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BICATEGORICAL FIBRATION STRUCTURES AND STACKS

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Abstract. In this paper we introduce two notions —systems of fibrant objects and fibration structures— which will allow us to associate to a bicategory $\mathcal{B}$ a homotopy bicategory $\mathbf{Ho}(\mathcal{B})$ in such a way that $\mathbf{Ho}(\mathcal{B})$ is the universal way to add pseudo-inverses to weak equivalences in $\mathcal{B}$. Furthermore, $\mathbf{Ho}(\mathcal{B})$ is locally small when $\mathcal{B}$ is and $\mathbf{Ho}(\mathcal{B})$ is a 2-category when $\mathcal{B}$ is. We thereby resolve two of the problems with known approaches to bicategorical localization.

As an important example, we describe a fibration structure on the 2-category of prestacks on a site and prove that the resulting homotopy bicategory is the 2-category of stacks. We also show how this example can be restricted to obtain algebraic, differentiable and topological (respectively) stacks as homotopy categories of algebraic, differential and topological (respectively) prestacks.

Introduction

It is widely known that Quillen's [19] notion of *model structure* on a category $\mathcal{C}$ provides a technical tool for forming the localization of $\mathcal{C}$ with respect to a class of weak equivalences: weak equivalences are inverted in a universal way in the passage to the homotopy category $\mathbf{Ho}(\mathcal{C})$ of $\mathcal{C}$. Consequently, it is possible to invert weak equivalences in this setting without having to resort to the Gabriel-Zisman [7] calculus of fractions. One advantage of using model structures for localization is that the resulting homotopy category will be locally small when $\mathcal{C}$ is. This is not necessarily the case for the calculus of fractions.

In the bicategorical setting, one might like to be able to invert a collection of weak equivalences in the sense of turning them into equivalences. In [17, 18], the first author gave a bicategorical generalization of the Gabriel-Zisman calculus of fractions which accomplishes this goal:

0.1. Theorem. [Pronk [18]] *Given a collection of arrows $\mathfrak{W}$ in a bicategory $\mathcal{C}$ satisfying certain conditions, there exists an explicitly constructed bicategory $\mathcal{C}(\mathfrak{W}^{-1})$ (called the* bicategory of fractions for $\mathfrak{W}$*) and a homomorphism $I\colon \mathcal{C} \to \mathcal{C}(\mathfrak{W}^{-1})$ such that I sends arrows in $\mathfrak{W}$ to equivalences in $\mathcal{C}(\mathfrak{W}^{-1})$ and I is universal with this property.*

Like the ordinary category of fractions, this construction suffers from the technical defect that $\mathcal{C}(\mathfrak{W}^{-1})$ will not in general have small hom-categories even when $\mathcal{C}$ does.

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Moreover, $\mathcal{C}(\mathfrak{W}^{-1})$ will be a bicategory even when $\mathcal{C}$ is a 2-category. However, in concrete examples one can often find a strict 2-category which is biequivalent to the bicategory of fractions; but this is not in general the case. For instance, in the case of étale groupoids there are the 2-categories of étendues and representable stacks [18]. In this case we are also reassured that the resulting 2-categories are locally small after all. Recent work by Roberts [20] gives a criterion for (essential) local smallness when the localization is in terms of a Grothendieck pretopology. However, there are so far no general constructions that guarantee both local smallness and strictness after localization and it is the goal of the present paper to remedy this.

In this paper, we introduce the notion of a *system of fibrant objects* (Definition 2.3) in a bicategory $\mathcal{C}$ and the notion of a *fibration structure* (Definition 2.4) on a bicategory which will allow us to form the localization of a bicategory $\mathcal{C}$ with respect to a class of weak equivalences in such a way that the result will both have small hom-categories when $\mathcal{C}$ does and will be a 2-category when $\mathcal{C}$ is. Moreover, the setting we describe is arguably a more convenient setting for forming and studying bicategorical localizations than bicategories of fractions. In particular, the setting described here allows one to employ the kind of "factorization system" reasoning and argumentation familiar from the 1-categorical setting. We believe that, where applicable, the approach to bicategorical localization presented here offers a similar kind of improvement over bicategories of fractions to that afforded by model categories over categories of fractions.

A system of fibrant objects consists of a collection $\mathfrak{W}$ (*weak equivalences*) of maps in $\mathcal{C}$, a pseudofunctor $R\colon \mathcal{C} \to \mathcal{C}$ (fibrant replacement) and a pseudonatural transformation $\eta\colon 1_{\mathcal{C}} \to R$ (whose components are weak equivalences) satisfying certain factorization conditions. To each bicategory $\mathcal{C}$ with a system of fibrant objects, there is an associated bicategory $\mathbf{Ho}(\mathcal{C})$, called the *homotopy bicategory of* $\mathcal{C}$, and a pseudofunctor $I\colon \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$. By construction, $\mathbf{Ho}(\mathcal{C})$ has small hom-categories when $\mathcal{C}$ does and it is a 2-category when $\mathcal{C}$ is. Our first main result is as follows:

0.2. Theorem. [Theorem 2.9 below] *The pseudofunctor I inverts weak equivalences and is universal with this property.*

Once this result has been established, the remainder of this paper is concerned with investigating specific examples of bicategories with systems of fibrant objects. Our leading example is the 2-category $\mathbf{St}(\mathcal{C})$ of stacks on a site $\mathcal{C}$, which we obtain (Corollary 4.7) as the homotopy 2-category $\mathbf{Ho}(\mathbf{PreSt}(\mathcal{C}))$ of the 2-category of prestacks on $\mathcal{C}$. This result is made possible using a characterization of the fibrations of prestacks which is inspired by an earlier result of Joyal and Tierney [11] (cf. also [9]).

The system of fibrant objects on $\mathbf{PreSt}(\mathcal{C})$ is notable in that it exhibits a number of additional features making it more closely resemble the notion of a model structure. These additional features are sufficiently interesting that we introduce the notion of a *fibration structure* on a bicategory to capture them. A category $\mathcal{C}$ has a fibration structure when there are stronger lifting and factorization conditions in place which among other things imply that the category has path objects and that the factorization lemma holds, so that one can construct generalized universal bundles.

In [18], the first author gave a number of examples of bicategorical equivalences between well-known 2-categories and bicategories of fractions. These examples include topological, differentiable and algebraic stacks and we show that these examples can also be captured in our setting. Note however that the characterizations given here of these 2-categories differ from those in *ibid.* In *ibid* these 2-categories were characterized as bicategories of fractions of certain categories of groupoids with respect to Morita equivalences. Here we will view them as homotopy categories of certain categories of prestacks with respect to local weak equivalences. Note that for our results here we assume that the stacks and prestacks are fibered in groupoids. David Roberts shows in forthcoming work [21] that this localization can be extended to arbitrary representable prestacks and stacks.

Ultimately we would like to extend the axiomatization given here to the lax setting (we are always working in a "pseudo" setting) and to relate the results presented here to Street's notion of 2-topos [23]. Intuitively, every 2-topos should arise as a homotopy 2-category by analogy with the way Grothendieck toposes arise as localizations of presheaf categories.

SUMMARY. In Section 1, we recall basic definitions and results on bicategories, pseudofunctors, pseudonatural transformations, and so forth. In Section 2, we introduce systems of fibrant objects and fibration structures on bicategories and we prove our main result (Theorem 2.9). In Section 3, we introduce a fibered notion of stack: *local fibrations.* Let $(\mathcal{C}, J)$ be a site and let pseudofunctors $E, B \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ and a pseudonatural transformation $p \colon E \to B$ be given. For each cover $\mathcal{S}$ of an object U of $\mathcal{C}$ we introduce the category $\mathbf{Desc}(p, \mathcal{S})$ of *descent data with respect to* p *and* $\mathcal{S}$. This category, like the usual category of descent data $\mathbf{Desc}(E, \mathcal{S})$, can be defined as a pseudo-limit (although here we give a direct description). (One obtains the usual notion of descent when one applies this construction to a morphism into the terminal object.) We define p to be a local fibration when it satisfies an effective descent condition with respect to $\mathbf{Desc}(p, \mathcal{S})$ analogous to the usual descent condition for stacks. In Section 4, we describe a fibration structure on the 2-category of prestacks $\mathbf{PreSt}(\mathcal{C})$ and prove that the resulting homotopy bicategory is the 2-category $\mathbf{St}(\mathcal{C})$ of stacks. In particular, we introduce the local weak equivalences (which are already known in the literature on stacks) and prove, using the Axiom of Choice, that the local fibrations are exactly those maps having a bicategorical version of the right lifting property with respect to the local weak equivalences. Further examples (algebraic, differentiable and topological prestacks) of systems of fibrant objects are considered in Section 5.

1. Basics and notation

We want to alert the reader to the fact that we make free use of the Axiom of Choice. As such, we do not distinguish between *strong* and *weak* equivalences of categories. Recall that for a strong equivalence the existence of a pseudo-inverse is required, whereas a weak equivalence only needs to be essentially surjective on objects and fully faithful (see [4] for more on the difference between strong and weak equivalences). We assume that the reader is familiar with the basic theory of 2-categories and refer the reader to [12] for further details. For more information regarding stacks we refer the reader to [8] and [15].

1.1. Bicategories. We briefly review the definitions of bicategories, pseudofunctors, pseudonatural transformations and modifications, in order to set our notation and spell out some of the technical details of bicategories we will be using later in this paper.

1.2. Definition. [Bénabou [2]] *A* **bicategory** $\mathcal{C}$ *consists of a collection of objects* $A, B, \ldots$ *together with the following data:*

- *Categories* $\mathcal{C}(A, B)$ *for objects* A *and* B *of* $\mathcal{C}$. *The objects of* $\mathcal{C}(A, B)$ *are called* arrows *and the arrows are called* 2-cells. *When* α *and* β *are composable 2-cells in* $\mathcal{C}(A, B)$ *we denote their composite by* $\beta \cdot \alpha$.

- *For objects* A, B *and* C *of* $\mathcal{C}$, *a functor* $c_{A,B,C} : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$. *We denote* $c_{A,B,C}(f, g)$ *by* $g \circ f$, *for arrows* $f \in \mathcal{C}(A, B)$ *and* $g \in \mathcal{C}(B, C)$, *and we denote* $c_{A,B,C}(\alpha, \beta)$ *by* $\beta \circ \alpha$, *for 2-cells* $\alpha \in \mathcal{C}(A, B)$ *and* $\beta \in \mathcal{C}(B, C)$. *When no confusion will result we omit the subscripts and write* c *instead of* $c_{A,B,C}$.

- *For each object* A *of* $\mathcal{C}$, *an arrow* $1_A \in \mathcal{C}(A, A)$.

- *For objects* A, B, C *and* D *of* $\mathcal{C}$, *a natural isomorphism:*

$$\begin{tikzcd}
\mathcal{C}(A,B) \times \mathcal{C}(B,C) \times \mathcal{C}(C,D) \arrow[rr, "c \times \mathcal{C}(C,D)"] \arrow[d, "\mathcal{C}(A,B) \times c"'] \arrow[drr, phantom, "\overset{\alpha_{A,B,C,D}}{\Rightarrow}"] & & \mathcal{C}(A,C) \times \mathcal{C}(C,D) \arrow[d, "c"] \\
\mathcal{C}(A,B) \times \mathcal{C}(B,D) \arrow[rr, "c"'] & & \mathcal{C}(A,D).
\end{tikzcd}$$

  As with the composition functors c, *we will omit subscripts and write* α *instead of* $\alpha_{A,B,C,D}$.

- *For objects* A *and* B *of* $\mathcal{C}$, *natural isomorphisms* $\lambda_{A,B}$ *and* $\rho_{A,B}$ *as indicated in the following diagrams:*

$$\begin{tikzcd}
1 \times \mathcal{C}(A,B) \arrow[rr, "1_A \times \mathcal{C}(A,B)"] \arrow[dr, "\pi_1"'] \arrow[rr, phantom, shift right=6ex, "\overset{\rho_{A,B}}{\Leftarrow}"] & & \mathcal{C}(A,A) \times \mathcal{C}(A,B) \arrow[dl, "c"] \\
& \mathcal{C}(A,B) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(A,B) \times 1 \arrow[rr, "\mathcal{C}(A,B) \times 1_B"] \arrow[dr, "\pi_0"'] \arrow[rr, phantom, shift right=6ex, "\overset{\lambda_{A,B}}{\Leftarrow}"] & & \mathcal{C}(A,B) \times \mathcal{C}(B,B) \arrow[dl, "c"] \\
& \mathcal{C}(A,B). &
\end{tikzcd}$$

  We again omit subscripts and simply write λ *and* ρ.

These data are required to satisfy the familiar coherence conditions (cf. [2, 13]).

The following definition also involves coherence data which should technically carry subscripts. These are indicated explicitly the first time they appear, but afterwards we adopt a policy of omitting subscripts wherever possible as in Definition 1.2.

1.3. Definition. *Given bicategories $\mathcal{C}$ and $\mathcal{D}$, a* **pseudofunctor** $F\colon \mathcal{C} \to \mathcal{D}$ *is given by the following data:*

- *An assignment of an object FC of $\mathcal{D}$ to each object C of $\mathcal{C}$.*
- *For all objects A and B of $\mathcal{C}$, a functor $F_{A,B}\colon \mathcal{C}(A,B) \to \mathcal{C}(FA,FB)$.*
- *For all objects A, B and C of $\mathcal{C}$, natural isomorphisms as indicated in the following diagrams:*

$$\begin{tikzcd}
\mathcal{C}(A,B) \times \mathcal{C}(B,C) \arrow[r, "c"] \arrow[d, "F\times F"'] \arrow[dr, phantom, "\overset{\varphi_{A,B,C}}{\Rightarrow}"] & \mathcal{C}(A,C) \arrow[d, "F"] \\
\mathcal{D}(FA,FB) \times \mathcal{D}(FB,FC) \arrow[r, "c"'] & \mathcal{D}(FA,FC)
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[rr, "1_A"] \arrow[dr, "1_{FA}"'] & \overset{\upsilon_A}{\Rightarrow} & \mathcal{C}(A,A) \arrow[dl, "F"] \\
& \mathcal{D}(FA,FA). &
\end{tikzcd}$$

These data are required to satisfy the familiar coherence conditions. The coherence conditions can be found in [2, 13], where our pseudofunctors are called homomorphisms.

