$
F(1_A \odot J) \overset{F_{\odot}}{\Longrightarrow} F1_A \odot FJ = 1_{FA} \odot FJ \overset{\mathfrak{l}}{\Rightarrow} FJ \tag{7}
$$

coincides with $F\mathfrak{l}$, and similar for $J \odot 1_B$.

3.3. EXAMPLE. Any functor $F\colon \mathcal{D} \to \mathcal{E}$ between categories with pullbacks induces a normal oplax double functor $\mathsf{Span}(F)\colon \mathsf{Span}(\mathcal{D}) \to \mathsf{Span}(\mathcal{E})$, simply by applying F to the spans of $\mathcal{D}$. The compositor of $\mathsf{Span}(F)$ is induced by the universal property of pullbacks; in particular, if F preserves pullbacks then $\mathsf{Span}(F)$ is a normal pseudo double functor.

3.4. DEFINITION. A *double transformation* $\xi\colon F \Rightarrow G$ between lax double functors F and $G\colon \mathcal{K} \to \mathcal{L}$ is given by natural transformations $\xi_{\mathrm{v}}\colon F_{\mathrm{v}} \Rightarrow G_{\mathrm{v}}$ and $\xi_{\mathrm{c}}\colon F_{\mathrm{c}} \Rightarrow G_{\mathrm{c}}$ (both will be denoted ξ), with $L\xi_{\mathrm{c}} = \xi_{\mathrm{v}}L$ and $R\xi_{\mathrm{c}} = \xi_{\mathrm{v}}R$, such that the following diagrams commute, where $(J, H) \in \mathcal{K}_{\mathrm{c}} \ {}_R\times_L \ \mathcal{K}_{\mathrm{c}}$ and $A \in \mathcal{K}_{\mathrm{v}}$.

$$
\begin{array}{ccc}
FJ \odot FH & \overset{F_\odot}{\Longrightarrow} & F(J \odot H) \\
{\scriptstyle \xi_J \odot \xi_H}\big\Downarrow & & \big\Downarrow{\scriptstyle \xi_{J \odot H}} \\
GJ \odot GH & \underset{G_\odot}{\Longrightarrow} & G(J \odot H)
\end{array}
\qquad\qquad
\begin{array}{ccc}
1_{FA} & \overset{F_1}{\Longrightarrow} & F1_A \\
{\scriptstyle 1_{\xi_A}}\big\Downarrow & & \big\Downarrow{\scriptstyle \xi_{1_A}} \\
1_{GA} & \underset{G_1}{\Longrightarrow} & G1_A
\end{array}
$$

Analogously, in the definition of a double transformation $\xi\colon F \Rightarrow G$ between oplax double functors F and $G\colon \mathcal{K} \to \mathcal{L}$ the directions of the coherence cells in the diagrams above are reversed; in both cases we shall call the commuting of these diagrams respectively the composition and unit axiom for ξ.

Double categories, lax double functors and double transformations form a 2-category which we denote $\mathsf{Dbl}_{\mathrm{l}}$; analogously $\mathsf{Dbl}_{\mathrm{o}}$ denotes the 2-category of double categories, oplax double functors and double transformations. We denote by $\mathsf{Dbl}_{\mathrm{nl}} \subset \mathsf{Dbl}_{\mathrm{l}}$ and $\mathsf{Dbl}_{\mathrm{no}} \subset \mathsf{Dbl}_{\mathrm{o}}$ the sub-2-categories consisting of normal double functors. Notice that the unit axiom for a double transformation $\xi\colon F \Rightarrow G$ between normal double functors reduces to the identity $\xi_{1_A} = 1_{\xi_A}$, for all objects A of $\mathcal{K}$.

3.5. EXAMPLE. Every transformation $\xi\colon F \Rightarrow G$, between functors of categories that have pullbacks, induces a double transformation $\mathsf{Span}(\xi)\colon \mathsf{Span}(F) \Rightarrow \mathsf{Span}(G)$ between normal oplax double functors, that is given by $A \mapsto \xi_A$ on objects and $(A \leftarrow J \to B) \mapsto \xi_J$ on spans. We conclude that the assignment $\mathcal{E} \mapsto \mathsf{Span}(\mathcal{E})$, that maps a category $\mathcal{E}$ with pullbacks to the double category $\mathsf{Span}(\mathcal{E})$ of spans in $\mathcal{E}$, extends to a 2-functor $\mathsf{Span}\colon \mathsf{pbCat} \to \mathsf{Dbl}_{\mathrm{no}}$, where pbCat denotes the 2-category of categories with pullbacks, all functors between such categories and their transformations.

The following propositions record some useful properties of double functors and their transformations. Remember that every pseudo double category $\mathcal{K}$ contains a vertical 2-category $V(\mathcal{K})$; in the proposition below 2-Cat denotes the 2-category of 2-categories, strict 2-functors and 2-transformations.

3.6. PROPOSITION. *The assignment* $\mathcal{K} \mapsto V(\mathcal{K})$ *extends to a strict* 2-*functor*

$$V\colon \mathsf{Dbl}_{\mathrm{no}} \to 2\text{-}\mathsf{Cat}.$$

PROOF (SKETCH). The image $V(F)\colon V(\mathcal{K}) \to V(\mathcal{L})$ of a double functor $F\colon \mathcal{K} \to \mathcal{L}$ is simply the restriction of F to objects, vertical morphisms and vertical cells; that $V(F)$ is a strict 2-functor follows from the unit axiom for F. Likewise the vertical part $\xi_{\mathrm{v}}\colon F_{\mathrm{v}} \Rightarrow G_{\mathrm{v}}$ of any double transformation $\xi\colon F \Rightarrow G$ forms a 2-transformation $V(\xi)\colon V(F) \Rightarrow V(G)$. ∎

The following is Proposition 6.8 of [Shu08].

3.7. PROPOSITION. [Shulman] *Every lax double functor between equipments preserves cartesian cells, while every oplax double functor between equipments preserves opcartesian cells.*

Having introduced double functors and their transformations we turn to double monads.

3.8. DEFINITION. An *oplax double monad* on a double category $\mathcal{K}$ is a monad $T = (T, \mu, \eta)$ on $\mathcal{K}$ in the 2-category Dbl_o, consisting of an oplax double endofunctor $T \colon \mathcal{K} \to \mathcal{K}$, a multiplication $\mu \colon T^2 \Rightarrow T$ and a unit $\eta \colon \mathrm{id}_\mathcal{K} \Rightarrow T$, which satisfy the associativity axiom $\mu \circ T\mu = \mu \circ \mu T$ and unit axiom $\mu \circ T\eta = \mathrm{id}_T = \mu \circ \eta T$.

Dually a *lax double monad* is a monad in Dbl_l. An oplax, or lax, double monad is called *normal* or *pseudo* whenever its underlying double endofunctor is a normal or pseudo double functor.

Notice that, Proposition 3.6, any normal oplax double monad T on $\mathcal{K}$ restricts to a strict 2-monad $V(T)$ on the vertical 2-category $V(\mathcal{K})$. The following example describes the 'free monoid'-monad on $\mathcal{V}$-Mat. In Example 3.11 below we will see that it induces the 'free strict monoidal $\mathcal{V}$-category'-monad on $\mathcal{V}$-Prof, which we shall use throughout.

For many other examples of lax double monads we refer to Section 3 of [CS10], where 'monads on virtual double categories' are considered. Analogous to the generalisation of the notion of monoidal category to that of 'multicategory', a 'virtual double category' consists of objects, vertical morphisms and horizontal morphisms, like a double category, but has cells of the form $(J_1, \ldots, J_n) \Rightarrow K$, where $(J_1, \ldots, J_n)$ is a sequence of horizontal morphisms $J_1 \colon A_0 \nrightarrow A_1, \ldots, J_n \colon A_{n-1} \nrightarrow A_n$. Virtual double categories that have all 'composites' and 'units', in the sense of Section 5 of [CS10], can be identified with double categories in our sense, and 'monads' on such virtual double categories correspond to the lax double monads in our sense.

3.9. EXAMPLE. Let $\mathcal{V} = (\mathcal{V}, \otimes, 1, \mathfrak{s})$ be a symmetric monoidal category that has coproducts preserved by $\otimes$ on both sides, so that $\mathcal{V}$-matrices form a double category $\mathcal{V}$-Mat; see Example 2.5. The pseudo double endofunctor T underlying the *'free monoid'-monad* on $\mathcal{V}$-Mat assigns to a set A the free monoid $TA = \coprod_{n \geq 0} A^n$ and, accordingly, maps a function $f \colon A \to C$ to $Tf = \coprod_{n \geq 0} f^n$. The elements of TA are (possibly empty) sequences $\underline{x} = (x_1, \ldots, x_n)$ of elements in A; we write $|\underline{x}| = n$. The function Tf simply applies f coordinatewise.

The assignments $A \mapsto TA$ and $f \mapsto Tf$ extend to a pseudo double endofunctor T on $\mathcal{V}$-Mat as follows. Its image of a $\mathcal{V}$-matrix $J \colon A \nrightarrow B$ is given by

$$TJ(\underline{x}, \underline{y}) = \begin{cases} \bigotimes_{i=1}^n J(x_i, y_i) & \text{if } |\underline{x}| = n = |\underline{y}|; \\ \emptyset & \text{otherwise.} \end{cases}$$

Likewise the image $T\phi$ of the cell on the left below is given by the family of $\mathcal{V}$-maps on the right.

$$
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
f\downarrow & \Downarrow\phi & \downarrow g \\
C & \xrightarrow[K]{} & D
\end{array}
\qquad
(T\phi)_{(\underline{x},\underline{y})} = \begin{cases} \bigotimes_{i=1}^{n} \phi_{(x_i,y_i)} & \text{if } |\underline{x}| = n = |\underline{y}|; \\ \text{id}_\emptyset & \text{otherwise.} \end{cases}
$$

Next the compositor $T_\odot\colon TJ \odot TH \cong T(J \odot H)$ is given, on sequences $\underline{x}$ and $\underline{z}$ of equal length n, by the canonical isomorphisms

$$
(TJ \odot TH)(\underline{x},\underline{z}) = \coprod_{\underline{y}\in TB} TJ(\underline{x},\underline{y}) \otimes TH(\underline{y},\underline{z}) \cong \coprod_{\underline{y}\in B^n} \Big(\bigotimes_{i=1}^{n} J(x_i,y_i)\Big) \otimes \Big(\bigotimes_{i=1}^{n} H(y_i,z_i)\Big)
$$

$$
\cong \coprod_{\underline{y}\in B^n} \bigotimes_{i=1}^{n} \big(J(x_i,y_i) \otimes H(y_i,z_i)\big) \cong \bigotimes_{i=1}^{n} \coprod_{y\in B} \big(J(x_i,y) \otimes H(y,z_i)\big) = T(J \odot H)(\underline{x},\underline{z}),
$$

where the first and the third isomorphism follow from the fact that $\otimes$ preserves coproducts on both sides, while the second isomorphism is given by reordering the factors $J(x_i,y_i)$ and $H(y_i,z_i)$; of course $(T_\odot)_{(\underline{x},\underline{z})} = \text{id}_\emptyset$ if $|\underline{x}| \neq |\underline{z}|$. On a set A the unitor $T_1\colon 1_{TA} \cong T1_A$ is given by the isomorphisms $1_{TA}(\underline{x},\underline{y}) = 1 \cong 1^{\otimes n} = (T1_A)(\underline{x},\underline{y})$ if $\underline{x} = \underline{y}$ and $\text{id}_\emptyset$ otherwise. This completes the definition of the pseudo double endofunctor T on $\mathcal{V}$-Mat.

The multiplication $\mu\colon T^2 \Rightarrow T$ and unit $\eta\colon \text{id} \Rightarrow T$, that make T into a double monad, restrict to sets as concatenation of double sequences and insertion as one-element sequences respectively. On a $\mathcal{V}$-matrix $J\colon A \nrightarrow B$ the multiplication $\mu_J\colon T^2J \Rightarrow TJ$ is given as follows: if $\underline{\underline{x}} = (\underline{x}_1,\ldots,\underline{x}_n)$ and $\underline{\underline{y}} = (\underline{y}_1,\ldots,\underline{y}_n)$ are double sequences of equal length, such that $|\underline{x}_i| = m_i = |\underline{y}_i|$ for all $i = 1,\ldots,n$, then $(\mu_J)_{(\underline{\underline{x}},\underline{\underline{y}})}$ is the $\mathcal{V}$-isomorphism

$$
(T^2J)(\underline{\underline{x}},\underline{\underline{y}}) = \bigotimes_{i=1}^{n}\Big(\bigotimes_{j=1}^{m_i} J(x_{ij},y_{ij})\Big) \cong \bigotimes_{i=1}^{m_1+\cdots+m_n} J\big((\mu\underline{\underline{x}})_i,(\mu\underline{\underline{y}})_i\big) = (TJ)(\mu\underline{\underline{x}},\mu\underline{\underline{y}}); \qquad (8)
$$

in all other cases it is the unique $\mathcal{V}$-map $\emptyset \to (TJ)(\mu\underline{\underline{x}},\mu\underline{\underline{y}})$. Finally, the unit $\eta_J\colon J \Rightarrow TJ$ is given by the isomorphisms $J(x,y) \cong (TJ)(\eta x,\eta y)$; this completes the description of the pseudo double monad T.

We remark that T can be modified into the following lax double monad on $\mathcal{V}$-Mat, which we denote T_Σ, by introducing 'twists' in the images TJ, as follows:

$$
T_\Sigma J(\underline{x},\underline{y}) = \begin{cases} \coprod_{\sigma\in\Sigma_n} \bigotimes_{i=1}^{n} J(x_{\sigma i},y_i) & \text{if } |\underline{x}| = n = |\underline{y}|; \\ \emptyset & \text{otherwise,} \end{cases}
$$

where Σ_n denotes the group of permutations of the set $\{1,\ldots,n\}$. The double monad T_Σ induces the 'free symmetric strict monoidal $\mathcal{V}$-category'-monad on $\mathcal{V}$-Prof; see Example 3.11.

Remember that monoids and bimodules in an equipment $\mathcal{K}$ form again an equipment $\mathsf{Mod}(\mathcal{K})$ whenever $\mathcal{K}$ has local reflexive coequalisers, that is each category $H(\mathcal{K})(A, B)$ has reflexive coequalisers which are preserved by $\odot$ on both sides; see Proposition 2.13. In the proposition below, which is Proposition 11.12 of [Shu08], $\mathsf{qEquip_l} \subset \mathsf{Dbl_l}$ and $\mathsf{qEquip_{nl}} \subset \mathsf{Dbl_{nl}}$ denote the full sub-2-categories consisting of equipments that have local reflexive coequalisers.

3.10. PROPOSITION. [Shulman] *The assignment* $\mathcal{K} \mapsto \mathsf{Mod}(\mathcal{K})$ *extends to a 2-functor* $\mathsf{Mod}: \mathsf{qEquip_l} \to \mathsf{qEquip_{nl}}$, *whose image of a pseudo double functor* $F: \mathcal{D} \to \mathcal{E}$ *is again pseudo whenever* F *preserves local reflexive coequalisers.*

By applying Mod to the 'free monoid'-monad on $\mathcal{V}$-Mat, of Example 3.9, we obtain the 'free strict monoidal $\mathcal{V}$-category'-monad on $\mathcal{V}$-Prof, as follows; this double monad, whose algebras are monoidal $\mathcal{V}$-categories (see Example 3.20), will be our main example.

3.11. EXAMPLE. Let $\mathcal{V}$ be a cocomplete symmetric monoidal category whose tensor product preserves colimits on both sides. We denote the image $\mathsf{Mod}(T)$ of the 'free monoid'-monad on $\mathcal{V}$-Mat, of Example 3.9, again by T; it is the *'free strict monoidal $\mathcal{V}$-category'-monad*, which is given as follows. Its image of a $\mathcal{V}$-category A is the free strict monoidal $\mathcal{V}$-category generated by A, that is $TA = \coprod_{n \geq 0} A^{\otimes n}$; likewise its image of a $\mathcal{V}$-functor $f: A \to C$ is given by $Tf = \coprod_{n \geq 0} f^{\otimes n}$. In detail, the objects of TA are (possibly empty) sequences $\underline{x} = (x_1, \ldots, x_n)$ of objects in A, while its hom-objects $(TA)(\underline{x}, \underline{y})$ are given by the tensor products $\bigotimes_{i=1}^{n} A(x_i, y_i)$ if $|\underline{x}| = |\underline{y}|$, and $\emptyset$ otherwise. The functor Tf simply applies f coordinatewise.

The assignments $A \mapsto TA$ and $f \mapsto Tf$ extend to a double functor T on $\mathcal{V}$-Prof as follows. Its image of a profunctor $J: A \nrightarrow B$ is given by

$$TJ(\underline{x}, \underline{y}) = \begin{cases} \bigotimes_{i=1}^{n} J(x_i, y_i) & \text{if } |\underline{x}| = n = |\underline{y}|; \\ \emptyset & \text{otherwise,} \end{cases}$$

on which the hom-objects of TA and TB act coordinatewise. It is clear that T is normal, that is $T1_A = 1_{TA}$ for all $\mathcal{V}$-categories A. The invertible compositors $T_\odot: TJ \odot TH \cong T(J \odot H)$, for composable $\mathcal{V}$-profunctors J and H, are induced by those of the 'free monoid'-monad on $\mathcal{V}$-Mat; that they are again invertible follows, by Proposition 3.10, from the fact that the latter preserves local reflexive coequalisers. This is a direct consequence of the fact that a tensor product of reflexive coequalisers in $\mathcal{V}$ forms again a reflexive coequaliser; see e.g. Corollary A1.2.12 of [Joh02] for the dual result. This completes the description of the normal pseudo double endofunctor T. The multiplication and unit, that make T into a double monad, are induced by those of the 'free monoid'-monad in the obvious way.

