Theory and Applications of Categories, Vol. 29, No. 23, 2014, pp. 635–653.

CONSTRUCTING MODEL CATEGORIES WITH PRESCRIBED FIBRANT OBJECTS

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ABSTRACT. We present a weak form of a recognition principle for Quillen model categories due to J.H. Smith. We use it to put a model category structure on the category of small categories enriched over a suitable monoidal simplicial model category. The proof uses a part of the model structure on small simplicial categories due to J. Bergner. We give an application of the weak form of Smith's result to left Bousfield localizations of categories of monoids in a suitable monoidal model category.

1. Introduction

There are nowadays several recognition principles that allow one to put a Quillen model category structure on a given category. For the purposes of this work we divide them into those that make use of the small object argument and those that don't. A recognition principle that makes use of the small object argument is the following theorem of J.H. Smith [2, Theorem 1.7].

1.1. THEOREM. *Let $\mathcal{E}$ be a locally presentable category,* W *a full accessible subcategory of* $\mathrm{Mor}(\mathcal{E})$*, and I a set of morphisms of $\mathcal{E}$. Suppose they satisfy:*

C1: W *has the two out of three property.*

C2: $\mathrm{inj}(I) \subset \mathrm{W}$.

C3: The class $\mathrm{cof}(I) \cap \mathrm{W}$ *is closed under transfinite composition and under pushout.*

Then setting weak equivalences:$=\mathrm{W}$*, cofibrations:*$=\mathrm{cof}(I)$ *and fibrations:*$=\mathrm{inj}(\mathrm{cof}(I) \cap \mathrm{W})$*, one obtains a cofibrantly generated model structure on $\mathcal{E}$.*

We can say that (a) in practice, it is condition C3 above that is often the most difficult to check and (b) the result gives *no* description of the fibrations of the resulting model structure. Another recognition principle that makes use of the small object argument is a result of D.M. Kan [9, Theorem 11.3.1], [10, Theorem 2.1.19]. We can say that Kan's result gives a *full* description of the fibrations of the resulting model structure. In this paper we

Supported by the project CZ.1.07/2.3.00/20.0003 of the Operational Programme Education for Competitiveness of the Ministry of Education, Youth and Sports of the Czech Republic

Received by the editors 2013-07-31 and, in revised form, 2014-06-03.

Transmitted by Myles Tierney. Published on 2014-10-23.

2010 Mathematics Subject Classification: 18G55, 18D20.

Key words and phrases: model category, enriched category.

(1) advertise (see Proposition 2.3) an abstraction of a technique due to D.-C. Cisinski [5, Proof of Théorème 1.3.22] and A. Joyal (unpublished, but present in his proof, circa 1996, of the model structure for quasi-categories) that addresses both (a) and (b) above, in the sense that it makes C3 easier to check and it gives a *partial* description of the fibrations of the resulting model structure—namely the fibrant objects and the fibrations between them are described—provided that other assumptions hold, and

(2) give an application of this technique to the homotopy theory of categories enriched over a suitable monoidal simplicial model category (see Theorem 3.5) and to left Bousfield localizations of categories of monoids in a suitable monoidal model category (see Theorem 5.7).

The paper is organized as follows. In Section 2 we detail the above mentioned technique. The two out of six property of a class of maps of Dwyer et al. [6] plays an important role. In Section 3 we prove that the category of small categories enriched over a monoidal simplicial model category that satisfies some assumptions, admits a certain model category structure. Our proof uses one result of the non-formal part of the proof of the analogous model structure for categories enriched over the category of simplicial sets, due to J. Bergner [3]. We modify one of the steps in Bergner's proof; this modification is a key point in our approach and it enables us to apply the technique from Section 2. We also fix (see Remark 3.10), in an appropriate way, a mistake in [16]. The idea to use the model structure for categories enriched over the category of simplicial sets is due to G. Tabuada [18]. In Section 4 we extend a result of R. Fritsch and D.M. Latch [8, Proposition 5.2] to enriched categories; this is needed in the proof of the main result of Section 3. The section is self contained. Motivated by considerations from [12], we apply in Section 5 the technique from Section 2 to the study of left Bousfield localizations of categories of monoids. Precisely, let $L\mathcal{M}$ be a left Bousfield localization of a monoidal model category $\mathcal{M}$. We consider the problem of putting a model category structure on the category of monoids in $\mathcal{M}$, somehow related to $L\mathcal{M}$.