1.4. Definition. *Given pseudofunctors $F, G\colon \mathcal{C} \to \mathcal{D}$, a* **pseudonatural transformation** $\psi\colon F \to G$ *consists of the following data:*

- *For each object C of $\mathcal{C}$, an arrow $\psi_C\colon FC \to GC$ in $\mathcal{D}$.*
- *For objects A and B of $\mathcal{C}$, a natural isomorphism*

$$\begin{tikzcd}
\mathcal{C}(A,B) \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{\psi_{A,B}}{\Rightarrow}"] & \mathcal{D}(FA,FB) \arrow[d, "\mathcal{D}(FA,\psi_B)"] \\
\mathcal{D}(GA,GB) \arrow[r, "\mathcal{D}(\psi_A,GB)"'] & \mathcal{D}(FA,GB).
\end{tikzcd}$$

which for each arrow $f\colon A \to B$ gives an invertible 2-cell,

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\psi_A"'] \arrow[dr, phantom, "\overset{\psi_f}{\Rightarrow}"] & FB \arrow[d, "\psi_B"] \\
GA \arrow[r, "Gf"'] & GB.
\end{tikzcd}$$

These data are required to be such that the following diagrams commute:

$$\begin{tikzcd}
(Gg \circ Gf) \circ \psi \arrow[r, "\alpha"] \arrow[d, "\varphi \circ \psi"'] & Gg \circ (Gf \circ \psi) \arrow[r, "Gg \circ \psi"] & Gg \circ (\psi \circ Ff) \arrow[r, "\alpha^{-1}"] & (Gg \circ \psi) \circ Ff \arrow[d, "\psi \circ Ff"] \\
G(g \circ f) \circ \psi \arrow[r, "\psi"'] & \psi \circ F(g \circ f) & \psi \circ (Fg \circ Ff) \arrow[l, "\psi \circ \varphi"] & (\psi \circ Fg) \circ Ff \arrow[l, "\alpha"]
\end{tikzcd}$$

$$\begin{tikzcd}
1_{GA} \circ \psi \arrow[r, "\lambda"] \arrow[d, "\upsilon \circ \psi"'] & \psi \arrow[r, "\rho^{-1}"] & \psi \circ 1_{FA} \arrow[d, "\psi \circ \upsilon"] \\
G1_A \circ \psi \arrow[rr, "\psi"'] & & \psi \circ F1_A.
\end{tikzcd}$$

In [13], our pseudonatural transformations are called *strong transformations.*

1.5. Definition. *A* **modification** $\omega\colon \psi \to \psi'$, *for* ψ *and* ψ' *pseudonatural transformations* $F \to G$, *consists of an assignment of 2-cells* $\omega_C\colon \psi_C \to \psi'_C$ *to each object* C *of* $\mathcal{C}$ *such that*

$$\begin{tikzcd}
Gf \circ \psi \arrow[r, "Gf \circ \omega"] \arrow[d, "\psi"'] & Gf \circ \psi' \arrow[d, "\psi'"] \\
\psi \circ Ff \arrow[r, "\omega \circ Ff"'] & \psi' \circ Ff
\end{tikzcd}$$

commutes, for each $f\colon A \to B$ *in* $\mathcal{C}$.

For further details, see [2, 13].

1.6. Pseudofunctor bicategories. Given bicategories $\mathcal{C}$ and $\mathcal{D}$, we denote by $[\mathcal{C}, \mathcal{D}]$ the bicategory which has as objects pseudofunctors $\mathcal{C} \to \mathcal{D}$, as arrows pseudonatural transformations, and as 2-cells modifications. Note that $[\mathcal{C}, \mathcal{D}]$ is a 2-category when $\mathcal{D}$ is.

1.7. Example. [Arrow bicategories] Given a bicategory $\mathcal{C}$ the arrow bicategory $\mathcal{C}^{\to}$ is the bicategory $[\mathbf{2}, \mathcal{C}]$, where $\mathbf{2}$ is the bicategory with two objects and one non-identity arrow between them. Since this bicategory will play an important role in this paper, we write out its objects, arrows and 2-cells explicitly.

Objects An object is an arrow $f\colon A \to B$ in $\mathcal{C}$.

Arrows Given objects $f\colon A \to B$ and $g\colon C \to D$, an arrow $f \to g$ is given by arrows $h\colon A \to C$ and $k\colon B \to D$ together with an invertible 2-cell γ as indicated in the following diagram:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\gamma"] & C \arrow[d, "g"] \\
B \arrow[r, "k"'] & D.
\end{tikzcd}$$

2-cells Given objects f and g, and arrows (h,k,γ) and (h',k',γ') from f to g, a 2-cell $\varphi\colon (h,k,\gamma) \to (h',k',\gamma')$ consists of 2-cells $\varphi_0\colon h \to h'$ and $\varphi_1\colon k \to k'$ in $\mathcal{C}$ such that

$$\gamma' \cdot (g \circ \varphi_0) = (\varphi_1 \circ f) \cdot \gamma.$$

1.8. Definition. *An arrow $f\colon A \to B$ in a bicategory $\mathcal{C}$ is an* **equivalence** *if there exists an arrow $f'\colon B \to A$ together with invertible 2-cells $f \circ f' \cong 1_B$ and $1_A \cong f' \circ f$.*

It is a well-known fact that an equivalence of categories can always be altered to give an adjoint equivalence. The same fact holds in an arbitrary bicategory:

1.9. Lemma. *If $f\colon A \to B$ is an equivalence in a bicategory $\mathcal{C}$, then there exists an arrow $f'\colon B \to A$ together with invertible 2-cells $\eta\colon 1_A \cong f' \circ f$ and $\epsilon\colon f \circ f' \cong 1_B$ which are the unit and counit of an adjunction $f \dashv f'$.*

1.10. Lemma. *Given pseudofunctors $F, G\colon \mathcal{C} \to \mathcal{D}$ between bicategories $\mathcal{C}$ and $\mathcal{D}$, if $\xi\colon F \to G$ is a pseudonatural transformation such that, for each A in $\mathcal{C}$, $\xi_A\colon FA \to GA$ is an equivalence, then there exists a pseudonatural transformation $\xi'\colon G \to F$ such that ξ' is an adjoint pseudoinverse of ξ in the bicategory $[\mathcal{C},\mathcal{D}]$.*

Proof. By Lemma 1.9, we may choose ξ'_A together with $\eta_A\colon 1_{FA} \cong \xi'_A \circ \xi_A$ and $\epsilon_A\colon \xi_A \circ \xi'_A \cong 1_{GA}$ making $\xi_A \dashv \xi'_A$. Then, for $f\colon A \to B$ in $\mathcal{C}$, the isomorphism $Ff \circ \xi'_A \cong \xi'_B \circ Gf$ is constructed by composing the isomorphisms

$$Ff \circ \xi'_A \;\cong\; \xi'_B \circ (\xi_B \circ Ff) \circ \xi'_A \;\cong\; \xi'_B \circ (Gf \circ \xi_A) \circ \xi'_A \;\cong\; \xi'_B \circ Gf$$

where the first isomorphism is a result of the coherence isomorphisms together with η_B, the second isomorphism is by ξ_f and the third is by coherence and ϵ_A. The coherence conditions on pseudonatural transformations follow from pseudonaturality of ξ and the triangle laws for adjunctions. ■

1.11. Definition. *Given bicategories $\mathcal{C}$ and $\mathcal{D}$, a pseudofunctor $F\colon \mathcal{C} \to \mathcal{D}$ is an* **equivalence of bicategories** *if there exists a pseudofunctor $G\colon \mathcal{D} \to \mathcal{C}$ together with maps $\eta\colon 1_{\mathcal{C}} \to G \circ F$ and $\epsilon\colon F \circ G \to 1_{\mathcal{D}}$ which are equivalences in the bicategories $[\mathcal{C},\mathcal{C}]$ and $[\mathcal{D},\mathcal{D}]$, respectively.*

1.12. Definition. *A pseudofunctor $F\colon \mathcal{C} \to \mathcal{D}$ is a* **weak equivalence of bicategories** *if the following conditions are satisfied:*

- *For each object D of $\mathcal{D}$, there exists an object C of $\mathcal{C}$ and an equivalence $FC \to D$ in $\mathcal{D}$.*
- *For all objects C and C' of $\mathcal{C}$, the map $\mathcal{C}(C, C') \to \mathcal{D}(FC, FC')$ is an equivalence of categories.*

Note that, in the presence of the Axiom of Choice, the notions of equivalence and weak equivalence of bicategories coincide.

1.13. Pseudofunctors into **Cat**. Since our primary example of a homotopy 2-category will be the category of stacks, we want to introduce some special notation for pseudofunctors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ where $\mathcal{C}$ is a category understood as having a trivial 2-category structure. Such a functor gives us for each object U of $\mathcal{C}$ a category $F(U)$, and for each map $f\colon V \to U$ a functor $F(f)\colon F(U) \to F(V)$. We will often denote the action of $F(f)$ on an object $x \in F(U)$ by $x \cdot f$ or, when the map f is understood, by $x|_V$. For each object U there is then a distinguished natural isomorphism $\upsilon_U\colon F(1_U) \to 1_{F(U)}$ as in the third part of Definition 1.3. Finally, for $f\colon V \to U$ and $g\colon E \to V$, the other isomorphism in that part of Definition 1.3 gives a distinguished natural isomorphism $\varphi_{f,g}\colon F(g \circ f) \to F(f) \circ F(g)$. When f and g are understood we omit subscripts and simply write φ. Similarly, we sometimes write υ instead of υ_U.

Assume given a fixed object U of $\mathcal{C}$ together with x in $F(U)$ and a family of arrows $f_\alpha\colon U_\alpha \to U$. In this situation we often denote the object $x|_{U_\alpha}$ by $x|_\alpha$ (and think of these as restrictions of the object x). When there is a second family $g_\beta\colon U_\beta \to U$, we will consider the common restrictions $x|_{U_\alpha \times_U U_\beta}$ and denote these by $x|_{\alpha\beta}$. In this situation, we can also restrict in two steps, by first restricting to U_α or U_β. However, the structure isomorphisms φ associated to the pseudo functor F give us canonical isomorphism from $x|_\alpha|_{\alpha\beta}$ to $x|_\beta|_{\alpha\beta}$, for which we introduce the notation $\sigma_{\beta\alpha}(x)$. Explicitly, $\sigma_{\beta\alpha}(x)$ is defined to be the composite

$$x|_\alpha|_{\alpha\beta} \xrightarrow{\varphi^{-1}(x)} x|_{\alpha\beta} \xrightarrow{\varphi(x)} x|_\beta|_{\alpha\beta}.$$

We also remark that $\sigma_{\alpha\beta}$ is the inverse of $\sigma_{\beta\alpha}$. When there is a third family $h_\gamma\colon U_\gamma \to U$, we similarly write $\sigma_{\alpha,\beta\gamma}(x)$ for the map $x|_{\alpha\gamma}|_{\alpha\beta\gamma} \to x|_{\alpha\beta}|_{\alpha\beta\gamma}$ which is defined in the same way as the composite

$$x|_{\alpha\gamma}|_{\alpha\beta\gamma} \xrightarrow{\varphi^{-1}(x)} x|_{\alpha\beta\gamma} \xrightarrow{\varphi(x)} x|_{\alpha\beta}|_{\alpha\beta\gamma},$$

for x an object of $F(U_\alpha)$. We could also view this as $\sigma_{\alpha\beta,\alpha\gamma}(x)$.

2. Fibrant objects and fibration structures

We will now axiomatize two bicategorical notions: bicategories with systems of fibrant objects and bicategories with fibration structures. The former suffices for the construction of the homotopy bicategory. However, the latter concept, which is a refinement of the former, captures additional structure present in certain examples and provides additional structure such as path objects for the homotopy category.

2.1. SYSTEMS OF FIBRANT OBJECTS. We will now turn to consider our first axiomatic structure on a bicategory which will allow us to form the homotopy bicategory and prove that it possesses the correct universal property. To begin, we need a 2-categorical version of the lifting property.

2.2. DEFINITION. *For arrows* $f\colon A \to B$ *and* $g\colon C \to D$ *in a bicategory* $\mathcal{C}$*, we write* $f \pitchfork g$ *to indicate that for any square of the form*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\gamma"] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd} \tag{1}$$

with γ *an invertible 2-cell, there exists a map* $l\colon B \to C$ *together with invertible 2-cells* $\lambda\colon h \cong l \circ f$ *and* $\rho\colon g \circ l \cong k$ *such that*

$$\begin{tikzcd} g \circ h \arrow[r, "g\circ\lambda"] \arrow[d, "\gamma"'] & g \circ (l \circ f) \arrow[d, "\alpha^{-1}"] \\ k \circ f & (g \circ l) \circ f \arrow[l, "\rho\circ f"] \end{tikzcd}$$

commutes in $\mathcal{C}(A, D)$*.*

Given a class $\mathfrak{M}$ *of maps in* $\mathcal{C}$ *we write* $\mathfrak{M} \pitchfork g$ *to indicate that* $f \pitchfork g$ *for all* f *in* $\mathfrak{M}$*. For* C *an object of* $\mathcal{C}$*, we write* $\mathfrak{M} \pitchfork C$ *to indicate that, for all maps* $f\colon A \to B$ *and* $h\colon A \to C$*, if* f *is in* $\mathfrak{M}$*, then there exists a map* $l\colon B \to C$ *and an invertible 2-cell* $h \Rightarrow l \circ f$*.*

Observe that if a bicategory $\mathcal{C}$ has a terminal object 1, then $\mathfrak{M} \pitchfork C$ if and only if $\mathfrak{M} \pitchfork (C \to 1)$. For such an object C, this is the bicategorical analogue of being fibrant or injective with respect to the class $\mathfrak{M}$.

2.3. DEFINITION. *A* **system of fibrant objects** *on a bicategory* $\mathcal{C}$ *consists of a collection of maps* $\mathfrak{W}$ *(*weak equivalences*) of* $\mathcal{C}$ *together with a pseudofunctor* $R\colon \mathcal{C} \to \mathcal{C}$ *(*fibrant replacement*) and a pseudonatural transformation* $\eta\colon 1_{\mathcal{C}} \to R$ *such that the following axioms are satisfied:*

Identities *All identity arrows* $1_A\colon A \to A$ *are in* $\mathfrak{W}$.