Recall that, at the end of Example 3.9, we briefly considered a lax double monad T_Σ on $\mathcal{V}$-Mat, that was obtained by introducing twists in the images TJ of the 'free monoid'-monad. Applying Mod to T_Σ, we obtain the *'free symmetric strict monoidal $\mathcal{V}$-category'-monad* on $\mathcal{V}$-Prof, whose algebras are symmetric monoidal $\mathcal{V}$-categories; see

Example 3.21. Again denoting it T_Σ, its image on a $\mathcal{V}$-category A is given as follows: the objects of $T_\Sigma A$ are (possibly empty) sequences of objects in A, while its hom-objects are given by

$$T_\Sigma A(\underline{x}, \underline{y}) = \begin{cases} \coprod_{\sigma \in \Sigma_n} \bigotimes_{i=1}^n A(x_{\sigma i}, y_i) & \text{if } |\underline{x}| = n = |\underline{y}|; \\ \emptyset & \text{otherwise.} \end{cases}$$

Its action on $\mathcal{V}$-profunctors is given analogously; that T_Σ is a normal pseudo double monad on $\mathcal{V}$-Prof is shown in Proposition 3.23 of [Kou13].

In closing this subsection we briefly consider monoidal double categories, and study pointwise Kan extensions along tensor products of horizontal morphisms. The following is Definition 9.1 of [Shu08].

3.12. DEFINITION. A *monoidal double category* is a double category $\mathcal{K}$ equipped with a pseudo double functor $\otimes \colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and an object $1 \in \mathcal{K}$, together with invertible double transformations $\otimes \circ (\otimes \times \mathrm{id}) \cong \otimes \circ (\mathrm{id} \times \otimes)$ and $1 \otimes - \cong \mathrm{id} \cong - \otimes 1$ which satisfy the usual coherence axioms.

We abbreviate by $1 = 1_1 \colon 1 \nrightarrow 1$ the horizontal unit for 1. For an unraveling of the above definition we refer to Section 9 of [Shu08]; for us it suffices to notice the 'interchange' isomorphisms

$$\otimes_\odot \colon (J \otimes H) \odot (K \otimes L) \cong (J \odot K) \otimes (H \odot L),$$

which form the invertible compositors of $\otimes$.

3.13. EXAMPLE. Let $\mathcal{V}$ be a cocomplete monoidal category whose tensor product preserves colimits on both sides, so that $\mathcal{V}$-profunctors form a double category $\mathcal{V}$-Prof as we saw in Example 2.15. If $\mathcal{V}$ is symmetric monoidal then we can form tensor products of $\mathcal{V}$-categories and $\mathcal{V}$-functors; see e.g. Section 1.4 of [Kel82]. It is straightforward to extend these tensor products to a monoidal structure on $\mathcal{V}$-Prof, in which the tensor product of $\mathcal{V}$-profunctors $J \otimes H \colon A \otimes C \nrightarrow B \otimes D$ is given by the $\mathcal{V}$-objects $(J \otimes H)\big((x,z)(y,w)\big) = J(x,y) \otimes H(z,w)$, and whose unit is the $\mathcal{V}$-category 1 consisting of a single object $*$ and hom-object $1(*,*) = 1$.

We consider pointwise Kan extensions along tensor products of horizontal morphisms. The following is a direct consequence of Proposition 2.19.

3.14. PROPOSITION. *Consider horizontally composable cells*

$$
\begin{array}{ccccc}
A \otimes C & \xrightarrow{J \otimes 1_C} & B \otimes C & \xrightarrow{1_B \otimes H} & B \otimes D \\
{\scriptstyle s}\downarrow & \Downarrow\gamma & \downarrow{\scriptstyle r} & \Downarrow\varepsilon & \downarrow{\scriptstyle d} \\
M & = & M & = & M
\end{array}
$$

in a monoidal double category and suppose that ε defines r as the pointwise right Kan extension of d along $1_B \otimes H$. Then γ defines s as the pointwise right Kan extension of r

along $J \otimes 1_C$ precisely if the composite

$$J \otimes H \overset{\mathfrak{r}^{-1} \otimes \mathfrak{l}^{-1}}{\cong} (J \odot 1_B) \otimes (1_C \odot H) \overset{\otimes_{\odot}^{-1}}{\cong} (J \otimes 1_C) \odot (1_B \otimes H) \overset{\gamma \odot \varepsilon}{\Longrightarrow} 1_M$$

defines s as the pointwise right Kan extension of d along $J \otimes H$.

Lastly we consider Kan extensions along $\mathcal{V}$-profunctors of the form $J \otimes 1_C$, in the monoidal double category $\mathcal{V}$-Prof.

$$
\begin{array}{ccc}
A \otimes C & \overset{J \otimes 1_C}{\longrightarrow} & B \otimes C \\
{\scriptstyle s} \downarrow & \Downarrow \gamma & \downarrow {\scriptstyle d} \\
M & = & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \overset{J}{\longrightarrow} & B \\
{\scriptstyle s(-,z)} \downarrow & \Downarrow \gamma_z & \downarrow {\scriptstyle d(-,z)} \\
M & = & M
\end{array}
$$

Given a transformation γ in $\mathcal{V}$-Prof as on the left above and an object $z \in C$, we write γ_z for the transformation on the right that is given by the composite below; here we have identified z with the corresponding $\mathcal{V}$-functor $z \colon 1 \to C$.

$$J \cong J \otimes 1 \overset{\mathrm{id} \otimes 1_z}{\Longrightarrow} J \otimes 1_C \overset{\gamma}{\Rightarrow} 1_M.$$

3.15. PROPOSITION. *Let $\mathcal{V}$ be a cocomplete symmetric monoidal category whose tensor product preserves colimits on both sides, so that $\mathcal{V}$-profunctors form a monoidal double category $\mathcal{V}$-Prof. A transformation γ, as on the left above, defines s as the pointwise right Kan extension of d along $J \otimes 1_C$ precisely if the transformations γ_z on the right, for each $z \in C$, define $s(-,z)$ as the pointwise right Kan extension of $d(-,z)$ along J. An analogous result holds for pointwise right Kan extensions along the $\mathcal{V}$-profunctor $1_C \otimes J$.*

PROOF. For any $x \in A$ we write δ_x for the cartesian cell defining the restriction $J(x, \mathrm{id})$. Likewise for any $z \in C$ we write ε_z for the cartesian cell defining the companion $z_* \colon 1 \nrightarrow C$ of $z \colon 1 \to C$; remember from the discussion following Example 2.8 that the unit cell $1_z \colon 1 \Rightarrow 1_C$ factors as $1_z = \varepsilon_z \circ \eta_z$, where $\eta_z \colon 1 \Rightarrow z_*$ is an opcartesian cell that defines z_* as the extension of $1 \colon 1 \nrightarrow 1$ along id_1 and z.

Notice that, by Proposition 3.7, the tensor product $\delta_x \otimes \varepsilon_z \colon J(x, \mathrm{id}) \otimes z_* \Rightarrow J \otimes 1_C$ is a cartesian cell defining the restriction of $J \otimes 1_C$ along $x \otimes z \colon 1 \otimes 1 \to A \otimes C$. It follows from Proposition 2.24 that we may equivalently prove that, for each pair (x, z), the composite $\gamma_{(x,z)} = \gamma \circ (\delta_x \otimes \varepsilon_z)$ defines $s(x, z)$ as the $J(x, \mathrm{id}) \otimes z_*$-weighted limit of d if and only if the composite $\gamma_z \circ \delta_x$ defines $s(x, z)$ as the $J(x, \mathrm{id})$-weighted limit of $d(-, z)$.

To see this we use that $\mathrm{id}_{J(x,\mathrm{id})} \otimes \eta_z$ is pointwise initial, because tensor products of opcartesian cells are opcartesian (Proposition 3.7) and by Proposition 2.34(a). By regarding weighted limits as pointwise Kan extensions (Proposition 2.24), it follows that each $\gamma_{(x,z)}$ defines $s(x, z)$ as the $J(x, \mathrm{id}) \otimes z_*$-weighted limit of d precisely if the composite $\gamma_{(x,z)} \circ (\mathrm{id} \otimes \eta_z)$ defines $s(x, z)$ as the $J(x, \mathrm{id}) \otimes 1$-weighted limit of $d(-, z)$. To complete the proof we notice that $\gamma_{(x,z)} \circ (\mathrm{id} \otimes \eta_z) = \gamma \circ (\delta_x \otimes 1_z)$ which, after composition with $J(x, \mathrm{id}) \cong J(x, \mathrm{id}) \otimes 1$, coincides with $\gamma_z \circ \delta_x$. ■

3.16. EILENBERG-MOORE DOUBLE CATEGORIES. In this subsection we define, for a normal oplax double monad T, several weakenings of the 'Eilenberg-Moore double category' associated to T, that was introduced by Grandis and Paré in [GP04]. We start by recalling from [Str74] the definitions of lax algebras for strict 2-monads, as well as several notions of weak morphism between such algebras, and the cells between those. The strict 2-monads of interest to us are those of the form $V(T)$, where T is a normal oplax double monad.

Consider a strict 2-monad $T = (T, \mu, \eta)$ on a 2-category $\mathcal{C}$, consisting of a strict 2-functor $T\colon \mathcal{C} \to \mathcal{C}$ and 2-natural transformations $\mu\colon T^2 \Rightarrow T$ and $\eta\colon \mathrm{id}_\mathcal{C} \Rightarrow T$ satisfying the usual axioms: $\mu \circ T\mu = \mu \circ \mu T$ and $\mu \circ T\eta = \mathrm{id}_T = \mu \circ \eta T$.

3.17. DEFINITION. A *lax T-algebra* is a quadruple $A = (A, a, \bar{a}, \hat{a})$ consisting of an object A equipped with a morphism $a\colon TA \to A$, its *structure map*, and cells $\bar{a}\colon a \circ Ta \Rightarrow a \circ \mu_A$ and $\hat{a}\colon \mathrm{id}_A \Rightarrow a \circ \eta_A$, its *associator* and *unitor*, that satisfy the following three coherence conditions.

Dually, in the notion of a *colax T-algebra* $A = (A, a, \bar{a}, \hat{a})$ the directions of the associator $\bar{a}\colon a \circ \mu_A \Rightarrow a \circ Ta$ and the unitor $\hat{a}\colon a \circ \eta_A \Rightarrow \mathrm{id}_A$ are reversed. A lax or colax T-algebra A is called a *pseudo T-algebra* if the cells $\bar{a}$ and $\hat{a}$ are isomorphisms; if they are identities then A is called *strict*.

3.18. DEFINITION. Given lax T-algebras $A = (A, a, \bar{a}, \hat{a})$ and $C = (C, c, \bar{c}, \hat{c})$, a *lax T-morphism* $A \to C$ is a morphism $f\colon A \to C$ that is equipped with a *structure cell* $\bar{f}\colon c \circ Tf \Rightarrow f \circ a$, which satisfies the following associativity and unit axioms.

Dually a *colax T-morphism* $f\colon A \to C$ is equipped with a structure cell $\bar{f}\colon f \circ a \Rightarrow c \circ Tf$ such that

$$(\bar{c}.T^2 f) \circ (c.T\bar{f}) \circ (\bar{f}.Ta) = (\bar{f}.\mu_A) \circ (f.\bar{a}) \qquad \text{and} \qquad (\bar{f}.\eta_A) \circ (f.\hat{a}) = \hat{c}.f.$$

Lax or colax T-morphisms whose structure cell $\bar{f}$ is invertible are called *pseudo T-morphisms*; notice that, for an invertible cell $\bar{f}\colon c \circ Tf \Rightarrow f \circ a$, the pair $(f, \bar{f})$ forms a lax T-morphism if and only if $(f, \bar{f}^{-1})$ forms a colax T-morphism. Pseudo T-morphisms are called *strict* whenever $\bar{f}$ is the identity cell.

This leaves the definition of cells between T-morphisms.

3.19. DEFINITION. Given lax T-morphisms f and $g\colon A \to C$, a *T-cell* between f and g is a cell $\phi\colon f \Rightarrow g$ satisfying

Likewise a T-cell between colax T-morphisms f and $g\colon A \to C$ is a cell $\phi\colon f \Rightarrow g$ satisfying $(c.T\phi) \circ \bar{f} = \bar{g} \circ (\phi.a)$.

For any strict 2-monad T lax T-algebras, lax T-morphisms and T-cells form a 2-category $\mathsf{Alg}_l(T)$, and we denote by $\mathsf{Alg}_{ps}(T)$ the sub-2-category of lax T-algebras, pseudo T-morphisms and all T-cells between them. Likewise lax T-algebras, colax T-morphisms and T-cells form a 2-category $\mathsf{Alg}_c(T)$.

3.20. EXAMPLE. Let T be the 'free strict monoidal $\mathcal{V}$-category'-monad described in Example 3.11. A lax $V(T)$-algebra $A = (A, \otimes, \mathfrak{a}, \mathfrak{i})$ is a *lax monoidal $\mathcal{V}$-category*: it consists of a $\mathcal{V}$-category A equipped with a $\mathcal{V}$-functor $\otimes\colon TA \to A$ that defines the tensor product $(x_1 \otimes \cdots \otimes x_n) = \otimes(\underline{x})$ of each (possibly empty) sequence $\underline{x}$ in TA, as well as $\mathcal{V}$-natural maps

$$\mathfrak{a}\colon \big((x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{n1} \otimes \cdots \otimes x_{nm_n})\big) \to (x_{11} \otimes \cdots \otimes x_{nm_n}),$$

for each double sequence $\underline{\underline{x}} \in T^2A$, and $\mathfrak{i}\colon x \to (x)$, where $x \in A$. These transformations satisfy certain coherence conditions, see Definition 3.1.1 of [Lei04] for example. A lax monoidal $\mathcal{V}$-category A whose structure transformations $\mathfrak{a}$ and $\mathfrak{i}$ are invertible is simply called a *monoidal $\mathcal{V}$-category*; if they are identities then A is called a *strict* monoidal $\mathcal{V}$-category. These notions of monoidal category are often called *unbiased*, referring to the fact that tensor products of all arities are part of their structure, in contrast to the classical notion, which is biased towards the nullary (unit) and binary tensor products.

Lax $V(T)$-morphisms are *lax monoidal $\mathcal{V}$-functors*, that is $\mathcal{V}$-functors $f\colon A \to C$ equipped with $\mathcal{V}$-natural maps

$$f_\otimes\colon (fx_1 \otimes \cdots \otimes fx_n) \to f(x_1 \otimes \cdots \otimes x_n),$$

its *compositors*, that are compatible with the coherence transformations of A and C; see Definition 3.1.3 of [Lei04]. Dually, in the notion of colax $V(T)$-morphism the direction of the compositors is reversed; such $\mathcal{V}$-functors are called *colax monoidal $\mathcal{V}$-functors*. Lax or colax monoidal $\mathcal{V}$-functors whose compositors are invertible are simply called *monoidal $\mathcal{V}$-functors*; those with identities as compositors are called *strict* monoidal $\mathcal{V}$-functors.

A $V(T)$-cell $f \Rightarrow g$ between lax monoidal $\mathcal{V}$-functors f and g, called a *monoidal transformation*, is simply a $\mathcal{V}$-natural transformation $\phi\colon f \Rightarrow g$ that is compatible with the compositors $f_\otimes$ and $g_\otimes$: the diagrams below commute for each sequence $\underline{x} \in TA$. Monoidal transformations between colax monoidal $\mathcal{V}$-functors are defined analogously.

$$
\begin{array}{ccc}
(fx_1 \otimes \cdots \otimes fx_n) & \xrightarrow{\;\;f_\otimes\;\;} & f(x_1 \otimes \cdots \otimes x_n) \\
{\scriptstyle (\phi_{x_1} \otimes \cdots \otimes \phi_{x_n})}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi_{(x_1 \otimes \cdots \otimes x_n)}} \\
(gx_1 \otimes \cdots \otimes gx_n) & \xrightarrow[\;\;g_\otimes\;\;]{} & g(x_1 \otimes \cdots \otimes x_n)
\end{array}
$$

3.21. EXAMPLE. Consider the 'free symmetric strict monoidal $\mathcal{V}$-category'-monad T_Σ, that was briefly considered at the end of Example 3.11. The notions of lax $V(T_\Sigma)$-algebra and lax $V(T_\Sigma)$-morphism are that of symmetric lax monoidal category and symmetric lax

monoidal functor, as follows. A *symmetric lax monoidal $\mathcal{V}$-category* A is a lax monoidal $\mathcal{V}$-category A that is equipped with $\mathcal{V}$-natural symmetries

$$\mathfrak{s}_\sigma \colon (x_1 \otimes \cdots \otimes x_n) \to (x_{\sigma 1} \otimes \cdots \otimes x_{\sigma n}),$$

for all $\sigma \in \Sigma_n$ and $\underline{x} \in A^{\otimes n}$, that are functorial in the sense that $\mathfrak{s}_\tau \circ \mathfrak{s}_\sigma = \mathfrak{s}_{\tau \circ \sigma}$ and $\mathfrak{s}_{\mathrm{id}} = \mathrm{id}$, and that make the following diagrams commute. Firstly, given permutations $\sigma_i \in \Sigma_{m_i}$, the diagrams

$$
\begin{array}{ccc}
((x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{n1} \otimes \ldots \otimes x_{nm_n})) & \xrightarrow{\;\mathfrak{a}\;} & (x_{11} \otimes \cdots \otimes x_{nm_n}) \\
{\scriptstyle(\mathfrak{s}_{\sigma_1} \otimes \cdots \otimes \mathfrak{s}_{\sigma_n})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathfrak{s}_{(\sigma_1,\ldots,\sigma_n)}} \\
((x_{1\sigma_1 1} \otimes \cdots \otimes x_{1\sigma_1 m_1}) \otimes \cdots \otimes (x_{n\sigma_n 1} \otimes \cdots \otimes x_{n\sigma_n m_n})) & \xrightarrow[\;\mathfrak{a}\;]{} & (x_{1\sigma_1 1} \otimes \cdots \otimes x_{n\sigma_n m_n})
\end{array}
$$

commute where, writing $N = m_1 + \cdots + m_n$, the permutation $(\sigma_1, \ldots, \sigma_n) \in \Sigma_N$ denotes the disjoint union of the σ_i. Secondly, for each $\tau \in \Sigma_n$, the diagrams

$$
\begin{array}{ccc}
((x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{n1} \otimes \ldots \otimes x_{nm_n})) & \xrightarrow{\;\mathfrak{a}\;} & (x_{11} \otimes \cdots \otimes x_{nm_n}) \\
{\scriptstyle\mathfrak{s}_\tau}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathfrak{s}_{\tau_{(m_1,\ldots,m_n)}}} \\
((x_{\tau 11} \otimes \cdots \otimes x_{\tau 1 m_{\tau 1}}) \otimes \cdots \otimes (x_{\tau n1} \otimes \cdots \otimes x_{\tau n m_{\tau n}})) & \xrightarrow[\;\mathfrak{a}\;]{} & (x_{\tau 11} \otimes \cdots \otimes x_{\tau n m_{\tau n}})
\end{array}
$$

commute, where the 'block permutation' $\tau_{(m_1,\ldots,m_n)} \in \Sigma_N$ permutes the n blocks of the $(m_{\tau 1}, \ldots, m_{\tau n})$-partition of $\{1, \ldots, N\}$ exactly like τ permutes the elements of $\{1, \ldots, n\}$, while preserving the ordering of the elements in each block.

