Theory and Applications of Categories, Vol. 30, No. 47, 2015, pp. 1624–1646.

THE WAVES OF A TOTAL CATEGORY

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ABSTRACT. For any *total* category $\mathscr{K}$, with defining adjunction $\bigvee \dashv Y : \mathscr{K} \to \mathbf{set}^{\mathscr{K}^{\mathrm{op}}}$, the expression $W(A)(K) = \mathbf{set}^{\mathbf{set}^{\mathscr{K}^{\mathrm{op}}}}(\mathscr{K}(A, \bigvee -), [K, -])$, where $[K, -]$ is evaluation at K, provides a well-defined functor $W : \mathscr{K} \to \widetilde{\mathscr{K}} = \mathbf{set}^{\mathscr{K}^{\mathrm{op}}}$. Also, there are natural transformations $\beta : W\bigvee \to 1_{\widetilde{\mathscr{K}}}$ and $\gamma : \bigvee W \to 1_{\mathscr{K}}$ satisfying $\bigvee\beta = \gamma\bigvee$ and $\beta W = W\gamma$. A total $\mathscr{K}$ is *totally distributive* if $\bigvee$ has a left adjoint. We show that $\mathscr{K}$ is totally distributive iff γ is invertible iff $W \dashv \bigvee$. The elements of $W(A)(K)$ are called *waves from K to A*.

Write $\widetilde{\mathscr{K}}(K, A)$ for $W(A)(K)$. For any total $\mathscr{K}$ there is an associative composition of waves. Composition becomes an arrow $\bullet : \widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}} \to \widetilde{\mathscr{K}}$. Also, there is an augmentation $(\bar{-}) : \widetilde{\mathscr{K}}(-, -) \to \mathscr{K}(-, -)$ corresponding to a natural $\delta : W \to Y$ constructed via β. We show that if $\mathscr{K}$ is totally distributive then $\bullet : \widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}} \to \widetilde{\mathscr{K}}$ is invertible and then $\widetilde{\mathscr{K}}$ supports an idempotent comonad structure. In fact, $\widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}} = \widetilde{\mathscr{K}} \circ_{\widetilde{\mathscr{K}}} \widetilde{\mathscr{K}}$ so that $\bullet$ is the coequalizer of $\bullet\widetilde{\mathscr{K}}$ and $\widetilde{\mathscr{K}}\bullet$, making $\widetilde{\mathscr{K}}$ a *taxon* in the sense of Koslowski [KOS]. For a small taxon $\mathscr{T}$, the category of interpolative modules $\mathbf{iMod}(\mathbf{1}, \mathscr{T})$ is totally distributive [MRW]. Here we show, for any totally distributive $\mathscr{K}$, that there is an equivalence $\mathscr{K} \to \mathbf{iMod}(\mathbf{1}, \widetilde{\mathscr{K}})$.

1. Preliminaries

1.1. Most of our notations and conventions are carried over from [MRW]. For a category $\mathscr{K}$, we write $|\mathscr{K}|$ for the set of objects of $\mathscr{K}$. If K and A are objects of $\mathscr{K}$, $\mathscr{K}(K, A)$ denotes the set of arrows from K to A. We assume the existence of full categories of sets called **set** and **SET**, both toposes, with **set** a (full) subcategory of **SET**, and $|\mathbf{set}|$ an object of **SET**. The sets in **set** are called *small* sets. We assume that **set** has all sums indexed by objects of **set** and **SET** has all sums indexed by objects of **SET**. We will write $i : \mathbf{set} \to \mathbf{SET}$ for the inclusion. We write **CAT** for the 2-category of category objects in **SET**. If a category $\mathscr{K}$ in **CAT** has all its hom-sets $\mathscr{K}(K, A)$ in **set**, we say that $\mathscr{K}$ is *locally small*. For $\mathscr{K}$ and $\mathscr{A}$ in **CAT**, we write $\mathbf{PRO}(\mathscr{K}, \mathscr{A})$ for the category

The author gratefully acknowledges financial support from the Canadian NSERC.
Diagrams typeset using M. Barr's diagxy package
Received by the editors 2015-07-14 and, in revised form, 2015-11-09.
Transmitted by Ross Street. Published on 2015-11-30.
2010 Mathematics Subject Classification: 18A30.
Key words and phrases: adjunction, totally cocomplete, totally distributive, taxon, i-module, proarrow equipment.

of all functors $\mathscr{A}^{\mathrm{op}} \times \mathscr{K} \to \mathbf{SET}$ We write $P : \mathscr{K} \nrightarrow \mathscr{A}$ for an object of $\mathbf{PRO}(\mathscr{K}, \mathscr{A})$ and say that it is *small* if P factors through $i : \mathbf{set} \to \mathbf{SET}$. With the usual compositions, $\mathbf{PRO}$ is a bicategory and we have the usual proarrow equipment $(-)_* : \mathbf{CAT} \to \mathbf{PRO}$.

2. The Yoneda functor and total categories

2.1. The following subsection elaborates on the $(-)_*$ notation, which is used to provide precision in many of the proofs in the paper.

A locally small category $\mathscr{K}$ has a Yoneda functor $Y : \mathscr{K} \to \mathbf{CAT}(\mathscr{K}^{\mathrm{op}}, \mathbf{set})$. We will usually write $\widehat{(-)} : \mathbf{CAT}^{\mathrm{coop}} \to \mathbf{CAT}$ for $\mathbf{CAT}((-)^{\mathrm{op}}, \mathbf{set})$. Note that if $\mathscr{K}$ is locally small then $\widehat{\mathscr{K}}$ is locally small if and only if $\mathscr{K}$ is equivalent to a small category — the "only if" clause being an important result of [F&S] — but, in any event, $\widehat{\mathscr{K}}$ is in $\mathbf{CAT}$. The objects of $\widehat{\mathscr{K}}$ are small profunctors $P : \mathbf{1} \nrightarrow \mathscr{K}$. Henceforth, $\mathscr{K}$ will denote a locally small category. For A in $\mathscr{K}$, $A : \mathbf{1} \to \mathscr{K}$ gives rise to the profunctor $A_* : \mathbf{1} \nrightarrow \mathscr{K}$ and $A_* = \mathscr{K}(-, A) = Y(A)$. Note that $A^* = \mathscr{K}(A, -) : \mathscr{K} \nrightarrow \mathbf{1}$ is right adjoint to A_* as a profunctor. Also, $K : \mathbf{1} \to \mathscr{K}$ gives rise to $\widehat{K} : \widehat{\mathscr{K}} \to \widehat{\mathbf{1}} = \mathbf{set}$, which is easily seen to be *evaluation at* K, and since $\mathbf{1}$ is small and $\mathscr{K}$ is locally small, $\widehat{K}$ has both left and right adjoints. We will write $K_+ : \mathbf{set} \to \widehat{\mathscr{K}}$ for the right adjoint of $\widehat{K}$. We have $K_+(X)(A) = X^{\mathscr{K}(K,A)}$. Since $\mathbf{CAT}$ is (cartesian) closed and $\widehat{\mathscr{K}} = \mathbf{CAT}(\mathscr{K}^{\mathrm{op}}, \mathbf{set}) = \mathbf{set}^{\mathscr{K}^{\mathrm{op}}}$ we have *evaluation* $\mathscr{K}^{\mathrm{op}} \times \widehat{\mathscr{K}} \to \mathbf{set}$ which sends a pair (K, P) to $\widehat{K}(P) = P(K)$. However it is convenient for us to write $[K, P]$, functorially, for the common value $\widehat{K}(P) = P(K)$ and denote the elements of $[K, P]$ by arrows $e : K \dashrightarrow P$. Such arrows are composable with arrows of the form $k : L \to K$ in $\mathscr{K}$, where the composite $ek : L \dashrightarrow P$ is $P(k)(e) = [k, P](e)$, and with arrows of the form $t : P \to Q$ in $\widehat{\mathscr{K}}$, where the composite $te : K \dashrightarrow Q$ is $t(K)(e) = [K, t](e)$. It is routine that these composites associate so that tek is well defined, making the arrows $e : K \dashrightarrow P$ elements of a module. In fact it may be useful to observe that $[-, -] : \widehat{\mathscr{K}} \nrightarrow \mathscr{K}$ is an isomorph of the small profunctor Y^* (because $Y^*(K, P) = \widehat{\mathscr{K}}(K_*, P)$) but $[-, -]$ exists as a small profunctor, for any $\mathscr{K}$, irrespective of local smallness of $\mathscr{K}$.

Yoneda's Lemma says that the function $\xi_{K,P} : \widehat{\mathscr{K}}(K_*, P) \to [K, P]$ given by $\xi_{K,P}(t) = t(K)(1_K)$ is an isomorphism, natural in K and P. We will write $\zeta_{K,P} : [K, P] \to \widehat{\mathscr{K}}(K_*, P)$ for the inverse of $\xi_{K,P}$ and $\zeta_{K,P}(e) = e_*$. Thus e_* is the unique natural transformation that corresponds to element e. We have $[K, A_*] = \mathscr{K}(K, A)$ and now our e_* notation specializes to the equality:

$$\begin{tikzcd} {[K, A_*]} \arrow[dr, "\zeta_{K,A_*}"] \arrow[d, equal] & \\ \mathscr{K}(K, A) \arrow[r, "Y_{K,A} = (-)_{*K,A}"'] & \widehat{\mathscr{K}}(K_*, A_*) \end{tikzcd}$$

2.2. Following [S&W] we say that the locally small $\mathscr{K}$ is *totally cocomplete* (usually abbreviated to *total*) if Y has a left adjoint, which will then be called X. Furthermore, η:

$1_{\widehat{\mathscr{K}}} \to YX$ and $\epsilon : XY \to 1_{\mathscr{K}}$ will be used consistently for the unit and counit respectively, so that we speak of the adjunction $(\eta, \epsilon : X \dashv Y)$. However, just as we write $Y(A) = A_* = \mathscr{K}(-, A)$ so also we frequently will write $X(P) = \bigvee P$. Since Y is fully faithful, ϵ is invertible. In terms of the notations introduced so far, we have components $\eta P : P \to (\bigvee P)_*$ in $\widehat{\mathscr{K}}$ and invertible components $\epsilon A : \bigvee A_* \to A$ in $\mathscr{K}$. So for $e : K \rightsquigarrow P$, $(\eta P)(K)(e) = \eta P.e$ can safely be abbreviated to $\eta e : K \rightsquigarrow (\bigvee P)_*$ which is simply an arrow $\eta e : K \to \bigvee P$ in $\mathscr{K}$. The adjunction equations for $X \dashv Y$ are:

$$\begin{tikzcd} A_* \arrow[r, "\eta A_*"] \arrow[dr, "1_{A_*}"'] & (\bigvee A_*)_* \arrow[d, "(\epsilon A)_*"] \\ & A_* \end{tikzcd} \quad \text{in } \widehat{\mathscr{K}}, \text{ for all } A \text{ in } \mathscr{K} \text{ and} \qquad \begin{tikzcd} \bigvee(\bigvee P)_* \arrow[d, "\epsilon \bigvee P"'] & \bigvee P \arrow[l, "\bigvee \eta P"'] \arrow[dl, "1_{\bigvee P}"] \\ \bigvee P & \end{tikzcd} \quad \text{in } \mathscr{K}, \text{ for all } P \text{ in } \widehat{\mathscr{K}}$$

But ϵA is an isomorphism and since $(\epsilon A)_*^{-1} = \eta A_*$ and ηA_* *is* a_* for a unique $a : A \to \bigvee A_*$ in $\mathscr{K}$, it is consistent with our other abbreviations, to write ηA for this a so that $(\eta A)_* = \eta A_*$. Now, the first of the triangle equations can be rewritten as

$$\begin{tikzcd} A \arrow[r, "\eta A"] \arrow[dr, "1_A"'] & \bigvee A_* \arrow[d, "\epsilon A"] \\ & A \end{tikzcd} \quad \text{in } \mathscr{K}, \text{ for all } A \text{ in } \mathscr{K}$$

From the second of the triangle equations, we deduce further that the special case $\eta \bigvee P$ of the abbreviation of the last paragraph satisfies $\eta \bigvee P = \bigvee \eta P$, for all P in $\widehat{\mathscr{K}}$.

2.3. Remark. For *any* adjunction $(\eta, \epsilon; X \dashv Y) : \mathscr{K} \to \mathscr{L}$, it is classical (see, for example, Theorem 1, page 90 of [MAC]) that ϵ is invertible if and only if Y is fully faithful. In this event, $\eta Y = Y\epsilon^{-1}$. In fact, if Y is also dense as in the case of the Yoneda functor (meaning that $1_{\mathscr{L}}$ is the left Kan extension of Y along Y), then η is uniquely determined by the requirement that $\eta Y = Y\epsilon^{-1}$.

2.4. For P in $\widehat{\mathscr{K}}$, $\bigvee P$ is the colimit of the discrete fibration, $\bar{P} : \mathrm{el}P \to \mathscr{K}$ determined by P and is easily seen to be given by $\int^K [K, P] \cdot K$, where $\cdot$ denotes *multiple*, so that $[K, P] \cdot K$ is the $[K, P]$-fold sum of copies of K, and the integral $\int$ is the familiar quotient of the corresponding sum $\sum$. Thus, for each element $e : K \rightsquigarrow P$, we have a colimit injection $i_e : K \to \bigvee P$, in $\mathscr{K}$. Since P is the colimit of $\mathrm{el}P \xrightarrow{\bar{P}} \mathscr{K} \xrightarrow{(-)_*} \widehat{\mathscr{K}}$, with colimit injections $K_* \xrightarrow{e_*} P$, in $\widehat{\mathscr{K}}$, we have

$$\begin{tikzcd} K_* \arrow[r, "e_*"] \arrow[rr, bend right, "(i_e)_*"'] & P \arrow[r, "\eta P"] & (\bigvee P)_* \end{tikzcd}$$

In fact, for any adjunction with invertible counit, colimit comparisons for the right adjoint are given by instances of the unit. From these observations and the identity above, the following identities follow easily:

For each pair $K \overset{e}{\dashrightarrow} P \xrightarrow{t} Q$, we have

$$\begin{tikzcd} K \arrow[r, "i_e"] \arrow[rr, bend right, "i_{te}"'] & \bigvee P \arrow[r, "\bigvee t"] & \bigvee Q \end{tikzcd}$$

For an element $f : K \dashrightarrow A_*$, one and the same as an arrow $f : K \to A$, we have

$$\begin{tikzcd} K \arrow[r, "i_f"] \arrow[rr, bend right, "f"'] & \bigvee A_* \arrow[r, "\epsilon A"] & A \end{tikzcd}$$

For any element $e : K \dashrightarrow P$, we have

$$\begin{tikzcd} K \arrow[r, "\eta K"] \arrow[rr, bend right, "i_e"'] & \bigvee K_* \arrow[r, "\bigvee e_*"] & \bigvee P \end{tikzcd}$$

2.5. Remark. Our "sup"-notation while suggestive for those familiar with (constructively) complete distributive latices is less well-adapted when we want to speak of preservation by a functor $F : \mathscr{K} \to \mathscr{L}$ of the colimits we call $\bigvee P$ (unless $\widehat{F} : \widehat{\mathscr{L}} \to \widehat{\mathscr{K}}$ has a left adjoint). The colimit $\bigvee P$ is more customarily denoted $P \cdot 1_{\mathscr{K}}$, a particular weighted colimit, and is said to be preserved by F when the arrows $Fi_x : FK \to F \bigvee K$ provide an isomorphism $P \cdot F \overset{\simeq}{\to} F(P \cdot 1_{\mathscr{K}})$ of weighted colimits. Note that, for total $\mathscr{K}$ and any functor $F : \mathscr{K} \to \mathscr{L}$ for which $P \cdot F$ exists in $\mathscr{L}$, the colimit comparison $P \cdot F \to F(\bigvee P)$ is given by the natural transformation $P \to \mathscr{L}(F-, F(\bigvee P))$ determined by the family $\langle Fi_x : FK \to F(\bigvee P)\rangle_{x \in [K,P]}$.

2.6. Remark. In the next section, in Theorem 3.1, we make passing use of the fact that **set** is *cototal* (meaning that $\mathbf{set}^{\mathrm{op}}$ is total). This follows from the fact that, for any functor $E : \mathbf{set} \to \mathbf{set}$, $\mathbf{set}^{\mathbf{set}}(E, 1_{\mathbf{set}})$ is small. To prove the latter, proceed as in the proof of Lemma 2.9 of [MRW] which establishes an apparently more complicated result.

3. The waves of a total category

3.1. Theorem. *For any total category $\mathscr{K}$, there is a functor $W : \mathscr{K} \to \widehat{\mathscr{K}}$ given by*

$$W(A)(K) = \mathbf{set}^{\widehat{\mathscr{K}}}(\mathscr{K}(A,-).\bigvee, [K,-]) = \mathbf{set}^{\widehat{\mathscr{K}}}(A^*.\bigvee, \widehat{K})$$

whose transpose $\mathscr{K}^{\mathrm{op}} \times \mathscr{K} \to$ **set** *will be denoted $\widetilde{\mathscr{K}}(-,-)$ so that $W(A)(K) = \widetilde{\mathscr{K}}(K,A)$. We adapt [J&J] and call $\widetilde{\mathscr{K}}(K,A)$ the set of* waves *from K to A and write $\omega : K \rightsquigarrow A$ for a* *typical wave. Moreover, there is a natural transformation* $\delta : W \to Y$, *whose* A*-component* $\delta(A) : W(A) \to Y(A)$ *is a natural transformation whose* K*-component,* $\delta(A)(K)$*, is denoted*

$$\delta(K,A) : W(A)(K) = \widetilde{\mathscr{K}}(K,A) \to \mathscr{K}(K,A) = Y(A)(K)$$

defined by $\delta(K,A)(K \overset{\omega}{\rightsquigarrow} A) = \omega A_*(\eta A) \in \mathscr{K}(K,A)$ *and we write* $\delta(K,A)(\omega) = \overline{\omega}$.

Proof. The definition of W, equivalently of $\widetilde{\mathscr{K}}$, is manifestly functorial. The force of the claim is that $\widetilde{\mathscr{K}}(K,A)$ is *small*, meaning that it is an object of **set**. In the following calculation we use the fact that **set** is cototal and that its identity functor is represented by $\mathbf{set}(1,-)$. Note that we use $\bigwedge$ for a functor that provides cototality.

$$\begin{aligned}
\widetilde{\mathscr{K}}(K,A) &= \mathbf{set}^{\widehat{\mathscr{K}}}(A^*.\bigvee, \widehat{K}) \\
&\cong \mathbf{set}^{\mathbf{set}}(A^*.\bigvee.K_+, 1_{\mathbf{set}}) \\
&\cong (\mathbf{set}^{\mathbf{set}})^{\mathrm{op}}(\mathbf{set}(1,-), A^*.\bigvee.K_+) \\
&\cong \mathbf{set}(1, \bigwedge(A^*.\bigvee.K_+)) \\
&\cong \bigwedge(A^*.\bigvee.K_+)
\end{aligned}$$

The second to last set is small, since **set** is a locally small category, hence $\widetilde{\mathscr{K}}(K,A)$ is small.