Pseudo 3-for-2 *Given a diagram*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\Downarrow\gamma"] & C \\
& B \arrow[ur, "h"'] &
\end{tikzcd}$$

with γ *an isomorphism, if any two of* f, g *and* h *are weak equivalences, then so is the third.*

Fibrant Replacement *The components of* η *are weak equivalences and* $\mathfrak{W} \pitchfork R(A)$ *for any object* A *of* $\mathcal{C}$.

The notion of a fibration structure on a bicategory $\mathcal{C}$ is a slight refinement of the notion of a system of fibrant objects:

2.4. Definition. *A* **fibration structure** *on a bicategory* $\mathcal{C}$ *with terminal object* 1 *is given by collections of maps* $\mathfrak{W}$ *(*weak equivalences*) and* $\mathfrak{F}$ *(*fibrations*) of* $\mathcal{C}$ *such that* $\mathfrak{W}$ *satisfies the identities and pseudo 3-for-2 conditions from Definition 2.3 above and such that the following additional axioms are satisfied:*

Lifting $p\colon E \to B$ *is a fibration if and only if* $\mathfrak{W} \pitchfork p$.

Factorization *There exists a pseudofunctor* $R\colon \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ *together with a pseudonatural transformation* $\eta\colon 1_{\mathcal{C}^{\to}} \to R$ *such that* $\partial_1 \circ R = \partial_1$, $\partial_1 \circ \eta = \partial_1$, *and, for each* $f\colon A \to B$ *in* $\mathcal{C}$, *the domain arrow part of* η_f *is a weak equivalence and* $R(f)$ *is a fibration. Here* ∂_1 *is the pseudofunctor* $\mathcal{C}^{\to} \to \mathcal{C}$ *which projects onto the codomain.*

In particular, the pseudofunctor R, by the conditions described, gives for each $f\colon A \to B$ in $\mathcal{C}$, a factorization

$$\begin{tikzcd}
A \arrow[rr, "\eta_f"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Rightarrow"] & \partial_0(R(f)) \arrow[dl, "R(f)"] \\
& B &
\end{tikzcd}$$

with the 2-cell invertible and $\partial_0(\eta_f)$ a weak equivalence. Note that we are simplifying notation here by denoting the arrow $A \to \partial_0(R(f))$ as η_f whereas η_f consists of this arrow, the 2-cell and the identity arrow $B \to B$.

Note that every fibration structure on a bicategory $\mathcal{C}$ determines a corresponding system of fibrant objects.

2.5. Remark. When we apply the factorization condition to the diagonal $\Delta_A \colon A \to A \times A$, we obtain a diagram

$$\begin{tikzcd}
A \arrow[rr, "\eta_{\Delta_A}"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Rightarrow"] & \partial_0(R(\Delta_A)) \arrow[dl, "R(\Delta_A)"] \\
& A \times A &
\end{tikzcd}$$

with the 2-cell invertible. Here, η_{Δ_A} is a weak equivalence and $R(\Delta_A)$ is a fibration. So we find that we can take $A^I = \partial_0 R\Delta_A$ as a path object for A and the classical factorization lemma holds up to an invertible 2-cell. Furthermore, since the two projections $\pi_i \colon A \times A \to A$ are always fibrations, we may take the composites $d_i = \pi_i \circ R\Delta_A \colon A^I \to A$ to obtain fibrations with the property that $d_i \circ \eta_{\Delta_A} \cong 1_A$.

For the remainder of this section we assume that we are working in a bicategory $\mathcal{C}$ with a system of fibrant objects.

2.6. Definition. *An object A of $\mathcal{C}$ is* **fibrant** *when $\mathfrak{W} \pitchfork A$.*

Note that, when combined with Definition 2.3, it follows that R maps all objects into fibrant objects.

2.7. Lemma. *If $f \colon A \to B$ is a weak equivalence between fibrant objects, then f is an equivalence.*

Proof. First, since A is fibrant there exists a map $f' \colon B \to A$ and an invertible 2-cell

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & A. \\
B \arrow[ur, "f'"'] &
\end{tikzcd}$$

It follows from the pseudo 3-for-2 property that f' is also a weak equivalence. Therefore, since B is fibrant, there exists another map $f'' \colon A \to B$ and an invertible 2-cell

$$\begin{tikzcd}
B \arrow[r, "1_B"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Downarrow"] & B. \\
A \arrow[ur, "f''"'] &
\end{tikzcd}$$

Now, the 2-cells above, together with the coherence 2-cells of $\mathcal{C}$, give us an isomorphism $f \cong f''$ and therefore f' is the pseudo-inverse of f, as required. ■

2.8. Definition. *The* **homotopy bicategory $\mathbf{Ho}(\mathcal{C})$ of** $\mathcal{C}$ *is the full sub-bicategory of fibrant objects of* $\mathcal{C}$.

We denote by $I\colon \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ the pseudofunctor induced by $R\colon \mathcal{C} \to \mathcal{C}$. It is an immediate consequence of Lemma 2.7 and the pseudo 3-for-2 property that I sends weak equivalences to equivalences. For any bicategory $\mathcal{D}$, let $[\mathcal{C},\mathcal{D}]_{\mathfrak{W}}$ denote the full sub-bicategory of $[\mathcal{C},\mathcal{D}]$ consisting of those pseudofunctors which send maps in $\mathfrak{W}$ to equivalences. Let $J\colon \mathbf{Ho}(\mathcal{C}) \to \mathcal{C}$ be the inclusion and observe that $R = J \circ I$.

We will now prove that I is the universal map from $\mathcal{C}$ to a bicategory which sends weak equivalences to equivalences.

2.9. Theorem. *For any bicategory* $\mathcal{D}$ *with a system of fibrant objects,* $I\colon \mathcal{D} \to \mathbf{Ho}(\mathcal{D})$ *induces an equivalence of bicategories*

$$[\mathbf{Ho}(\mathcal{C}),\mathcal{D}] \xrightarrow{[I,\mathcal{D}]} [\mathcal{C},\mathcal{D}]_{\mathfrak{W}}.$$

Proof. Precomposition with the inclusion J gives a pseudofunctor $[\mathcal{C},\mathcal{D}]_{\mathfrak{W}} \to [\mathbf{Ho}(\mathcal{C}),\mathcal{D}]$ which we denote by $[J,\mathcal{D}]$. The pseudonatural transformation $\eta\colon 1_{\mathbf{Ho}(\mathcal{C})} \to I \circ J$ (obtained by restricting η to $\mathbf{Ho}(\mathcal{C})$) induces a pseudonatural transformation $[\eta,\mathcal{D}]\colon 1_{[\mathbf{Ho}(\mathcal{C}),\mathcal{D}]} \to [J,\mathcal{D}] \circ [I,\mathcal{D}]$. Observe that, by Lemmas 1.10 and 2.7, $\eta\colon 1_{\mathbf{Ho}(\mathcal{C})} \to I \circ J$ is an equivalence. Therefore the induced $[\eta,\mathcal{D}]$ is also an equivalence.

On the other hand, for F in $[\mathcal{C},\mathcal{D}]_{\mathfrak{W}}$, Lemma 1.10 exhibits $F\eta\colon F \to FR$ as an adjoint equivalence. Let ϑ^F denote the adjoint pseudoinverse of $F\eta$. Allowing F to vary, we have that ϑ is an equivalence $[I,\mathcal{D}] \circ [J,\mathcal{D}] \to 1_{[\mathcal{C},\mathcal{D}]_{\mathfrak{W}}}$. ∎

2.10. Example. [Localizations of categories] Let $\mathbf{C}$ be a category, considered as a 2-category with just identity 2-cells, and let W be a class of arrows which contains all identities, satisfies the 3-for-2 property, with a system of fibrant objects and fibrant replacement functor $R\colon \mathbf{C} \to \mathbf{C}$. We want to compare the resulting localization with the traditional localizations such as the category of fractions $\mathbf{C}[W^{-1}]$ and the bicategory of fractions $\mathbf{C}(W^{-1})$. The universal properties of these localizations give us the following sequence of equivalences of bicategories:

$$\mathbf{C}[W^{-1}] \simeq \mathbf{C}(W^{-1}) \simeq \mathbf{Ho}(\mathbf{C}). \tag{2}$$

Note that in this case both the category of fractions $\mathbf{C}[W^{-1}]$ and the homotopy category $\mathbf{Ho}(\mathbf{C})$ are ordinary categories. By construction, $\mathbf{Ho}(\mathbf{C})$ has small hom-sets. But the equivalence of bicategories (2) gives us then a bijection on the hom-sets, so we conclude that if $\mathcal{C}$ is a category with a system of fibrant objects with respect to W, the ordinary category of fractions will also have small hom-sets. We see that the fact that we have the system of fibrant objects provides us also with additional information about the ordinary category of fractions.

3. Stacks and local fibrations

We will now begin developing the machinery required to explain our first example of a fibration structure in a bicategory (actually, in this case a 2-category): the 2-category of prestacks. In this section we recall some of the basic notions involved and we also introduce a fibered version of the usual category of descent data that will allow us to describe the maps, which we call *local fibrations*, that provide the fibrations in the fibration structure for the 2-category of prestacks.

3.1. COVERINGS AND SITES. Throughout we assume given a fixed site $(\mathcal{C}, J)$ for $\mathcal{C}$ a category with finite limits. Given an object U of $\mathcal{C}$, recall that a *sieve on* U is a family of maps with codomain U which is a right ideal for composition. To say that $(\mathcal{C}, J)$ is a site then means that J assigns to each object U of $\mathcal{C}$ a collection $J(U)$ of sieves on U (called *covering sieves*, *covering families* or *covers*) in such a way that the following conditions are satisfied:

1. The maximal sieve on U, which consists of all arrows with codomain U, is in $J(U)$.
2. For a cover $\mathcal{S}$ in $J(U)$ and $g\colon V \to U$, the sieve $g^*(\mathcal{S}) := \{f\colon E \to V \mid g \circ f \in \mathcal{S}\}$ is in $J(V)$.
3. Given $\mathcal{S}$ in $J(U)$ and a sieve $\mathcal{R}$ on U, if $f^*(\mathcal{R})$ is in $J(V)$ for all $f\colon V \to U$ in $\mathcal{S}$, then $\mathcal{R}$ is also in $J(U)$.

We will sometimes also work with the notion of a *basis* for covers (also known as a pretopology). A basis consists of an operation K which assigns to objects U of $\mathcal{C}$ a collection $K(U)$ of families of maps with codomain U such that

1. The singleton family $(f\colon U' \to U)$ is in $K(U)$ when f is an isomorphism.
2. If $U_\alpha \to U$ is in $K(U)$ and $V \to U$ is any map, then $V \times_U U_\alpha \to V$ is in $K(V)$.
3. If $(f_\alpha\colon U_\alpha \to U)$ is a family of maps in $K(U)$ and $(f^\alpha_\beta\colon U^\alpha_\beta \to U_\alpha)$ is in $K(U_\alpha)$, then the family of maps $(f_\alpha \circ f^\alpha_\beta)$ is in $K(U)$.

If K is a basis, then K generates a site $(\mathcal{C}, J)$ by letting S be in $J(U)$ if and only if there exists a family T in $K(U)$ such that $T \subseteq S$.

Readers unfamiliar with sites and sheaves may consult [14].

3.2. STACKS. Given a site $(\mathcal{C}, J)$, a pseudofunctor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ is a **stack** when, for any cover $(f_\alpha\colon U_\alpha \to U)_\alpha$ of U, the canonical map

$$F(U) \to \varprojlim_{\alpha} F(U_\alpha) \tag{3}$$

is a weak equivalence of categories. Note that here $\varprojlim_\alpha F(U_\alpha)$ indicates the *pseudolimit* and *not* the strict limit (an elementary description can be found below for which we refer the reader to Example 3.11). $\mathbf{St}(\mathcal{C})$ denotes the full subcategory of the 2-category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]$ of pseudofunctors, consisting of stacks.

Note that if a basis K generates the covering sieves of a site $(\mathcal{C}, J)$, then it suffices, in order to tell whether F is a stack, to test on the families of maps $\mathcal{U}$ in $K(U)$.

3.3. Prestacks. Given a pseudofunctor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ and objects a and b of $F(U)$, there is an induced functor

$$\begin{tikzcd}(\mathcal{C}/U)^{\mathrm{op}} \arrow[r, "F(a,b)"] & \mathbf{Set}\end{tikzcd}$$

which is defined on objects by

$$\begin{tikzcd}f\colon V \to U \arrow[r, mapsto] & \mathrm{Hom}_{FV}(a \cdot f, b \cdot f).\end{tikzcd}$$

(We will also use $a|_V$ for $a \cdot f$ when f is obvious from the context.) To define this on arrows, we use the structure isomorphisms for F. For instance, for a commutative triangle

$$\begin{tikzcd}V \arrow[rr, "h"] \arrow[dr, "f"'] & & W \arrow[dl, "k"] \\ & U & \end{tikzcd}$$

let $\varphi_{k,h}\colon F(h) \circ F(k) \overset{\sim}{\Rightarrow} F(k \circ h) = F(f)$. Then $F(a,b)(h)(\alpha\colon a \cdot k \to b \cdot k)$ is the composition of the following arrows

$$\begin{tikzcd}a \cdot f \arrow[r, "(\varphi_{k,h}^{-1})_a"] & (a \cdot k) \cdot h \arrow[r, "\alpha \cdot h"] & (b \cdot k) \cdot h \arrow[r, "(\varphi_{k,h}^{-1})_b"] & b \cdot f\end{tikzcd}$$

It follows from the coherence conditions on the structure isomorphisms for F that $F(a,b)$ is indeed a strict functor, rather than a pseudofunctor.

3.4. Definition. *A pseudofunctor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ is a* **prestack** *if, for any object U in $\mathcal{C}$ and $a, b \in F(U)$, $F(a,b)$ is a sheaf.*

Note that in a prestack it is possible to construct arrows in the categories $F(U)$ locally (i.e., on a cover): an arrow $z\colon a \to b$ in $F(U)$ is completely determined by its restrictions $z|_\alpha\colon a|_\alpha \to b|_\alpha$ in $F(U_\alpha)$ on a cover $(U_\alpha \to U)_\alpha$, and a family of matching arrows like this gives rise to a unique arrow $a \to b$ in $F(U)$. Note that every stack is a prestack.