Acknowledgements. We are deeply indebted to André Joyal for many useful discussions and suggestions.

2. Constructing model categories with prescribed fibrant objects

We recall from [6] the following definitions. Let $\mathcal{E}$ be an arbitrary category and W a class of maps of $\mathcal{E}$. W is said to satisfy the *two out of six property* if for every three maps r, s, t of $\mathcal{E}$ for which the two compositions sr and ts are defined and are in W, the four maps r, s, t and tsr are in W. The class W is said to satisfy the *weak invertibility property* if every map s of $\mathcal{E}$ for which there exist maps r and t such that the compositions sr and ts exist and are in W, is itself in W. The two out of six property implies the two out of three property. The converse holds in the presence of the weak invertibility property.

The terminal object of a category, when it exists, is denoted by 1.

Let $\mathcal{E}$ be a locally presentable category and J a set of maps of $\mathcal{E}$. Then the pair

$(\mathrm{cof}(J), \mathrm{inj}(J))$ is a weak factorization system on $\mathcal{E}$ [2, Proposition 1.3]. We call a map of $\mathcal{E}$ that belongs to $\mathrm{inj}(J)$ a *naive fibration*, and say that an object X of $\mathcal{E}$ is *naively fibrant* if $X \to 1$ is a naive fibration. We denote the class of naive fibrations between naively fibrant objects by $\mathrm{inj}_0(J)$.

2.1. Lemma. (D.-C. Cisinski, A. Joyal) *Let $\mathcal{E}$ be a locally presentable category, $(\mathcal{A}, \mathcal{B})$ a weak factorisation system on $\mathcal{E}$, W a class of maps of $\mathcal{E}$ satisfying the two out of six property and J a set of maps of $\mathcal{E}$.*

(1) *Suppose that cell$(J) \subset \mathrm{W}$. Then a map that has the left lifting property with respect to maps in $\mathrm{inj}_0(J)$ belongs to W.*

(2) *Suppose that cell$(J) \subset \mathrm{W}$ and that $\mathrm{inj}_0(J) \cap \mathrm{W} \subset \mathcal{B}$. Then a map in $\mathcal{A}$ belongs to W if and only if it has the left lifting property with respect to the maps in $\mathrm{inj}_0(J)$. In particular, $\mathcal{A} \cap \mathrm{W}$ is closed under pushouts and transfinite compositions.*

(3) *Suppose that cell$(J) \subset \mathcal{A} \cap \mathrm{W}$ and that $\mathrm{inj}_0(J) \cap \mathrm{W} \subset \mathcal{B}$. Then an object X of $\mathcal{E}$ is naively fibrant if and only if the map $X \to 1$ is in $\mathrm{inj}(\mathcal{A} \cap \mathrm{W})$. Also, a map between naively fibrant objects is in $\mathrm{inj}(\mathcal{A} \cap \mathrm{W})$ if and only if it is a naive fibration.*

Proof. (1) Let $i\colon A \to B$ be a map which has the left lifting property with respect to the naive fibrations between naively fibrant objects. Factorize (see, for example, [2, Proposition 1.3]) the map $B \to 1$ as $B \to \bar{B} \to 1$, where $B \to \bar{B}$ is in *cell*(J) and $\bar{B}$ is naively fibrant. Next, factorize the composite map $A \to \bar{B}$ as a map $A \to \bar{A}$ in *cell*(J) followed by a naive fibration $\bar{A} \to \bar{B}$. The resulting commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & \bar{A} \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & \bar{B} \end{array}$$

has then a diagonal filler, and so the hypothesis and the two out of six property of W imply that i is in W.