For $k : L \to K$, $\omega : A^*.\bigvee \to \widehat{K}$, and $a : A \to B$, the following unambiguous composite of natural transformations:

$$B^*.\bigvee \xrightarrow{a^*.\bigvee} A^*.\bigvee \xrightarrow{\omega} \widehat{K} \xrightarrow{\widehat{k}} \widehat{L}$$

defines the wave

$$a\omega k = \widetilde{\mathscr{K}}(k,a)(\omega) = (L \xrightarrow{k} K \overset{\omega}{\rightsquigarrow} A \xrightarrow{a} B)$$

(Both $(-)^*$ and $\widehat{(-)}$ have variance $(-)^{\mathrm{coop}}$.) For naturality of δ it suffices to show that $\overline{a\omega k} = a\overline{\omega}k$. Here, and elsewhere, the following configuration diagram is helpful. For a P in $\widehat{\mathscr{K}}$ and an arrow $f : A \to \bigvee P$ in $\mathscr{K}$, $(\omega P)(f)$ is a K-element of P as in

$$\begin{tikzcd}
K \arrow[r, squiggly, "\omega"] \arrow[d, dotted, "(\omega P)(f)"'] & A \arrow[d, "f"] \\
P & \bigvee P
\end{tikzcd} \tag{1}$$

(Observe that the right hand vertical arrow of Diagram 1 can equally be displayed as

$f : A \dashrightarrow (\bigvee P)_*$.) In terms of Diagram 1 we see $((a\omega k)(P))(f)$ as:

$$\begin{tikzcd} L \arrow[r, "k"] \arrow[d, dotted, "((a\omega k)(P))(f)=((\omega P)(fa))k"'] & K \arrow[r, squiggly, "\omega"] \arrow[d, dotted, "(\omega P)(fa)"] & A \arrow[r, "a"] \arrow[d, "fa"] & B \arrow[d, "f"] \\ P & P & \bigvee P & \bigvee P \end{tikzcd}$$

Since $\overline{a\omega k} = (\omega B_*)(\eta B.a)k$, to show $\overline{a\omega k} = a\overline{\omega}k$ is to show

$$((\omega B_*)(\eta B.a))k = a((\omega A_*)(\eta A))k$$

for which the case $k = 1_K$ clearly suffices. By naturality of η (here standing for ϵ^{-1}) we can rewrite the condition as

$$(\omega B_*)(\bigvee a_*.\eta A) = a((\omega A_*)(\eta A))$$

which follows from

$$\begin{tikzcd} \mathscr{K}(A, \bigvee A_*) \arrow[r, "\mathscr{K}(A,\bigvee a_*)"] \arrow[d, "\omega A_*"'] & \mathscr{K}(A, \bigvee B_*) \arrow[d, "\omega B_*"] \\ {[K, A_*]} \arrow[r, "{[K,a_*]}"'] & {[K, B_*]} \end{tikzcd}$$

an instance of naturality of ω, evaluated at ηA. ∎

3.2. Remark. The definition of a wave $\omega : K \rightsquigarrow A$ as a natural transformation $A^*.\bigvee \to \widehat{K}$ provides that it has, for each $P \in \widehat{\mathscr{K}}$, a P-component which is a function $\omega P : \mathscr{K}(A, \bigvee P) \to [K, P]$. If, instead of a total category $\mathscr{K}$, we were speaking of a complete lattice $\mathscr{K}$ with elements K and A, we would have K *totally below* A, written $K << A$, if and only if,

$$(\forall P \in \widehat{\mathscr{K}})(A \leq \bigvee P \text{ implies } K \in P)$$

where now $\widehat{\mathscr{K}}$ is the lattice of down-sets of $\mathscr{K}$. The reader familiar with [LAW] will see the connection between waves and the totally below relation. In the study of continuous posets the latter relation, with universal quantification restricted to up-directed down-sets, has long been called the *way below* relation. In their paper on continuous categories, [J&J], Johnstone and Joyal generalized the way below relation to *wavy arrows* with a definition analogous to ours. Since the word *wave* already connotes a directional phenomenon and physical wavy arrows would not fly very well, we speak more simply of *waves.* ∎

Whenever possible we will abbreviate $(\omega P)(f)$ by $\omega(f)$. We illustrate this by exhibiting naturality of ω in P in terms of Diagram 1:

$$\begin{tikzcd}
K \arrow[r, squiggly, "\omega"] \arrow[d, dotted, "\omega(f)"] \arrow[dd, dotted, bend right, "\omega((\bigvee t)f)"'] & A \arrow[d, "f"] \\
P \arrow[d, "t"] & \bigvee P \arrow[d, "\bigvee t"] \\
Q & \bigvee Q
\end{tikzcd}$$

for a $t : P \to Q$ in $\widehat{\mathscr{K}}$.

The reader is encouraged to reprove the equation $\overline{a\omega} = a\overline{\omega}$ (part of the proof of Theorem 3.1) in terms of Diagram 1.

3.3. Remark. Any full set of natural transformations between a parallel pair of functors is the value of an end and here we see that

$$\widetilde{\mathscr{K}}(K, A) = \int_P \mathbf{set}(\mathscr{K}(A, -).\bigvee P, [K, P]) \cong \int_P [K, P]^{\mathscr{K}(A, \bigvee P)}$$

where the end is over P in $\widehat{\mathscr{K}}$. Of course, we can see $\widetilde{\mathscr{K}}$ as a small profunctor from $\mathscr{K}$ to $\mathscr{K}$ and the display above reveals the following diagram as a right extension diagram in the bicategory of profunctors. (Recall that $\bigvee_*(K, P) = \mathscr{K}(K, \bigvee P)$. We noted in subsection 2.1 that $[-,-] \cong Y^*$.)

$$\begin{tikzcd}
\widehat{\mathscr{K}} \arrow[rr, "\bigvee_*", "\shortmid"{marking}] \arrow[dr, "Y^* \cong {[-,-]}"', "\shortmid"{marking}] & \arrow[d, phantom, "\leftarrow"] & \mathscr{K} \arrow[dl, "\widetilde{\mathscr{K}}", "\shortmid"{marking}] \\
& \mathscr{K} &
\end{tikzcd}$$

The totally below relation for a sup-lattice is an order ideal and is given by an analogous right extension.

3.4. Proposition. *For any wave $\omega : K \rightsquigarrow A$ in a total category $\mathscr{K}$, any P in $\widehat{\mathscr{K}}$, and any arrow $f : A \to \bigvee P$ in $\mathscr{K}$, we have the following equation:*

$$\begin{tikzcd}
K \arrow[r, "\overline{\omega}"] \arrow[d, dotted, "\omega(f)"'] & A \arrow[d, dotted, "f"] \\
P \arrow[r, "\eta P"'] & (\bigvee P)_*
\end{tikzcd} \tag{2}$$

PROOF.

$$
\begin{tikzcd}
K \arrow[r, squiggly, "\omega"] \arrow[d, dotted, "\omega(f)"] \arrow[dd, dotted, bend right, "\omega(\bigvee \eta P.f)"'] & A \arrow[d, "f"'] \arrow[dr, "\eta A"] & \\
P \arrow[d, "\eta P"] & \bigvee P \arrow[d, shift right, "\bigvee \eta P"'] \arrow[d, shift left, "\eta \bigvee P"] & \bigvee A_* \arrow[dl, "\bigvee f_*"] \\
(\bigvee P)_* & \bigvee(\bigvee P)_* &
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, squiggly, "\omega"] \arrow[d, dotted, "\omega(\eta A)"] \arrow[dd, dotted, bend right, "\omega(\bigvee f_*.\eta A)"'] & A \arrow[d, "\eta A"] \\
A_* \arrow[d, "f_*"] & \bigvee A_* \arrow[d, "\bigvee f_*"] \\
(\bigvee P)_* & \bigvee(\bigvee P)_*
\end{tikzcd}
$$

Consider the diagram on the left above. Amongst other things, it shows by naturality of ω that $\eta P.\omega(f) = \omega(\bigvee \eta P.f)$. But it also shows that $\bigvee \eta P.f = \bigvee f_*.\eta A$. So we have $\eta P.\omega(f) = \omega(\bigvee f_*.\eta A)$. Turning to the diagram on the right, we see again by naturality of ω that $\omega(\bigvee f_*.\eta A) = f_*.\omega(\eta A)$. But $\omega(\eta A) = \overline{\varpi}$ and the element $f_*.\overline{\varpi}$ is equally the arrow $f\overline{\varpi}$ so that we have $\eta P.\omega(f) = f.\overline{\varpi}$ as required. ∎

3.5. DEFINITION. *For any total category $\mathscr{K}$, there are natural transformations*

$$\beta : WX \to 1_{\widehat{\mathscr{K}}} \quad \textit{and} \quad \gamma : XW \to 1_{\mathscr{K}}$$

where $\beta P : \widetilde{\mathscr{K}}(-, \bigvee P) \to P$ has $\beta(K,P) = \beta P(K) : \widetilde{\mathscr{K}}(K, \bigvee P) \to [K,P]$ given by

$$\beta(K,P)(\omega : K \rightsquigarrow \bigvee P) = \omega P(1_{\bigvee P}) = \omega(1) : K \dashrightarrow P$$

and $\gamma A : \bigvee \widetilde{\mathscr{K}}(-,A) \to A$ in $\mathscr{K}$ is defined to be $\epsilon A.\bigvee \delta A$ so that

$$\gamma = \epsilon.X\delta$$

∎

3.6. REMARK. It should be remarked that, for any wave $\omega : K \rightsquigarrow A$, any P, and any $f : A \to \bigvee P$, we have

$$\omega P(f) = \beta P(f.\omega) \tag{3}$$

It follows that, for parallel waves ψ and ω, $\psi = \omega$ if and only if, for all P and all $f : A \to \bigvee P$, $\beta P(f.\psi) = \beta P(f.\omega)$.