A lax monoidal functor $f \colon A \to C$ between symmetric lax monoidal categories is called *symmetric* whenever the diagrams below commute. Transformations between symmetric lax monoidal functors are simply monoidal transformations.

$$
\begin{array}{ccc}
(fx_1 \otimes \cdots \otimes fx_n) & \xrightarrow{\;f_\otimes\;} & f(x_1 \otimes \cdots \otimes x_n) \\
{\scriptstyle\mathfrak{s}_\sigma}\Big\downarrow & & \Big\downarrow{\scriptstyle f\mathfrak{s}_\sigma} \\
(fx_{\sigma 1} \otimes \cdots \otimes fx_{\sigma n}) & \xrightarrow[\;f_\otimes\;]{} & f(x_{\sigma 1} \otimes \cdots \otimes x_{\sigma n})
\end{array}
\qquad (9)
$$

Having recalled, for a normal oplax double monad T on $\mathcal{K}$, the notions of lax $V(T)$-algebra, weak $V(T)$-morphism and $V(T)$-cell in the 2-category $V(\mathcal{K})$, we now introduce the notion of horizontal T-morphism between lax $V(T)$-algebras and that of a cell between such morphisms. Both are straightforward generalisations of notions that were introduced by Grandis and Paré in Section 7.1 of [GP04], where $V(T)$-algebras were assumed to be strict. Afterwards, in Proposition 3.26, we will show that lax $V(T)$-algebras, weak $V(T)$-morphisms, horizontal T-morphisms and their cells form a double category.

3.22. DEFINITION. Let $T = (T, \mu, \eta)$ be a normal oplax double monad on a double category $\mathcal{K}$. Given lax $V(T)$-algebras $A = (A, a, \bar{a}, \hat{a})$ and $B = (B, b, \bar{b}, \hat{b})$, a *horizontal T-morphism* $A \nrightarrow B$ is a horizontal morphism $J \colon A \nrightarrow B$ equipped with a *structure cell*

$$
\begin{array}{ccc}
TA & \xrightarrow{TJ} & TB \\
a \downarrow & \Downarrow \bar{J} & \downarrow b \\
A & \xrightarrow[J]{} & B
\end{array}
$$

satisfying the following associativity and unit axioms.

$$
\begin{array}{ccccc}
T^2A \xrightarrow{T^2J} T^2B == T^2B & & T^2A == T^2A \xrightarrow{T^2J} T^2B \\
\end{array}
$$

3.23. EXAMPLE. Consider the double monad T of free strict monoidal $\mathcal{V}$-categories described in Example 3.11; we will call a horizontal T-morphism $A \nrightarrow B$ between lax monoidal $\mathcal{V}$-categories A and B a *monoidal $\mathcal{V}$-profunctor*. It consists of a $\mathcal{V}$-profunctor $J \colon A \nrightarrow B$ equipped with $\mathcal{V}$-maps

$$
J_\otimes \colon J(x_1, y_1) \otimes \cdots \otimes J(x_n, y_n) \to J\big((x_1 \otimes \cdots \otimes x_n), (y_1 \otimes \cdots \otimes y_n)\big)
$$

which are compatible with the actions of TA and TB and satisfy the following coherence conditions. The unit axiom states that all diagrams of the form

$$
\begin{array}{ccc}
J(x, y) & \xrightarrow{\rho_i} & J\big(x, (y)\big) \\
& J_\otimes \searrow \quad \nearrow \lambda_i & \\
& J\big((x), (y)\big) &
\end{array}
$$

commute, while the associativity axiom means that those of the form

$$\big(J(x_{11},y_{11}) \otimes \cdots \otimes J(x_{1m_1},y_{1m_1})\big) \otimes \cdots \otimes \big(J(x_{n1},y_{n1}) \otimes \cdots \otimes J(x_{nm_n},y_{nm_n})\big)$$

$$J_\otimes \otimes \cdots \otimes J_\otimes \qquad\qquad\qquad \cong$$

$$J\big((x_{11} \otimes \cdots \otimes x_{1m_1}),(y_{11} \otimes \cdots \otimes y_{1m_1})\big) \otimes \cdots \\ \otimes J\big((x_{n1} \otimes \cdots \otimes x_{nm_n}),(y_{n1} \otimes \cdots \otimes y_{nm_n})\big)$$

$$J(x_{11},y_{11}) \otimes \cdots \otimes J(x_{nm_n},y_{nm_n})$$

$$J_\otimes \qquad\qquad\qquad\qquad J_\otimes$$

$$J\Big(\big((x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{n1} \otimes \cdots \otimes x_{nm_n})\big), \\ \big((y_{11} \otimes \cdots \otimes y_{1m_1}) \otimes \cdots \otimes (y_{n1} \otimes \cdots \otimes y_{nm_n})\big)\Big)$$

$$J\big((x_{11} \otimes \cdots \otimes x_{nm_n}),(y_{11} \otimes \cdots \otimes y_{nm_n})\big)$$

$$\rho_\mathfrak{a} \qquad\qquad\qquad \lambda_\mathfrak{a}$$

$$J\Big(\big((x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{n1} \otimes \cdots \otimes x_{nm_n})\big),(y_{11} \otimes \cdots \otimes y_{nm_n})\Big)$$

commute. If A and B are symmetric lax monoidal $\mathcal{V}$-categories then we call J a *symmetric monoidal* $\mathcal{V}$-*profunctor* whenever the diagrams below commute.

$$J(x_1,y_1) \otimes \cdots \otimes J(x_n,y_n) \xrightarrow{\ \sigma\ } J(x_{\sigma 1},y_{\sigma 1}) \otimes \cdots \otimes J(x_{\sigma n},y_{\sigma n})$$

$$J_\otimes \qquad\qquad\qquad\qquad J_\otimes$$

$$J\big((x_1 \otimes \cdots \otimes x_n),(y_1 \otimes \cdots \otimes y_n)\big) \qquad J\big((x_{\sigma 1} \otimes \cdots \otimes x_{\sigma n}),(y_{\sigma 1} \otimes \cdots \otimes y_{\sigma n})\big)$$

$$\rho_{\mathfrak{s}_\sigma} \qquad\qquad \lambda_{\mathfrak{s}_\sigma}$$

$$J\big((x_1 \otimes \cdots \otimes x_n),(y_{\sigma 1} \otimes \cdots \otimes y_{\sigma n})\big)$$

The following definition introduces general T-cells; we will see in Proposition 3.26 that it generalises the vertical $V(T)$-cells of Definition 3.19.

3.24. DEFINITION. Let T be a normal oplax double monad on a double category $\mathcal{K}$. Consider a cell ϕ as on the left below, where f and g are lax $V(T)$-morphisms while J and K are horizontal T-morphisms. We call ϕ a T-*cell* whenever the identity on the right is satisfied.

$$
\begin{array}{ccc}
A \xrightarrow{J} B & TA \xrightarrow{TJ} TB == TB & TA == TA \xrightarrow{TJ} TB \\
f \downarrow\ \Downarrow\phi\ \downarrow g & Tf \downarrow\ \Downarrow T\phi\ \downarrow Tg\quad \downarrow b & Tf \downarrow\quad a \downarrow\ \Downarrow\bar{J}\ \downarrow b \\
C \xrightarrow{K} D & TC \xrightarrow{TK} TD\ \Downarrow\bar{g}\ B\ =\ TC\ \Downarrow\bar{f}\ A \xrightarrow{J} B \\
& c \downarrow\ \Downarrow\bar{K}\ \downarrow d\quad \downarrow g & c \downarrow\quad f \downarrow\ \Downarrow\phi\ \downarrow g \\
& C \xrightarrow{K} D == D & C == C \xrightarrow{K} D
\end{array}
$$

Similarly, in the case that f and g are colax $V(T)$-morphisms, we call ϕ a T-cell whenever $\bar{f} \odot (\bar{K} \circ T\phi) = (\phi \circ \bar{J}) \odot \bar{g}$.

3.25. EXAMPLE. Let T be the double monad of 'free strict monoidal categories' described in Example 3.11, and consider a transformation ϕ in $\mathcal{V}$-Prof, as on the left above, where f and g are lax monoidal $\mathcal{V}$-functors (Example 3.20) while J and K are monoidal $\mathcal{V}$-profunctors (Example 3.23). Remember that ϕ is given by a family of $\mathcal{V}$-maps $\phi\colon J(x,y) \to K(fx, gy)$ that is compatible with the actions of A and B. The transformation ϕ is a T-cell if the following diagram commutes; in that case we call ϕ a *monoidal transformation*.

$$\begin{array}{ccc}
J(x_1, y_1) \otimes \cdots \otimes J(x_n, y_n) & \xrightarrow{\phi \otimes \cdots \otimes \phi} & K(fx_1, gy_1) \otimes \cdots \otimes K(fx_n, gy_n) \\
\downarrow^{J_\otimes} & & \downarrow^{K_\otimes} \\
J\big((x_1 \otimes \cdots \otimes x_n), (y_1 \otimes \cdots \otimes y_n)\big) & & K\big((fx_1 \otimes \cdots \otimes fx_n), (gy_1 \otimes \cdots \otimes gy_n)\big) \\
\downarrow^{\phi} & & \downarrow^{\rho_{g_\otimes}} \\
K\big(f(x_1 \otimes \cdots \otimes x_n), g(y_1 \otimes \cdots \otimes y_n)\big) & \xrightarrow[\lambda_{f_\otimes}]{} & K\big((fx_1 \otimes \cdots \otimes fx_n), g(y_1 \otimes \cdots \otimes y_n)\big)
\end{array}$$

A monoidal transformation $\phi\colon J \Rightarrow K(f, g)$ where f and g are colax monoidal functors is defined similarly.

Given a normal oplax double monad T, we now show that lax $V(T)$-algebras, any choice of weak $V(T)$-morphisms, horizontal T-morphisms and T-cells form a double category $\mathsf{Alg}_{\mathrm{w}}(T)$, such that $V(\mathsf{Alg}_{\mathrm{w}}(T)) = \mathsf{Alg}_{\mathrm{w}}(V(T))$. Consequently we will call lax $V(T)$-algebras simply *lax T-algebras*, while weak $V(T)$-morphisms will be called *weak vertical T-morphisms*.

3.26. PROPOSITION. *Let T be a normal oplax double monad on a double category $\mathcal{K}$, and let 'weak' mean either 'colax' or 'lax'. The double category structure of $\mathcal{K}$ lifts to make lax $V(T)$-algebras, weak $V(T)$-morphisms, horizontal T-morphisms and T-cells into a double category $\mathsf{Alg}_{\mathrm{w}}(T)$, such that $V(\mathsf{Alg}_{\mathrm{w}}(T)) = \mathsf{Alg}_{\mathrm{w}}(V(T))$.*

We denote by $\mathsf{Alg}_{\mathrm{ps}}(T)$ the sub-double category of $\mathsf{Alg}_{\mathrm{l}}(T)$ consisting of lax T-algebras, pseudo $V(T)$-morphisms, horizontal T-morphisms and all T-cells whose vertical source and target are pseudo $V(T)$-morphisms. Clearly $V(\mathsf{Alg}_{\mathrm{l}}(T)) = \mathsf{Alg}_{\mathrm{l}}(V(T))$ implies that $V(\mathsf{Alg}_{\mathrm{ps}}(T)) = \mathsf{Alg}_{\mathrm{ps}}(V(T))$.

PROOF. We shall show that the double category structure of $\mathcal{K}$ lifts to make lax $V(T)$-algebras, lax $V(T)$-morphisms, horizontal T-morphisms and T-cells into a double category $\mathsf{Alg}_{\mathrm{l}}(T)$; the case of $\mathsf{Alg}_{\mathrm{c}}(T)$ is similar. First notice that the equality $V(\mathsf{Alg}_{\mathrm{w}}(T)) = \mathsf{Alg}_{\mathrm{w}}(V(T))$ determines the structure on the category $(\mathsf{Alg}_{\mathrm{w}}(T))_{\mathrm{v}}$ completely; in particular the algebra structure on the vertical composite $h \circ f$ of lax $V(T)$-morphisms $f\colon A \to C$ and $h\colon C \to E$ must be given by the composite $\overline{h \circ f} = (\bar{h} \circ 1_{Tf}) \odot (1_h \circ \bar{f})$.

To also lift the horizontal structure of $\mathcal{K}$ to $\mathsf{Alg}_{\mathrm{l}}(T)$ it remains to define structure cells for each composite $J \odot H$, of horizontal T-morphisms $J\colon A \nrightarrow B$ and $H\colon B \nrightarrow C$, as well as for the horizontal units $1_A\colon A \nrightarrow A$, in a way such that any composite of T-cells forms

again a T-cell, the horizontal units 1_f form T-cells, and the associators and unitors of $\mathcal{K}$ form T-cells. We define the structure cell of $J \odot H$ by

$$\overline{J \odot H} = \begin{array}{ccc} TA \xrightarrow{T(J \odot H)} & & TC \\ \| & \Downarrow T_\odot & \| \\ TA \xrightarrow{TJ} TB \xrightarrow{TH} & & TC \\ a\downarrow \quad \Downarrow \bar{J} \quad \downarrow b \quad \Downarrow \bar{H} \quad \downarrow c \\ A \xrightarrow{\quad J \quad} B \xrightarrow{\quad H \quad} C. \end{array} \tag{10}$$

That this composite satisfies the associativity axiom of Definition 3.22 is shown by the following equation of composites where, to save space, only the non-identity cells are depicted while objects and morphisms are left out. The cells $\bar{c}$ and $\bar{a}$ here are the structure cells of C and A respectively (see Definition 3.17) and the identities follow from the naturality of $T_\odot$; the associativity axioms of H and J and the fact that $(T^2)_\odot = T_\odot \circ TT_\odot$; the composition axiom of μ (see Definition 3.4). The unit axiom for $\overline{J \odot H}$ follows similarly from that for $\bar{H}$ and $\bar{J}$, as well as the composition axiom for η.

The structure cell of a horizontal unit 1_A we take to be $T1_A = 1_{TA} \overset{1_a}{\Rightarrow} 1_A$, where $a \colon TA \to A$ is the structure map of A. The associativity and unit axioms follow from the unit axioms for μ and η.

It is now easily checked that vertical and horizontal composites of T-cells form again T-cells and that, for a lax T-morphism $f \colon A \to C$, the unit cell 1_f is a T-cell. It is also readily seen that the components of the associator and unitors of $\mathcal{K}$ form T-cells, using the associativity and unit axioms for T; we conclude that, with the chosen algebra structures above, the structure of double category $\mathcal{K}$ lifts to form a double category $\mathsf{Alg}_l(T)$.

Finally notice that the T-cell axiom for a vertical cell $\phi \colon f \Rightarrow g$ coincides with the $V(T)$-cell axiom for ϕ (see Definition 3.19). We conclude that $V(\mathsf{Alg}_l(T)) = \mathsf{Alg}_l(V(T))$ as asserted. ∎

4. Kan extensions in $\mathsf{Alg}_w(T)$

In this section we study Kan extensions in the double category $\mathsf{Alg}_w(T)$. Firstly, in light of Proposition 2.18, we wonder whether $\mathsf{Alg}_w(T)$ has restrictions, so that Kan extensions in $\mathsf{Alg}_w(V(T)) = V(\mathsf{Alg}_w(T))$ correspond to Kan extensions in $\mathsf{Alg}_w(T)$. Secondly we

wonder whether $\mathsf{Alg}_{\mathrm{w}}(T)$ has opcartesian tabulations, so that we can obtain analogues of Theorem 2.29 and Proposition 2.30 for $\mathsf{Alg}_{\mathrm{w}}(T)$. As we shall see, restrictions and tabulations can often be 'lifted' along the forgetful double functors $\mathsf{Alg}_{\mathrm{w}}(T) \to \mathcal{K}$.

We start with the lifting of restrictions. Given a double functor $F \colon \mathcal{K} \to \mathcal{L}$ and morphisms $f \colon A \to C$, $K \colon C \nrightarrow D$ and $g \colon B \to D$ in $\mathcal{K}$, we say that F *lifts the restriction* $K(f, g)$ if for any cartesian cell ϕ in $\mathcal{L}$, as on the left below, there exists a unique cell ϕ' in $\mathcal{K}$, as on the right, such that $F\phi' = \phi$ and, moreover, ϕ' is cartesian.

$$
\begin{array}{ccc}
FA \xrightarrow{\;J\;} FB & \qquad\qquad & A \xrightarrow{\;J'\;} B \\
{\scriptstyle Ff}\big\downarrow \quad \Downarrow\phi \quad \big\downarrow{\scriptstyle Fg} & & {\scriptstyle f}\big\downarrow \quad \Downarrow\phi' \quad \big\downarrow{\scriptstyle g} \\
FC \xrightarrow[\;FK\;]{} FD & & C \xrightarrow[\;K\;]{} D
\end{array}
$$

4.1. PROPOSITION. *For a normal oplax double monad T on a double category $\mathcal{K}$ the following hold for the forgetful double functors $\mathsf{Alg}_{\mathrm{w}}(T) \to \mathcal{K}$, where* $\mathrm{w} \in \{\mathrm{c}, \mathrm{l}, \mathrm{ps}\}$:

(a) $\mathsf{Alg}_{\mathrm{c}}(T) \to \mathcal{K}$ *lifts restrictions* $K(f, g)$ *where g is a pseudo T-morphism;*

(b) $\mathsf{Alg}_{\mathrm{l}}(T) \to \mathcal{K}$ *lifts restrictions* $K(f, g)$ *where f is a pseudo T-morphism;*

(c) $\mathsf{Alg}_{\mathrm{ps}}(T) \to \mathcal{K}$ *lifts all restrictions.*

We conclude that if $\mathcal{K}$ is an equipment then so is $\mathsf{Alg}_{\mathrm{ps}}(T)$. Moreover, in that case $\mathsf{Alg}_{\mathrm{c}}(T)$ has all companions while, in general, it has only conjoints of pseudo T-morphisms; dually $\mathsf{Alg}_{\mathrm{l}}(T)$ has all conjoints but only companions of pseudo T-morphisms.