(2) Let

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{v} & Y \end{array}$$

be a commutative diagram with i in $\mathcal{A} \cap \mathrm{W}$ and p in $\mathrm{inj}_0(J)$. Factorize v as a map $B \to \bar{B}$ in *cell*(J) followed by a naive fibration $\bar{B} \to Y$. Next, factorize the canonical map $A \to \bar{B} \times_Y X$ as a map $A \to \bar{A}$ in *cell*(J) followed by a naive fibration $\bar{A} \to \bar{B} \times_Y X$. It suffices to show that the square

$$\begin{array}{ccc} A & \longrightarrow & \bar{A} \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & \bar{B} \end{array}$$

has a diagonal filler. The map $\bar{A} \to \bar{B}$ is a naive fibration between naively fibrant objects. It also belongs to W by the two out of three property, and so by hypothesis it is in $\mathcal{B}$. Therefore the diagonal filler exists. The converse follows from (1). Thus, in order to detect if an element of $\mathcal{A}$ is in W one can use the left lifting property with respect to a class of maps, namely $\mathrm{inj}_0(J)$. In particular, $\mathcal{A} \cap \mathrm{W}$ is closed under pushouts and transfinite compositions.

(3) This is straightforward from (2). ∎

2.2. Remark. One can make variations in Lemma 2.1. For example, the path object argument devised by Quillen shows that the conclusion of (1) remains valid if instead of $cell(J) \subset \mathrm{W}$ one requires that $\mathcal{E}$ *has a functorial naively fibrant replacement functor and every naively fibrant object has a naive path object*. This new requirement implies that $cell(J) \subset \mathrm{W}$.

The following result makes the connection between Smith's Theorem and Lemma 2.1.

2.3. Proposition. *Let $\mathcal{E}$ be a locally presentable category,* W *a full accessible subcategory of* $\mathrm{Mor}(\mathcal{E})$ *and I and J be two sets of morphisms of $\mathcal{E}$. Let us call a map of $\mathcal{E}$ that belongs to* $\mathrm{inj}(J)$ *a* naive fibration*, and an object X of $\mathcal{E}$* naively fibrant *if $X \to 1$ is a naive fibration. Suppose the following conditions are satisfied:*

- *C1:* W *has the two out of three property.*
- *C2:* $\mathrm{inj}(I) \subset \mathrm{W}$.
- *C3:* W *has the weak invertibility property.*
- *C4:* $cell(J) \subset \mathrm{cof}(I) \cap \mathrm{W}$.
- *C5: A map between naively fibrant objects that is both a naive fibration and in* W *is in* $\mathrm{inj}(I)$.

Then the triple $(\mathrm{W}, \mathrm{cof}(I), \mathrm{inj}(\mathrm{cof}(I) \cap \mathrm{W}))$ *is a model structure on $\mathcal{E}$. Moreover, an object of $\mathcal{E}$ is fibrant if and only if it is naively fibrant, and the fibrations between fibrant objects are the naive fibrations.*

Proof. All assumptions of Theorem 1.1 hold, except possibly condition C3. To check that C3 holds we apply the last part of Lemma 2.1(2) to the weak factorization system $(\mathcal{A}, \mathcal{B}) = (\mathrm{cof}(I), \mathrm{inj}(I))$. It follows that the triple $(\mathrm{W}, \mathrm{cof}(I), \mathrm{inj}(\mathrm{cof}(I) \cap \mathrm{W}))$ is a model structure. The characterization of fibrant objects and of the fibrations between fibrant objects is then a consequence of Lemma 2.1(3) applied to the weak factorization system $(\mathrm{cof}(I), \mathrm{inj}(I))$. ∎

The following result is a variation of Proposition 2.3, essentially due to A.K. Bousfield [4, Proof of Theorem 9.3]. We leave the proof to the interested reader.