3.5. Example. Let $\mathbf{T}$ be a category internal to the category of topological spaces $\mathbf{Top}$. This gives rise to a prestack $H_{\mathbf{T}}$ on $\mathbf{Top}$, regarded as a site with the topology generated by open surjections as follows. $H_{\mathbf{T}}(X) = \mathrm{Hom}(X, \mathbf{T})$, the Hom is taken in the category of topological categories with internal functors and a space is viewed as a topological category with only identity arrows and all identity structure arrows. To see that this is indeed a prestack, consider $f, g \in H_{\mathbf{T}}(X)$, i.e., $f, g\colon X \rightrightarrows \mathbf{T}$. A matching family of arrows from f to g on a cover corresponds in this case to an open surjection $u\colon U \to X$ with a continuous map $\alpha\colon U \to \mathbf{T}_1$ representing a natural transformation from f to g and such that the composites of α with the two projections $U \times_X U \rightrightarrows U$ are equal. It follows then from the fact that every open surjection is the coequalizer of its kernel pair (open surjections are regular epimorphisms and so the basis described is subcanonical) that α factors through u to give a natural transformation from f to g.

For further examples of stacks and prestacks we refer the reader to [8, 15, 16].

3.6. Descent data. In this section we want to generalize the familiar definition of the category of descent data by making this data vary relative to a fixed morphism. In order to do this we will first consider the morphisms between pseudofunctors, i.e., pseudonatural transformations, in a bit more detail. Let $p\colon E \to B$ be a pseudonatural transformation between pseudofunctors $E, B\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$. This has components $p_U\colon E(U) \to B(U)$, which are functors, and for each arrow $f_\alpha\colon U_\alpha \to U$, there is an invertible natural transformation as in the following square,

$$\begin{tikzcd}
E(U) \arrow[r, "-|_\alpha = E(f_\alpha)"] \arrow[d, "p_U"'] \arrow[dr, phantom, "\overset{p_{f_\alpha}}{\Rightarrow}"] & E(U_\alpha) \arrow[d, "p_{U_\alpha}"] \\
B(U) \arrow[r, "-|_\alpha = B(f_\alpha)"'] & B(U_\alpha)
\end{tikzcd} \tag{4}$$

We will generally omit the subscripts of the functors p_U and p_{U_α}, and we will write $\wp$ for the invertible natural transformations p_{f_α} (omitting their subscripts) and $\wp(e)\colon p(e|_\alpha) \to p(e)|_\alpha$ for the component at $e \in E(U)$.

3.7. Definition. *Given* $p\colon E \to B$ *in* $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]$ *and a cover* $\mathcal{S} = (f_\alpha\colon U_\alpha \to U)_\alpha$ *of some* U *we define the category* $\mathbf{Desc}(p, \mathcal{S})$ *as follows*

Objects *An object is a 4-tuple* $(b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ *where* $b \in B(U)$, $e_\alpha \in E(U_\alpha)$, *and each* $\psi_\alpha\colon p(e_\alpha) \to b|_\alpha$ *is an isomorphism, i.e., the* e_α *can be thought of as isomorphic to the restrictions of* b *relative to* p*. Furthermore, we require the* e_α *to be locally compatible in the sense that there are isomorphisms* $\vartheta_{\alpha\beta}\colon e_\beta|_{\alpha\beta} \to e_\alpha|_{\alpha\beta}$*. This data is furthermore required to be compatible with the structure isomorphisms of* p*,* E *and* B *in the sense that it needs to satisfy the conditions that*

$$\vartheta_{\alpha\alpha} = 1_{e_\alpha}$$

and that the diagrams

$$\begin{tikzcd}
e_\gamma|_{\alpha\gamma}|_{\alpha\beta\gamma} \arrow[d, "\vartheta_{\alpha\gamma}|_{\alpha\beta\gamma}"'] & e_\gamma|_{\beta\gamma}|_{\alpha\beta\gamma} \arrow[l, "\sigma_{\gamma,\alpha\beta}(e_\gamma)"'] \arrow[r, "\vartheta_{\beta\gamma}|_{\alpha\beta\gamma}"] & e_\beta|_{\beta\gamma}|_{\alpha\beta\gamma} \arrow[d, "\sigma_{\beta,\alpha\gamma}(e_\beta)"] \\
e_\alpha|_{\alpha\gamma}|_{\alpha\beta\gamma} \arrow[r, "\sigma_{\alpha,\beta\gamma}(e_\alpha)"'] & e_\alpha|_{\alpha\beta}|_{\alpha\beta\gamma} & e_\beta|_{\alpha\beta}|_{\alpha\beta\gamma} \arrow[l, "\vartheta_{\alpha\beta}|_{\alpha\beta\gamma}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
p(e_\beta|_{\alpha\beta}) \arrow[r, "\wp(e_\beta)"] \arrow[d, "p(\vartheta_{\alpha\beta})"'] & p(e_\beta)|_{\alpha\beta} \arrow[r, "\psi_\beta|_{\alpha\beta}"] & b|_\beta|_{\alpha\beta} \arrow[d, "\sigma_{\alpha\beta}(b)"] \\
p(e_\alpha|_{\alpha\beta}) \arrow[r, "\wp(e_\alpha)"'] & p(e_\alpha)|_{\alpha\beta} \arrow[r, "\psi_\alpha|_{\alpha\beta}"'] & b|_\alpha|_{\alpha\beta}
\end{tikzcd}$$

commute.

Arrows *An arrow* $(b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta})) \to (b', (e'_\alpha), (\psi'_\alpha), (\vartheta'_{\alpha\beta}))$ *is given by a pair* $(g, (g_\alpha))$ *such that* $g\colon b \to b'$ *in* $B(U)$ *and* $g_\alpha\colon e_\alpha \to e'_\alpha$ *in* $E(U_\alpha)$*. This data is subject to the requirements that the diagrams*

$$\begin{tikzcd}
e_\beta|_{\alpha\beta} \arrow[r, "\vartheta_{\alpha\beta}"] \arrow[d, "g_\beta|_{\alpha\beta}"'] & e_\alpha|_{\alpha\beta} \arrow[d, "g_\alpha|_{\alpha\beta}"] \\
e'_\beta|_{\alpha\beta} \arrow[r, "\vartheta'_{\alpha\beta}"'] & e'_\alpha|_{\alpha\beta}
\end{tikzcd}$$

and

$$\begin{tikzcd}
p(e_\alpha) \arrow[r, "\psi_\alpha"] \arrow[d, "p(g_\alpha)"'] & b|_\alpha \arrow[d, "g|_\alpha"] \\
p(e'_\alpha) \arrow[r, "\psi'_\alpha"'] & b'|_\alpha
\end{tikzcd}$$

commute.

There is a projection functor $\pi\colon \mathbf{Desc}(p, \mathcal{S}) \to B(U)$. When p is the canonical map $E \to 1$ into the terminal object we write $\mathbf{Desc}(E, \mathcal{S})$ instead of $\mathbf{Desc}(p, \mathcal{S})$ and observe that this is the usual category of descent data. There is also an evident functor

$$\begin{tikzcd}
E(U) \arrow[rr, "\Phi_{\mathcal{S}}"] \arrow[dr, "p"'] & & \mathbf{Desc}(p, \mathcal{S}) \arrow[dl, "\pi"] \\
& B(U) &
\end{tikzcd}$$

which sends an object e of $E(U)$ to the tuple

$$\big(p(e), (e|_\alpha), (1_{p(e|_\alpha)}), (\sigma_{\alpha\beta}(e))\big)$$

where $\sigma_{\alpha\beta}$ is as in Section 1.13.

Given a commutative triangle

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]$ and a covering family $\mathcal{S}$ of U, there is a corresponding commutative diagram

$$\begin{tikzcd} \mathbf{Desc}(p, \mathcal{S}) \arrow[rr, "f_*"] \arrow[dr, "\pi"'] & & \mathbf{Desc}(p', \mathcal{S}) \arrow[dl, "\pi"] \\ & B(U) & \end{tikzcd}$$

in $\mathbf{Cat}$. Here the functor f_* acts as follows:

On objects f_* sends $(b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ in $\mathbf{Desc}(p, \mathcal{U})$ to $(b, (f(e_\alpha)), (\psi_\alpha), \xi_{\alpha\beta})$ where $\xi_{\alpha\beta}$ is the composite

$$\begin{tikzcd} f(e_\beta)|_{\alpha\beta} \arrow[r] & f(e_\beta|_{\alpha\beta}) \arrow[r, "f(\vartheta_{\alpha\beta})"] & f(e_\alpha|_{\alpha\beta}) \arrow[r] & f(e_\alpha)|_{\alpha\beta} \end{tikzcd}$$

where the unnamed arrows are the isomorphisms associated to the pseudonatural transformation f as in (4).

On arrows An arrow $(g, (g_\alpha))$ in $\mathbf{Desc}(p, \mathcal{U})$ is sent to $(g, (f(g_\alpha)))$.

Moreover, this construction is functorial in the sense that $\mathbf{Desc}(-, \mathcal{S})$ is a functor from $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]/B$ to $\mathbf{Cat}/B(U)$. This fact is a special case of a more general result to which we now turn.

3.8. Lemma. *For a fixed object U of $\mathcal{C}$ and a covering family $\mathcal{S}$ of U,* $\mathbf{Desc}(-, \mathcal{S})$ *is a functor* $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]^{\rightarrow} \to \mathbf{Cat}$.

Proof. Given a square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] \arrow[dr, phantom, "\Downarrow\gamma"] & E \arrow[d, "p"] \\ C \arrow[r, "k"'] & B \end{tikzcd} \qquad (5)$$

in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]$ with γ invertible, the induced functor $(h,k,\gamma)_*\colon \mathbf{Desc}(i,\mathcal{S}) \to \mathbf{Desc}(p,\mathcal{S})$ sends descent data $(c,(a_\alpha),(\psi_\alpha),(\vartheta_{\alpha\beta}))$ in $\mathbf{Desc}(i,\mathcal{S})$ to $(k(c),(h(a_\alpha)),(\hat{\psi}_\alpha),(\hat{\vartheta}_{\alpha\beta}))$ in $\mathbf{Desc}(p,\mathcal{S})$ where $\hat{\psi}_\alpha$ is the composite

$$p(h(a_\alpha)) \xrightarrow{\gamma(a_\alpha)} k(i(a_\alpha)) \xrightarrow{k(\psi_\alpha)} k(c|_\alpha) \xrightarrow{\quad} k(c)|_\alpha$$

and $\hat{\vartheta}_{\alpha\beta}$ is the composite

$$h(a_\beta)|_{\alpha\beta} \xrightarrow{\quad} h(a_\beta|_{\alpha\beta}) \xrightarrow{h(\vartheta_{\alpha\beta})} h(a_\alpha|_{\alpha\beta}) \xrightarrow{\quad} h(a_\alpha)|_{\alpha\beta}.$$

Here the unnamed arrows are from the isomorphisms associated to the pseudonatural transformations as in (4). Next, given another arrow $(h',k',\gamma')\colon i \to p$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]$ together with a 2-cell $(\varphi_0,\varphi_1)\colon (h,k,\gamma) \to (h',k',\gamma')$, the corresponding natural transformation $(\varphi_0,\varphi_1)_*$ is obtained at an object $(c,(a_\alpha),(\psi_\alpha),(\vartheta_{\alpha\beta}))$ as the arrow $(\varphi_1(c),(\varphi_0(a_\alpha)))$. ■

Observe that, given a square (5) in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]$ and a cover $\mathcal{S}$ of some U, the following diagram commutes on the nose:

$$\begin{tikzcd}
\mathbf{Desc}(i,\mathcal{S}) \arrow[r, "(h,k,\gamma)_*"] \arrow[d, "\pi"'] & \mathbf{Desc}(p,\mathcal{S}) \arrow[d, "\pi"] \\
C(U) \arrow[r, "k"'] & B(U).
\end{tikzcd}$$

In particular, the functors $\mathbf{Desc}(-,\mathcal{S})$ from Lemma 3.8 factor as indicated in the following diagram:

$$\begin{tikzcd}
{[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]^{\to}} \arrow[rr] \arrow[dr, "\mathbf{Desc}(-{,}\mathcal{S})"'] & & \mathbf{Cat}^{\to} \arrow[dl, "\partial_1"] \\
& \mathbf{Cat}. &
\end{tikzcd} \tag{6}$$

On the other hand, we merely have a natural isomorphism $\hat{\gamma}$ as indicated in the following diagram:

$$\begin{tikzcd}
A(U) \arrow[r, "h"] \arrow[d, "\Phi_{\mathcal{S}}"'] \arrow[dr, phantom, "\Downarrow\hat{\gamma}"] & E(U) \arrow[d, "\Phi_{\mathcal{S}}"] \\
\mathbf{Desc}(i,\mathcal{S}) \arrow[r, "(h,k,\gamma)_*"'] & \mathbf{Desc}(p,\mathcal{S})
\end{tikzcd}$$

which, for a an object of $A(U)$, is the map of descent data

$$\Big(\gamma(a)\colon ph(a) \to ki(a), \big(h(a)|_\alpha \to h(a|_\alpha)\big)\Big)\colon \Phi_{\mathcal{S}}\big(h(a)\big) \to (h,k,\gamma)_*\big(\Phi_{\mathcal{S}}(a)\big).$$

This has the property that

$$\begin{tikzcd}
A(U) \arrow[r, "h"] \arrow[d, "\Phi_{\mathcal{S}}"'] \arrow[dr, phantom, "\Downarrow\hat{\gamma}"] & E(U) \arrow[d, "\Phi_{\mathcal{S}}"] \\
\mathbf{Desc}(i,\mathcal{S}) \arrow[r, "(h,k,\gamma)_*"'] \arrow[d, "\pi"'] & \mathbf{Desc}(p,\mathcal{S}) \arrow[d, "\pi"] \\
C(U) \arrow[r, "k"'] & B(U)
\end{tikzcd}
=
\begin{tikzcd}
A(U) \arrow[r, "h"] \arrow[d, "i"'] \arrow[dr, phantom, "\Downarrow\gamma"] & E(U) \arrow[d, "p"] \\
C(U) \arrow[r, "k"'] & B(U)
\end{tikzcd}$$

The construction of the category of descent data is also functorial in the second argument in the sense that if $\mathcal{S}$ and $\mathcal{R}$ are both covers of some U with $\mathcal{R} \subseteq \mathcal{S}$, then there exists an associated restriction functor $\cdot|_{\mathcal{R}}\colon \mathbf{Desc}(p,\mathcal{S}) \to \mathbf{Desc}(p,\mathcal{R})$ which acts by restricting descent data to the maps in $\mathcal{R}$. These restrictions satisfy the functoriality condition $(\cdot|_{\mathcal{U}}) \circ (\cdot|_{\mathcal{R}}) = \cdot|_{\mathcal{U}}$ and are well-behaved with respect to the associated maps $\Phi_{\mathcal{S}}\colon E(U) \to \mathbf{Desc}(p,\mathcal{S})$, in the sense that the diagram

$$\begin{tikzcd}
E(U) \arrow[rr, "\Phi_{\mathcal{S}}"] \arrow[dr, "\Phi_{\mathcal{R}}"'] & & \mathbf{Desc}(p,\mathcal{S}) \arrow[dl, "\cdot|_{\mathcal{R}}"] \\
& \mathbf{Desc}(p,\mathcal{R}) &
\end{tikzcd}$$

commutes for any $\mathcal{R} \subseteq \mathcal{S}$.