4.2. EXAMPLE. If $f \colon A \to C$ is a monoidal $\mathcal{V}$-functor (Example 3.20), $K \colon C \nrightarrow D$ a monoidal $\mathcal{V}$-profunctor (Example 3.23) and $g \colon B \to D$ a lax monoidal $\mathcal{V}$-functor then the restriction $K(f, g)$ admits a canonical monoidal structure that is given by the composites

$$
K(fx_1, gy_1) \otimes \cdots \otimes K(fx_n, gy_n) \xrightarrow{K_\otimes} K\big((fx_1 \otimes \cdots \otimes fx_n), (gy_1 \otimes \cdots \otimes gy_n)\big)
$$

$$
\xrightarrow{\rho_{g_\otimes} \circ \lambda_{f_\otimes^{-1}}} K\big(f(x_1 \otimes \cdots \otimes x_n), g(y_1 \otimes \cdots \otimes y_n)\big).
$$

PROOF OF PROPOSITION 4.1. We will prove (b). The inclusion $\mathsf{Alg}_{\mathrm{ps}}(T) \to \mathsf{Alg}_{\mathrm{l}}(T)$ lifts all restrictions so that (c) follows, while proving (a) is similar to proving (b).

Consider a cartesian cell ϕ in $\mathcal{K}$ as on the left below, where f is a pseudo T-morphism, g is a lax T-morphism and K is a horizontal T-morphism. We claim that the unique factorisation $\bar{J}$ of the composite on the right below, through the cartesian cell ϕ, as

shown, forms a structure cell for J.

$$
\begin{array}{ccc}
A \xrightarrow{J} B & \quad &
\end{array}
$$

(11)

We have to show that $\bar{J}$ satisfies the associativity and unit axiom of Definition 3.22. To show that the first holds consider the following equation of composites in $\mathcal{K}$, whose identities are described below. Since its left-hand and right-hand side are the left-hand and right-hand side of the associativity axiom for $\bar{J}$ postcomposed with ϕ, it implies the associativity axiom for $\bar{J}$, because factorisations through cartesian cells are unique.

The identity (i) above follows from the factorisation (11); (ii) follows from the T-image of (11), where the T-image of the left-hand side is rewritten as $T\bar{f}^{-1} \odot T(\bar{K} \circ T\phi) \odot T\bar{g}$ by using the unit axioms for T (see (7)); (iii) follows from the associativity axiom for $\bar{g}$; (iv) from the associativity axiom for $\bar{K}$; (v) from the naturality of μ and the associativity axiom for $\bar{f}$; (vi) from the factorisation (11). Showing that $\bar{J}$ satisfies the unit axiom is done analogously: it is equivalent to the identity obtained by composing it with ϕ, which follows from the factorisation (11) and the unit axioms for $\bar{g}$, $\bar{K}$ and $\bar{f}$. This completes the proof that $\bar{J}$ forms a structure cell making J into a horizontal T-morphism.

Composing (11) on the left with $\bar{f}$ we see that ϕ forms a T-cell $(J, \bar{J}) \Rightarrow (K, \bar{K})$, showing that ϕ lifts along $\mathsf{Alg}_l(T) \to \mathcal{K}$. In fact, because the factorisation (11) is unique, this lift of ϕ is unique as well. To complete the proof we have to show that ϕ forms a

cartesian T-cell. Hence consider a T-cell ψ as on the left-hand side below, where h and k are lax T-morphisms while H is a horizontal T-morphism.

$$
\begin{array}{ccc}
X \xrightarrow{H} Y & & X \xrightarrow{H} Y \\
h\downarrow \quad \downarrow k & & h\downarrow \ \ \Downarrow\psi' \ \downarrow k \\
A \ \Downarrow\psi \ B & = & A \xrightarrow{J} B \\
f\downarrow \quad \downarrow g & & f\downarrow \ \ \Downarrow\phi \ \downarrow g \\
C \xrightarrow{K} D & & C \xrightarrow{K} D
\end{array}
$$

Since ϕ is cartesian in $\mathcal{K}$ we obtain a unique factorisation ψ' of ψ as shown; we have to prove that ψ' is a T-cell. The following equation shows that the T-cell axiom for ψ' holds after composing it with ϕ, where the identities follow from (11), the factorisation above, the T-cell axiom for ψ and again the factorisation above.

Since factorisations through ϕ are unique the T-cell axiom for ψ' follows, completing the proof. ∎

We now turn to tabulations, see Definition 2.26.

4.3. PROPOSITION. *Let T be a normal oplax double monad on an equipment $\mathcal{K}$. The forgetful functors $\mathsf{Alg}_w(T) \to \mathcal{K}$ lift tabulations in the following sense. If the cell π below forms the tabulation of J in $\mathcal{K}$, where J is a horizontal T-morphism, then there exists a unique lax T-algebra structure on $\langle J\rangle$ with respect to which π_A and π_B are strict T-morphisms and π is a T-cell. Thus a T-cell, π forms the tabulation of J in $\mathsf{Alg}_w(T)$, which is opcartesian as soon as π is opcartesian in $\mathcal{K}$.*

$$
\begin{array}{ccc}
\langle J\rangle & == & \langle J\rangle \\
\pi_A\downarrow & \Downarrow\pi & \downarrow\pi_B \\
A & \xrightarrow{J} & B
\end{array}
$$

4.4. EXAMPLE. The tabulation $\langle J\rangle$ (see Example 2.27) of a monoidal (unenriched) profunctor $J\colon A \nrightarrow B$ (see Example 3.23) admits a canonical lax monoidal structure whose tensor product on objects and morphisms is given by applying the tensor products of A, J and B coordinatewise:

$$
\big((x_1, u_1, y_1) \otimes \cdots \otimes (x_n, u_n, y_n)\big) = \big((x_1 \otimes \cdots \otimes x_n), (u_1 \otimes \cdots \otimes u_n), (y_1 \otimes \cdots \otimes y_n)\big);
$$
$$
\big((p_1, q_1) \otimes \cdots \otimes (p_n, q_n)\big) = \big((p_1 \otimes \cdots \otimes p_n), (q_1 \otimes \cdots \otimes q_n)\big).
$$

Similarly its coherence maps are given by those of A and B: $\mathfrak{a}_{(\underline{x},\underline{u},\underline{y})} = (\mathfrak{a}_{\underline{x}}, \mathfrak{a}_{\underline{y}})$ for any double sequence $(\underline{x}, \underline{u}, \underline{y})$ of objects in $\langle J \rangle$, and $\mathfrak{i}_{(x,u,y)} = (\mathfrak{i}_x, \mathfrak{i}_y)$ for any object (x, u, y) in $\langle J \rangle$.

For the proof of Proposition 4.3 it is useful to record the following simple property of tabulations.

4.5. LEMMA. *The projections π_A and π_B of a tabulation π, as on the left below, are jointly monic in the following sense. Any two cells ϕ and ψ as on the right below coincide as soon as $1_{\pi_A} \circ \phi = 1_{\pi_A} \circ \psi$ and $1_{\pi_B} \circ \phi = 1_{\pi_B} \circ \psi$.*

$$
\begin{array}{ccc}
\langle J \rangle = \!\!= \langle J \rangle & \qquad X \xrightarrow{H} Y & \qquad X \xrightarrow{H} Y \\
\pi_A \downarrow \quad \Downarrow\pi \quad \downarrow \pi_B & f \downarrow \quad \Downarrow\phi \quad \downarrow g & f \downarrow \quad \Downarrow\psi \quad \downarrow g \\
A \xrightarrow{J} B & \langle J \rangle = \!\!= \langle J \rangle & \langle J \rangle = \!\!= \langle J \rangle
\end{array}
$$

PROOF. Applying the 2-dimensional universal property of π to the identity

$$
\begin{array}{c}
\boxed{\phi} \\
\quad \boxed{\pi}
\end{array}
\quad = \quad
\begin{array}{c}
\qquad \boxed{\phi} \\
\boxed{\pi}
\end{array}
$$

we find that there is exactly one cell χ such that $1_{\pi_A} \circ \chi = 1_{\pi_A} \circ \phi$ and $1_{\pi_B} \circ \chi = 1_{\pi_B} \circ \phi$. Since both $\chi = \phi$ and $\chi = \psi$ satisfy these identities we must have $\phi = \psi$. ∎

PROOF OF PROPOSITION 4.3. Given a horizontal T-morphism $(J, \bar{J}) \colon A \nrightarrow B$, we assume that there exists a cell π in $\mathcal{K}$, as on the left above, that defines the tabulation of its underlying horizontal morphism $J \colon A \nrightarrow B$. We will construct a lax T-algebra structure on $\langle J \rangle$ and check that it is unique such that π_A and π_B become strict T-morphisms, and that π becomes a T-cell. Following this we prove that as a T-cell π satisfies the universal properties of a tabulation, and that it is opcartesian whenever π is opcartesian in $\mathcal{K}$.

Lax T-algebra structure on $\langle J \rangle$. The universal properties of the tabulation π induce a lax T-algebra structure on $\langle J \rangle$ as follows. For the structure map $w \colon T\langle J \rangle \to \langle J \rangle$ consider the composite on the left-hand side below: by the 1-dimensional universal property of tabulations it factors uniquely through π as a vertical morphism $w \colon T\langle J \rangle \to \langle J \rangle$ as shown.

$$
\begin{array}{ccc}
T\langle J \rangle = \!\!= T\langle J \rangle & & T\langle J \rangle = \!\!= T\langle J \rangle \\
T\pi_A \downarrow \quad \Downarrow T\pi \quad \downarrow T\pi_B & & w \downarrow \qquad\qquad \downarrow w \\
TA \xrightarrow{TJ} TB & = & \langle J \rangle = \!\!= \langle J \rangle \\
a \downarrow \quad \Downarrow \bar{J} \quad \downarrow b & & \pi_A \downarrow \quad \Downarrow \pi \quad \downarrow \pi_B \\
A \xrightarrow{J} B & & A \xrightarrow{J} B
\end{array}
\tag{12}
$$

To obtain the vertical associator cell $\bar{w} \colon w \circ Tw \Rightarrow w \circ \mu_{\langle J \rangle}$ we consider the identity below, which is the associativity axiom for J precomposed with $T^2\pi$. Notice that the

second column of the left-hand side coincides with $\pi \circ 1_w \circ 1_{\mu_{\langle J \rangle}}$, and the first of the right-hand side with $\pi \circ 1_w \circ 1_{Tw}$. Invoking the 2-dimensional universal property of π we obtain a unique cell $\bar{w} \colon w \circ Tw \Rightarrow w \circ \mu_{\langle J \rangle}$ such that $1_{\pi_A} \circ \bar{w} = \bar{a} \circ 1_{T^2 \pi_A}$ and $1_{\pi_B} \circ \bar{w} = \bar{b} \circ 1_{T^2 \pi_B}$; we take $\bar{w}$ as the associator of $\langle J \rangle$.

$$
\begin{array}{ccc}
T^2\langle J \rangle === T^2\langle J \rangle & \qquad & T^2\langle J \rangle === T^2\langle J \rangle
\end{array}
$$

To prove the coherence axiom for $\bar{w}$ (see Definition 3.17) we use the lemma above as follows. It is clear from the definitions of w and $\bar{w}$ that, by postcomposing either side of the coherence axiom for $\bar{w}$ with 1_{π_A}, we obtain the corresponding side of the coherence axiom for $\bar{a}$ precomposed with $1_{T^3 \pi_A}$. Likewise, postcomposing them with 1_{π_B} we obtain the sides of the coherence axiom for $\bar{b}$, precomposed with $1_{T^3 \pi_B}$. Since the coherence axioms for $\bar{a}$ and $\bar{b}$ hold, and because 1_{π_A} and 1_{π_B} are jointly monic, the coherence axiom for $\bar{w}$ follows.

The vertical unitor cell $\hat{w} \colon \mathrm{id}_{\langle J \rangle} \Rightarrow w \circ \eta_{\langle J \rangle}$ is obtained in a similar way, by applying the 2-dimension universal property of π to the unit axiom for $\bar{J}$ precomposed with π; it is unique such that $1_{\pi_A} \circ \hat{w} = \hat{a} \circ 1_{\pi_A}$ and $1_{\pi_B} \circ \hat{w} = \hat{b} \circ 1_{\pi_B}$. The coherence axioms for $\hat{w}$ follow from those for $\hat{a}$ and $\hat{b}$, as well as the joint monicity of 1_{π_A} and 1_{π_B}, analogous to how the coherence axiom for $\bar{w}$ followed from that for $\bar{a}$ and $\bar{b}$ above. This completes the definition of the lax T-algebra $\langle J \rangle = (\langle J \rangle, w, \bar{w}, \hat{w})$.

Uniqueness of the algebra structure on $\langle J \rangle$. Summarising the above, the algebra structure on $\langle J \rangle$ is determined as follows: the structure map w is uniquely determined by the identity (12), while the associator $\bar{w}$ is uniquely determined by the identities $1_{\pi_A} \circ \bar{w} = \bar{a} \circ 1_{T^2 \pi_A}$ and $1_{\pi_B} \circ \bar{w} = \bar{b} \circ 1_{T^2 \pi_B}$, and the unitor $\hat{w}$ by the identities $1_{\pi_A} \circ \hat{w} = \hat{a} \circ 1_{\pi_A}$ and $1_{\pi_B} \circ \hat{w} = \hat{b} \circ 1_{\pi_B}$. It follows from (12) that π becomes a T-cell if we take the structure cells of the projections π_A and π_B to be trivial, while the identities determining $\bar{w}$ and $\hat{w}$ are precisely the coherence axioms for these trivial structure cells. Conversely, by the uniqueness of w, $\bar{w}$ and $\hat{w}$, any lift of π to a T-cell with strict vertical T-morphisms must have $(w, \bar{w}, \hat{w})$ as the algebraic structure on $\langle J \rangle$.

Universal properties of π as a T-cell. We will show that π forms a tabulation of J in $\mathsf{Alg}_{\mathrm{l}}(T)$ and in $\mathsf{Alg}_{\mathrm{ps}}(T)$; proving that it forms one in $\mathsf{Alg}_{\mathrm{c}}(T)$ is similar to the proof for $\mathsf{Alg}_{\mathrm{l}}(T)$. For the 1-dimensional universal property consider a T-cell ϕ as on the left below, where $X = (X, x, \bar{x}, \hat{x})$ is a lax T-algebra and where ϕ_A and ϕ_B are lax T-morphisms. Since π is a tabulation in $\mathcal{K}$ the cell ϕ factors uniquely through π as a vertical morphism

ϕ' as shown; we have to show that ϕ' can be lifted to a lax T-morphism.

$$
\begin{array}{ccc}
\begin{array}{c}
X \mathrel{=\!=} X \\
\phi_A \downarrow \quad \Downarrow\phi \quad \downarrow \phi_B \\
A \xrightarrow{\ J\ } B
\end{array}
& = &
\begin{array}{c}
X \mathrel{=\!=} X \\
\phi' \downarrow \qquad \downarrow \phi' \\
\langle J \rangle \mathrel{=\!=} \langle J \rangle \\
\pi_A \downarrow \quad \Downarrow\pi \quad \downarrow \pi_B \\
A \xrightarrow{\ J\ } B
\end{array}
\end{array}
\qquad
\begin{array}{c}
\bar{\phi}_A \;\;\phi
\end{array}
\;=\;
\begin{array}{c}
T\phi \;\; \bar{\phi}_B \\
\bar{J}
\end{array}
$$

To obtain a structure cell for ϕ' we consider the T-cell axiom for ϕ, as on the right above. Notice that $\bar{J} \circ T\phi = \pi \circ 1_w \circ 1_{T\phi'}$ and $\phi \circ 1_x = \pi \circ 1_{\phi'} \circ 1_x$ here, by the identity on the left above and (12), so that by invoking the 2-dimensional universal property of π we obtain a unique vertical cell $\bar{\phi}' \colon w \circ T\phi' \Rightarrow \phi' \circ x$ such that $1_{\pi_A} \circ \bar{\phi}' = \bar{\phi}_A$ and $1_{\pi_B} \circ \bar{\phi}' = \bar{\phi}_B$. From these identities it follows that the coherence axioms for $\bar{\phi}_A$ and $\bar{\phi}_B$ coincide with those of $\bar{\phi}'$ postcomposed with 1_{π_A} and 1_{π_B} respectively. Since the latter are jointly monic the coherence axioms for $\bar{\phi}'$ follow, and we conclude that $(\phi', \bar{\phi}')$ forms a unique lift of ϕ'; this completes the proof of π satisfying the 1-dimensional universal property in $\mathsf{Alg}_l(T)$.

To see that the same property is satisfied by π in $\mathsf{Alg}_{\mathrm{ps}}(T)$ assume that, in the above, ϕ_A and ϕ_B are pseudo T-morphisms. Then, using the 2-dimensional universal property of π again, we can obtain a unique vertical cell $\chi \colon \phi' \circ x \Rightarrow w \circ T\phi'$ such that $1_{\pi_A} \circ \chi = \bar{\phi}_A^{\,-1}$ and $1_{\pi_B} \circ \chi = \bar{\phi}_B^{\,-1}$. Using the joint monicity of π_A and π_B again, it follows that χ forms the inverse of $\bar{\phi}'$ and we conclude that the lift $(\phi', \bar{\phi}')$ of ϕ' is a pseudo T-morphism.