2.4. Proposition. *Let $\mathcal{E}$ be a category that is closed under limits and colimits and let* W *be a class of maps of $\mathcal{E}$ that has the two out of three property. If I and J are two sets of morphisms of $\mathcal{E}$ such that*

(1) *both I and J permit the small object argument [9, Definition 10.5.15],*

(2) $\mathrm{inj}(I) \subset \mathrm{W}$*,*

(3) *cell*$(J) \subset \mathrm{cof}(I) \cap \mathrm{W}$*,*

(4) $\mathrm{inj}_0(J) \cap \mathrm{W} \subset \mathrm{inj}(I)$*, and*

(5) *the class* W *is stable under pullback along maps in* $\mathrm{inj}_0(J)$*,*

then the triple $(\mathrm{W}, \mathrm{cof}(I), \mathrm{inj}(\mathrm{cof}(I) \cap \mathrm{W}))$ *is a right proper model structure on $\mathcal{E}$. Moreover, an object of $\mathcal{E}$ is fibrant if and only if it is naively fibrant, and the fibrations between fibrant objects are the naive fibrations.*

Here is an application of Lemma 2.1. Let $\mathcal{E}$ be a locally presentable closed category with initial object $\emptyset$. We denote by $\otimes$ the monoidal product of $\mathcal{E}$ and for two objects X, Y of $\mathcal{E}$ we write Y^X for their internal hom. In the language of Lemma 2.1 we have

2.5. Proposition. *Let* W *be a class of maps of $\mathcal{E}$ having the two out of six property and let I and J be two sets of maps of $\mathcal{E}$. Suppose that the domains of the elements of I are in* $\mathrm{cof}(I)$*, that cell*$(J) \subset \mathrm{cof}(I) \cap \mathrm{W}$ *and that a map between naively fibrant objects which is both a naive fibration and in* W *is in* $\mathrm{inj}(I)$*. Then the following are equivalent:*

(a) *for any maps $A \to B$ and $K \to L$ of* $\mathrm{cof}(I)$*, the canonical map*

$$A \otimes L \sqcup_{A \otimes K} B \otimes K \to B \otimes L$$

is in $\mathrm{cof}(I)$*, which is in* W *if either one of the given maps is in* W*;*

(b) *for any maps $A \to B$ and $K \to L$ of* $\mathrm{cof}(I)$*, the canonical map*

$$A \otimes L \sqcup_{A \otimes K} B \otimes K \to B \otimes L$$

is in $\mathrm{cof}(I)$ and *for every element $A \to B$ of I and every naive fibration $X \to Y$ between naively fibrant objects, the canonical map*

$$X^B \to Y^B \times_{Y^A} X^A$$

is a naive fibration between naively fibrant objects.

Proof. The fact that the domains of the elements of I are in $\mathrm{cof}(I)$ means that for every element $A \to B$ of I, the map $\emptyset \to A$ is in $\mathrm{cof}(I)$ (and therefore so is $\emptyset \to B$).

We prove that (a) implies (b). Let $A \to B$ be an element of I, $X \to Y$ a naive fibration between naively fibrant objects and $C \to D$ an element of J. A commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X^B \arrow[d] \\ D \arrow[r] & Y^B \times_{Y^A} X^A \end{tikzcd}$$

has a diagonal filler if and only if its adjoint transpose

$$\begin{tikzcd} C \otimes B \sqcup_{C \otimes A} D \otimes A \arrow[r] \arrow[d] & X \arrow[d] \\ D \otimes B \arrow[r] & Y \end{tikzcd}$$

has one. The latter is true by Lemma 2.1(2) applied to the weak factorization system $(\mathrm{cof}(I), \mathrm{inj}(I))$. It follows that $X^B \to Y^B \times_{Y^A} X^A$ is a naive fibration. A similar adjunction argument shows that $X^A \to Y^A$ and $X^B \to Y^B$ are naive fibrations between naively fibrant objects, therefore $Y^B \times_{Y^A} X^A$ is naively fibrant.