3.7. PROPOSITION. *For any wave $\omega : K \rightsquigarrow A$, seen as an element $\omega : K \dashrightarrow WA$, in any total category $\mathscr{K}$, we have the following equation:*

$$
\begin{tikzcd}
K \arrow[r, "i_\omega"] \arrow[dr, "\bar{\varpi}"'] & \bigvee WA \arrow[d, "\gamma A"] \\
& A
\end{tikzcd}
$$

Proof.

$$\begin{tikzcd}
K \arrow[r, "i_\omega"] \arrow[dr, "i_{\delta.\omega}=i_{\bar\omega}"] \arrow[ddr, bend right, "\bar\omega"'] & \bigvee WA \arrow[d, "\bigvee \delta A"] \arrow[dd, bend left, "\gamma A"] \\
 & \bigvee YA \arrow[d, "\epsilon A"] \\
 & A
\end{tikzcd}$$

We have used the second and third identities of sub-section 2.4 and $\delta.\omega = \bar\omega$. ■

3.8. Lemma. *For any total category,*

$$\begin{tikzcd}
W \arrow[r, "W\epsilon^{-1}"] \arrow[dr, "\delta"'] & WXY \arrow[d, "\beta Y"] \\
 & Y
\end{tikzcd}$$

Proof. For a total category $\mathscr{K}$, the equation of the statement is equivalent to the following equation, for all K and A in $\mathscr{K}$,

$$\begin{tikzcd}
\widetilde{\mathscr{K}}(K,A) \arrow[r, "\widetilde{\mathscr{K}}(K{,}\epsilon^{-1}A)"] \arrow[dr, "\delta(K{,}A)"'] & \widetilde{\mathscr{K}}(K, \bigvee A_*) \arrow[d, "\beta(K{,}A_*)"] \\
 & \mathscr{K}(K,A) = [K, A_*]
\end{tikzcd}$$

which follows *immediately* from the definitions. ■

3.9. Lemma. *For any total category $\mathscr{K}$,*

$$\begin{tikzcd}
WX \arrow[r, "\beta"] \arrow[dr, "\delta X"'] & 1_{\widehat{\mathscr{K}}} \arrow[d, "\eta"] \\
 & YX
\end{tikzcd}$$

Proof. Apply the equation of Lemma 3.8 to X, note $\epsilon^{-1}X = \eta X$, and invoke naturality of β. ■

3.10. PROPOSITION. *For any total category $\mathscr{K}$, $\beta : WX \to 1_{\widehat{\mathscr{K}}}$ and $\gamma : XW \to 1_{\mathscr{K}}$ satisfy*

$$X\beta = \gamma X : XWX \longrightarrow X \qquad \text{and} \qquad \beta W = W\gamma : WXW \longrightarrow W$$

PROOF. To show the first equation, observe that $X\beta$ is the unique natural transformation satisfying $YX\beta.\eta WX = \eta.\beta$. From Lemma 3.9 we have $\eta.\beta = \delta X$ so it suffices to show that $Y\gamma X.\eta WX = \delta X$. From Definition 3.5 it follows that we must show $Y\epsilon X.YX\delta X.\eta WX = \delta X$, which is shown by

$$\begin{tikzcd}
YXWX \arrow[r, "YX\delta X"] & YXYX \arrow[r, "Y\epsilon X"] & YX \\
WX \arrow[u, "\eta WX"] \arrow[rr, "\delta X"'] & & YX \arrow[ul, "\eta YX"'] \arrow[u, "1_{YX}"']
\end{tikzcd}$$

For the second equation, observe that the description of β gives $\beta W : WXW \longrightarrow W$, with $\beta W(A) = \beta(WA)$ so that we have

$$\beta WA(K) = \beta(K, WA) : \widetilde{\mathscr{K}}(K, \bigvee(WA)) \to [K, WA] = \widetilde{\mathscr{K}}(K, A)$$

and, for a wave $\omega : K \rightsquigarrow \bigvee(WA)$, we have the wave

$$\beta(K, WA)(\omega) = \omega(WA)(1_{\bigvee(WA)}) = \omega(1) : K \rightsquigarrow A \tag{4}$$

Also, we have

$$W\gamma A(K) = \widetilde{\mathscr{K}}(K, \gamma A) : \widetilde{\mathscr{K}}(K, \bigvee(WA)) \to \widetilde{\mathscr{K}}(K, A)$$

and hence, for a wave $\omega : K \rightsquigarrow \bigvee(WA)$, we have the wave

$$\widetilde{\mathscr{K}}(K, \gamma A)(\omega) = \gamma A.\omega : K \rightsquigarrow A \tag{5}$$

We must show that the parallel waves $\omega(1)$ and $\gamma A.\omega$, of Equations 4 and 5 respectively, are equal. So take P in $\widehat{\mathscr{K}}$, $f : A \to \bigvee$ in $\mathscr{K}$, and consider the following diagram:

$$\begin{tikzcd}
K \arrow[r, rightsquigarrow, "\omega"] \arrow[d, dotted, "\omega(1)"'] & \bigvee(WA) \arrow[d, "1"'] \arrow[ddr, "\gamma A"] & \\
WA \arrow[dd, "Wf"'] & \bigvee(WA) \arrow[dd, "\bigvee(Wf)"] & \\
 & & A \arrow[ddl, "f"] \\
W(\bigvee P) \arrow[d, "\beta P"'] & \bigvee(W(\bigvee P)) \arrow[d, shift right, "\bigvee(\beta P)"'] \arrow[d, shift left, "\gamma \bigvee P"] & \\
P & \bigvee P &
\end{tikzcd}$$

Observe that we have used the first, established, equation $X\beta = \gamma X$ to assert $\bigvee(\beta P) = \gamma \bigvee P$. We now interpret the diagram in the spirit of Diagram 1 and conclude:

$$\beta P.Wf.\omega(1) = \omega(\bigvee(\beta P).\bigvee(Wf)) = \omega(f.\gamma A) = (\gamma A.\omega)(f)$$

The element $\omega(1){:}\, K \dashrightarrow WA = \widetilde{\mathscr{K}}(-,A)$ is of course the wave $\omega(1){:}\, K \rightsquigarrow A$ and, since $Wf = \widetilde{\mathscr{K}}(-,f)$, $Wf.\omega(1)$ is the wave $f.\omega(1){:}\, K \rightsquigarrow \bigvee P$. It follows that the left side of the display above is $\beta(f.\omega(1))$ and, by Equation 3 of Remark 3.6, this is simply $(\omega(1))(f)$ so that we have shown

$$(\omega(1))(f) = (\gamma A.\omega)(f)$$

which proves that $\omega(1) = \gamma A.\omega$. This completes the proof of $\beta W = W\gamma$. ∎

Our somewhat ponderous treatment of *elements* and the $(-)_*$ notation was developed to provide precision for the somewhat subtle notation in the proof of the next Proposition.

3.11. Proposition. *For waves $K \overset{\omega}{\rightsquigarrow} L \overset{\psi}{\rightsquigarrow} A$ in a total category $\widehat{\mathscr{K}}$,*

$$\overline{\psi}\omega = \psi\overline{\omega}$$

Proof. We must show that, for any P in $\widehat{\mathscr{K}}$ and any $f : A \to \bigvee P$,

$$((\overline{\psi}\omega)P)(f) = ((\psi\overline{\omega})P)(f)$$

Consider

$$\begin{tikzcd}
K \arrow[r, squiggly, "\omega"] & L \arrow[r, "\overline{\psi}"] \arrow[d, dotted, "\psi(f)"'] \arrow[dr, dotted, "e"] & A \arrow[d, dotted, "f"] \\
 & P \arrow[r, "\eta P"] & (\bigvee P)_*
\end{tikzcd}
\qquad
\begin{tikzcd}
L_* \arrow[r, "\psi(f)_*"] \arrow[d, "\eta L_* = (\eta L)_*"'] \arrow[dr, "e_*"] & P \arrow[d, "\eta P"] \\
(\bigvee L_*)_* \arrow[r, "(\bigvee \psi(f)_*)_*"'] & (\bigvee P)_*
\end{tikzcd}$$

$$\begin{tikzcd}
 & K \arrow[dl, "\overline{\omega}"'] \arrow[d, dotted, "\overline{\omega}"] \arrow[r, squiggly, "\omega"] & L \arrow[d, "\eta L"] \\
L \arrow[dr, dotted, "\psi(f)"'] & L_* \arrow[d, "\psi(f)_*"] & \bigvee L_* \arrow[d, "\bigvee \psi(f)_*"] \\
 & P & \bigvee P
\end{tikzcd}$$

The first diagram provides the configuration for calculating $(\overline{\psi}\omega)(f)$ and the extra information needed to apply Proposition 3.4. It names the element $e = f\overline{\psi}$ and the next diagram provides another calculation for e_* enabling us to also conclude $e = \bigvee \psi(f)_*.\eta L$. In the third diagram we use naturality of ω to recalculate $\omega(e)$ and also note $\psi(f)_*.\overline{\omega} = \psi(f).\overline{\omega}$. But $\psi(f).\overline{\omega} = (\psi\overline{\omega})(f)$. The step by step details are:

$$(\overline{\psi}\omega)(f) \;=\; \omega(f\overline{\psi}) \quad \text{by definition of } \overline{\psi}\omega$$

$$
\begin{aligned}
&= \omega(\eta P.\psi(f)) \quad \text{by Proposition 3.4} \\
&= \omega(\bigvee \psi(f)_*.\eta L) \quad \text{as noted in the second diagram above} \\
&= \psi(f)_*.\overline{\omega} \quad \text{by naturality of } \omega \\
&= \psi(f).\overline{\omega} \quad \text{as noted in the third diagram above} \\
&= (\psi\overline{\omega})(f) \quad \text{by definition of } \psi\overline{\omega}
\end{aligned}
$$

■

3.12. Theorem. *The waves of a total category $\mathscr{K}$ admit an associative composition $\psi \bullet \omega$ given by the common value $\overline{\psi}\omega = \psi\overline{\omega}$ and this composition is preserved by $\overline{(-)}$. In somewhat informal terminology, $\widetilde{\mathscr{K}}$ is a semicategory with the same objects as $\mathscr{K}$ and $\overline{(-)} : \widetilde{\mathscr{K}} \to \mathscr{K}$ is an identity-on-objects semifunctor.*

Proof. For composable waves we have

$$(\chi \bullet \psi) \bullet \omega = (\chi \bullet \psi)\overline{\omega} = (\overline{\chi}\psi)\overline{\omega} = \overline{\chi}\psi\overline{\omega} = \overline{\chi}(\psi\overline{\omega}) = \overline{\chi}(\psi \bullet \omega) = \chi \bullet (\psi \bullet \omega)$$

Moreover, by definition, $\overline{\overline{\psi}\omega} = \overline{\psi \bullet \omega} = \overline{\psi\overline{\omega}}$ and, by Theorem 3.1, $\overline{\overline{\psi}\omega} = \overline{\psi}.\overline{\omega} = \overline{\psi\overline{\omega}}$ so that $\overline{\psi \bullet \omega} = \overline{\psi}.\overline{\omega}$. ■

3.13. Remark. It follows immediately that an n-fold composite of waves $\omega_1 \bullet \cdots \bullet \omega_i \bullet \cdots \bullet \omega_n$ is given by each of the n expressions, $\overline{\omega}_1. \cdots .\omega_i. \cdots .\overline{\omega}_n$.