The 2-dimensional universal property for π, as a T-cell, follows easily from the corresponding property for π as a cell in $\mathcal{K}$. Indeed, consider an identity $\xi_A \odot \psi = \phi \odot \xi_B$ of T-cells, as in Definition 2.26. By the 2-dimensional universal property of π there exists a unique cell ξ' in $\mathcal{K}$ such that $1_{\pi_A} \circ \xi' = \xi_A$ and $1_{\pi_B} \circ \xi' = \xi_B$. It follows that the T-cell axiom for ξ', postcomposed with 1_{π_A} and 1_{π_B}, coincides with the T-cell axioms for ξ_A and ξ_B so that, by the joint monicity of 1_{π_A} and 1_{π_B}, the T-cell axiom for ξ' follows. We conclude that π as a T-cell forms the tabulation of the horizontal T-morphism J, both in $\mathsf{Alg}_l(T)$ and $\mathsf{Alg}_{\mathrm{ps}}(T)$.

The T-cell π is opcartesian as soon as π is opcartesian in $\mathcal{K}$. If π is an opcartesian cell in $\mathcal{K}$ then so is $T\pi$, by Proposition 3.7. To prove that this ensures that π is opcartesian as a T-cell we consider a T-cell ϕ as on the left below; in $\mathcal{K}$ it factors through the opcartesian cell π as a cell ϕ', as shown.

$$
\begin{array}{ccc}
\begin{array}{c}
\langle J \rangle \mathrel{=\!=} \langle J \rangle \\
\pi_A \downarrow \qquad \downarrow \pi_B \\
A \quad \Downarrow\phi \quad B \\
f \downarrow \qquad \downarrow g \\
C \xrightarrow{\ K\ } D
\end{array}
& = &
\begin{array}{c}
\langle J \rangle \mathrel{=\!=} \langle J \rangle \\
\pi_A \downarrow \quad \Downarrow\pi \quad \downarrow \pi_B \\
A \xrightarrow{\ J\ } B \\
f \downarrow \quad \Downarrow\phi' \quad \downarrow g \\
C \xrightarrow{\ K\ } D
\end{array}
\end{array}
$$

To show that ϕ' satisfies the T-cell axiom consider the following equation of composites in $\mathcal{K}$, where the identities are given by the factorisation above, the T-cell axiom for ϕ, again the factorisation above, and the T-cell axiom for π.

$$
\begin{array}{ccccccccc}
\boxed{\begin{array}{c} T\pi \\ T\phi' \\ \bar{K} \end{array}\ \bar{g}} & = & \boxed{\begin{array}{c} T\phi \\ \bar{K} \end{array}\ \bar{g}} & = & \boxed{\bar{f}}\ \boxed{\phi} & = & \boxed{\bar{f}}\ \boxed{\begin{array}{c} \pi \\ \phi' \end{array}} & = & \boxed{\bar{f}}\ \boxed{\begin{array}{c} T\pi \\ \bar{J} \\ \phi' \end{array}}
\end{array}
$$

Since the two sides of the equation above are the sides of the T-cell axiom for ϕ' pre-composed with $T\pi$, the axiom itself follows because $T\pi$ is opcartesian, and factorisations through opcartesian cells are unique. This completes the proof. ∎

Combining Proposition 4.1 and Proposition 4.3 we conclude that if T is a normal oplax double monad on an equipment $\mathcal{K}$ that has opcartesian tabulations, then $\mathsf{Alg}_{\mathrm{ps}}(T)$ is again an equipment that has opcartesian tabulations. It follows that pointwise Kan extensions in $\mathsf{Alg}_{\mathrm{ps}}(T)$ can be defined in terms of Kan extensions (by Theorem 2.29) so that pointwise right Kan extensions in the 2-category $\mathsf{Alg}_{\mathrm{ps}}(V(T))$ correspond to pointwise right Kan extensions along conjoints in $\mathsf{Alg}_{\mathrm{ps}}(T)$ (by Proposition 2.30).

Even though in the cases w = c and w = l the double categories $\mathsf{Alg}_{\mathrm{w}}(T)$ fail to be equipments in general, pointwise Kan extensions in $\mathsf{Alg}_{\mathrm{w}}(T)$ can still be defined in terms of Kan extensions by the following variant of Theorem 2.29. Remember that, for a pseudo T-morphism $f\colon C \to A$ and a horizontal T-morphism $J\colon A \nrightarrow B$, the restriction $J(f, \mathrm{id})$ exists in each of the double categories $\mathsf{Alg}_{\mathrm{w}}(T)$, by Proposition 4.1.

4.6. THEOREM. *Let T be a normal oplax double monad on an equipment $\mathcal{K}$ and let* w $\in \{\mathrm{c}, \mathrm{l}, \mathrm{ps}\}$. *Consider a cell ε in $\mathsf{Alg}_{\mathrm{w}}(T)$ as on the left below.*

$$
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
r\downarrow & \Downarrow\varepsilon & \downarrow d \\
M & = & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
C & \xrightarrow{J(f,\,\mathrm{id})} & B \\
f\downarrow & \mathrm{cart} & \| \\
A & \xrightarrow{J} & B \\
r\downarrow & \Downarrow\varepsilon & \downarrow d \\
M & = & M
\end{array}
$$

For the following conditions the implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) hold, while (c) $\Rightarrow$ (a) holds as soon as $\mathcal{K}$ has opcartesian tabulations.

  (a) *The cell ε defines r as the pointwise right Kan extension of d along J;*

  (b) *for all pseudo T-morphisms $f\colon C \to A$ the composite on the right above defines $r \circ f$ as the pointwise right Kan extension of d along $J(f, \mathrm{id})$;*

  (c) *for all strict T-morphisms $f\colon C \to A$ the composite on the right above defines $r \circ f$ as the right Kan extension of d along $J(f, \mathrm{id})$.*

PROOF. The proof given in [Kou14] for its Theorem 5.11, which has been restated in the present paper as Theorem 2.29, applies to the double categories $\mathsf{Alg}_\mathrm{w}(T)$ without needing any adjustment. This is because all conjoints and companions used in that proof exist in $\mathsf{Alg}_\mathrm{w}(T)$, as all of them are conjoints or companions of pseudo T-morphisms, and because, in the proof of the implication (c) $\Rightarrow$ (a), the projections of the lifted tabulations in $\mathsf{Alg}_\mathrm{w}(T)$ are strict T-morphisms; see Proposition 4.3. ∎

Just like the proof of Theorem 2.29 applies to the double category $\mathsf{Alg}_\mathrm{l}(T)$, so does the proof of Proposition 2.30 (which is Proposition 5.12 of [Kou14]). We thus obtain the following result, which describes pointwise right Kan extensions along conjoints in $\mathsf{Alg}_\mathrm{l}(T)$ in terms of right Kan extensions in the 2-category $\mathsf{Alg}_\mathrm{l}(V(T))$, in a sense that is analogous to Street's definition of pointwise Kan extension given in [Str74].

In the diagram on the right below f/j denotes the 'comma object' of $f\colon C \to A$ and $j\colon B \to A$ in the 2-category $\mathsf{Alg}_\mathrm{l}(V(T))$, see e.g. Definition 5.2 of [Kou14]. In terms of the double category $\mathsf{Alg}_\mathrm{l}(T)$, the comma object f/j can be defined as the tabulation of the restriction $A(f, j)$; in particular f/j exists whenever f is a pseudo T-morphism and $\mathcal{K}$ is an equipment that has tabulations, by Proposition 4.1 and Proposition 4.3.

4.7. PROPOSITION. *Let T be a normal oplax double monad on an equipment $\mathcal{K}$ that has opcartesian tabulations. Consider a vertical T-cell ε between lax T-morphisms, as on the left below, as well as its factorisation through the opcartesian cell defining j^*, as shown.*

The factorisation ε' defines r as the pointwise right Kan extension of d along j^* in $\mathsf{Alg}_\mathrm{l}(T)$ precisely if, for each strict T-morphism $f\colon C \to A$, the composite on the right above defines $r \circ f$ as the right Kan extension of $d \circ \pi_B$ along π_C in $\mathsf{Alg}_\mathrm{l}(V(T))$.

4.8. EXAMPLE. The 'free strict monoidal category'-monad (Example 3.11) on the equipment Prof of (unenriched) profunctors satisfies the hypotheses of the proposition above. Thus a monoidal transformation $\varepsilon\colon r \circ j \Rightarrow d$ of lax monoidal functors, as on the left above, defines r as the pointwise Kan extension of d along j if, for every strict monoidal functor $f\colon C \to A$, the composite on the right above defines $r \circ f$ as the right Kan extension of $d \circ \pi_B$ along π_C. In this case f/j is the usual 'comma category' that has triples (z, u, y) as objects, where $z \in C$, $y \in B$ and $u\colon fz \to jy$ in A. Its monoidal structure, that is induced by that of C and B, maps a sequence $(\underline{z}, \underline{u}, \underline{y})$ to the triple consisting of the tensor products $(z_1 \otimes \cdots \otimes z_n)$ and $(y_1 \otimes \cdots \otimes y_n)$ as well as the map

$$f(z_1 \otimes \cdots \otimes z_n) = (fz_1 \otimes \cdots \otimes fz_n) \xrightarrow{(u_1 \otimes \cdots \otimes u_n)} (jy_1 \otimes \cdots \otimes jy_n) \xrightarrow{j_\otimes} j(y_1 \otimes \cdots \otimes y_n).$$

5. Lifting algebraic Kan extensions

In this section we prove our main result, which gives conditions allowing Kan extensions to be lifted along the forgetful double functors $\mathsf{Alg_w}(T) \to \mathcal{K}$. As an application we recover and generalise a result of Getzler, given in [Get09], on the lifting of symmetric monoidal pointwise left Kan extensions.

We start by making the idea of 'lifting Kan extensions' precise.

5.1. DEFINITION. Consider a normal oplax double functor $F\colon \mathcal{K} \to \mathcal{L}$ as well as morphisms $J\colon A \nrightarrow B$ and $d\colon B \to M$ in $\mathcal{K}$. We say that F *lifts* the (pointwise) right Kan extension of d along J if, given any cell ε in $\mathcal{L}$, as on the left below, that defines r as the (pointwise) right Kan extension of Fd along FJ, there exists a unique cell ε' in $\mathcal{K}$ as on the right such that $F\varepsilon' = \varepsilon$ and, moreover, ε' defines r' as the (pointwise) right Kan extension of d along J.

$$
\begin{array}{ccc}
FA \xrightarrow{FJ} FB & \qquad\qquad & A \xrightarrow{J} B \\
{\scriptstyle r}\big\downarrow \quad \Downarrow{\scriptstyle \varepsilon} \quad \big\downarrow{\scriptstyle Fd} & & {\scriptstyle r'}\big\downarrow \quad \Downarrow{\scriptstyle \varepsilon'} \quad \big\downarrow{\scriptstyle d} \\
FM = \!\!= FM & & M = \!\!= M
\end{array}
$$

To state the main theorem we need Definitions 5.2 and 5.5.

5.2. DEFINITION. Let $d\colon B \to M$, $f\colon M \to N$ and $J\colon A \nrightarrow B$ be morphisms in a double category $\mathcal{K}$. We say that f *preserves* the (pointwise) right Kan extension of d along J if, for any cell ε on the left below that defines r as the (pointwise) right Kan extension of d along J, the composite $1_f \circ \varepsilon$ in the middle defines $f \circ r$ as the (pointwise) right Kan extension of $f \circ d$ along J.

$$
\begin{array}{ccccc}
 & A \xrightarrow{J} B & & A \xrightarrow{J} B & & TA \xrightarrow{TJ} TB \\
A \xrightarrow{J} B & {\scriptstyle r}\big\downarrow \Downarrow{\scriptstyle \varepsilon} \big\downarrow{\scriptstyle d} & & {\scriptstyle r}\big\downarrow \Downarrow{\scriptstyle \varepsilon} \big\downarrow{\scriptstyle d} & & {\scriptstyle Tr}\big\downarrow \Downarrow{\scriptstyle T\varepsilon} \big\downarrow{\scriptstyle Td} \\
{\scriptstyle r}\big\downarrow \Downarrow{\scriptstyle \varepsilon} \big\downarrow{\scriptstyle d} & M = \!\!= M & & M = \!\!= M & & TM = \!\!= TM \\
M = \!\!= M & {\scriptstyle f}\big\downarrow \quad \big\downarrow{\scriptstyle f} & & {\scriptstyle m}\big\downarrow \quad \big\downarrow{\scriptstyle m} \\
 & N = \!\!= N & & M = \!\!= M
\end{array}
$$

Next let T be a normal oplax double monad on $\mathcal{K}$ and $M = (M, m, \bar{m}, \hat{m})$ a lax T-algebra. As a variation on the above, we say that *the algebraic structure of M preserves* the (pointwise) right Kan extension of d along J if for any cell ε as on the left above, that defines r as the (pointwise) right Kan extension of d along J, the composite $1_m \circ T\varepsilon$ on the right defines $m \circ Tr$ as the (pointwise) right Kan extension of $m \circ Td$ along TJ.

Recall from Proposition 2.24 that, in the double category $\mathcal{V}\text{-}\mathsf{Prof}$, pointwise right Kan extensions along $\mathcal{V}$-weights $J\colon 1 \nrightarrow B$ coincide with J-weighted limits. Consequently we say that a $\mathcal{V}$-functor $f\colon M \to N$ *preserves* J-weighted limits if it preserves all pointwise

right Kan extensions along J. That this coincides with the classical notion of 'preserving weighted limits', see e.g. Section 3.2 in [Kel82], in the case that $\mathcal{V}$ is complete and closed symmetric monoidal, follows from Proposition 2.23. Notice that a $\mathcal{V}$-functor preserves all pointwise right Kan extensions whenever it preserves weighted limits, again by Proposition 2.24.

5.3. EXAMPLE. In anticipation of Proposition 5.9 we describe monoidal $\mathcal{V}$-categories whose monoidal structure preserves pointwise *left* Kan extensions. Consider a monoidal $\mathcal{V}$-category $M = (M, \otimes, \mathfrak{a}, \mathfrak{i})$ (Example 3.20), a $\mathcal{V}$-functor $d\colon A \to M$ and a $\mathcal{V}$-profunctor $J\colon A \nrightarrow B$; we claim that the algebraic structure of M preserves the pointwise left Kan extension of d along J whenever the n-ary tensor products $\otimes_n = [M^{\otimes n} \xrightarrow{j_n} TM \xrightarrow{\otimes} M]$ preserve it in every variable, for each $n \geq 1$, where $j_n\colon (-)^{\otimes n} \Rightarrow T$ denotes the double transformation that includes the nth tensor powers into T. Of course, because the components of $\mathfrak{a}$ and $\mathfrak{i}$ are isomorphisms, it suffices that the tensor product $\otimes_2$ does so. Moreover, by the discussion above, it also suffices that $\otimes_2$ preserves weighted colimits.

To prove the claim we consider a transformation η, as on the left below, that defines l as the pointwise left Kan extension of d along J (see Definition 2.20). We have to show that the composite $1_\otimes \circ T\eta$ defines $\otimes \circ Tl$ as a pointwise left Kan extension. By Lemma 5.4 below we may equivalently prove that the composites $1_{\otimes_n} \otimes \eta^{\otimes n}$ in the middle define $\otimes_n \circ l^{\otimes n}$ as pointwise left Kan extensions, for each $n \geq 1$.

$$
\begin{array}{ccc}
A \xrightarrow{J} B & \quad &
\begin{array}{c}
A^{\otimes n} \xrightarrow{J^{\otimes n}} B^{\otimes n} \\
d^{\otimes n} \downarrow \quad \Downarrow\eta^{\otimes n} \quad \downarrow l^{\otimes n} \\
M^{\otimes n} = M^{\otimes n} \\
\otimes_n \downarrow \qquad \downarrow \otimes_n \\
M === M
\end{array}
\end{array}
$$

To see that they do consider, for each $0 \leq i < n$, the transformation on the right above, where $i' = i + 1$. That it defines $\otimes_n \circ (l^{\otimes i'} \otimes d^{\otimes(n-i')})$ as a pointwise left Kan extension follows, by using Proposition 3.15 twice, from the fact that for all objects $y_1, \ldots, y_i \in B$ and $x_{i+2}, \ldots, x_n \in A$ the transformations

$$
(ly_1 \otimes \cdots \otimes ly_i \otimes - \otimes dx_{i+2} \otimes \cdots \otimes dx_n) \circ \eta
$$

define pointwise Kan extensions, by the assumption on $\otimes_n$. Therefore, by repeatedly applying Proposition 3.14 to the horizontal composite of the transformations on the right above, we can conclude that the composite $1_{\otimes_n} \circ \eta^{\otimes n}$ defines $\otimes_n \circ l^{\otimes n}$ as a pointwise left Kan extension, as needed.

Let $\mathcal{V}$ be a cocomplete symmetric monoidal category whose tensor product preserves colimits in both variables, so that $\mathcal{V}$-profunctors form a monoidal double category $\mathcal{V}$-Prof (Example 3.13), and let T denote the 'free strict monoidal $\mathcal{V}$-category'-monad on $\mathcal{V}$-Prof

(Example 3.11). For each $n \geq 0$ we denote by $j_n \colon (-)^{\otimes n} \Rightarrow T$ the double transformation that includes the nth tensor powers into T.

5.4. LEMMA. *Let $\mathcal{V}$ and $j_n \colon (-)^{\otimes n} \Rightarrow T$ be as above. A cell*

$$
\begin{array}{ccc}
TA & \xrightarrow{TJ} & TB \\
\scriptstyle d \downarrow & \Downarrow\eta & \downarrow \scriptstyle l \\
M & = & M
\end{array}
$$

defines l as the pointwise left Kan extension of d along TJ if and only if, for each $n \geq 0$, the composite $J^{\otimes n} \xRightarrow{(j_n)_J} TJ \xRightarrow{\eta} 1_M$ defines $l \circ j_n$ as the pointwise left Kan extension of $d \circ j_n$ along $J^{\otimes n}$.