In addition to the functorial behavior of $\mathbf{Desc}(-,-)$ described above, if we are given a fixed $p\colon E \to B$, a cover $\mathcal{U}$ of U and a map $g\colon V \to U$ in the site, we obtain a further restriction functor $g^*\colon \mathbf{Desc}(p,\mathcal{U}) \to \mathbf{Desc}(p,g^*(\mathcal{U}))$ which sends descent data

$$(b,(e_\alpha),(\psi_\alpha),(\vartheta_{\alpha\beta}))$$

to the descent data given by:

- the object $b|_V$ of $B(V)$;
- the family of objects (e_α) (this makes sense by virtue of the definition of $g^*(\mathcal{U})$);
- the family of maps given by the composites

$$p(e_\alpha) \xrightarrow{\psi_\alpha} b|_\alpha \longrightarrow (b|_V)|_\alpha,$$

  which we denote by $g^*(\psi)_\alpha$ when no confusion will result; and

- the family of maps given by the composites

  $$\begin{tikzcd} e_\beta|_{U_\alpha \times_V U_\beta} \arrow[r] & e_\beta|_{U_\alpha \times_U U_\beta}|_{U_\alpha \times_V U_\beta} \arrow[r, "\vartheta_{\alpha\beta}|_{U_\alpha \times_V U_\beta}"] & e_\alpha|_{U_\alpha \times_U U_\beta}|_{U_\alpha \times_V U_\beta} \arrow[r] & e_\alpha|_{U_\alpha \times_V U_\beta} \end{tikzcd}$$

  where $\vartheta_{\alpha\beta}$ is here restricted along the induced map $U_\alpha \times_V U_\beta \to U_\alpha \times_U U_\beta$ and the unlabeled maps are the structural isomorphisms associated with pseudofunctoriality of E;

and which acts on arrows by sending $(g, (g_\alpha))$ to $(g|_V, (g_\alpha))$.

3.9. LOCAL FIBRATIONS. We are now in a position to describe the maps which will be the fibrations in our fibration structure on the 2-category of prestacks.

3.10. DEFINITION. *A map $p\colon E \to B$ of prestacks is a* **local fibration** *if and only if, for every U and cover $\mathcal{S}$ of U, the map*

$$\Phi_{\mathcal{S}}\colon E(U) \to \mathbf{Desc}(p, \mathcal{S})$$

described in Section 3.6 above is a weak equivalence.

3.11. EXAMPLE. When p is the canonical map $F \to 1$, $\mathbf{Desc}(p, (U_\alpha))$ is the pseudolimit from (3) and this map is a local fibration if and only if F is a stack.

3.12. EXAMPLE. Let $\mathbf{2}$ be as in Example 1.7, then, for $A\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ a pseudofunctor, $[\mathbf{2}, A]$ denotes the cotensor with $\mathbf{2}$. I.e., $[\mathbf{2}, A](U) = A(U)^{\mathbf{2}}$. A is a prestack if and only if the induced map $\langle \partial_0, \partial_1 \rangle\colon [\mathbf{2}, A] \to A \times A$ is a local fibration.

Notice that the map $\Phi_{\mathcal{S}}$ is always faithful and that we have the following characterization of local fibrations between prestacks:

3.13. LEMMA. *If E and B are prestacks, then $p\colon E \to B$ is a local fibration if and only if, for each U and cover $\mathcal{S}$, $\Phi_{\mathcal{S}}$ is essentially surjective on objects.*

PROOF. By hypothesis, it remains to show that $\Phi_{\mathcal{S}}$ is full. Given a map $(f, f_\alpha)\colon \Phi_{\mathcal{S}}(e) \to \Phi_{\mathcal{S}}(e')$ in $\mathbf{Desc}(p, \mathcal{S})$ it follows from the fact that E is a prestack that the f_α possess a unique amalgamation $g\colon e \to e'$. Since B is a prestack we may test locally to see that $p(g) = f$. ■

3.14. DEFINITION. *An internal functor $\varphi\colon \mathbf{T} \to \mathbf{S}$ between internal categories in* **Top** *is said to be an* **essential equivalence** *when it is full and faithful, and when it is* essentially surjective on objects *in the sense that the composite $t \circ p_1$ in*

$$\begin{tikzcd} \mathbf{T}_0 \times_{\mathbf{S}_0} \mathbf{S}_1 \arrow[r, "p"] \arrow[d] & \mathbf{S}_1 \arrow[r, "t"] \arrow[d, "s"] & \mathbf{S}_0 \\ \mathbf{T}_0 \arrow[r, "\varphi_0"'] & \mathbf{S}_0 & \end{tikzcd}$$

is an open surjection.

3.15. Example. An internal functor $\varphi\colon \mathbf{T} \to \mathbf{S}$ between categories in **Top** induces a local fibration $\varphi_*\colon H_{\mathbf{T}} \to H_{\mathbf{S}}$ when for each open surjection $u\colon U \to X$ and each diagram

$$\begin{tikzcd}
U \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{\psi}{\Rightarrow}"] & \mathbf{T} \arrow[d, "\varphi"] \\
X \arrow[r, "f"'] & \mathbf{S}
\end{tikzcd}$$

with ψ invertible and where g satisfies the local compatibility condition, $g\pi_1 = g\pi_2$ for

$$\begin{tikzcd}
U \times_X U \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & U,
\end{tikzcd}$$

there is a lifting $h\colon X \to \mathbf{T}$ with invertible 2-cells as in

$$\begin{tikzcd}
U \arrow[r, "g"] \arrow[d, "u"'] \arrow[r, phantom, shift right=4, "\overset{\rho}{\Rightarrow}"'] & \mathbf{T} \arrow[d, "\varphi"] \\
X \arrow[r, "f"'] \arrow[ur, "h"] \arrow[r, phantom, shift left=4, "\overset{\tau}{\Rightarrow}"] & \mathbf{S}
\end{tikzcd}$$

such that the pasting of ρ and τ is ψ_u.

An open surjection $u\colon U \to X$ of topological spaces gives rise to an essential equivalence of topological groupoids in the following way. Let $\Delta_X(U)$ be the groupoid with space of objects U and space of arrows $U \times_X U$ where the structure maps are all the obvious ones. We can extend u to a map of groupoids $u'\colon \Delta_X(U) \to X$ (where we view X as the groupoid with only identity arrows) with $u' = u$ on the objects and u' sends all arrows to identities. Also, since $g\pi_1 = g\pi_2$ there is an induced map $g'\colon \Delta_X(U) \to \mathbf{T}$ which is g on the space of objects and sends each arrow in $U \times_X U$ to an identity in $\mathbf{T}$.

The map from U to the space of arrows of $\mathbf{S}$ representing ψ_u in the first square also represents an invertible 2-cell $\psi_{u'}$ as in the following diagram,

$$\begin{tikzcd}
\Delta_X(U) \arrow[r, "g'"] \arrow[d, "u'"'] \arrow[dr, phantom, "\overset{\psi_{u'}}{\Rightarrow}"] & \mathbf{T} \arrow[d, "\varphi"] \\
X \arrow[r, "f"'] & \mathbf{S}.
\end{tikzcd}$$

Furthermore, a lifting as in the second diagram above, corresponds precisely to a lifting

$$\begin{tikzcd}
\Delta_X(U) \arrow[r, "g'"] \arrow[d, "u'"'] \arrow[r, phantom, shift right=4, "\overset{\rho'}{\Rightarrow}"'] & \mathbf{T} \arrow[d, "\varphi"] \\
X \arrow[r, "f"'] \arrow[ur, "h"] \arrow[r, phantom, shift left=4, "\overset{\tau'}{\Rightarrow}"] & \mathbf{S}
\end{tikzcd}$$

where the invertible ρ' and τ' are represented by the same maps of topological spaces as ρ and τ, respectively. So a groupoid homomorphism gives rise to a local fibration between prestacks when it satisfies a somewhat restricted version of the pseudo right lifting property with respect to these special essential equivalences of the form $\Delta_X(U) \to X$.

In [5], Colman and Costoya describe a Quillen model structure on orbifold groupoids. Their fibrations are required to have the pseudo right lifting property with respect to all essential equivalences. (They phrase it as a right lifting property, but they work with isomorphism classes of maps.) So we see that all fibrations in their terminology are local fibrations in our terminology.

4. The fibration structure on prestacks

We will now describe the fibration structure on $\mathbf{PreSt}(\mathcal{C})$ for a site $(\mathcal{C}, J)$ such that the topology J is subcanonical. We begin by defining what we will call *local weak equivalences* (this definition can be found in [16] and similar definitions appear throughout the literature on stacks and homotopy theory such as in, e.g., [3, 10]).

4.1. Definition. *A map $h\colon A \to B$ in $\mathbf{PreSt}(\mathcal{C})$ is said to be* **locally essentially surjective on objects** *if and only if for any U and $b \in B(U)$ there exists a cover $\mathcal{S} = (f_\alpha\colon U_\alpha \to U)$ of U together with, for each α, an element $\tilde{b}_\alpha \in A(U_\alpha)$ and an isomorphism $\psi_\alpha\colon h(\tilde{b}_\alpha) \to b|_\alpha$.*

4.2. Example. An internal functor $\varphi\colon \mathbf{T} \to \mathbf{S}$ between categories in **Top** induces a map of prestacks $\varphi_*\colon H_{\mathbf{T}} \to H_{\mathbf{S}}$ which is locally essentially surjective on objects if for all $f\colon X \to \mathbf{S}$ there is an open surjection $u\colon U \to X$ with an arrow $f_u\colon U \to \mathbf{T}$ and invertible natural transformation ψ_u as in the following diagram,

$$\begin{tikzcd} U \arrow[r, "f_u"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{\psi_u}{\Rightarrow}"] & \mathbf{T} \arrow[d, "\varphi"] \\ X \arrow[r, "f"'] & \mathbf{S} \end{tikzcd} \tag{7}$$

4.3. Definition. *A map $h\colon A \to B$ in $\mathbf{PreSt}(\mathcal{C})$ is a* **local weak equivalence** *if it is full, faithful and locally essentially surjective on objects.*

Here being full and faithful means being *pointwise* full and faithful.

4.4. Example. An internal functor $\varphi\colon \mathbf{T} \to \mathbf{S}$ between categories in **Top** induces a map of prestacks $\varphi_*\colon H_{\mathbf{T}} \to H_{\mathbf{S}}$ which is full and faithful if for any parallel arrows $f, g\colon X \rightrightarrows \mathbf{T}$ and any 2-cell $\alpha\colon \varphi \circ f \Rightarrow \varphi \circ g$ there is a unique $\tilde{\alpha}\colon f \Rightarrow g$ such that $\alpha = \varphi \circ \tilde{\alpha}$.

The examples 4.2 and 4.4 lead us to the following result about prestacks on **Top**.

4.5. Proposition. *For groupoids* $\mathbf{S}$ *and* $\mathbf{T}$ *in* $\mathbf{Top}$*, an internal functor* $\varphi\colon \mathbf{T} \to \mathbf{S}$ *induces a local weak equivalence* $\varphi_*\colon H_{\mathbf{T}} \to H_{\mathbf{S}}$ *of prestacks if and only if it is an essential equivalence of internal groupoids in* $\mathbf{Top}$.

Proof. Let $\mathbf{S}$ and $\mathbf{T}$ be topological groupoids and let $\varphi\colon \mathbf{T} \to \mathbf{S}$ be a functor such that $\varphi_*\colon H_{\mathbf{T}} \to H_{\mathbf{S}}$ is a local weak equivalence of prestacks. We need to show that φ is essentially surjective on objects and fully faithful on arrows. To show that φ is essentially surjective on objects we need to show that the composite of the two top arrows in the following diagram

$$\begin{tikzcd}
T_0 \times_{S_0} S_1 \arrow[r] \arrow[d] & S_1 \arrow[r, "t"] \arrow[d, "s"] & S_0 \\
T_0 \arrow[r, "\varphi"'] & S_0 &
\end{tikzcd}$$

is an open surjection. Since φ_* is locally essentially surjective on objects, we can apply the description in Example 4.2 with $X = S_0$ and $f_0 = \mathrm{id}_{S_0}$ on objects and $s_0 = u$ on arrows. This gives us an open surjection $u\colon U \to S_0$ with a map $f_u\colon U \to T_0$ and a map $\psi_u\colon U \to S_1$ such that $s\psi_u = \varphi_0 f_u$ and $t\psi_u = u$. By the universal property of the pullback there is a unique map $U \to T_0 \times_{S_0} S_1$ that makes the following diagram commute:

$$\begin{tikzcd}
U \arrow[dr, "!"] \arrow[drr, bend left=15, "\psi_u"] \arrow[drrr, bend left=25, "u"] \arrow[ddr, bend right=20, "f_u"'] & & & \\
& T_0 \times_{S_0} S_1 \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & S_1 \arrow[r, "t"'] \arrow[d, "s"] & S_0 \\
& T_0 \arrow[r, "\varphi_0"'] & S_0 &
\end{tikzcd}$$

Since u is an open surjection, and so is the composite $t\pi_2$, as required.