3.14. Lemma. *For data $K \overset{\omega}{\rightsquigarrow} M \xrightarrow{f} L \overset{\psi}{\rightsquigarrow} A$ in a total category,*

$$\psi f \bullet \omega = \psi \bullet f\omega$$

Proof.

$$\psi f \bullet \omega = \overline{\psi f}\omega = (\overline{\psi} f)\omega = \overline{\psi}(f\omega) = \psi\overline{f\omega} = \psi \bullet f\omega$$

■

If follows immediately that a composable pair of waves $K \overset{\omega}{\rightsquigarrow} M \overset{\psi}{\rightsquigarrow} A$ has the same composite as a composable pair of waves $K \overset{\tau}{\rightsquigarrow} L \overset{\sigma}{\rightsquigarrow} A$, if there exists $M \xrightarrow{f} L$ such that

$$
\begin{tikzcd}
& M \arrow[dd, "f"] \arrow[dr, squiggly, "\psi"] & \\
K \arrow[ur, squiggly, "\omega"] \arrow[dr, squiggly, "\tau"'] & & A \\
& L \arrow[ur, squiggly, "\sigma"'] &
\end{tikzcd}
$$

Temporarily, denote by R the relation on composable pairs of waves from K to A given by $\langle\omega,\psi\rangle R\langle\tau,\sigma\rangle$ if there exists an arrow f as above. Write $\psi\otimes\omega$ for the equivalence class of the pair $\langle\omega,\psi\rangle$ with respect to the equivalence relation generated by R. It is clear that composition of waves can be seen as a function defined on such equivalence classes so that $\bullet(\psi\otimes\omega)=\psi\bullet\omega$. The equivalence relation generated by R is the one whose quotient is given by:

$$\int^{L}\widetilde{\mathscr{K}}(K,L)\times\widetilde{\mathscr{K}}(L,A)$$

This integral is of course the value of the *composite* profunctor $\widetilde{\mathscr{K}}\circ_{\mathscr{K}}\widetilde{\mathscr{K}}:\mathscr{K}\relbar\joinrel\mapstochar\joinrel\rightarrow\mathscr{K}$ at the pair $\langle K,A\rangle$, although we have no reason to suppose, for a general total $\mathscr{K}$, that these values are in **set**. Regardless, we have $\bullet:\widetilde{\mathscr{K}}\circ_{\mathscr{K}}\widetilde{\mathscr{K}}\to\widetilde{\mathscr{K}}$.

4. Waves and total distributivity

4.1. Lemma. *For $\mathscr{K}$ a total category, the following are equivalent:*

i) The functor $X:\widehat{\mathscr{K}}\to\mathscr{K}$ has a left adjoint;

ii) The natural transformation $\gamma:XW\to 1_{\mathscr{K}}$ of Lemma (3.10) is invertible;

iii) The $\beta:WX\to 1_{\widehat{\mathscr{K}}}$ of Lemma (3.10) is the counit of an adjunction $\alpha,\beta:W\dashv X$.

Proof. *ii) implies iii)* Let $\alpha=\gamma^{-1}:1_{\mathscr{K}}\to XW$. Then from $\beta W=W\gamma$ we have $\beta W.W\alpha=1_W$ and from $X\beta=\gamma X$ we have $X\beta.\alpha X=1_X$ so that $\alpha,\beta:W\dashv X$ is an adjunction with counit β.

iii) implies i) is trivial.

i) implies ii) From $L\dashv X=\bigvee$ we have

$$\begin{array}{rcll} W(A)(K) & = & \widetilde{\mathscr{K}}(K,A) & \\ & = & \mathbf{set}^{\mathscr{K}}(\mathscr{K}(A,\bigvee -),\widehat{K}) & \\ & \cong & \mathbf{set}^{\widehat{\mathscr{K}}}(\widehat{\mathscr{K}}(L(A),-),\widehat{K}) & \text{by the hypothesized adjunction} \\ & \cong & \widehat{K}(L(A)) & \text{by Yoneda's Lemma} \\ & = & L(A)(K) & \end{array}$$

In more detail, suppose that we have an adjunction $\langle\iota,\kappa;L\dashv X:\widehat{\mathscr{K}}\to\mathscr{K}\rangle$, with unit ι. Since we then have $L\dashv X\dashv Y$ and Y is fully faithful, it follows that L is fully faithfull and $\iota:1_{\mathscr{K}}\to XL$ is an isomorphism. Define $\mu:W\to L$ by the following display:

$$\begin{tikzcd} W \arrow[rr, "\mu"] \arrow[dr, "W\iota"'] & & L \\ & WXL \arrow[ur, "\beta L"'] & \end{tikzcd}$$

where $\beta : WX \to 1_{\widehat{\mathscr{K}}}$ is as described in Definition 3.5.

We evaluate $\mu(A)(K)$:

$$\widetilde{\mathscr{K}}(K,A) \xrightarrow{\widetilde{\mathscr{K}}(K,\iota A)} \widetilde{\mathscr{K}}(K,\bigvee LA) \xrightarrow{\beta(K,LA)} [K,LA]$$

and see that, for $\omega : K \to A$, $\mu(\omega) = \omega(\iota A)$. Next, define $\nu : L \to W$, for $\lambda : K \dashrightarrow L(A)$ and $f : A \to \bigvee P$ by $\nu(\lambda)(f) = \kappa P.Lf.\lambda$. Clearly

$$\mu(\nu(\lambda)) = \nu(\lambda)(\iota A) = \kappa LA.L\iota A.\lambda = \lambda$$

while

$$\nu(\mu(\omega))(f) = \kappa P.Lf.\omega(\iota A) = \omega(\bigvee \kappa P.\bigvee Lf.\iota A) = \omega(\bigvee \kappa P.\iota \bigvee P.f) = \omega(f)$$

shows that $\nu(\mu(\omega)) = \omega$ so that μ and ν are inverse isomorphisms.

Consider the following diagram:

$$\begin{tikzcd}
XW \arrow[r, "XW\iota"] \arrow[rr, bend left, "X\mu"] \arrow[dr, "\gamma"'] & XWXL \arrow[r, shift left, "X\beta L"] \arrow[r, shift right, "\gamma XL"'] & XL \\
& 1_{\mathscr{K}} \arrow[ur, "\iota"'] &
\end{tikzcd}$$

We have equality in the top region by the definition of μ. The parallel pair of arrows are equal by Proposition 3.10. We have equality in the lower region by naturality. It follows that we have $X\mu = \iota.\gamma$ and since μ and ι are invertible, so too is γ. ∎

In [MRW] a total category $\mathscr{K}$ is said to be *totally distributive* if $X : \widehat{\mathscr{K}} \to \mathscr{K}$ has a left adjoint. We prefer to restate Lemma 4.1 as

4.2. Theorem. *For $\mathscr{K}$ a total category, the following are equivalent:*

i) The total category $\mathscr{K}$ is totally distributive;

ii) The natural transformation $\gamma : XW \to 1_{\mathscr{K}}$ is invertible;

iii) The natural transformation $\beta : WX \to 1_{\widehat{\mathscr{K}}}$ provides the counit for an adjunction $\alpha, \beta : W \dashv X$.

∎

Henceforth, we reserve α for γ^{-1} when $\mathscr{K}$ is totally distributive. Observe now that if $\mathscr{K}$ is totally distributive and $\omega : K \rightsquigarrow A$ is a wave in $\mathscr{K}$ then at first glance we seem to have also $\omega(\alpha) : K \rightsquigarrow A$, where expanding the notation $\omega(\alpha) = \omega(WA)(\alpha A)$ as below

$$\begin{tikzcd} K \arrow[r, squiggly, "\omega"] \arrow[d, dotted, "\omega(WA)(\alpha A)"'] & A \arrow[d, "\alpha A"] \\ WA & \bigvee WA \end{tikzcd} ,$$

since K-elements of WA are waves from K to A. However,

4.3. Lemma. *For any wave $\omega : K \rightsquigarrow A$ in a totally distributive category, $\omega(\alpha) = \omega$*

Proof. We have

$$\omega(\alpha) = \alpha.\omega(1_{\bigvee WA}) = \gamma(\alpha.\omega) = (\gamma\alpha).\omega = 1.\omega = \omega$$

where the second equality is an instance of the equation $\beta W = W\gamma$ of Proposition 3.10. ∎

We should also point out that the isomorphisms $\alpha A : A \to \bigvee WA$, for all A in a totally distributive category, can be interpreted as saying that every object A is the colimit of all waves $\omega : K \rightsquigarrow A$. This follows from the fact that each WA, like any object in $\widehat{\mathscr{K}}$, is the (not-necessarily small) colimit of its elements and that $\bigvee WA$ is the colimit *of the same diagram* in $\mathscr{K}$. Each element $\omega : K \rightsquigarrow A$ gives rise to a colimit injection $i_\omega : K \to \bigvee \widetilde{\mathscr{K}}(-, A)$.

4.4. Theorem. *In a totally distributive category, every wave $\omega : K \rightsquigarrow A$ can be factored as a composite of waves:*

$$\begin{tikzcd} K \arrow[r, squiggly, "\omega_1"] \arrow[rr, squiggly, bend right, "\omega"'] & M \arrow[r, squiggly, "\omega_0"] & A \end{tikzcd}$$

and the factorization is unique up to tensor, in the sense that if also $\omega = \tau_0 \bullet \tau_1$ then $\tau_0 \otimes \tau_1 = \omega_0 \otimes \omega_1$.