PROOF. For each $\underline{y} \in TB$ of length n consider the identity below.

$$
\begin{array}{ccccc}
A^{\otimes n} \xrightarrow{TJ(j_n, \underline{y})} 1 & & & A^{\otimes n} \xrightarrow{J^{\otimes n}(\mathrm{id}, \underline{y})} 1 \\
\scriptstyle j_n \downarrow \quad \mathrm{cart} \quad \| & & & \| \quad \mathrm{cart} \quad \downarrow \scriptstyle \underline{y} \\
TA \xrightarrow[TJ(\mathrm{id}, \underline{y})]{} 1 & & & A^{\otimes n} \xrightarrow[J^{\otimes n}]{} B^{\otimes n} \\
\| \quad \mathrm{cart} \quad \downarrow \scriptstyle \underline{y} & = & & \scriptstyle j_n \downarrow \quad \Downarrow (j_n)_J \quad \downarrow \scriptstyle j_n \\
TA \xrightarrow[TJ]{} TB & & & TA \xrightarrow[TJ]{} TB \\
\scriptstyle d \downarrow \quad \Downarrow\eta \quad \downarrow \scriptstyle l & & & \scriptstyle d \downarrow \quad \Downarrow\eta \quad \downarrow \scriptstyle l \\
M = M & & & M = M
\end{array}
$$

In light of Proposition 2.24 it suffices to prove that the composite of the bottom two cells on the left-hand side defines $l(\underline{y})$ as a pointwise left Kan extension precisely if the full composite on the right-hand side defines $(l \circ j_n)(\underline{y})$ as one. It is readily checked that the top cartesian cell on the left-hand side is opcartesian, so that this equivalence follows from the horizontal dual of Proposition 2.34(a). ∎

5.5. DEFINITION. A double monad $T = (T, \mu, \eta)$ is called *(pointwise) right exact* if the cells μ_J and η_J are (pointwise) right exact for each horizontal morphism $J \colon A \nrightarrow B$ (see Definition 2.32). (Pointwise) left exact double monads are defined analogously.

5.6. EXAMPLE. The 'free strict monoidal $\mathcal{V}$-category'-monad T of Example 3.11 is both pointwise left and right exact. For example, to show that the cell μ_J, where $J \colon A \nrightarrow B$ is a $\mathcal{V}$-profunctor, is pointwise left exact it suffices, by the horizontal dual of Proposition 2.34(b), to show that the factorisation μ'_J of μ_J through the cartesian cell defining

$TJ(\mathrm{id}, \mu_B)$, as in the right-hand side below, is opcartesian.

$$
\begin{array}{ccc}
T^2A \xrightarrow{T^2J} T^2B & \qquad & T^2A \xrightarrow{T^2J} T^2B \\
\mu_A \downarrow \qquad \downarrow \qquad & \qquad & \mu_A \downarrow \quad \Downarrow \mu'_J \quad \| \\
TA \quad \Downarrow \chi \quad g & = & TA \xrightarrow[TJ(\mathrm{id}, \mu_B)]{} T^2B \\
f \downarrow \qquad \downarrow \qquad & & f \downarrow \quad \Downarrow \chi' \quad \downarrow g \\
C \xrightarrow[K]{} D & & C \xrightarrow[K]{} D
\end{array}
$$

To see this consider a cell χ as above, we have to show that it factors uniquely through μ'_J as shown. Given $\underline{x} \in TA$ and $\underline{\underline{y}} \in T^2B$ notice that, if $|\underline{x}| = |\mu \underline{\underline{y}}|$, then there exists a unique $\underline{\underline{x}} \in T^2A$ with $\mu\underline{\underline{x}} = \underline{x}$, $|\underline{\underline{x}}| = |\underline{\underline{y}}|$ and $|\underline{\underline{x}}_i| = |\underline{\underline{y}}_i|$ for all $1 \le i \le |\underline{\underline{x}}|$; in this case we take as the $\mathcal{V}$-map $\chi'_{(\underline{x},\underline{\underline{y}})} \colon TJ(\underline{x}, \mu\underline{\underline{y}}) \to K(f\underline{x}, g\underline{\underline{y}})$ the composite

$$
TJ(\underline{x}, \mu\underline{\underline{y}}) \cong T^2J(\underline{\underline{x}}, \underline{\underline{y}}) \xrightarrow{\chi} K(f\underline{x}, g\underline{\underline{y}}),
$$

where the isomorphism is the inverse of $(\mu_J)_{(\underline{\underline{x}}, \underline{\underline{y}})}$; see (8). In the case that $|\underline{x}| \ne |\mu\underline{\underline{y}}|$ we must take $\chi'_{(\underline{x}, \underline{\underline{y}})}$ to be the unique map $\emptyset \to K(f\underline{x}, g\underline{\underline{y}})$. It is clear that, defined like this, χ' forms the unique factorisation of χ through μ'_J.

We are now ready to state and prove our main result. Its horizontal dual, Theorem 1.1, which concerns (pointwise) left Kan extensions between colax T-algebras, has already been stated in the introduction where, to simplify exposition, it has been restricted to pseudo T-algebras and pointwise left Kan extensions.

5.7. THEOREM. *Let T be a right exact, normal oplax double monad on a double category $\mathcal{K}$ and let 'weak' mean either 'colax', 'lax' or 'pseudo'. Given lax T-algebras A, B and M, consider the following conditions on a horizontal T-morphism $J \colon A \nrightarrow B$ and a weak vertical T-morphism $d \colon B \to M$:*

(p) *the algebraic structure of M preserves the right Kan extension of d along J;*

(e) *the structure cell of J is right d-exact;*

(l) *the forgetful double functor $\mathsf{Alg}_{\mathrm{w}}(T) \to \mathcal{K}$ lifts the right Kan extension of d along J.*

The following hold:

(a) *if 'weak' means 'colax' then (p) implies (l);*

(b) *if 'weak' means 'lax' then (e) implies (l);*

(c) *if 'weak' means 'pseudo' then any two of (p), (e) and (l) imply the third.*

If T is a pointwise right exact, normal pseudo double monad then analogous results hold for pointwise Kan extensions, obtained by replacing 'right Kan extension' in (p) and (l) by 'pointwise right Kan extension', and 'right d-exact' in (e) by 'pointwise right d-exact'.

5.8. REMARK. We remark that, in the "pointwise part" of the above theorem, the condition that T be a pseudo double monad can be dispensed with whenever $\mathcal{K}$ is an equipment that has opcartesian tabulations.

Indeed, consider a cell ε as on the left below and assume that it defines r as a pointwise right Kan extension in $\mathcal{K}$. Using the arguments given in the proof below it can be lifted to form a T-cell in $\mathsf{Alg_w}(T)$. Now, instead of showing that this lift defines a pointwise right Kan extension in $\mathsf{Alg_w}(T)$ in the usual sense (Definition 2.17), we may equivalently show that, for every strict T-morphism $f\colon C \to A$, the composite on the right below defines $r \circ f$ as an ordinary right Kan extension in $\mathsf{Alg_w}(T)$, by Theorem 4.6. Proving the latter amounts to showing that certain factorisations in $\mathcal{K}$, through the composites on the right below, lift to $\mathsf{Alg_w}(T)$; compare the lifting of the factorisation (14) in the proof below. However, in contrast to (14), these factorisations will be vertical cells, so that requiring that the compositor $T_\odot$ be invertible is not necessary.

$$
\begin{array}{ccc}
A & \xrightarrow{\ J\ } & B \\
r \downarrow & \Downarrow \varepsilon & \downarrow d \\
M & = & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
C & \xrightarrow{J(f,\mathrm{id})} & B \\
f \downarrow & \mathrm{cart} & \| \\
A & \xrightarrow{\ J\ } & B \\
r \downarrow & \Downarrow \varepsilon & \downarrow d \\
M & = & M
\end{array}
$$

PROOF OF THEOREM 5.7. We treat the ordinary and pointwise case simultaneously. As usual we denote the algebra structure on A by $A = (A, a, \bar{a}, \hat{a})$, where $a\colon TA \to A$ is the structure map, $\bar{a}\colon a \circ Ta \Rightarrow a \circ \mu_A$ is the associator and $\hat{a}\colon \mathrm{id}_A \Rightarrow a \circ \eta_A$ is the unitor; likewise $B = (B, b, \bar{b}, \hat{b})$ and $M = (M, m, \bar{m}, \hat{m})$, while the structure cell $\bar{J}$ of J is of the form as on the left below. To start we consider a cell ε in $\mathcal{K}$, as on the right below, and assume that it defines r either as the ordinary right Kan extension of d along J or as the pointwise right Kan extension of d along J.

$$
\begin{array}{ccc}
TA & \xrightarrow{TJ} & TB \\
a \downarrow & \Downarrow \bar{J} & \downarrow b \\
A & \xrightarrow{\ J\ } & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{\ J\ } & B \\
r \downarrow & \Downarrow \varepsilon & \downarrow d \\
M & = & M
\end{array}
$$

Part (a). If 'weak' means 'colax' then the algebraic structure on d is given by a cell $\bar{d}\colon d \circ b \Rightarrow m \circ Td$ as in the composite on the left-hand side below.

$$
\begin{array}{ccccccccc}
TA & \xrightarrow{TJ} & TB & = & TB & & TA & = & TA & \xrightarrow{TJ} & TB \\
a \downarrow & \Downarrow \bar{J} & \downarrow b & & \downarrow Td & & a \downarrow & & Tr \downarrow & \Downarrow T\varepsilon & \downarrow Td \\
A & \xrightarrow{\ J\ } & B & \Downarrow \bar{d} & TM & = & A & \Downarrow \bar{r} & TM & = & TM \\
r \downarrow & \Downarrow \varepsilon & \downarrow d & & \downarrow m & & r \downarrow & & m \downarrow & & \downarrow m \\
M & = & M & = & M & & M & = & M & = & M
\end{array}
\tag{13}
$$

Assuming that condition (p) holds, the second column in the right-hand side above defines $m \circ Tr$ as the (pointwise) right Kan extension of $m \circ Td$ along TJ, see Definition 5.2, so that in both the ordinary and the pointwise case the composite on the left factors uniquely as a vertical cell $\bar{r}\colon r \circ a \Rightarrow m \circ Tr$, as shown.

We claim that $\bar{r}$ makes r into a colax T-morphism, that is it satisfies the associativity and unit axioms of Definition 3.18. To see that it satisfies the former consider the equation of composites below, whose left-hand and right-hand sides are given by the corresponding sides of the associativity axiom for $\bar{r}$, composed with $T^2\varepsilon$ on the right.

$$
\begin{array}{cccccccc}
\boxed{\begin{array}{c} T\bar{r} \\ \bar{r} \quad \bar{m} \end{array}\ T^2\varepsilon} & = & \boxed{\begin{array}{c} T\bar{J} \ T\bar{d} \\ \bar{r}\ T\varepsilon\ \bar{m} \end{array}} & = & \boxed{\begin{array}{c} T\bar{J}\ T\bar{d} \\ \bar{J}\ \bar{d}\ \bar{m} \\ \varepsilon \end{array}} & = & \boxed{\begin{array}{c} T\bar{J}\ \bar{b} \\ \bar{J}\ \bar{d} \\ \varepsilon \end{array}}
\end{array}
$$

$$
\begin{array}{ccccc}
= & \boxed{\begin{array}{c} \mu_J \\ \bar{a}\ \bar{J}\ \bar{d} \\ \varepsilon \end{array}} & = & \boxed{\begin{array}{c} \mu_J \\ \bar{a}\ T\varepsilon \\ \bar{r} \end{array}} & = & \boxed{\begin{array}{c} \bar{a}\ T^2\varepsilon \\ \bar{r} \end{array}}
\end{array}
$$

The identities here follow from the T-image the of factorisation (13); the factorisation (13) itself; the associativity axiom for $\bar{d}$; the associativity axiom for $\bar{J}$; again (13); the naturality of μ. Now notice that the fourth column of the left and right-hand sides equals $1_m \circ 1_{\mu_M} \circ T^2\varepsilon = 1_m \circ T\varepsilon \circ \mu_J$ which, because μ_J is assumed to be (pointwise) right exact, defines $m \circ Tr \circ \mu_A$ as the (pointwise) right Kan extension of $m \circ Td \circ \mu_B$ along T^2J. We conclude that the associativity axiom for $\bar{r}$ holds after it is composed on the right with a cell defining a right Kan extension; since factorisations through such cells are unique the axiom itself follows.

$$
\begin{array}{ccccc}
\boxed{\begin{array}{c} \varepsilon \\ \hat{m} \end{array}} & = & \boxed{\begin{array}{c} \hat{m} \\ \varepsilon \end{array}} & = & \boxed{\begin{array}{c} \hat{b} \\ \bar{d} \\ \varepsilon \end{array}}
\end{array}
$$

$$
\begin{array}{ccccc}
= & \boxed{\begin{array}{c} \eta_J \\ \hat{a}\ \bar{J}\ \bar{d} \\ \varepsilon \end{array}} & = & \boxed{\begin{array}{c} \eta_J \\ \hat{a}\ T\varepsilon \\ \bar{r} \end{array}} & = & \boxed{\begin{array}{c} \hat{a}\ \varepsilon \\ \bar{r} \end{array}}
\end{array}
$$

The unit axiom for $\bar{r}$ is proved similarly: we consider the equation above, whose left and right-hand sides coincide with those of the unit axiom for $\bar{r}$ composed with ε on the right. The identities follow from the unit axiom for $\bar{d}$, the unit axiom for $\bar{J}$, the factorisation (13) and the naturality of η. As before, the unit axiom for $\bar{r}$ follows

from the fact that the last columns in the left and right-hand sides below are given by $1_m \circ 1_{\eta_M} \circ \varepsilon = 1_m \circ T\varepsilon \circ \eta_J$, which defines $m \circ Tr \circ \eta_A$ as the (pointwise) right Kan extension of $m \circ Td \circ \eta_B$ along J, because η_J is (pointwise) right exact. We conclude that the cell $\bar{r}$ forms a colax structure cell for $r \colon B \to M$. Moreover, notice that with this algebra structure on r the factorisation (13) forms the T-cell axiom for ε. In fact, since this factorisation is unique, this is the only way to lift the $\mathcal{K}$-cell ε to $\mathsf{Alg}_c(T)$.

Having obtained an algebra structure on r it remains to show that ε, as a T-cell, defines r as the (pointwise) right Kan extension of d along J. To see this consider a T-cell ϕ as on the left below where, in the case of an ordinary Kan extension, we assume $H \colon C \nrightarrow A$ to be the horizontal unit 1_A. Since ε defines r as the (pointwise) right Kan extension in $\mathcal{K}$ we obtain a unique factorisation ϕ' as shown below; we have to prove that ϕ' is a T-cell.

$$
\begin{array}{ccc}
\begin{array}{ccc}
C & \xrightarrow{H} A \xrightarrow{J} & B \\
{\scriptstyle s}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle d} \\
M & =\!=\!=\!=\!= & M
\end{array}
& = &
\begin{array}{ccccc}
C & \xrightarrow{H} A & \xrightarrow{J} & B \\
{\scriptstyle s}\downarrow & \Downarrow\phi' \;\downarrow{\scriptstyle r} & \Downarrow\varepsilon & \downarrow{\scriptstyle d} \\
M & =\!= M & =\!= & M
\end{array}
\end{array}
\tag{14}
$$

To see this first notice that the compositor $T_\odot \colon T(H \odot J) \Rightarrow TH \odot TJ$ is invertible in either case. Indeed if r is an ordinary Kan extension then $H = 1_A$, so that this follows from the unit axiom for T (see (7)). On the other hand, if r is a pointwise Kan extension then T is assumed to be a pseudo double monad.

Next consider the equation below, whose identities follow from the naturality of $T_\odot$ and the factorisation above; the T-cell axiom for ϕ (remember that $\overline{H \odot J} = (\bar{H} \odot \bar{J}) \circ T_\odot$, see (10)); the factorisation above; the factorisation (13).

Precomposing the equation above with $T_\odot^{-1}$ we find that both sides of the T-cell axiom for ϕ' coincide after composition with $1_m \circ T\varepsilon$ on the right. Since the latter defines a (pointwise) right Kan extension the T-cell axiom itself follows. This finishes the proof of part (a).

Part (b). In the case that d is a lax T-morphism, with structure cell $\bar{d} \colon m \circ Td \Rightarrow d \circ b$, and the structure cell $\bar{J}$ of J is (pointwise) right exact, so that the second column in the right-hand side below defines $r \circ a$ as the (pointwise) right Kan extension of $d \circ b$ along

TJ, then the composite on the left below factors uniquely as a cell $\bar{r}\colon m \circ Tr \Rightarrow r \circ a$ as shown.

$$
\begin{array}{ccc}
\begin{array}{ccc}
TA \xrightarrow{TJ} TB = TB \\
{\scriptstyle Tr}\downarrow \quad \Downarrow T\varepsilon \quad \downarrow {\scriptstyle Td} \quad\quad \downarrow {\scriptstyle b} \\
TM = TM \quad \Downarrow \bar{d} \quad B \\
{\scriptstyle m}\downarrow \quad\quad \downarrow {\scriptstyle m} \quad\quad \downarrow {\scriptstyle d} \\
M = M = M
\end{array}
&
=
&
\begin{array}{ccc}
TA = TA \xrightarrow{TJ} TB \\
{\scriptstyle Tr}\downarrow \quad\quad {\scriptstyle a}\downarrow \quad \Downarrow \bar{J} \quad \downarrow {\scriptstyle b} \\
TM \quad \Downarrow \bar{r} \quad A \xrightarrow{\;J\;} B \\
{\scriptstyle m}\downarrow \quad\quad {\scriptstyle r}\downarrow \quad \Downarrow \varepsilon \quad \downarrow {\scriptstyle d} \\
M = M = M
\end{array}
\end{array}
\tag{15}
$$

Analogous to the proof of part (a) above, we can show that the cell $\bar{r}$ satisfies the associativity and unit axioms, so that it makes $r\colon A \to M$ into a lax T-morphism.

Indeed, the associativity axiom follows from the identity below, which itself follows from the associativity axiom for $\bar{J}$, the factorisation above, the associativity axiom for $\bar{d}$ and the naturality of μ. As before notice that the identity below is the associativity axiom for $\bar{r}$ composed on the right with the composite $\varepsilon \circ \bar{J} \circ \mu_J$. This composite defines a (pointwise) right Kan extension, since both μ_J and $\bar{J}$ are assumed to be (pointwise) right exact, so that the associativity axiom itself follows.