To show that φ is fully faithful, we need to show that

$$\begin{tikzcd}
T_1 \arrow[r, "\varphi_1"] \arrow[d, "{(s,t)}"'] & S_1 \arrow[d, "{(s,t)}"] \\
T_0 \times T_0 \arrow[r, "\varphi_0 \times \varphi_0"'] & S_0 \times S_0
\end{tikzcd} \tag{8}$$

is a pullback square. So let X be a space with maps $(f, g)\colon X \to T_0 \times T_0$ and $\alpha\colon X \to S_1$, such that the following diagram commutes,

$$\begin{tikzcd}
X \arrow[drr, bend left=20, "\alpha"] \arrow[ddr, bend right=20, "{(f,g)}"'] & & \\
& T_1 \arrow[r, "\varphi_1"] \arrow[d, "{(s,t)}"] & S_1 \arrow[d, "{(s,t)}"] \\
& T_0 \times T_0 \arrow[r, "\varphi_0 \times \varphi_0"'] & S_0 \times S_0
\end{tikzcd}$$

Since X represents a groupoid with only identity arrows, commutativity of this diagram implies that α represents a natural transformation $\varphi f \Rightarrow \varphi g$. Since φ_* is fully faithful this implies by the result in Example 4.4 that there is a unique natural transformation $\tilde{\alpha}\colon f \Rightarrow g$ such that $\varphi \circ \tilde{\alpha} = \alpha$. But this is equivalent to saying that there is a unique map $\tilde{\alpha}\colon X \to T_1$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[dr, "\tilde{\alpha}"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "{(f,g)}"'] & & \\
& T_1 \arrow[r, "\varphi_1"] \arrow[d, "{(s,t)}"] & S_1 \arrow[d, "{(s,t)}"] \\
& T_0 \times T_0 \arrow[r, "\varphi_0 \times \varphi_0"'] & S_0 \times S_0
\end{tikzcd}$$

We conclude that (8) is indeed a pullback and hence φ is an essential equivalence of internal groupoids in **Top**.

Conversely, let $\varphi\colon \mathbf{T} \to \mathbf{S}$ be an essential equivalence of groupoids in **Top**. We will first show that $\varphi_*\colon H_{\mathbf{T}} \to H_{\mathbf{S}}$ is locally essentially surjective on objects. So let $f \in H_{\mathbf{S}}(X)$. Consider the pseudo pullback of topological groupoids,

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "\bar{\varphi}"] \arrow[d, "\bar{f}"'] \arrow[dr, phantom, "\overset{\psi}{\Rightarrow}"] & X \arrow[d, "f"] \\
\mathbf{T} \arrow[r, "\varphi"'] & \mathbf{S}.
\end{tikzcd}$$

Since φ is an essential equivalence of groupoids, so is $\bar{\varphi}$. Since X is discrete as a groupoid, this implies that $\bar{\varphi}_0$ is an open surjection. So this diagram induces the following diagram of topological groupoids, where P_0 represents the discrete groupoid on P_0,

$$\begin{tikzcd}
P_0 \arrow[r, "\bar{\varphi}_0"] \arrow[d, "\bar{f}_0"'] \arrow[dr, phantom, "\overset{\psi}{\Rightarrow}"] & X \arrow[d, "f"] \\
\mathbf{T} \arrow[r, "\varphi"'] & \mathbf{S}.
\end{tikzcd}$$

which shows that φ_* is locally essentially surjective. The proof that φ_* is fully faithful is just the reverse of the proof given above to show that φ is fully faithful when φ_* is. ∎

The remainder of this section is devoted to giving a proof of the following result:

4.6. THEOREM. *There is a fibration structure on* $\mathbf{PreSt}(\mathcal{C})$ *with fibrations the local fibrations and weak equivalences the local weak equivalences.*

Consequently, the fibrant objects in this case are precisely the stacks.

4.7. Corollary. *There is an equivalence of 2-categories* $\mathbf{St}(\mathcal{C}) \simeq \mathbf{Ho}\big(\mathbf{PreSt}(\mathcal{C})\big)$.

Throughout the remainder of this section we denote by $\mathfrak{W}$ the class of local weak equivalences and by $\mathfrak{F}$ the class of maps p such that $\mathfrak{W} \pitchfork p$.

4.8. Pseudo three-for-two. We will now show that the local weak equivalences satisfy the "pseudo three-for-two" condition:

4.9. Proposition. *Given a diagram*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "g"'] & \Downarrow\gamma & C \\
& B \arrow[ur, "h"'] &
\end{tikzcd}$$

with γ an isomorphism, if any two of f, g and h are local weak equivalences, then so is the third.

Proof. If h and g are local weak equivalences, then it is trivial to verify that f is also a local weak equivalence.

When f and g are local weak equivalences it is easily seen that h is locally essentially surjective since f is. To see that h is full, suppose given a map $j\colon h(x) \to h(y)$ in $C(U)$. Because g is locally essentially surjective on objects we can find a cover $\mathcal{S}$ of U and isomorphisms $\varphi_\alpha\colon g(a_\alpha) \to x|_\alpha$ and $\psi_\alpha\colon g(b_\alpha) \to y|_\alpha$ in $B(U_\alpha)$ for each $U_\alpha \to U$ in the cover. We can then construct composites

$$\begin{tikzcd}
f(a_\alpha) \arrow[d, "\gamma_{a_\alpha}"'] & & & & & & f(b_\alpha) \\
h(g(a_\alpha)) \arrow[r, "h(\varphi_\alpha)"] & h(x|_\alpha) \arrow[r] & h(x)|_\alpha \arrow[r, "j|_\alpha"] & h(y)|_\alpha \arrow[r] & h(y|_\alpha) \arrow[r, "h(\psi_\alpha^{-1})"] & h(g(b_\alpha)) \arrow[ur, phantom] & \\
\end{tikzcd}$$

$$\begin{tikzcd}
& & & & & & f(b_\alpha) \\
h(g(a_\alpha)) & & & & & & h(g(b_\alpha)) \arrow[u, "\gamma_{b_\alpha}^{-1}"']
\end{tikzcd}$$

where the unlabeled arrows are the isomorphisms associated to h as in (4). Since f is full and faithful there exists a unique lift $u_\alpha\colon a_\alpha \to b_\alpha$ in $A(U_\alpha)$ for each $U_\alpha \to U$ in the cover $\mathcal{S}$. Using these lifts we similarly obtain maps $v_\alpha\colon x|_\alpha \to y|_\alpha$ defined as $\psi_\alpha \circ g(u_\alpha) \circ \varphi_\alpha^{-1}$. These constitute a matching family for the presheaf $B(x, y)$. To see this it suffices to show that, for each $U_\alpha \to U$ and $U_\beta \to U$ in $\mathcal{S}$, the diagram

$$\begin{tikzcd}
g(a_\alpha)|_{\alpha\beta} \arrow[r, "\varphi_\alpha|_{\alpha\beta}"] & x|_\alpha|_{\alpha\beta} \arrow[r] & x|_{\alpha\beta} \arrow[r] & x|_\beta|_{\alpha\beta} \arrow[r, "\varphi_\beta^{-1}|_{\alpha\beta}"] & g(a_\beta)|_{\alpha\beta} \arrow[r, "g(u_\beta)|_{\alpha\beta}"] & g(b_\beta)|_{\alpha\beta} \arrow[d] \\
g(a_\alpha|_{\alpha\beta}) \arrow[u] \arrow[d] & & & & & g(b_\beta|_{\alpha\beta}) \\
g(a_\alpha)|_{\alpha\beta} \arrow[r, "g(u_\alpha)|_{\alpha\beta}"'] & g(b_\alpha)|_{\alpha\beta} \arrow[r, "\psi_\alpha"'] & y|_\alpha|_{\alpha\beta} \arrow[r] & y|_{\alpha\beta} \arrow[r] & y|_\beta|_{\alpha\beta} \arrow[r, "\psi_\beta^{-1}"'] & g(b_\beta)|_{\alpha\beta} \arrow[u]
\end{tikzcd}$$

commutes, where the unnamed arrows are the evident coherence isomorphisms. Since g is full and faithful both ways around this diagram induces unique lifts $\xi, \zeta\colon a_\alpha|_{\alpha\beta} \to b_\beta|_{\alpha\beta}$. It suffices by faithfulness of f to show that $f(\xi) = f(\zeta)$, which holds by a straightforward diagram chase. Since the v_α are a matching family it follows from the fact that B is a prestack that there exists a canonical amalgamation $v\colon x \to y$ in $B(U)$. This map clearly has the property that $h(v) = j$, as required.

To see that h is faithful one uses roughly the same kind of approach. Given $j, k\colon x \to y$ in $B(U)$ with $h(j) = h(k)$ we use local essential surjectivity of g to obtain a cover $\mathcal{S}$ and isomorphisms $g(a_\alpha) \cong x|_\alpha$ and $g(b_\alpha) \cong y|_\alpha$. Conjugation of $j|_\alpha$ and $k|_\alpha$ by these isomorphisms gives two families of maps $g(a_\alpha) \to g(b_\alpha)$ and since g is full and faithful these induce canonical lifts $u_\alpha, v_\alpha\colon a_\alpha \to b_\alpha$ in $A(U_\alpha)$. Using the fact that $h(j) = h(k)$ we can then show that $f(u_\alpha) = f(v_\alpha)$ so that $u_\alpha = v_\alpha$. It then follows by the fact that B is a prestack that $j = k$.

The proof that g is a local weak equivalence when f and h are is similar and is left to the reader. ∎

4.10. Characterization of the fibrations. We now turn to providing a characterization of the fibrations $\mathfrak{F}$. This result is analogous to an earlier result of Joyal and Tierney [11] in which they characterize stacks as weakly fibrant objects. The differences between our result and theirs are as follows. First, they consider a Grothendieck topos $\mathcal{E}$ with the canonical topology and they characterize those groupoids G in $\mathcal{E}$ such that the externalization $\mathcal{E}(-, G)$ is a stack. In our case, the site is an arbitrary subcanonical site and our prestacks are fibered in categories rather than groupoids. In the setting of *ibid* it is not necessary to consider prestacks and it is not necessary to make use of the Axiom of Choice. Because we work in a more general setting we must restrict first to prestacks and we also appeal to the Axiom of Choice. Finally, the characterization we give is of local fibrations in general and not just stacks, which are the locally fibrant objects.

4.11. Lemma. *For $i\colon A \to C$ in $\mathfrak{W}$ and U in $\mathcal{C}$, for every object c of $C(U)$ there exists a cover $\mathcal{S}$ and an object of $\mathbf{Desc}(i, \mathcal{S})$ which projects via $\pi\colon \mathbf{Desc}(i, \mathcal{S}) \to C(U)$ onto c.*

Proof. Let an object c of $C(U)$ be given. Because i is locally essentially surjective on objects there exists a family of isomorphisms $\psi_\alpha\colon i(\tilde{c}_\alpha) \to c|_\alpha$. We may form the composites

$$\begin{tikzcd} i(\tilde{c}_\beta|_{\alpha\beta}) \arrow[r] & i(\tilde{c}_\beta)|_{\alpha\beta} \arrow[r, "\psi_\beta|_{\alpha\beta}"] & c|_\beta|_{\alpha\beta} \arrow[r, "\sigma_{\alpha\beta}(c)"] & c|_\alpha|_{\alpha\beta} \arrow[r, "\psi_\alpha^{-1}|_{\alpha\beta}"] & i(\tilde{c}_\alpha)|_{\alpha\beta} \arrow[r] & i(\tilde{c}_\alpha|_{\alpha\beta}) \end{tikzcd}$$

where the unlabeled arrows are induced by the coherence 2-cell associated to the pseudo-natural transformation i. Since i is full and faithful these possess unique invertible lifts $\vartheta_{\alpha\beta}\colon \tilde{c}_\beta|_{\alpha\beta} \to \tilde{c}_\alpha|_{\alpha\beta}$ in $A(U_{\alpha\beta})$. It is routine to verify that $(c, (\tilde{c}_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ is an object of $\mathbf{Desc}(i, \mathcal{S})$. ∎

Note that the proof of the following lemma requires the Axiom of Choice.

4.12. Lemma. *If $p\colon E \to B$ is a local fibration, then $\mathfrak{W} \pitchfork p$.*

Proof. Suppose $p\colon E \to B$ is a local fibration and let a diagram of the form

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] \arrow[dr, phantom, "\Downarrow\gamma"] & E \arrow[d, "p"] \\ C \arrow[r, "k"'] & B \end{tikzcd}$$

be given with $i\colon A \to C$ in $\mathfrak{W}$ and γ invertible. Given an object c of $C(U)$ we may choose, by Lemma 4.11, a cover $\mathcal{S}$ together with descent data $(c, (a_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ in $\mathbf{Desc}(i, \mathcal{S})$. By Lemma 3.8 this gives descent data $\hat{c} := (k(c), (h(a_\alpha)), (\hat{\psi}_\alpha), (\hat{\vartheta}_{\alpha\beta}))$ in $\mathbf{Desc}(p, \mathcal{S})$. Thus, we choose, in order to define an arrow $l\colon C \to E$, $l(c)$ to be an amalgamation of this descent data.

Given $f\colon c \to d$ in $C(U)$ assume that $\mathcal{S}$ and $\mathcal{R}$ are the covers chosen in order to define $l(c)$ and $l(d)$ and assume that $(c, (a_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ and $(d, (b_\alpha), (\varphi_\alpha), (\omega_{\alpha\beta}))$ are the descent data chosen in the definition of $l(c)$ and $l(d)$, respectively. Let $\mathcal{W}$ be the common refinement $\mathcal{S} \cap \mathcal{R}$ of $\mathcal{S}$ and $\mathcal{R}$ and observe that, for $U_\alpha \to U$ in $\mathcal{W}$, we have isomorphisms $\chi\colon \Phi_{\mathcal{S}}(l(c)) \cong \hat{c}$ and $\mu\colon \Phi_{\mathcal{R}}(l(d)) \cong \hat{d}$. We also have

$$i(a_\alpha) \xrightarrow{\psi_\alpha} c|_\alpha \xrightarrow{f|_\alpha} d|_\alpha \xrightarrow{\varphi_\alpha^{-1}} i(b_\alpha).$$

As such, since $i\colon A \to C$ is full and faithful, there exists a unique map $f_\alpha\colon a_\alpha \to b_\alpha$ which is mapped by i onto this composite. This gives a map of descent data

$$(f, f_\alpha)\colon \big(c, (a_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta})\big)|_{\mathcal{W}} \to \big(d, (b_\alpha), (\varphi_\alpha), (\omega_{\alpha\beta})\big)|_{\mathcal{W}}$$

and by Lemma 3.8, we have that $(k(f), (h(f_\alpha)))\colon \hat{c}|_{\mathcal{W}} \to \hat{d}|_{\mathcal{W}}$ in $\mathbf{Desc}(p, \mathcal{W})$. Therefore we may form the composite

$$\Phi_{\mathcal{S}}(l(c))|_{\mathcal{W}} \xrightarrow{\chi|_{\mathcal{W}}} \hat{c}|_{\mathcal{W}} \xrightarrow{(k(f),(h(f_\alpha)))} \hat{d}|_{\mathcal{W}} \xrightarrow{\mu^{-1}|_{\mathcal{W}}} \Phi_{\mathcal{V}}(l(d))|_{\mathcal{W}}$$

which gives us a family of maps $l(c)|_\alpha \to l(d)|_\alpha$ for $U_\alpha \to U$ in $\mathcal{W}$. This family constitutes a matching family for the presheaf $E(l(c), l(d))$ and since E is a prestack there exists a canonical amalgamation $l(f)\colon l(c) \to l(d)$. Functoriality follows from the uniqueness of amalgamations.