Proof. Let A be in $\mathscr{K}$, totally distributive. So γA is invertible and we have

$$W \bigvee WA \xrightarrow[\simeq]{W\gamma A} WA$$

which in a more convenient notation is

$$\widetilde{\mathscr{K}}(-, \bigvee WA) \xrightarrow[\simeq]{\widetilde{\mathscr{K}}(-,\gamma A)} \widetilde{\mathscr{K}}(-, A)$$

and still more conveniently is

$$\widetilde{\mathscr{K}}(-, \int^L L \cdot \widetilde{\mathscr{K}}(L, A)) \xrightarrow[\simeq]{\widetilde{\mathscr{K}}(-,\gamma A)} \widetilde{\mathscr{K}}(-, A) \quad .$$

Because W is a left adjoint, $W = \widetilde{\mathscr{K}}(-,\square)$ preserves colimits and we have

$$\int^L \widetilde{\mathscr{K}}(-,L)\cdot\widetilde{\mathscr{K}}(L,A) \xrightarrow{\simeq} \widetilde{\mathscr{K}}(-,\bigvee WA) \xrightarrow[\simeq]{\widetilde{\mathscr{K}}(-,\gamma A)} \widetilde{\mathscr{K}}(-,A)$$

which, evaluating at K in $\mathscr{K}$, gives bijections

$$\int^L \widetilde{\mathscr{K}}(K,L)\times\widetilde{\mathscr{K}}(L,A) \xrightarrow{\simeq} \widetilde{\mathscr{K}}(K,\bigvee WA) \xrightarrow[\simeq]{\widetilde{\mathscr{K}}(K,\gamma A)} \widetilde{\mathscr{K}}(K,A) \quad .$$

This much establishes that $\widetilde{\mathscr{K}}\circ_{\mathscr{K}}\widetilde{\mathscr{K}}\cong\widetilde{\mathscr{K}}$ but we want to show that $\bullet\colon\widetilde{\mathscr{K}}\circ_{\mathscr{K}}\widetilde{\mathscr{K}}\to\widetilde{\mathscr{K}}$ is an isomorphism. To this end, consider a composable pair of waves:

$$K \overset{\psi}{\rightsquigarrow} L \overset{\chi}{\rightsquigarrow} A$$

The invertible integral comparison, as in 2.5 sends $\chi\otimes\psi$ to the wave

$$K \overset{\psi}{\rightsquigarrow} L \xrightarrow{i_\chi} \bigvee WA$$

To act on this wave by γA is to form $\gamma A(i_\chi\psi) = (\gamma A.i_\chi)\psi = \bar{\chi}\psi = \chi\bullet\psi$, where the second equality is Proposition 3.7. ∎

4.5. Remark. It might be wondered if the components of $\delta\colon W\to Y$ exhibit the wave sets $\widetilde{\mathscr{K}}(K,A)$ of a total category $\mathscr{K}$ as subsets of the arrow sets $\mathscr{K}(K,A)$. This is not true is general, even for **set**, the very best behaved total category according to [R&W]. For $\mathscr{K} = \mathbf{set}$, one can take $W(A)(K) = A$ and $\delta(a\colon 1\to A) = K \xrightarrow{!} 1 \xrightarrow{a} A$. Evidently, $\delta(A)(\emptyset)$ is not monic.

5. An equivalence

We continue with $\mathscr{K}$ a totally distributive category. By Theorem 4.4, $\bullet\colon\widetilde{\mathscr{K}}\circ_{\mathscr{K}}\widetilde{\mathscr{K}}\to\widetilde{\mathscr{K}}$ is invertible and, by Theorem 3.12, $\bullet$ is associative. It follows that $\bullet^{-1}\colon\widetilde{\mathscr{K}}\to\widetilde{\mathscr{K}}\circ_{\mathscr{K}}\widetilde{\mathscr{K}}$ is coassociative. Moreover, it follows immediately from the definition of $\bullet$ in terms of $\overline{(-)}\colon\widetilde{\mathscr{K}}\to\mathscr{K}$ that $\overline{(-)}$ provides a counit for $\bullet^{-1}$ and thus $(\widetilde{\mathscr{K}},\bullet^{-1},\overline{(-)})$ is a (small) idempotent comonad on the totally distributive $\mathscr{K}$ in the bicategory **PRO** of profunctors.

It will be convenient to make some further remarks in terms of the full and locally small subbicategory of **PRO**, which we call **MAT**, determined by the discrete objects. It follows that the objects of **MAT** are the objects of **SET**. For (possibly large) sets X and A, a profunctor $M\colon X \nrightarrow A$ amounts to a *matrix* of (possibly large) sets $M(a,x)$, for all pairs of elements a in A and x in X. We write $\mathbf{Mat}(X,A)$ for the full subcategory of $\mathbf{MAT}(X,A)$ determined by the **set**-valued matrices. Given M and a composable matrix

$N : A \longrightarrow\!\!\!\!\!\!\!\!+\; Y$, the composite profunctor $N \circ M : X \longrightarrow\!\!\!\!\!\!\!\!+\; Y$ amounts to the matrix product NM given by:

$$NM(y,x) = \sum_a N(y,a) \times M(a,x)$$

Our reason for delving into **MAT** is that we first defined wave composition $\bullet$ at the level of composable pairs of waves and it is helpful here to temporarily return to that point of view. Indeed, if we simply regard $\widetilde{\mathscr{K}}$ as a matrix $\widetilde{\mathscr{K}} : |\mathscr{K}| \to |\mathscr{K}|$ then the matrix product $\widetilde{\mathscr{K}}\widetilde{\mathscr{K}}$ is evidently the matrix of composable pairs of waves and we can see $\bullet$ as $\bullet : \widetilde{\mathscr{K}}\widetilde{\mathscr{K}} \to \widetilde{\mathscr{K}}$

It is standard that a category $\mathscr{M}$ can be seen as a monad $\mathscr{M} : |\mathscr{M}| \longrightarrow\!\!\!\!\!\!\!\!+\; |\mathscr{M}|$in **MAT** and that a profunctor $\widetilde{Q} : \mathscr{M} \longrightarrow\!\!\!\!\!\!\!\!+\; \mathscr{L}$ can be seen as a matrix $\widetilde{Q} : |\mathscr{M}| \longrightarrow\!\!\!\!\!\!\!\!+\; |\mathscr{L}|$, together with associating, associative and unitary, actions $\rho : \widetilde{Q}\mathscr{M} \to \widetilde{Q}$ and $\lambda : \mathscr{L}\widetilde{Q} \to \widetilde{Q}$. 2-cells between parallel profunctors can then be seen as matrix 2-cells that are equivariant with respect to the actions.

The bicategory **MAT**, like the bicategory **PRO** admits local coequalizers that are stable under composition (from either side). Given $\widetilde{Q}$ and a composable profunctor $\widetilde{P} : \mathscr{L} \longrightarrow\!\!\!\!\!\!\!\!+\; \mathscr{K}$, the composite profunctor $\widetilde{P} \circ_{\mathscr{L}} \widetilde{Q} : \mathscr{M} \longrightarrow\!\!\!\!\!\!\!\!+\; \mathscr{K}$ can be computed in **MAT** as the local coequalizer (coequalizer in $\mathbf{MAT}(|\mathscr{M}|, |\mathscr{K}|)$):

$$\begin{tikzcd} \widetilde{P}\mathscr{L}\widetilde{Q} \arrow[r, shift left, "\rho\widetilde{Q}"] \arrow[r, shift right, "\widetilde{Q}\lambda"'] & \widetilde{P}\widetilde{Q} \arrow[r] & \widetilde{P} \circ_{\mathscr{L}} \widetilde{Q} \end{tikzcd}$$

where ρ is the right action of $\mathscr{L}$ on $\widetilde{P}$ and λ is the left action of $\mathscr{L}$ on $\widetilde{Q}$.

In particular, the composite of $\widetilde{\mathscr{K}}$ with itself, *as a profunctor*, $\widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}}$ is a local coequalizer in **MAT** and our $\bullet : \widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}}$ arises as below:

$$\begin{tikzcd} \widetilde{\mathscr{K}}\mathscr{K}\widetilde{\mathscr{K}} \arrow[r, shift left, "\rho\widetilde{\mathscr{K}}"] \arrow[r, shift right, "\widetilde{\mathscr{K}}\lambda"'] & \widetilde{\mathscr{K}}\widetilde{\mathscr{K}} \arrow[r] \arrow[dr, "\bullet"'] & \widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}} \arrow[d, "\bullet"] \\ & & \widetilde{\mathscr{K}} \end{tikzcd}$$

invertibility of $\bullet : \widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}} \to \widetilde{\mathscr{K}}$ is equivalent to saying that

$$\begin{tikzcd} \widetilde{\mathscr{K}}\mathscr{K}\widetilde{\mathscr{K}} \arrow[r, shift left, "\rho\widetilde{\mathscr{K}}"] \arrow[r, shift right, "\widetilde{\mathscr{K}}\lambda"'] & \widetilde{\mathscr{K}}\widetilde{\mathscr{K}} \arrow[r, "\bullet"] & \widetilde{\mathscr{K}} \end{tikzcd}$$

is a coequalizer.

Below on the left, we recall the relation on composable pairs $\langle \omega, \psi \rangle$ that generates the equivalence relation whose quotient is $\widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}}$. We usually denote the equivalence class of $\langle \omega, \psi \rangle$ by $\psi \otimes \omega$ but, provisionally, we will now call it $\psi \otimes_{\mathscr{K}} \omega$. While $\widetilde{\mathscr{K}}$ is not a monad on $|\mathscr{K}|$ we *can* can regard $\bullet : \widetilde{\mathscr{K}}\widetilde{\mathscr{K}} \to \widetilde{\mathscr{K}}$ as both a left and a right associative action of $\widetilde{\mathscr{K}}$ on itself. Accordingly, we *can*, as on the right below, contemplate the relation on composable pairs of waves determined by the existence of a wave ϕ rather than an arrow f making commutative triangles. For the relation on the right we will write, provisionally, $\psi \otimes_{\widetilde{\mathscr{K}}} \omega$ for the equivalence class of $\langle \omega, \psi \rangle$ in the generated equivalence relation.

$$\begin{tikzcd}
& M \arrow[dd, "f"] \arrow[dr, rightsquigarrow, "\psi"] & & & & M \arrow[dd, rightsquigarrow, "\phi"] \arrow[dr, rightsquigarrow, "\psi"] & \\
K \arrow[ur, rightsquigarrow, "\omega"] \arrow[dr, rightsquigarrow, "\tau"'] & & A & & K \arrow[ur, rightsquigarrow, "\omega"] \arrow[dr, rightsquigarrow, "\tau"'] & & A \\
& L \arrow[ur, rightsquigarrow, "\sigma"'] & & & & L \arrow[ur, rightsquigarrow, "\sigma"'] &
\end{tikzcd} \tag{6}$$

It should be clear that just as composition of profunctors can be computed as a local coequalizer in **MAT**, so too the following **MAT** local coequalizer

$$\begin{tikzcd}
\widetilde{\mathscr{K}}\widetilde{\mathscr{K}}\widetilde{\mathscr{K}} \arrow[r, shift left, "\bullet\widetilde{\mathscr{K}}"] \arrow[r, shift right, "\widetilde{\mathscr{K}}\bullet"'] & \widetilde{\mathscr{K}}\widetilde{\mathscr{K}} \arrow[r] & \widetilde{\mathscr{K}} \circ_{\widetilde{\mathscr{K}}} \widetilde{\mathscr{K}}
\end{tikzcd}$$

computes the matrix whose value $\widetilde{\mathscr{K}} \circ_{\widetilde{\mathscr{K}}} \widetilde{\mathscr{K}}(K, A)$ is the set of composable pairs of waves modulo the equivalence relation with equivalence classes given by the $\psi \otimes_{\widetilde{\mathscr{K}}} \omega$.