The unit axiom for $\bar{r}$ follows in the same way from the identity on the left below, which itself follows from the unit axioms for $\bar{J}$ and $\bar{d}$, the factorisation above and the naturality of η.

Having given $r\colon A \to M$ a lax T-morphism structure, we notice that the factorisation (15) forms the T-cell axiom for ε. It remains to show that, as a T-cell, ε defines r as the (pointwise) right Kan extension of d along J. To do so we consider a T-cell ϕ as in (14), where now s and d are lax T-morphisms. Because ε defines r as a (pointwise) right Kan extension, ϕ factors uniquely through ε as ϕ', as in (14), and we have to show that ϕ' is a T-cell. That the T-cell axiom for ϕ' holds when composed with $\varepsilon \circ \bar{J}$ on the right, and then with $T_\odot\colon T(H \odot J) \Rightarrow TH \odot TJ$ on top, as shown on the right above, follows from the T-cell axiom for ε (15), the factorisation $\phi = \phi' \odot \varepsilon$, the naturality of $T_\odot$ and the T-cell axiom for ϕ. Since $T_\odot$ is invertible and $\varepsilon \circ \bar{J}$ defines a (pointwise) right Kan extension the T-cell axiom for ϕ' follows. This concludes the proof of part (b).

Part (c). In this case d is a pseudo T-morphism, with invertible structure cell $\bar{d}\colon m \circ Td \Rightarrow d \circ b$. If two of (p), (e) and (l) hold then one of (p) and (e) must hold.

Suppose that (e) holds; we will show that (p) and (l) are equivalent. By part (b) the cell ε, that defines r as a right Kan extension, lifts along $\mathsf{Alg}_l(T) \to \mathcal{K}$ to a T-cell defining $(r, \bar{r})$ as a right Kan extension in $\mathsf{Alg}_l(T)$, where $\bar{r} \colon m \circ Tr \Rightarrow r \circ a$ is uniquely determined by the identity (15). If (p) holds then, using the fact that $\bar{d}$ is invertible, the left-hand side of (15) defines the right Kan extension of $d \circ b$ along TJ. Because the second column in the right hand side of (15) does too, it follows that the factorisation $\bar{r}$ is invertible, so that the lift $(r, \bar{r})$ of r is a pseudomorphism; we conclude that condition (l) holds.

On the other hand, if (l) holds then r can be given the structure of a pseudo T-morphism such that ε forms a T-cell that defines r as a right Kan extension in $\mathsf{Alg}_{\mathrm{ps}}(T)$. Since the lift of ε that we obtained by applying part (b) is unique, the structure cell $m \circ Tr \Rightarrow r \circ a$ of the lift of r to $\mathsf{Alg}_{\mathrm{ps}}(T)$ must coincide with the structure cell $\bar{r}$ that we obtained in (15). Thus $\bar{r}$ is invertible and hence, by composing (15) on the right with $\bar{d}^{-1}$, we see that $1_m \circ T\varepsilon$ defines $m \circ Tr$ as a right Kan extension. This means that the algebraic structure of M preserves the right Kan extension of d along J, that is (p) holds. We conclude that if (e) holds then (p) and (l) are equivalent. By applying analogous arguments to the factorisation (13) one shows that the equivalence of (e) and (l) follows from (p). This completes the proof of part (c).   ∎

In the proposition below we apply Theorem 1.1, which is the horizontal dual of the theorem above, to the double monad of free strict monoidal $\mathcal{V}$-categories. Consider a (symmetric) monoidal $\mathcal{V}$-profunctor $J \colon A \nrightarrow B$ (Example 3.23). Horizontally dual to the factorisation considered in Proposition 2.34, its structure transformation $J_\otimes$ factors uniquely as a transformation $(J_\otimes)^* \colon \otimes_A^* \odot TJ \Rightarrow J(\mathrm{id}, \otimes_B)$, of $\mathcal{V}$-profunctors $A \nrightarrow TB$, that is induced by the $\mathcal{V}$-maps

$$A\big(x, (y_1 \otimes \cdots \otimes y_n)\big) \otimes J(y_1, z_1) \otimes \cdots \otimes J(y_n, z_n) \xrightarrow{\mathrm{id} \otimes J_\otimes}$$

$$A\big(x, (y_1 \otimes \cdots \otimes y_n)\big) \otimes J\big((y_1 \otimes \cdots \otimes y_n), (z_1 \otimes \cdots \otimes z_n)\big) \xrightarrow{\lambda} J\big(x, (z_1 \otimes \cdots \otimes z_n)\big)$$

where $x \in A$, $\underline{y} \in TA$ and $\underline{z} \in TB$. We say that J satisfies the *left Beck-Chevalley condition* if the transformation $(J_\otimes)^*$ is an isomorphism.

In the proposition below $\mathcal{V}\text{-}\mathsf{MonProf}_w$ (respectively $\mathcal{V}\text{-}\mathsf{sMonProf}_w$) denotes the double category of (symmetric) monoidal $\mathcal{V}$-categories, (symmetric) weak monoidal $\mathcal{V}$-functors, (symmetric) monoidal $\mathcal{V}$-profunctors and their transformations. For a double functor $F \colon \mathcal{K} \to \mathcal{L}$, a horizontal morphism $J \colon A \nrightarrow B$ and an object M in $\mathcal{L}$, we say that F *lifts all pointwise left Kan extensions along J into M* if, for every vertical morphism $d \colon A \to M$ in $\mathcal{L}$, F lifts the pointwise left Kan extension of d along J.

5.9. PROPOSITION. *Let $\mathcal{V}$ be a cocomplete symmetric monoidal category whose tensor product preserves colimits on both sides, let A, B and M be monoidal $\mathcal{V}$-categories and let $J \colon A \nrightarrow B$ be a monoidal $\mathcal{V}$-profunctor. Consider the following conditions:*

(p) *the tensor product $\otimes_2 \colon M \otimes M \to M$ preserves pointwise left Kan extensions along J in each variable;*

(e) J satisfies the left Beck-Chevalley condition.

If condition (e) holds then the forgetful double functor $\mathcal{V}$-MonProf$_c$ → $\mathcal{V}$-Prof lifts all pointwise left Kan extensions along J into M. If (p) holds then so does $\mathcal{V}$-MonProf$_l$ → $\mathcal{V}$-Prof; in the case that both conditions hold the same follows for $\mathcal{V}$-MonProf$_{ps}$ → $\mathcal{V}$-Prof.

Analogous assertions hold for the double functors $\mathcal{V}$-sMonProf$_w$ → $\mathcal{V}$-Prof, for each $w \in \{c, l, ps\}$, *in the case that A, B and M are symmetric monoidal $\mathcal{V}$-categories and $J\colon A \nrightarrow B$ is a symmetric monoidal $\mathcal{V}$-profunctor.*

PROOF. The assertions for the forgetful double functors $\mathcal{V}$-MonProf$_w$ → $\mathcal{V}$-Prof follow immediately from applying Theorem 1.1 to the 'free strict monoidal $\mathcal{V}$-category'-monad T of Example 3.11. Indeed, T is pointwise left exact (Example 5.6) and the conditions (p) and (e) above imply the corresponding conditions in Theorem 1.1, by Example 5.3 and the horizontal dual of Proposition 2.34(b).

To see that the assertions remain true in the symmetric case we claim that, for each choice of $w \in \{c, l, ps\}$, any lift of a pointwise left Kan extension along the forgetful double functor $\mathcal{V}$-MonProf$_w$ → $\mathcal{V}$-Prof can be further lifted along the forgetful double functor $\mathcal{V}$-sMonProf$_w$ → $\mathcal{V}$-MonProf$_w$. We will prove this claim in the case that $w = l$, in the other cases it can be proved analogously. Hence consider symmetric monoidal $\mathcal{V}$-categories A, B and M, a symmetric lax monoidal $\mathcal{V}$-functor $d\colon A \to M$ and a symmetric monoidal $\mathcal{V}$-profunctor $J\colon A \nrightarrow B$, and assume that in $\mathcal{V}$-Prof the transformation η on the left below defines a $\mathcal{V}$-functor $l\colon B \to M$ as the pointwise left Kan extension of d along J.

$$(16)$$

If the condition (p) holds then $1_\otimes \circ T\eta$ defines $\otimes \circ Tl$ as a pointwise left Kan extension and, horizontally dual to (13), the lax monoidal structure $l_\otimes\colon \otimes \circ Tl \Rightarrow l \circ \otimes$ that makes l into a lax monoidal $\mathcal{V}$-functor is obtained as the unique factorisation on the right above.

We claim that $l_\otimes$ is compatible with the symmetries on B and M, in the sense that it makes a diagram like (9) commute. Rewritten in terms of vertical cells in $\mathcal{V}$-Prof, this means that the identity above holds; here $l_{\otimes_n} = l_\otimes \circ 1_{j_n}$, where $j_n\colon (-)^{\otimes n} \Rightarrow T$ denotes the double transformation that includes the nth tensor powers into T. The following equation, where $d_{\otimes_n} = d_\otimes \circ 1_{j_n}$, $J_{\otimes_n} = J_\otimes \circ (j_n)_J$ and whose identities are described below, shows that the two sides above coincide after composition on the left with $1_{\otimes_n} \circ \eta^{\otimes n}$. The latter defines a pointwise left Kan extension because $1_\otimes \circ T\eta$ does, by Lemma 5.4, so that the identity above follows.

The identities above follow from the composite of (16) with $(j_n)_J$; the symmetry axiom for J (see Example 3.23); the symmetry axiom for d; (16) composed with $(j_n)_J$ again; the naturality of σ.

We conclude that l, as a lax monoidal $\mathcal{V}$-functor, has a unique lift along the forgetful double functor $U\colon \mathcal{V}\text{-sMonProf}_\mathrm{w} \to \mathcal{V}\text{-MonProf}_\mathrm{w}$. That η, as a cell in $\mathcal{V}\text{-sMonProf}_\mathrm{w}$, defines this lift as the pointwise left Kan extension of d along J follows readily from the fact that U restricts to the identity on cells. This completes the proof. ∎

As a corollary to the previous proposition we can now recover Proposition 2.3 of [Get09]. Given a symmetric monoidal $\mathcal{V}$-functor $j\colon A \to B$ recall that the representable $\mathcal{V}$-profunctor $j_*\colon A \nrightarrow B$ is monoidal, by Proposition 4.1. It satisfies the left Beck-Chevalley condition if the transformation $\big((j_*)_\otimes\big)^*\colon \otimes_A^* \odot Tj_* \Rightarrow B(j, \otimes_B)$ of profunctors $A \nrightarrow TB$, that is induced by the compositions

$$A\big(x, (y_1 \otimes \cdots \otimes y_n)\big) \otimes B(jy_1, z_1) \otimes \cdots \otimes B(jy_n, z_n)$$
$$\xrightarrow{\mathrm{id} \otimes \otimes_B} A\big(x, (y_1 \otimes \cdots \otimes y_n)\big) \otimes B\big((jy_1 \otimes \cdots \otimes jy_n), (z_1 \otimes \cdots \otimes z_n)\big)$$
$$\xrightarrow{j \otimes \lambda_{j_\otimes}{}^{-1}} B\big(jx, j(y_1 \otimes \cdots \otimes y_n)\big) \otimes B\big(j(y_1 \otimes \cdots \otimes y_n), (z_1 \otimes \cdots \otimes z_n)\big)$$
$$\xrightarrow{\mu} B\big(jx, (z_1 \otimes \cdots \otimes z_n)\big),$$

is invertible. To see what this means in the case of $\mathcal{V} = \mathsf{Set}$, we first notice that the composite profunctor $\otimes_A^* \odot Tj_*\colon A \nrightarrow TB$, see Example 2.3, is given by the sets $(\otimes^* \odot Tj_*)\big(x, (z_1, \ldots, z_n)\big)$ forming equivalence classes of sequences of morphisms

$$\big(x \xrightarrow{p} (y_1 \otimes \cdots \otimes y_n), jy_1 \xrightarrow{u_1} z_1, \ldots, jy_n \xrightarrow{u_n} z_n\big),$$

where $p \in A$ and $u_i \in B$, under the smallest equivalence relation $\sim$ that relates the pairs $(p, u_1, \ldots, u_n) \sim (p', u_1', \ldots, u_n')$ for which there exist maps $r_i \colon y_i \to y_i'$ in A that make the diagrams

$$
\begin{array}{ccc}
 & \xrightarrow{\;p\;} (y_1 \otimes \cdots \otimes y_n) & \\
x & \Big\downarrow {\scriptstyle (r_1 \otimes \cdots \otimes r_n)} & \qquad \text{and} \\
 & \xrightarrow{\;p'\;} (y_1' \otimes \cdots \otimes y_n') &
\end{array}
\qquad
\begin{array}{ccc}
jy_i & \xrightarrow{\;u_i\;} & \\
\Big\downarrow {\scriptstyle jr_i} & & z_i \\
jy_i' & \xrightarrow{\;u_i'\;} &
\end{array}
$$

commute, the latter for each $i = 1, \ldots, n$. In terms of $\sim$, the profunctor j_* satisfies the left Beck-Chevalley condition if every morphism $v \colon jx \to (z_1 \otimes \cdots \otimes z_n)$ in B can be represented as

$$
\begin{array}{ccc}
jx & \xrightarrow{\quad v \quad} & (z_1 \otimes \cdots \otimes z_n) \\
{\scriptstyle jp} \Big\downarrow & & \Big\uparrow {\scriptstyle (u_1 \otimes \cdots \otimes u_n)} \\
j(y_1 \otimes \cdots \otimes y_n) & \xrightarrow[{\; j_{\otimes}^{-1} \;}]{} & (jy_1 \otimes \cdots \otimes jy_n),
\end{array}
$$

where $p \in A$ and $u_i \in B$, and that any two such representations $(p, u_1, \ldots, u_n)$ and $(p', u_1', \ldots, u_n')$ are related under $\sim$.

In the corollary below we have denoted by $\mathcal{V}$-sMonCat the 2-category of symmetric monoidal $\mathcal{V}$-categories, symmetric monoidal $\mathcal{V}$-functors and monoidal $\mathcal{V}$-natural transformations.

5.10. COROLLARY. [Getzler] *Let $\mathcal{V}$ be a closed symmetric monoidal category that is both complete and cocomplete. Consider symmetric monoidal $\mathcal{V}$-categories A, B and M, as well as a symmetric monoidal $\mathcal{V}$-functor $j \colon A \to B$. If $j_* \colon A \nrightarrow B$ satisfies the left Beck-Chevalley condition and M is cocomplete, such that its tensor product preserves $\mathcal{V}$-weighted colimits in each variable, then the functor*

$$\lambda_j \colon \mathcal{V}\text{-sMonCat}(B, M) \to \mathcal{V}\text{-sMonCat}(A, M),$$

that is given by precomposition with j, admits a left adjoint that is given by pointwise left Kan extension along j.

PROOF. It is well known that, for any morphism $j \colon A \to B$ in a 2-category $\mathcal{C}$, the functor $\lambda_j \colon \mathcal{C}(B, M) \to \mathcal{C}(A, M)$, given by precomposition with j, has a left adjoint precisely if all left Kan extensions along j exist; see e.g. Section X.3 of [ML98] for the case $\mathcal{C} = \mathsf{Cat}$. It is also well known that all pointwise left Kan extensions into the $\mathcal{V}$-category M (in the sense of the final assertion of Proposition 2.24) exist whenever it is cocomplete; see Proposition 4.33 of [Kel82]. By the assumptions on M and j_* we may apply the previous proposition to these pointwise left Kan extensions, which shows that they are lifted to the double category $\mathcal{V}$-sMonProf$_{\mathrm{ps}}$. Therefore, to conclude the proof, it suffices to recall that (pointwise) left Kan extensions along j_*, in $\mathcal{V}$-sMonProf$_{\mathrm{ps}}$, correspond to left Kan extensions along j in $\mathcal{V}$-sMonCat, by (the horizontal dual of) Proposition 2.18. ∎

6. Application: free bicommutative Hopf monoids

In this last section we use Corollary 5.10 above to obtain a left adjoint to the forgetful functor

$$\mathsf{ubicHopf}(M) \to \mathsf{ucocComon}(M),$$

from the category of 'unbiased' bicommutative Hopf monoids, in a suitable symmetric monoidal category M, to the category of unbiased cocommutative comonoids in M. The idea is to regard these categories as categories of algebras of 'PROPs'. Informally, a PROP is an algebraic structure that defines a type of algebra that lives in a symmetric monoidal category, and whose operations may have multiple inputs and multiple outputs; the following formal definition is the unbiased variant of the original definition, that was given in Section 24 of [ML65].

6.1. DEFINITION. A *PROP* $\mathbb{P}$ is a symmetric strict monoidal category $\mathbb{P}$ whose monoid of objects equals $(\mathbb{N}, +, 0)$. A *morphism* $\phi \colon \mathbb{P} \to \mathbb{Q}$ of PROPs is a symmetric strict monoidal functor that restricts to the identity on objects.

Unraveling this, a PROP $\mathbb{P}$ consists of a category $\mathbb{P}$ with $\mathrm{ob}\, \mathbb{P} = \mathbb{N}$, equipped with functors $\otimes_k \colon \mathbb{P}^k \to \mathbb{P}$, one for each $k \geq 0$, and symmetries $\mathfrak{s}_\sigma \colon n \to n$, that are functorial in $\sigma \in \Sigma_n$, satisfying the following axioms. Firstly the tensor products restrict to addition $(n_1, \dots, n_k) \mapsto n_1 + \cdots + n_k$ on objects, with 0 as unit, and they are strictly associative and unital in the sense that

$$\big((\xi_{11} \otimes \cdots \otimes \xi_{1l_1}) \otimes \cdots \otimes (\xi_{k1} \otimes \cdots \otimes \xi_{kl_k})\big) = (\xi_{11} \otimes \cdots \otimes \xi_{kl_k}),$$

for any double sequence of morphisms $\xi_{ij} \colon m_{ij} \to n_{ij}$ in $\mathbb{P}$, and $(\xi) = \xi$ for each single morphism $\xi \colon m \to n$. Secondly $(\mathfrak{s}_{\sigma_1} \otimes \cdots \otimes \mathfrak{s}_{\sigma_k}) = \mathfrak{s}_{(\sigma_1, \dots, \sigma_k)}$, where $(\sigma_1, \dots, \sigma_k)$ denotes the disjoint union of the permutations $\sigma_i \in \Sigma_{n_i}$, while the symmetry

$$\mathfrak{s}_{\tau_{(n_1, \dots, n_k)}} \colon n_1 + \cdots + n_k \to n_{\tau 1} + \cdots + n_{\tau k},$$

for the block permutation $\tau_{(n_1, \dots, n_k)}$ that is induced by $\tau \in \Sigma_k$ (see Example 3.21), is required to be natural in each $n_1, \dots, n_k$.