We now construct the natural isomorphisms $\lambda\colon h \cong l \circ i$ and $\rho\colon p \circ l \cong k$. First, for λ, assume given an object a of $A(U)$. Assume that $\mathcal{S}$ is a cover of U and $(i(a), (a_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ is the descent data chosen in the construction of $l(i(a))$. Then $(1_{i(a)}, (\psi_\alpha^{-1}))$ is an isomorphism in $\mathbf{Desc}(i, \mathcal{S})$ from $\Phi_{\mathcal{S}}(u)$ to $(i(a), (a_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$. As such, we may form the following composite

$$\Phi_{\mathcal{S}}(h(a)) \xrightarrow{\hat{\gamma}(a)} (h, k, \gamma)_*(\Phi_{\mathcal{S}}(a)) \xrightarrow{(h,k,\gamma)_*(1_{i(a)},(\psi_\alpha^{-1}))} \widehat{i(a)} \longrightarrow \Phi_{\mathcal{S}}(l(i(a)))$$

in $\mathbf{Desc}(\mathcal{S}, p)$, where $\hat{\gamma}$ is as in the discussion of $(h, k, \gamma)_*$ from Section 3.6 and the unnamed map is the isomorphism associated to the definition of $l(i(a))$. Because Φ_U is full and faithful this gives a canonical isomorphism $\lambda(a)\colon h(a) \to l(i(a))$ with $\Phi_{\mathcal{S}}(\lambda(a))$ the composite above. Naturality of λ follows from faithfulness of the $\Phi_{\mathcal{S}}$ together with the definition of the action of l on arrows. Next, we define $\rho(c)\colon p(l(c)) \to k(c)$ to be the first component of the isomorphism $\Phi_{\mathcal{S}}(l(c)) \cong \hat{c}$ of descent data associated to the definition of $l(c)$. This is natural by definition of l. Finally, it is immediate from the definitions that γ can be recovered by composing the pasting diagram obtained from λ and ρ. ■

4.13. Theorem. *For a map $p\colon E \to B$ the following are equivalent:*

1. *p is a local fibration.*
2. *p is in $\mathfrak{F}$.*

Proof. By Lemma 4.12 it suffices to prove that if $p\colon E \to B$ is in $\mathfrak{F}$, then it is a local fibration. To this end, let U together with a cover $\mathcal{S}$ be given. Assume given descent data $(b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ in $\mathbf{Desc}(p, \mathcal{S})$. Then we have a square

$$\begin{tikzcd}
\hat{\mathcal{S}} \arrow[r, "e"] \arrow[d, "i"'] \arrow[dr, phantom, "\Downarrow\psi"] & E \arrow[d, "p"] \\
yU \arrow[r, "b"'] & B
\end{tikzcd}$$

where yU is the representable functor and $\hat{\mathcal{S}}$ is the subfunctor of yU induced by the cover $\mathcal{S}$ (note that both of these are prestacks). Also, e is the pseudonatural transformation representing the family (e_α) with coherence isomorphisms constructed using the $\vartheta_{\alpha\beta}$. Similarly, b is the pseudonatural transformation representing b. Finally, ψ is the invertible modification with component at $U_\alpha \to U$ in $\mathcal{S}$ given by ψ_α.

Notice that i is a local weak equivalence so that, since $\mathfrak{W} \pitchfork p$, it follows that there exists a lift $l\colon yU \to E$ together with isomorphisms $\lambda\colon e \cong l \circ i$ and $\rho\colon p \circ l \cong b$ such that the square above can be recovered from these. That is, we have an object $l\ E(U)$ together with $\rho_V\colon p(l)|_V \cong b|_V$ for every $V \to U$ and $\lambda_\alpha\colon e_\alpha \cong l|_\alpha$ for each $U_\alpha \to U$ in the cover. It is then routine to verify that l is an amalgamation of our descent data. ■

4.14. Corollary. *For any F, the canonical map $F \to 1$ is in $\mathfrak{F}$ if and only if F is a stack.*

4.15. Corollary. *If $p\colon E \to B$ is in $\mathfrak{F} \cap \mathfrak{W}$, then p is an equivalence (i.e., there exists a $p'\colon B \to E$ together with invertible $\eta\colon 1_B \to p \circ p'$ and $\epsilon\colon p' \circ p \to 1_E$).*

4.16. Corollary. *If $p\colon E \to B$ is in $\mathfrak{F} \cap \mathfrak{W}$ and $i\colon A \to C$ is any map, then $i \pitchfork p$.*

4.17. Corollary. *Theorem 4.13 is equivalent to the Axiom of Choice.*

PROOF. Consider the case where our site consists of the lattice $\mathcal{O}(\emptyset)$ of open subsets of the empty set with its canonical topology and the notion of covering family is given by the usual topological notion of covering family. In this case we are working directly in **Cat** and we can easily prove that every object is locally fibrant. Using this it is possible to construct pseudo-inverses of weak categorical equivalences. Therefore the Axiom of Choice holds. ∎

4.18. FACTORIZATION AND ISOCOMMA OBJECTS. We will now describe the factorizations in $\mathbf{PreSt}(\mathcal{C})$. To a map $f\colon A \to B$ we associate a prestack $\mathbf{Path}(f)$ by letting $\mathbf{Path}(f)(U)$ be the category with

Objects Tuples consisting of a cover $\mathcal{S}$ and an object $(b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ of $\mathbf{Desc}(f, \mathcal{S})$.

Arrows An arrow $(\mathcal{S}, b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta})) \to (\mathcal{V}, b', (e'_\gamma), (\psi'_\gamma), (\vartheta'_{\gamma\delta}))$ is an equivalence class of data given by a common refinement $\mathcal{W}$ of $\mathcal{S}$ and $\mathcal{V}$ together with a map

$$\big(b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta})\big)|_{\mathcal{W}} \to \big(b', (e'_\gamma), (\psi'_\gamma), (\vartheta'_{\gamma\delta})\big)|_{\mathcal{W}}$$

in $\mathbf{Desc}(f, \mathcal{W})$. We identify $(\mathcal{W}, g, (g_\alpha))$ and $(\mathcal{W}', g', (g'_\alpha))$ when there exists a common refinement of $\mathcal{W}$ and $\mathcal{W}'$ on which the maps of descent data agree.

Note that $g = g'$ when $(\mathcal{W}, g, (g_\alpha))$ and $(\mathcal{W}', g', (g'_\alpha))$ are identified in $\mathbf{Path}(f)(U)$.

There is, for $g\colon V \to U$, an obvious restriction map $\mathbf{Path}(f)(U) \to \mathbf{Path}(f)(V)$ which acts by pullback on both covers and descent data. This makes $\mathbf{Path}(f)$ into a pseudofunctor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$. We observe that we have the following lemma, the proof of which is straightforward, but tedious:

4.19. LEMMA. *If A and B are prestacks and $f\colon A \to B$, then $\mathbf{Path}(f)$ is a prestack.*

There is a projection map $\mathbf{Path}(f) \to B$ which sends $(\mathcal{S}, b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ to b and sends an arrow $[\mathcal{W}, g, (g_\alpha)]$ to g. We define $R\colon \mathbf{PreSt}(\mathcal{C})^{\to} \to \mathbf{PreSt}(\mathcal{C})^{\to}$ by letting $R(f)$, for $f\colon A \to B$ in $\mathbf{PreSt}(\mathcal{C})$, be the projection $\mathbf{Path}(f) \to B$. In particular, the action of R on arrows and 2-cells is induced by the action of the functors $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]^{\to} \to \mathbf{Cat}^{\to}$ as in (6) and arising from the descent functors $\mathbf{Desc}(-, \mathcal{S})$ from Lemma 3.8.

For the pseudonatural transformation $\eta\colon 1_{\mathbf{PreSt}(\mathcal{C})^{\to}} \to R$, note that there is a map $A \to \mathbf{Path}(f)$, which we denote by η_f, that sends an a in $A(U)$ to $(M_U, f(a), (a|_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ where M_U denotes the maximal sieve on U, the ψ_α are the coherence isomorphisms associated to f, and the $\vartheta_{\alpha\beta}$ are the coherence isomorphisms obtained from the structure of A as a pseudofunctor. It is straightforward to verify that $R(f) \circ \eta_f = f$ and this equation determines the rest of the data of the pseudonatural transformation η.

4.20. LEMMA. *For $f\colon A \to B$, in the factorization*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\eta_f"'] & & B \\
& \mathbf{Path}(f) \arrow[ur, "R(f)"'] &
\end{tikzcd}$$

$R(f)$ *is a local fibration and* η_f *is in* $\mathfrak{W}$.

PROOF. It is trivial that η_f is in $\mathfrak{W}$. To see that $R(f)$ is a local fibration let a cover $\mathcal{W} = (h^\gamma \colon U^\gamma \to U)_\gamma$ of U be given together with an object

$$\Big(b, \big(\mathcal{S}^\gamma, b^\gamma, (e^\gamma_\alpha), (\psi^\gamma_\alpha), (\vartheta^\gamma_{\alpha\beta})\big), (\varphi_\gamma), (\Theta_{\gamma\delta})\Big)$$

of $\mathbf{Desc}(R(f), \mathcal{W})$ where $\mathcal{S}^\gamma = (h^\gamma_\alpha \colon U^\gamma_\alpha \to U^\gamma)_\alpha$. Here $\Theta_{\gamma\delta} = (h_{\gamma\delta}, (h^\alpha_{\gamma\delta}))$ is an isomorphism

$$(b_\delta, (e^\delta_\alpha), (\psi^\delta_\alpha), (\vartheta^\delta_{\alpha\beta}))|_{\gamma\delta} \xrightarrow{\cong} (b_\gamma, (e^\gamma_\alpha), (\psi^\gamma_\alpha), (\vartheta^\gamma_{\alpha\beta}))|_{\gamma\delta}$$

of descent data in $\mathbf{Desc}(f, \mathcal{S}^{\gamma\delta})$. Define a new cover $\bar{\mathcal{W}}$ of U as the cover consisting of the maps of the form $h^\gamma \circ h^\gamma_\alpha \colon U^\gamma_\alpha \to U^\gamma \to U$ for h^γ in $\mathcal{W}$ and h^γ_α in $\mathcal{U}^\gamma$. We then have an object

$$(\bar{\mathcal{W}}, b, (e^\gamma_\alpha), (\varphi^\gamma_\alpha), (\chi^{\gamma\delta}_{\alpha\beta}))$$

of $\mathbf{Desc}(f, \mathcal{S})$ where φ^γ_α is the composite

$$f(e^\gamma_\alpha) \xrightarrow{\psi^\gamma_\alpha} b^\gamma|_{U^\gamma_\alpha} \xrightarrow{\varphi_\gamma|_{U^\gamma_\alpha}} b|_{U^\gamma_\alpha}$$

and $\chi^{\gamma\delta}_{\alpha\beta}$ is the composite

$$e^\delta_\beta|_{U^\gamma_\alpha \cap U^\delta_\beta} \xrightarrow{h^\beta_{\gamma\delta}} e^\gamma_\beta|_{U^\gamma_\alpha \cap U^\delta_\beta} \xrightarrow{\vartheta^\gamma_{\alpha\beta}} e^\gamma_\alpha|_{U^\gamma_\alpha \cap U^\delta_\beta}.$$

With these definitions, it is a (quite) lengthy but straightforward verification that we have described the amalgamation of the descent data. ∎

4.21. EXAMPLE. When f is the canonical map $A \to 1$ we see that $\mathbf{Path}(f)$ is the associated stack $\mathbf{a}(A)$ of A (cf. [15, 16] for more on the associated stack).

We note that when A is a stack it is possible to factor f in a more straightforward way using isocomma objects.

4.22. DEFINITION. *Given maps* $f \colon A \to B$ *and* $g \colon C \to B$ *in* $\mathbf{PreSt}(\mathcal{C})$, *the* **isocomma object** (f, g) *is the pseudofunctor given at* U *by the category* $(f, g)(U)$ *with*

Objects *Tuples consisting of objects* a *and* c *of* $A(U)$ *and* $C(U)$, *respectively, and an isomorphism* $\xi \colon f(a) \cong g(c)$.

Arrows *An arrow* $(a,c,\xi)\to(b,d,\zeta)$ *is given by maps* $i\colon a\to b$ *and* $j\colon c\to d$ *such that the diagram*

$$\begin{tikzcd}
f(a) \arrow[r, "f(i)"] \arrow[d, "\xi"'] & f(b) \arrow[d, "\zeta"] \\
g(c) \arrow[r, "g(j)"'] & g(d)
\end{tikzcd}$$

commutes.

The action of (f,g) *on arrows is simply by restriction of all of the aforementioned data.*

There is an invertible 2-cell χ as indicated in the following diagram:

$$\begin{tikzcd}
(f,g) \arrow[r] \arrow[d] & C \arrow[d, "g"] \arrow[dl, phantom, "\Downarrow\chi"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

where the unnamed arrows are the obvious projections. Here χ projects $(a,c,\xi)\mapsto\xi$. The universal property of (f,g) is that for any other diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & C \arrow[d, "g"] \arrow[dl, phantom, "\Downarrow\chi'"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

with χ' invertible, there exists a canonical map $z\colon Z\to(f,g)$ such that the diagram

$$\begin{tikzcd}
& Z \arrow[dl, bend right] \arrow[d, "z"] \arrow[dr, bend left] & \\
A & (f,g) \arrow[l] \arrow[r] & C
\end{tikzcd}$$

commutes and such that $\chi'=\chi\circ z$. It is straightforward to show that (f,g) is a prestack when A and C are.