5.1. THEOREM. *If $\mathscr{K}$ is a total category for which $\bullet : \widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}} \to \widetilde{\mathscr{K}}$ is invertible, in particular if $\mathscr{K}$ is a totally distributive category, then $\widetilde{\mathscr{K}} \circ_{\widetilde{\mathscr{K}}} \widetilde{\mathscr{K}} = \widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}}$ so that*

$$\begin{tikzcd}
\widetilde{\mathscr{K}}\widetilde{\mathscr{K}}\widetilde{\mathscr{K}} \arrow[r, shift left, "\bullet\widetilde{\mathscr{K}}"] \arrow[r, shift right, "\widetilde{\mathscr{K}}\bullet"'] & \widetilde{\mathscr{K}}\widetilde{\mathscr{K}} \arrow[r, "\bullet"] & \widetilde{\mathscr{K}}
\end{tikzcd}$$

is a local coequalizer in **MAT** *and $\widetilde{\mathscr{K}}$ is a* taxon *in the sense of Koslowski [KOS].*

PROOF. It suffices to show that, for any composable pair of waves $\langle \omega, \psi \rangle$, $\psi \otimes_{\widetilde{\mathscr{K}}} \omega = \psi \otimes_{\mathscr{K}} \omega$. If we consider the diagram on the right of (6) and replace the wave ϕ by the arrow $\overline{\phi}$, the definition of wave composition shows immediately that the result is a diagram as on the left of (6). Formally, we have $\psi \otimes_{\widetilde{\mathscr{K}}} \omega \subseteq \psi \otimes_{\mathscr{K}} \omega$. On the other hand, consider the diagram on the left of (6) and construct the following diagram by factoring σ as $\sigma_0 \bullet \sigma_1$:

$$\begin{tikzcd}
& M \arrow[dr, squiggly, "\psi"] \arrow[d, squiggly, "\sigma_1 f"] & \\
K \arrow[ur, squiggly, "\omega"] \arrow[r, squiggly, "\sigma_1 f\omega"] \arrow[dr, squiggly, "\tau"'] & N \arrow[r, squiggly, "\sigma_0"] & A \\
& L \arrow[u, squiggly, "\sigma_1"] \arrow[ur, squiggly, "\sigma"'] &
\end{tikzcd}$$

The construction shows that $\psi \otimes_{\mathscr{K}} \omega \subseteq \psi \otimes_{\widetilde{\mathscr{K}}} \omega$. ∎

In [KOS], Koslowski constructed not only a 2-category of taxons and semi-functors (with the main effort involving the 2-cells) but also a bicategory of taxons and *interpolative modules*. The latter when constructed within **MAT** are like profunctors between categories except that the unitary conditions for modules — which are not stateable for modules between taxons — are replaced by requiring that the actions be local coequalizers in **MAT**. These ideas were repeated in [MRW] from a perspective closer to ours here. Nevertheless, we attempt to make this account reasonably self-contained with respect to the definitions.

5.2. Definition. *A* taxon $\mathscr{T}$ *consists of a set* $|\mathscr{T}|$ *(possibly large) and a matrix* $\mathscr{T} : |\mathscr{T}| \mathrel{\mapstochar\rightarrow} |\mathscr{T}|$ *together with a (matrix) 2-cell* $\bullet : \mathscr{T}\mathscr{T} \to \mathscr{T}$ *for which*

$$\begin{tikzcd}
\mathscr{T}\mathscr{T}\mathscr{T} \arrow[r, shift left, "\bullet\mathscr{T}"] \arrow[r, shift right, "\mathscr{T}\bullet"'] & \mathscr{T}\mathscr{T} \arrow[r, "\bullet"] & \mathscr{T}
\end{tikzcd}$$

is a coequalizer. For taxons $\mathscr{S}$ *and* $\mathscr{T}$*, an* i-module *(short for* interpolative module*) from* $\mathscr{S}$ *to* $\mathscr{T}$ *is a matrix* $P : |\mathscr{S}| \mathrel{\mapstochar\rightarrow} |\mathscr{T}|$ *together with actions* $\rho : P\mathscr{S} \to P$ *and* $\lambda : \mathscr{T}P \to P$ *for which*

$$\begin{tikzcd}
P\mathscr{S}\mathscr{S} \arrow[r, shift left, "\rho\mathscr{S}"] \arrow[r, shift right, "P\bullet"'] & P\mathscr{S} \arrow[r, "\rho"] & P
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathscr{T}\mathscr{T}P \arrow[r, shift left, "\bullet P"] \arrow[r, shift right, "\mathscr{T}\lambda"'] & \mathscr{T}P \arrow[r, "\lambda"] & P
\end{tikzcd}$$

are coequalizers. A 2-cell $t : P \to Q : \mathscr{S} \mathrel{\mapstochar\rightarrow} \mathscr{T}$ *is a matrix 2-cell* $t : P \to Q : |\mathscr{S}| \mathrel{\mapstochar\rightarrow} |\mathscr{T}|$ *which is equivariant with respect to the actions. Given i-modules* $P : \mathscr{S} \mathrel{\mapstochar\rightarrow} \mathscr{T}$ *and* $Q : \mathscr{T} \mathrel{\mapstochar\rightarrow} \mathscr{U}$*, the composite i-module* $Q \circ_{\mathscr{T}} P : \mathscr{S} \mathrel{\mapstochar\rightarrow} \mathscr{U}$ *is given by the coequalizer*

$$\begin{tikzcd}
Q\mathscr{T}P \arrow[r, shift left, "\rho\mathscr{T}"] \arrow[r, shift right, "\mathscr{T}\lambda"'] & QP \arrow[r] & Q \circ_{\mathscr{T}} P
\end{tikzcd}$$

With the other, expected, composites and the obvious constraints, taxons, i-modules, and their 2-cells constitute a bicategory that we call **iMOD**. *We write* $\mathbf{iMod}(\mathscr{S}, \mathscr{T})$ *for the full subcategory of* $\mathbf{iMOD}(\mathscr{S}, \mathscr{T})$ *determined by the* **set***-valued i-modules.* ∎

Our intention now is to show that, for $\mathscr{K}$ a totally distributive category, there is an equivalence $\mathscr{K} \to \mathbf{iMod}(\mathbf{1}, \widetilde{\mathscr{K}})$. To give a small i-module $P : \mathbf{1} \nrightarrow \widetilde{\mathscr{K}}$ is to give a small matrix $P : 1 \nrightarrow |\widetilde{\mathscr{K}}|$ and a left action $\lambda : \widetilde{\mathscr{K}} P \to P$ for which

$$\begin{tikzcd} \widetilde{\mathscr{K}}\widetilde{\mathscr{K}}P \arrow[r, shift left, "\bullet P"] \arrow[r, shift right, "\widetilde{\mathscr{K}}\lambda"'] & \widetilde{\mathscr{K}}P \arrow[r, "\lambda"] & P \end{tikzcd}$$

is a coequalizer. On the other hand, to give a module $P : \mathbf{1} \nrightarrow \mathscr{K}$ is to give a matrix $P : 1 \nrightarrow |\mathscr{K}|$ and a left action $\mathscr{K} P \to P$ which is associative and unitary. (Of course such a module is but a functor $P : \mathscr{K}^{\mathrm{op}} \to \mathbf{set}$, equally a small profunctor $P : \mathbf{1} \nrightarrow \mathscr{K}$.) From an i-module $(P, \lambda) : \mathbf{1} \nrightarrow \widetilde{\mathscr{K}}$ we now construct a module $(P, \lambda_1) : \mathbf{1} \nrightarrow \mathscr{K}$. Consider the following diagram, in which we have written @ for the left action of $\mathscr{K}$ on $\widetilde{\mathscr{K}}$.

$$\begin{tikzcd} \mathscr{K}\widetilde{\mathscr{K}}\widetilde{\mathscr{K}}P \arrow[r, shift left, "\mathscr{K}\bullet P"] \arrow[r, shift right, "\mathscr{K}\widetilde{\mathscr{K}}\lambda"'] \arrow[d, "@\widetilde{\mathscr{K}}P"'] & \mathscr{K}\widetilde{\mathscr{K}}P \arrow[r, "\mathscr{K}\lambda"] \arrow[d, "@P"] & \mathscr{K}P \arrow[d, "\lambda_1"] \\ \widetilde{\mathscr{K}}\widetilde{\mathscr{K}}P \arrow[r, shift left, "\bullet P"] \arrow[r, shift right, "\widetilde{\mathscr{K}}\lambda"'] & \widetilde{\mathscr{K}}P \arrow[r, "\lambda"] & P \end{tikzcd}$$

The top row is $\mathscr{K}$ applied to the coequalizer diagram for (P, λ). It is also a coequalizer since coequalizers in **MAT** are preserved by composition. We have equality in the bottom square on the left by naturality of @. The definition of $\bullet$ in terms of the equivariant $\overline{(-)}$ provides equality in the top square on the left. These considerations uniquely determine λ_1 as displayed. Straightforward diagram chases show that λ_1 is unitary and associative. Thus the construction $(P, \lambda) \mapsto (P, \lambda_1)$ defines an object function from $\widetilde{\mathscr{K}}$-i-modules to $\mathscr{K}$-modules. Moreover, if (Q, λ) is another i-module from $\mathbf{1}$ to $\widetilde{\mathscr{K}}$ and $t : P \to Q$ is a matrix 2-cell which is equivariant for the $\widetilde{\mathscr{K}}$ actions then another, easy, diagram shows that t is also equivariant for the $\mathscr{K}$ actions.