6.2. EXAMPLE. We denote by $\mathbb{F}$ the PROP that is (isomorphic to) the skeletal category of finite sets. More precisely, we think of a morphism $\xi \colon m \to n$ in $\mathbb{F}$ as being a function $\xi \colon [m] \to [n]$ between the ordinals $[m] = \{0 < \cdots < m - 1\}$ and $[n] = \{0 < \cdots < n - 1\}$. The tensor product $(\xi_1 \otimes \cdots \otimes \xi_k)$ is given by disjoint union:

$$[m_1 + \cdots + m_k] \cong [m_1] \sqcup \cdots \sqcup [m_k] \xrightarrow{\xi_1 \sqcup \cdots \sqcup \xi_k} [n_1] \sqcup \cdots \sqcup [n_k] \cong [n_1 + \cdots + n_k],$$

where the bijections are unique such that they preserve order; here the order on the disjoint union $[m_1] \sqcup \cdots \sqcup [m_k]$ is induced by the orders of each $[m_i]$ and by the rule that, for each pair $i < j$, all elements of $[m_i]$ are less than those of $[m_j]$. Finally, the symmetries $\mathfrak{s}_\sigma \colon n \to n$ are simply given by the permutations of $[n]$.

6.3. EXAMPLE. We denote by $\mathbb{H}$ the PROP that is (isomorphic to) the skeletal category of free, finitely generated abelian groups. Precisely, a morphism $\xi\colon m \to n$ in $\mathbb{H}$ is a homomorphism $\xi\colon \mathbb{Z}^m \to \mathbb{Z}^n$, which we identify with an $n \times m$-matrix with coefficients in $\mathbb{Z}$. The monoidal structure on $\mathbb{H}$ is given by disjoint sum, so that the tensor product of morphisms $\xi_i\colon m_i \to n_i$ is identified with the block matrix

$$
\begin{pmatrix}
\xi_1 & 0 & \ldots & 0 \\
0 & \xi_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & \xi_k
\end{pmatrix}.
$$

The symmetries $\mathfrak{s}_\sigma\colon n \to n$ are given by permuting the generators of $\mathbb{Z}^n$.

The opposite $\mathbb{F}^{\mathrm{op}}$ of the PROP $\mathbb{F}$, which is again a PROP, can be embedded into $\mathbb{H}$ by mapping each morphism $m \to n$, corresponding to a function $\xi\colon [n] \to [m]$, to the $n \times m$-matrix (ξ_{ij}) that is given by $\xi_{ij} = 1$ if $\xi i = j$ and 0 otherwise. Thus the image of $\mathbb{F}^{\mathrm{op}}$ in $\mathbb{H}$ is the subPROP consisting of all matrices that contain precisely one non-zero entry in each row, whose coefficient is 1. This gives a morphism of PROPs $j\colon \mathbb{F}^{\mathrm{op}} \to \mathbb{H}$.

6.4. DEFINITION. Let $\mathbb{P}$ be a PROP and $M = (M, \otimes, \mathfrak{a}, \mathfrak{i})$ a symmetric monoidal category. An *algebra* A of $\mathbb{P}$ in M is a symmetric monoidal functor $A\colon \mathbb{P} \to M$. A *morphism* $f\colon A \to B$ of such algebras is a monoidal transformation $f\colon A \Rightarrow B$. In other words, the category of algebras of $\mathbb{P}$ in M is defined as

$$
\mathsf{Alg}(\mathbb{P}, M) = \mathsf{sMonCat}(\mathbb{P}, M),
$$

where $\mathsf{sMonCat}$ denotes the 2-category of symmetric monoidal categories, symmetric monoidal functors and monoidal transformations.

In describing algebras of PROPs it is useful to abbreviate $A^{\otimes n} = (\overbrace{A \otimes \cdots \otimes A}^{n \text{ times}})$.

6.5. EXAMPLE. An algebra of the PROP $\mathbb{F}$ in M is essentially an *unbiased commutative monoid*; that is, an object A of M equipped with a family μ of morphisms $\mu_n\colon A^{\otimes n} \to A$, one for each $n \geq 0$, that make the following diagrams commute, where on the right σ is any permutation in Σ_n.

$$
\begin{array}{ccc}
(A^{\otimes n_1} \otimes \cdots \otimes A^{\otimes n_k}) \xrightarrow{(\mu_{n_1} \otimes \cdots \otimes \mu_{n_k})} A^{\otimes k} \\
\mathfrak{a} \downarrow \qquad\qquad\qquad \downarrow \mu_k \\
A^{\otimes(n_1+\cdots+n_k)} \xrightarrow{\mu_{n_1+\cdots+n_k}} A
\end{array}
\qquad
\begin{array}{ccc}
A \xrightarrow{\mathfrak{i}} (A) \\
\text{id} \searrow \ \downarrow \mu_1 \\
A
\end{array}
\qquad
\begin{array}{ccc}
A^{\otimes n} \\
\mathfrak{s}_\sigma \downarrow \ \searrow \mu_n \\
A^{\otimes n} \xrightarrow{\mu_n} A
\end{array}
$$

To be precise, every algebra $A\colon \mathbb{F} \to M$ induces the structure of an unbiased commutative monoid on $A(1)$, using the maps

$$
A(1)^{\otimes n} \xrightarrow{A_\otimes} A(n) \xrightarrow{A(!\colon [n]\to[1])} A(1)
$$

where $!\colon [n] \to [1]$ denotes the unique map into the terminal set $[1]$. That this family satisfies the axioms above is easy to check. Moreover, it is not hard to show that the assignment $A \mapsto A(1)$ extends to an equivalence of categories

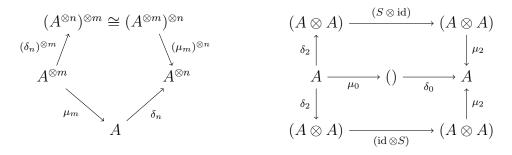$$\mathsf{Alg}(\mathbb{F}, M) \simeq \mathsf{ucMon}(M),$$

where $\mathsf{ucMon}(M)$ denotes the category of unbiased commutative monoids in M, in which a morphism $f\colon (A, \mu_A) \to (B, \mu_B)$ is simply a map $f\colon A \to B$ in M such that $\mu_B \circ f^{\otimes n} = f \circ \mu_A$. In proving this it is helpful to notice that each morphism $\xi\colon m \to n$ in $\mathbb{F}$ factors as

$$
\begin{array}{ccc}
m & \xrightarrow{\quad \xi \quad} & n, \\
& \searrow_{\mathfrak{s}_\sigma} \qquad \nearrow_{(! \otimes \cdots \otimes !)} & \\
& m_1 + \cdots + m_n &
\end{array}
$$

where $\sigma \in \Sigma_m$ is unique up to composition with disjoint unions $(\tau_1, \ldots, \tau_n)$ of symmetries $\tau_i \in \Sigma_{m_i}$.

Dual to the above, in the notion of an *unbiased cocommutative comonoid* A the structure is given by a family of maps $\delta_n\colon A \to A^{\otimes n}$. Analogous to the equivalence above we have $\mathsf{Alg}(\mathbb{F}^{\mathrm{op}}, M) \simeq \mathsf{ucocComon}(M)$.

6.6. EXAMPLE. An *unbiased bicommutative bimonoid* in M is a triple $A = (A, \mu, \delta)$ where (A, μ) forms an unbiased commutative monoid and (A, δ) forms an unbiased cocommutative comonoid, such that the compatibility diagrams on the left below commute. Here the isomorphism moves each copy of A in $(A^{\otimes n})^{\otimes m}$, that is indexed by $(i, j) \in [m] \times [n]$, to the copy in $(A^{\otimes m})^{\otimes n}$ that is indexed by $(j, i) \in [n] \times [m]$.

$$
\begin{array}{ccc}
(A^{\otimes n})^{\otimes m} \cong (A^{\otimes m})^{\otimes n} & \qquad & (A \otimes A) \xrightarrow{(S \otimes \mathrm{id})} (A \otimes A) \\
{\scriptstyle (\delta_n)^{\otimes m}} \nearrow \qquad \searrow {\scriptstyle (\mu_m)^{\otimes n}} & & {\scriptstyle \delta_2} \uparrow \qquad \downarrow {\scriptstyle \mu_2} \\
A^{\otimes m} \qquad\qquad A^{\otimes n} & & A \xrightarrow{\mu_0} () \xrightarrow{\delta_0} A \\
{\scriptstyle \mu_m} \searrow \qquad \nearrow {\scriptstyle \delta_n} & & {\scriptstyle \delta_2} \downarrow \qquad \uparrow {\scriptstyle \mu_2} \\
A & & (A \otimes A) \xrightarrow{(\mathrm{id} \otimes S)} (A \otimes A)
\end{array}
$$

A morphism $A \to B$ of unbiased bicommutative bimonoids is a map $A \to B$ in M that is simultaneously a map of monoids and comonoids. Furthermore, an unbiased bicommutative bimonoid A is called an *unbiased bicommutative Hopf monoid* whenever there exists a map $S\colon A \to A$ that makes the diagram on the right above commute. Such a map, which is necessarily unique (see e.g. Lemma 37 of [Por13]), is called the *antipode* of A.

The algebras of the PROP $\mathbb{H}$ are essentially the unbiased bicommutative Hopf monoids in M: each algebra $A\colon \mathbb{H} \to M$ induces the structure of such a Hopf monoid on $A(1)$

using the maps

$$\mu_n = \left[A(1)^{\otimes n} \xrightarrow{A_{\otimes}} A(n) \xrightarrow{A(1\ldots 1)} A(1) \right],$$

$$\delta_n = \left[A(1) \xrightarrow{A(1\ldots 1)^{\mathrm{t}}} A(n) \xrightarrow{A_{\otimes}^{-1}} A(1)^{\otimes n} \right]$$

and
$$S = \left[A(1) \xrightarrow{A(-1)} A(1) \right],$$

where $(1\ldots 1)^{\mathrm{t}}$ denotes the transpose of the $1 \times n$-matrix $(1\ldots 1)$. That these maps satisfy the bicommutative bimonoid axioms and the Hopf monoid axiom follows directly from various matrix equations. For example, the top of the diagram on the right above is the A-image of the matrix equation

$$(1 \quad 1) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (0).$$

One can show that the assignment $A \mapsto A(1)$ extends to an equivalence of categories

$$\mathsf{Alg}(\mathbb{H}, M) \simeq \mathsf{ubicHopf}(M),$$

where $\mathsf{ubicHopf}(M)$ denotes the category of unbiased bicommutative Hopf monoids in M. In fact it is shown in [Wad08] that, in the case that M is symmetric strict monoidal, the category of symmetric strict monoidal functors $\mathbb{H} \to M$ is isomorphic to the category of biased bicommutative Hopf monoids; it is straightforward to modify (parts of) the proof given there into a proof for the equivalence above.

As promised, we finish by showing that unbiased bicommutative Hopf monoids that are freely generated by unbiased cocommutative comonoids exist. For a much more detailed and thorough treatment of free, and cofree, non-commutative Hopf monoids we refer the reader to [Por13].

6.7. THEOREM. *Let M be a cocomplete symmetric monoidal category whose tensor product preserves colimits in each variable. The forgetful functor*

$$\mathsf{ubicHopf}(M) \to \mathsf{ucocComon}(M),$$

of unbiased bicommutative Hopf monoids to unbiased cocommutative comonoids, admits a left adjoint.

PROOF. We may just as well prove that the composite of the forgetful functor with the equivalences $\mathsf{Alg}(\mathbb{H}, M) \simeq \mathsf{ubicHopf}(M)$ and $\mathsf{ucocComon}(M) \simeq \mathsf{Alg}(\mathbb{F}^{\mathrm{op}}, M)$ admits a left adjoint. In light of Definition 6.4, this composite is simply the functor

$$\lambda_j \colon \mathsf{sMonCat}(\mathbb{H}, M) \to \mathsf{sMonCat}(\mathbb{F}^{\mathrm{op}}, M)$$

that is given by precomposition with the embedding of PROPs $j \colon \mathbb{F}^{\mathrm{op}} \to \mathbb{H}$. Hence the proof follows if we can show that we may apply Corollary 5.10, which means that we have to show that the companion $j_* \colon \mathbb{F}^{\mathrm{op}} \nrightarrow \mathbb{H}$ satisfies the left Beck-Chevalley condition.

Thus we consider a morphism ξ in $\mathbb{H}$ that is of the form $\xi\colon m \to n_1 + \cdots + n_k$; following the discussion preceding Corollary 5.10, we have to show that it can be uniquely represented as a composite $\xi = (\xi_1 \otimes \cdots \otimes \xi_k) \circ \zeta$, where the $\xi_i\colon l_i \to n_i$ are maps in $\mathbb{H}$ and $\zeta\colon m \to l_1 + \cdots + l_k$ is a map in $\mathbb{F}^{\mathrm{op}}$. Recalling the definitions of $\mathbb{H}$ and $j\colon \mathbb{F}^{\mathrm{op}} \to \mathbb{H}$ (see Example 6.3), this means that ξ is a $(n_1 + \cdots + n_k) \times m$-matrix with coefficients in $\mathbb{Z}$, the ξ_i are such $n_i \times l_i$-matrices, and ζ is an $(l_1 + \cdots + l_k) \times m$-matrix that contains precisely one non-zero entry in each row, whose coefficient is 1. It is easy to show that such a representation exists: we take $l_i = m$ and let ξ_i be the blocks of ξ as shown below. As the $km \times m$-matrix ζ we take the block matrix of identity matrices I_m of dimension m, as on the right.

$$
\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{pmatrix} = \begin{pmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \xi_k \end{pmatrix} \begin{pmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{pmatrix}
$$

It remains to prove that the representation $\xi = (\xi_1 \otimes \cdots \otimes \xi_k) \circ \zeta$ is unique, in the sense explained before Corollary 5.10. Hence consider another such representation $\xi = (\xi_1' \otimes \cdots \otimes \xi_k') \circ \zeta'$, where each ξ_i' is a $n_i \times l_i'$-matrix and ζ' is a $(l_1' + \cdots + l_k') \times m$-matrix in $\mathbb{F}^{\mathrm{op}} \subset \mathbb{H}$; it suffices to show that there exist $l_i' \times m$-matrices χ_i in $\mathbb{F}^{\mathrm{op}}$ satisfying $\zeta' = (\chi_1 \otimes \cdots \otimes \chi_k) \circ \zeta$ as well as $\xi_i = \xi_i' \circ \chi_i$, for each $i = 1, \ldots, k$. Using the same trick, we take the χ_i to be defined by the identity

$$
\zeta' = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_k \end{pmatrix} = \begin{pmatrix} \chi_1 & 0 & \cdots & 0 \\ 0 & \chi_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \chi_k \end{pmatrix} \begin{pmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{pmatrix}.
$$

Clearly $\zeta' \in \mathbb{F}^{\mathrm{op}}$ implies that each $\chi_i \in \mathbb{F}^{\mathrm{op}}$, while $\xi_i = \xi_i' \circ \chi_i$ follows by postcomposing both sides of the identity above with $(\xi_1' \otimes \cdots \otimes \xi_k')$. ∎

References

[CS10]　G. S. H. Cruttwell and M. A. Shulman. A unified framework for generalized multicategories. *Theory and Applications of Categories*, 24(21):580–655, 2010.

[DS97]　B. Day and R. Street. Monoidal bicategories and Hopf algebroids. *Advances in Mathematics*, 129(1):99–157, 1997.

[Get09]　E. Getzler. Operads revisited. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Volume I*, volume 269 of *Progress in Mathematics*, pages 675–698. Birkhäuser Boston Inc., 2009.

[GP99]　M. Grandis and R. Paré. Limits in double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40(3):162–220, 1999.

[GP04] M. Grandis and R. Paré. Adjoint for double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 45(3):193–240, 2004.

[GP08] M. Grandis and R. Paré. Kan extensions in double categories (on weak double categories, part III). *Theory and Applications of Categories*, 20(8):152–185, 2008.

[Gui80] R. Guitart. Relations et carrés exacts. *Les Annales des Sciences Mathématiques du Québec*, 4(2):103–125, 1980.

[Joh02] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium: Volume 1*. Number 43 in Oxford Logic Guides. Oxford Science Publications, 2002.

[Kel82] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982.

[Kou13] S. R. Koudenburg. *Algebraic weighted colimits*. PhD thesis, University of Sheffield, 2013.

[Kou14] S. R. Koudenburg. On pointwise Kan extensions in double categories. *Theory and Applications of Categories*, 29(27):781–818, 2014.

[Lei04] T. Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.

[ML65] S. Mac Lane. Categorical algebra. *Bulletin of the American Mathematical Society*, 71(1):40–106, 1965.

[ML98] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Por13] H.-E. Porst. The formal theory of Hopf algebras: Part I: Hopf monoids in a monoidal category. Preprint, 2013.

[Shu08] M. A. Shulman. Framed bicategories and monoidal fibrations. *Theory and Applications of Categories*, 20(18):650–738, 2008.

[Str74] R. Street. Fibrations and Yoneda's lemma in a 2-category. In *Proceedings of the Sydney Category Theory Seminar, 1972/1973*, volume 420 of *Lecture Notes in Mathematics*, pages 104–133. Springer, Berlin, 1974.

[Wad08] S. Wadsley, 2008. Unpublished note, available as `http://www.dpmms.cam.ac.uk/~sjw47/PROP2.pdf`.

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