We prove that (b) implies (a). Suppose first that $A \to B$ is an element of I and let $K \to L$ be a fixed map in $\mathrm{cof}(I) \cap \mathrm{W}$. Then the canonical map

$$A \otimes L \sqcup_{A \otimes K} B \otimes K \to B \otimes L$$

is in W by Lemma 2.1(2) applied to the weak factorization system $(\mathrm{cof}(I), \mathrm{inj}(I))$ and an adjunction argument. Thus, it suffices to show that the class of maps $A' \to B'$ of $\mathrm{cof}(I)$ such that

$$A' \otimes L \sqcup_{A' \otimes K} B' \otimes K \to B' \otimes L$$

is in $\mathrm{cof}(I) \cap \mathrm{W}$ is closed under pushout, transfinite composition and retracts. This is the case since by Lemma 2.1(2) applied to the weak factorization system $(\mathrm{cof}(I), \mathrm{inj}(I))$ the elements of $\mathrm{cof}(I)$ which are in W can be detected by the left lifting property with respect to a class of maps. ∎

3. Application: categories enriched over monoidal simplicial model categories

We denote by **S** the category of simplicial sets, regarded as having the standard model structure (due to Quillen). We let **Cat** be the category of small categories. We say that an arrow $f\colon C \to D$ of **Cat** is an *isofibration* if for any $x \in Ob(C)$ and any isomorphism $v\colon y' \to f(x)$ in D, there exists an isomorphism $u\colon x' \to x$ in C such that $f(u) = v$. The class of isofibrations is invariant under isomorphisms in the sense that given a commutative diagram in **Cat**

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd}$$

in which the horizontal arrows are isomorphisms, the map f is an isofibration if and only if g is so.

3.1. Monoidal simplicial model categories. Let $\mathcal{M}$ be a monoidal model category with cofibrant unit. We recall [10, Definition 4.2.20] that $\mathcal{M}$ is said to be a *monoidal* **S**-*model category* if it is given a Quillen pair $F\colon \mathbf{S} \rightleftarrows \mathcal{M}\colon G$ such that F is strong monoidal. Since F is strong monoidal, G becomes a monoidal functor.

3.2. Classes of $\mathcal{M}$-functors and the main result. Let $\mathcal{M}$ be a monoidal model category with cofibrant unit e. We denote by $\mathcal{M}$-**Cat** the category of small $\mathcal{M}$-categories. If S is a set, we denote by $\mathcal{M}$-$\mathbf{Cat}(S)$ (resp. $\mathcal{M}$-$\mathbf{Graph}(S)$) the category of small $\mathcal{M}$-categories (resp. $\mathcal{M}$-graphs) with fixed set of objects S. When S is a one element set $\{*\}$, $\mathcal{M}$-$\mathbf{Cat}(\{*\})$ is the category $Mon(\mathcal{M})$ of monoids in $\mathcal{M}$. There is a free-forgetful adjunction

$$F_S\colon \mathcal{M}\text{-}\mathbf{Graph}(S) \rightleftarrows \mathcal{M}\text{-}\mathbf{Cat}(S)\colon U_S$$

We denote by ε^S the counit of this adjunction. Every function $f\colon S \to T$ induces an adjoint pair

$$f_!\colon \mathcal{M}\text{-}\mathbf{Cat}(S) \rightleftarrows \mathcal{M}\text{-}\mathbf{Cat}(T)\colon f^*$$

If $\mathcal{K}$ is a class of maps of $\mathcal{M}$, an $\mathcal{M}$-functor $f\colon \mathcal{A} \to \mathcal{B}$ is said to be *locally in* $\mathcal{K}$ if for each pair $x, y \in \mathcal{A}$ of objects, the map $f_{x,y}\colon \mathcal{A}(x,y) \to \mathcal{B}(f(x), f(y))$ is in $\mathcal{K}$.