Now, the universal property gives us a map $i\colon A\to(f,1_B)$ such that

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\
& (f,1_B) \arrow[ur, "p"'] &
\end{tikzcd}$$

commutes, where p is the projection. Here it is clear that this gives a factorization $f=p\circ i$. In particular, $i(a)$ is $(a,f(a),1_{f(a)})$ and it is straightforward to verify that i is in $\mathfrak{W}$.

4.23. Lemma. *When A is a stack, $p\colon (f, 1_B) \to B$ is a local fibration.*

Proof. Given descent data $(b, (e_\alpha\colon f(a^\alpha) \cong b^\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta}))$ in $\mathbf{Desc}(\mathcal{S}, p)$, we have that $\vartheta_{\alpha\beta}$ is a commutative square

$$\begin{tikzcd}
f(a^\beta)|_{\alpha\beta} \arrow[r, "f(\chi_{\alpha\beta})"] \arrow[d, "e_\beta"'] & f(a^\alpha)|_{\alpha\beta} \arrow[d, "e_\alpha"] \\
b^\beta|_{\alpha\beta} \arrow[r, "\omega_{\alpha\beta}"'] & b^\alpha|_{\alpha\beta}
\end{tikzcd}$$

of isomorphisms. This gives us descent data $((a^\alpha), (\chi_{\alpha\beta}))$ for A and $\mathcal{S}$ and since A is a stack there is an amalgamating object a of $A(U)$. For each α, we have the isomorphism

$$\begin{tikzcd}
f(a)|_\alpha \arrow[r] & f(a|_\alpha) \arrow[r] & f(a^\alpha) \arrow[r, "e_\alpha"] & b^\alpha \arrow[r, "\psi_\alpha"] & b|_\alpha
\end{tikzcd}$$

and these are easily seen to constitute a matching family for $B(f(a), b)$. Therefore, since B is a prestack there is a canonical amalgam $e\colon f(a) \cong b$. We define this isomorphism to be the object of $(f, B)(U)$ corresponding to our descent data. It is routine to verify that this constitutes a pseudo-inverse to the map $(f, B)(U) \to \mathbf{Desc}(\mathcal{S}, p)$ satisfying the coherence conditions from the definition of local fibrations. ■

This completes the proof of Theorem 4.6.

5. Topological, differentiable and algebraic stacks

We will now show that the results of Section 4 can be used to give analogous characterizations of the 2-categories of topological, differentiable and algebraic stacks. These three cases are formal analogues. The categories of topological spaces, differentiable manifolds and schemes all have in common that quotients in them are not well-behaved. This gives rise to the situation, familiar from the theory of étendues from [1], in which one would like to form a "generalized quotient" of a space, manifold or scheme. (Indeed, there is an important connection with the theory of étendues as described in [18], but we do not describe it here.) Topological, differentiable and algebraic stacks are the appropriate "generalized quotients" of suitable equivalence relations in each of these situations. These three cases are formally analogous in the sense that topological, differentiable and algebraic stacks are by definition stacks X which appear in a suitable sense as "quotients" of topological spaces, differentiable manifolds, or schemes, respectively. This formal analogy permits us to give a single argument (here described in detail for topological stacks) which will show that each of these 2-categories can be described as the homotopy 2-category of the corresponding 2-category of prestacks.

It is worth emphasizing that the topological (differentiable, algebraic) (pre)stacks considered below are in fact (pre)stacks which are in a suitable sense *presentable.* As such, we are dealing with a slightly different situation from above where *all* (pre)stacks were considered.

We begin by defining what a topological prestack is and we will consider the full sub 2-category on these prestacks. We will then show that our system of fibrant objects can be restricted to this sub 2-category with the same local weak equivalences as before. And we will finally prove that the fibrant objects become precisely the topological stacks.

5.1. TOPOLOGICAL STACKS. We will briefly recall the definition of topological stacks, which are essentially the topological version of the *algebraic stacks* of Deligne and Mumford [6]. Throughout this section we will be working with the **topological site** which consists of a small category **Top** of sober topological spaces $U, V, \dots$ equipped with the étale Grothendieck topology. Here in saying that **Top** is a small category we assume the existence of a sufficiently large cardinal κ and consider only those topological spaces of cardinality less than κ (equivalently, we may assume the existence of a Grothendieck universe). We refer to a topological space of cardinality less than κ as a *small* space. Similar remarks apply in connection with the small categories of differentiable manifolds and schemes below. For those readers wishing to avoid universes, but still interested in having locally small 2-categories of prestacks we refer to [20] for a hint of how to proceed. The étale topology is generated by families $(f_i \colon U_i \to U)_i$ which are said to cover when the map $\sum_i U_i \to U$ is an étale surjection.

5.2. DEFINITION. *A map $f \colon A \to B$ of prestacks is* **representable** *if, for any space U in* **Top** *and map $g \colon yU \to B$, the isocomma object (f, g) is representable.*

We now consider pseudofunctors $[\mathbf{Top}, \mathbf{Gpd}]$ valued in groupoids. Throughout this section "prestack" means prestack valued in groupoids and similarly for "stack". Roughly, topological prestacks are those prestacks which arise as quotients of spaces.

5.3. DEFINITION. *A* **topological prestack** *is a prestack A such that the following conditions are satisfied:*

1. *The diagonal $\Delta \colon A \to A \times A$ is representable.*
2. *There exists a space U in* **Top** *and a map $q \colon yU \to A$ such that, for all spaces V in* **Top** *and maps $f \colon yV \to A$, the map $(f, q) \to V$ is an étale surjection.*

Notice that it makes sense in condition (2) to say that $(f, q) \to V$ is an étale surjection since the domain of this map is, by condition (1), representable. We will often refer to the map $q \colon yU \to A$ from condition (2) as a **chart for** A. Observe that representables are trivially topological prestacks. We will henceforth omit explicit mention of the Yoneda embedding y when no confusion will result.

A *topological stack* is a topological prestack which is also a stack (although the concept of topological stack predates the notion of topological prestack; they were studied in [18] for instance).

We denote by **TopPreSt** the 2-category of topological prestacks and we observe that it is an immediate consequence of Lemma 4.12 that if $p \colon E \to B$ is a local fibration between topological prestacks, then $\mathfrak{W} \pitchfork p$ where $\mathfrak{W}$ denotes the class of local weak equivalences in **TopPreSt**. We will now consider to what extent the additional structure of **PreSt**(**Top**) restricts to **TopPreSt**.

5.4. Lemma. *If $f\colon A \to B$ is an equivalence between prestacks and B has a representable diagonal, then so does A.*

Proof. Let maps $v\colon V \to A$ and $w\colon W \to A$ be given. Because B has a representable diagonal the isocomma object $(f \circ v, f \circ w)$ is a representable U. This gives us the following diagram of invertible 2-cells:

$$\begin{tikzcd}
U \arrow[rr] \arrow[dd] & & W \arrow[d, "w"] & \\
& \cong & A \arrow[d, "f"] \arrow[ddr, bend left, "1_A"] & \\
V \arrow[r, "v"'] & A \arrow[r, "f"] \arrow[drr, bend right, "1_A"'] & B \arrow[dr, "f'"] \arrow[r, phantom, "\cong"] & \\
& & \cong & A
\end{tikzcd}$$

where f' is a pseudoinverse of f. This is easily seen to exhibit U as (v, w). ∎

5.5. Lemma. *If $f\colon A \to B$ is an equivalence between prestacks and B is a topological prestack, then A is also a topological prestack.*

Proof. By Lemma 5.4 it suffices to prove that there exists a space U and an étale map $U \to A$. Because B is a topological prestack there exists an étale map $e\colon U \to B$. We claim that the map $f' \circ e\colon U \to A$ is étale, where f' is a pseudoinverse of f. Let another map $v\colon V \to A$ be given. Then the isocomma object $(f \circ v, e)$ is a representable W. We then obtain the diagram

$$\begin{tikzcd}
W \arrow[rr] \arrow[d] \arrow[drr, phantom, "\cong"] & & U \arrow[d, "e"] \\
V \arrow[r, "v"'] & A \arrow[r, "f"] \arrow[dr, bend right, "1_A"'] \arrow[r, phantom, shift right=4ex, "\cong"] & B \arrow[d, "f'"] \\
& & A
\end{tikzcd}$$

where the vertical map $W \to V$ is an étale surjection. It is straightforward to show that the diagram above exhibits U as the isocomma object $(v, f' \circ e)$ so that $f' \circ e$ is étale. ∎

Modifying a construction of [18], we associate to each topological prestack A and chart $e\colon U \to A$ the étale groupoid G^e with space of objects U and space of arrows the space representing the isocomma object (e, e). In *ibid* it is assumed that A is a topological stack, but it is in fact sufficient for A to be a topological prestack. Also in *ibid* it is shown how to associate to any étale groupoid G a topological stack $R(G)$. Combining these two procedures, we obtain, for each topological prestack A and chart $e\colon U \to A$, a topological stack $R(A, e)$ given by $R(G^e)$. In elementary terms, we have

$$R(A, e)_V := \mathbf{GeomMorph}\big(\mathbf{Sh}(V), \mathbf{Sh}(G^e)\big)$$

for V a space. Here the objects are geometric morphisms, arrows are invertible natural transformations, $\mathbf{Sh}(V)$ is the ordinary category of sheaves on the space V and $\mathbf{Sh}(G^e)$ is the category of equivariant sheaves on the groupoid G^e. Note that it is shown in *ibid* that there is a map $i\colon A \to R(A, e)$ which is a weak equivalence.

5.6. Lemma. *The associated stack* $\mathbf{a}(A)$ *of a topological prestack is a topological stack.*

Proof. It suffices by Lemma 5.5, and the fact that both $\mathbf{a}(A)$ and $R(A, e)$ are stacks, to construct a local weak equivalence $\mathbf{a}(A) \to R(A, e)$. Because the map $\eta\colon A \to \mathbf{a}(A)$ is a local weak equivalence and $R(A, e)$ is a stack there exists a map $\mathbf{a}(A) \to R(A, e)$ and an invertible 2-cell as indicated in the diagram:

$$\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dr, bend right, "\eta"'] & & R(A, e). \\
& \mathbf{a}(A) \arrow[ur, bend right] \arrow[u, phantom, "\cong"] &
\end{tikzcd}$$

By the pseudo three-for-two property for local weak equivalences it then follows that $\mathbf{a}(A) \to R(A, e)$ is also a local weak equivalence. ∎

Putting these lemmas together with Theorem 4.6 lets us establish the following:

5.7. Theorem. *There is a system of fibrant objects on* **TopPreSt** *given by taking the local weak equivalences and with fibrant replacement given by the associated stack.*

Proof. A full sub-bicategory of a bicategory with a fibration structure has a system of fibrant objects with the same weak equivalences if there is a fibrant replacement functor which restricts to the sub-bicategory. We proved in Lemma 5.6 that this applies to the associated stack functor. ∎

5.8. Corollary. *There is an equivalence of 2-categories* $\mathbf{TopSt} \simeq \mathbf{Ho}(\mathbf{TopPreSt})$.

5.9. Remark. Note that we require here that both the topological prestacks and stacks take their values in groupoids rather than categories. There is a generalization of this result to representable stacks with values in **Cat** in forthcoming work by Roberts [21]. This applies also to the next examples, of differentiable and algebraic stacks.

5.10. Differentiable stacks. We will now turn to differentiable stacks. As mentioned above, this case is proved in precisely the same manner as the topological case. In this case, we work with the site **Diff** of small differentiable manifolds with the étale topology (cf. the discussion above regarding the topological site).

5.11. Definition. *A* **differentiable prestack** *is a prestack A such that there exists a manifold U in* **Diff** *and a map $q\colon U \to A$ such that, for all manifolds V in* **Diff** *and maps $f\colon V \to A$, the isocomma object (f, q) is representable and the map $(f, q) \to V$ is an étale surjection.*

As in the topological case, we may associate to each differentiable prestack A and chart $e\colon U \to A$ a differentiable groupoid G^e. To such a differentiable groupoid we then have an associated differentiable stack $R(A, e)$ given by

$$R(A,e)_V := \mathbf{Ringed}\big((\mathbf{Sh}(V), C^\infty(V)), (\mathbf{Sh}(G^e), C^\infty(U))\big)$$

where the objects are morphisms of ringed toposes and the arrows are natural isomorphisms thereof.

5.12. Theorem. *There is a system of fibrant objects on* **DiffPreSt** *given by taking the local weak equivalences and with fibrant replacement given by the associated stack.*

Proof. By the differentiable analogues of Lemma 5.5 and the argument given in the proof of Theorem 5.7, it suffices to construct a local weak equivalence $A \to R(A, e)$ for any differentiable prestack A with chart $e\colon U \to A$. This was done in [18]. ■

5.13. Corollary. *There is an equivalence of 2-categories* $\mathbf{DiffSt} \simeq \mathbf{Ho}(\mathbf{DiffPreSt})$.

5.14. Algebraic stacks. The case of algebraic stacks is even closer to the topological case. In this case we work with the site **Sch** of small schemes with the étale topology.

5.15. Definition. *An* **algebraic prestack** *is a prestack A over* **Sch** *such that the following conditions are satisfied:*

1. *The diagonal $\Delta\colon A \to A \times A$ is representable and proper.*
2. *There exists a scheme U in* **Sch** *and a map $q\colon U \to A$ such that, for all schemes V in* **Sch** *and maps $f\colon V \to A$, the map $(f, q) \to V$ is an étale surjection.*

The algebraic prestacks of Definition 5.15 are Deligne-Mumford stacks and analogous results to those given below hold for Artin stacks where étale is replaced by smooth.

5.16. Theorem. *There is a system of fibrant objects on* **AlgPreSt** *given by taking the local weak equivalences and with fibrant replacement given by the associated stack.*

Proof. By the algebraic analogues of Lemmas 5.4 and 5.5, and the argument given in the proof of Theorem 5.7, it suffices to construct a local weak equivalence $A \to R(A, e)$ for any algebraic prestack A with chart $e\colon U \to A$. This was done in *ibid.* ■

5.17. Corollary. *There is an equivalence of 2-categories* $\mathbf{AlgSt} \simeq \mathbf{Ho}(\mathbf{AlgPreSt})$.

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