5.3. Proposition. *The construction $(P, \lambda) \mapsto (P, \lambda_1)$ defines a functor J satisfying the following identity with the evident forgetful functors.*

$$\begin{tikzcd} \mathbf{iMod}(\mathbf{1}, \widetilde{\mathscr{K}}) \arrow[rr, "J"] \arrow[dr] & & \mathbf{Pro}(\mathbf{1}, \mathscr{K}) \arrow[dl] \\ & \mathbf{Mat}(1, |\mathscr{K}|) & \end{tikzcd}$$

■

The functor $W : \mathscr{K} \to \widehat{\mathscr{K}}$ which sends A to $\widetilde{\mathscr{K}}(-,A)$ can equally be seen as $W : \mathscr{K} \to \mathbf{Pro}(\mathbf{1},\mathscr{K})$. The left $\mathscr{K}$ action on $\widetilde{\mathscr{K}}(-,A)$ is just the left action of arrows on waves with codomain A. An element of the matrix of actable pairs with codomain A, namely $\mathscr{K}\widetilde{\mathscr{K}}(-,A)$ is a pair $L \xrightarrow{f} K \overset{\omega}{\rightsquigarrow} A$. The effect of the action has been denoted ωf and can further be seen as the result of applying the function $\widetilde{\mathscr{K}}(f,A) : \widetilde{\mathscr{K}}(K,A) \to \widetilde{\mathscr{K}}(L,A)$ to the element $\omega \in \widetilde{\mathscr{K}}(K,A)$. It is clear that for a *wave* $L \overset{\psi}{\rightsquigarrow} K$ we can define $\widetilde{\mathscr{K}}(\psi,A) : \widetilde{\mathscr{K}}(K,A) \to \widetilde{\mathscr{K}}(L,A)$ by composition of waves and, at the risk of labouring the obvious, $A \mapsto \widetilde{\mathscr{K}}(-,A)$ defines *some* functor $\mathscr{K} \to \mathbf{iMod}(\mathbf{1},\widetilde{\mathscr{K}})$ that we will call $\widetilde{W} : \mathscr{K} \to \mathbf{iMod}(\mathbf{1},\widetilde{\mathscr{K}})$. (That $\widetilde{\mathscr{K}}(-,A)$ satisfies the requisite coequalizer condition is easily seen formally by recognizing it as the profunctor composite $\mathbf{1} \overset{A}{\nrightarrow} \mathscr{K} \overset{\widetilde{\mathscr{K}}}{\nrightarrow} \mathscr{K}$.) What does require a check, which we leave for the reader, is showing that J applied to the actions $\widetilde{\mathscr{K}}(\psi,A)$, for ψ a wave returns the actions given by the $\widetilde{\mathscr{K}}(f,A)$, for f an arrow. More formally:

5.4. Proposition. *The functor $\widetilde{W}$ satisfies the equality*

$$\begin{tikzcd} & \mathbf{iMod}(\mathbf{1},\widetilde{\mathscr{K}}) \arrow[d, "J"] \\ \mathscr{K} \arrow[ur, "\widetilde{W}"] \arrow[r, "W"'] & \mathbf{Pro}(\mathbf{1},\mathscr{K}) \end{tikzcd}$$

■

We further define $\widetilde{X} : \mathbf{iMod}(\mathbf{1},\widetilde{\mathscr{K}}) \to \mathscr{K}$ to be the composite XJ. In the sequel, when we refer to the matrix entries $P(K,*)$ of an i-module (P,λ) from $\mathbf{1}$ to $\widetilde{\mathscr{K}}$, we will write $P(K,*) = [K,(P,\lambda)] = [K,P]$ to conform with our usage in $\widehat{\mathscr{K}}$, equally in $\mathbf{Pro}(\mathbf{1},\mathscr{K})$.

5.5. Theorem. *For $\mathscr{K}$ totally distributive, the functors $\widetilde{W} : \mathscr{K} \to \mathbf{iMod}(\mathbf{1},\widetilde{\mathscr{K}})$ and $\widetilde{X}$ provide an adjoint equivalence of categories.*

Proof. We have the isomorphism $\widetilde{\alpha} = \alpha : 1_{\mathscr{K}} \to XW = XJ\widetilde{W} = \widetilde{X}\widetilde{W}$, by total distributivity of $\mathscr{K}$. We wish to construct a $\widetilde{\beta} : \widetilde{W}\widetilde{X} \to 1_{\mathbf{iMod}(\mathbf{1},\widetilde{\mathscr{K}})}$. To this end we require, for each (P,λ) in $\mathbf{iMod}(\mathbf{1},\widetilde{\mathscr{K}})$, a $\widetilde{\mathscr{K}}$-equivariant $\widetilde{\beta}(K,(P,\lambda)) : \widetilde{\mathscr{K}}(K,\bigvee(P,\lambda_1)) \to [K,(P,\lambda)]$. The requisite *functions* are provided by the $\beta(K,(P,\lambda_1)) : \widetilde{\mathscr{K}}(K,\bigvee(P,\lambda_1)) \to [K,(P,\lambda_1)]$, where we recall that for $\omega : K \rightsquigarrow \bigvee(P,\lambda_1)$, we have $\beta(\omega) = \omega(1)$. However, we need to show that these are equivariant with respect to the $\widetilde{\mathscr{K}}$ actions. So given $L \overset{\psi}{\rightsquigarrow} K$ we require that $\beta(\omega \bullet \psi) = \beta(\omega)\psi$. By the right side we refer to $L \overset{\psi}{\rightsquigarrow} K \overset{\beta(\omega)}{\dashrightarrow} P$ *in*

$[L, (P, \lambda)]$. However, writing $\beta(\omega) = p\chi$ by interpolativity of (P, λ) as in:

$$\begin{tikzcd} K \arrow[rr, dotted, "\beta(\omega)"] \arrow[dr, squiggly, "\chi"'] & & P \\ & M \arrow[ur, dotted, "p"'] & \end{tikzcd}$$

we have

$$\beta(\omega \bullet \psi) = \beta(\omega\overline{\psi}) = (\omega\overline{\psi})(1) \overset{(a)}{=} \omega(1)\overline{\psi} = \beta(\omega)\overline{\psi} = (p\chi)\overline{\psi} \overset{(b)}{=} p(\chi\overline{\psi}) = p(\chi \bullet \psi) = (p\chi)\psi = \beta(\omega)\psi$$

where (a) follows from the action of $\mathscr{K}$ on waves and (b) follows from the construction of a $\mathscr{K}$ module from a $\widetilde{\mathscr{K}}$ i-module. To complete the proof of the theorem it now suffices to show that the $\beta P : \widetilde{\mathscr{K}}(-, \bigvee P) \to [-, P]$ are invertible when P arises from a left $\widetilde{K}$ i-module via J. Because $\widetilde{\mathscr{K}}(-, \Box)$ is a left adjoint,it suffices to show that the composite:

$$\int^L \widetilde{\mathscr{K}}(-, L) \cdot [L, P] \xrightarrow{\simeq} \widetilde{\mathscr{K}}(-, \int^L L \cdot [L, P]) \xrightarrow{\beta P} [-, P]$$

is invertible, where the first isomorphism is the sup-comparison. Invertibility of this composite is equivalent to each evaluation at K:

$$\int^L \widetilde{\mathscr{K}}(K, L) \times [L, P] \xrightarrow{\simeq} \widetilde{\mathscr{K}}(K, \bigvee P) \xrightarrow{\beta P} [K, P]$$

being a bijection. Note that the domain of the composite can be written as the profunctor composite $\widetilde{\mathscr{K}} \circ_{\mathscr{K}} P$, evaluated at K. However, using the interpolativity of P, the argument in Theorem 5.1 shows that $\widetilde{\mathscr{K}} \circ_{\mathscr{K}} P$ is equally $\widetilde{\mathscr{K}} \circ_{\widetilde{\mathscr{K}}} P$, so that the composite above can be written:

$$(\widetilde{\mathscr{K}} \circ_{\widetilde{\mathscr{K}}} P)(K) \xrightarrow{\simeq} \widetilde{\mathscr{K}}(K, \bigvee P) \xrightarrow{\beta P} [K, P] \tag{7}$$

We will show that this composite is the evaluation at K of the invertible matrix arrow $\widetilde{\mathscr{K}} \circ_{\widetilde{\mathscr{K}}} P \to P$ corresponding to the **MAT** coequalizer $\lambda : \widetilde{\mathscr{K}} P \to P$. Consider a composable wave and element $K \overset{\omega}{\rightsquigarrow} L \overset{p}{\dashrightarrow} P$. Integral comparison sends the tensor $p \otimes \omega$ to the wave $K \overset{\omega}{\rightsquigarrow} L \xrightarrow{i_p} \bigvee P$ which β sends to $i_p\omega(1_{\bigvee P}) = \omega(i_p) : K \dashrightarrow P$. Using the expansion of i_p on the right below, as discussed in 2.4:

$$\begin{tikzcd} & K \arrow[dl, "\overline{\omega}"'] \arrow[d, dotted, "\overline{\omega}"] \arrow[r, squiggly, "\omega"] & L \arrow[d, "\eta L"] \arrow[dd, bend left, "i_p"] \\ L \arrow[dr, dotted, "p"'] & L_* \arrow[d, "p_*"] & \bigvee L_* \arrow[d, "\bigvee p_*"] \\ & P & \bigvee P \end{tikzcd}$$

we see that $\omega(i_p) = p\overline{\omega}$. However, we can write $p = q\psi$ (for unique $q \otimes \psi$) and now

$$p\overline{\omega} = (q\psi)\overline{\omega} = q(\psi\overline{\omega}) = q(\psi \bullet \omega) = (q\psi)\omega = p\omega$$

This shows that the composite in (7) is the required bijection and completes the proof that $\widetilde{\beta}$ is invertible. ∎

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