We have a functor $[\_]_{\mathcal{M}}\colon \mathcal{M}\text{-}\mathbf{Cat} \to \mathbf{Cat}$ obtained by change of base along the symmetric monoidal composite functor

$$\mathcal{M} \longrightarrow Ho(\mathcal{M}) \xrightarrow{Ho(\mathcal{M})(e,-)} Set$$

3.3. Definition. Let $f\colon \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathcal{M}$-**Cat**.

(1) The morphism f is a DK-*equivalence* if f is locally a weak equivalence of $\mathcal{M}$ and $[f]_{\mathcal{M}}\colon [\mathcal{A}]_{\mathcal{M}} \to [\mathcal{B}]_{\mathcal{M}}$ is essentially surjective.

(2) The morphism f is a DK-*fibration* if f is locally a fibration of $\mathcal{M}$ and $[f]_{\mathcal{M}}$ is an isofibration.

(3) The morphism f is called a *trivial fibration* if it is both a DK-equivalence and a DK-fibration.

(4) The morphism f is called a *cofibration* if it has the left lifting property with respect to the trivial fibrations.

It follows from Definition 3.3 that (a) an $\mathcal{M}$-functor f is a DK-equivalence if and only if $Ho(f)$ is an equivalence of $Ho(\mathcal{M})$-categories, and (b) an $\mathcal{M}$-functor is a trivial fibration if and only if it is surjective on objects and locally a trivial fibration of $\mathcal{M}$. In particular, the class of DK-equivalences has the two out of three and weak invertibility properties.

We denote by $\mathcal{I}$ the $\mathcal{M}$-category with a single object $*$ and $\mathcal{I}(*,*) = e$. For an object X of $\mathcal{M}$ we denote by 2_X the $\mathcal{M}$-category with two objects 0 and 1 and with $2_X(0,0) = 2_X(1,1) = e$, $2_X(0,1) = X$ and $2_X(1,0) = \emptyset$. When $\mathcal{M}$ is cofibrantly generated, an $\mathcal{M}$-functor is a trivial fibration if and only if it has the right lifting property with respect to the saturated class generated by $\{\emptyset \to \mathcal{I}\} \cup \{2_X \overset{2_i}{\to} 2_Y,\ i \text{ generating cofibration of } \mathcal{M}\}$, where $\emptyset$ denotes the initial object of $\mathcal{M}$-**Cat**. We have the following fundamental result of J. Bergner [3].

3.4. Theorem. *The category* **S**-**Cat** *of simplicial categories admits a cofibrantly generated model structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations. A generating set of trivial cofibrations consists of*

B1: $\{2_X \xrightarrow{2_j} 2_Y\}$, *where* j *is a horn inclusion, and*

B2: inclusions $\mathcal{I} \overset{\delta_y}{\to} \mathcal{H}$, *where* $\{\mathcal{H}\}$ *is a set of representatives for the isomorphism classes of simplicial categories on two objects which have countably many simplices in each function complex. Furthermore, each such* $\mathcal{H}$ *is required to be cofibrant and weakly contractible in* **S**-**Cat**$(\{x,y\})$. *Here* $\{x,y\}$ *is the set with elements* x *and* y *and* δ_y *omits* y.

Recall from [14, Definition 3.3] the monoid axiom. The main result of this section is

3.5. Theorem. *Let* $\mathcal{M}$ *be a cofibrantly generated monoidal* **S**-*model category having cofibrant unit and which satisfies the monoid axiom. Suppose furthermore that* $\mathcal{M}$ *is locally presentable and that a transfinite composition of weak equivalences of* $\mathcal{M}$ *is a weak equivalence.*

Then $\mathcal{M}$-**Cat** *admits a cofibrantly generated model category structure in which the weak equivalences are the DK-equivalences, the cofibrations are the elements of* $\mathrm{cof}(\{\emptyset \to \mathcal{I}\} \cup \{2_X \overset{2_i}{\to} 2_Y\})$, *where* i *is a generating cofibration of* $\mathcal{M}$, *the fibrant objects are the locally fibrant* $\mathcal{M}$-*categories and the fibrations between fibrant objects are the DK-fibrations.*

If the model structure on $\mathcal{M}$ *is right proper, then so is the one on* $\mathcal{M}$-**Cat**.

Proof. We shall apply Proposition 2.3. We take $\mathcal{E}$ to be $\mathcal{M}$-**Cat** and W to be the class of DK-equivalences. The fact that $\mathcal{M}$-**Cat** is locally presentable can be seen in a few ways, one is presented in [13]. The fact that the class of DK-equivalences is accessible follows essentially from the fact that the classes of weak equivalences of $\mathcal{M}$ and of essentially surjective functors are accessible. We take I to be the set $\{\emptyset \to \mathcal{I}\} \cup \{2_X \overset{2_i}{\to} 2_Y\})$, where i is a generating cofibration of $\mathcal{M}$. Let

$$F\colon \mathbf{S} \rightleftarrows \mathcal{M}\colon G$$

be the Quillen pair guaranteed by the definition. The adjoint pair (F, G) induces adjoint pairs

$$F'\colon \mathbf{S}\text{-}\mathbf{Cat} \rightleftarrows \mathcal{M}\text{-}\mathbf{Cat}\colon G'$$

and

$$F'\colon \mathbf{S}\text{-}\mathbf{Cat}(S) \rightleftarrows \mathcal{M}\text{-}\mathbf{Cat}(S)\colon G'$$

for every set S. The first G' functor preserves trivial fibrations and the $\mathcal{M}$-functors which are locally a fibration. The latter adjoint pair is a Quillen pair. Finally, we take J to be the set $F'(B2) \cup \{2_X \xrightarrow{2_i} 2_Y\}$, where i is a generating trivial cofibration of $\mathcal{M}$, where $B2$ is as in Theorem 3.4.

Step 1. Conditions C1, C2 and C3 from Proposition 2.3 were dealt with above.

Step 2. Since every map δ_y belonging to the set B2 from Theorem 3.4 has a retraction, one readily checks that an $\mathcal{M}$-category is naively fibrant if and only if it is locally fibrant. We claim that if an $\mathcal{M}$-functor between locally fibrant $\mathcal{M}$-categories is a naive fibration, then it is a DK-fibration. To see this, let first $\mathcal{M}\text{-}\mathbf{Cat}_f$ be the full subcategory of $\mathcal{M}\text{-}\mathbf{Cat}$ consisting of the locally fibrant $\mathcal{M}$-categories. By [9, Proposition 8.5.16] we have a natural isomorphism of functors

$$\eta\colon [\_]_{\mathbf{S}} G' \cong [\_]_{\mathcal{M}}\colon \mathcal{M}\text{-}\mathbf{Cat}_f \to \mathbf{Cat}$$

such that for all $\mathcal{A} \in \mathcal{M}\text{-}\mathbf{Cat}_f$, $\eta_{\mathcal{A}}$ is the identity on objects: indeed, for each pair $x, y \in \mathcal{A}$ of objects we have natural isomorphisms

$$\begin{aligned} [\mathcal{A}]_{\mathcal{M}}(x,y) &\cong Ho(\mathcal{M})(e, \mathcal{A}(x,y)) \cong Ho(\mathcal{M})(F1, \mathcal{A}(x,y)) \\ &\cong Ho(\mathbf{S})(1, G\mathcal{A}(x,y)) \cong [G'\mathcal{A}]_{\mathbf{S}}(x,y) \end{aligned}$$

Second, we use the following relaxed version of [3, Proposition 2.3]. Let f be a simplicial functor between categories enriched in Kan complexes such that f is locally a Kan fibration. If f has the right lifting property with respect to every element of the set B2, then f is a DK-fibration. (This is the only result from [3] that we need.) These facts, together with the observation that the class of isofibrations is invariant in **Cat** under isomorphisms, imply the claim. It is now clear that condition C5 from Proposition 2.3 holds.

Step 3. We check condition C4 from Proposition 2.3. Let $j\colon X \to Y$ be a trivial cofibration of $\mathcal{M}$. We show that for every $\mathcal{M}$-category $\mathcal{A}$, in the pushout diagram

$$\begin{tikzcd} 2_X \arrow[r, "2_j"] \arrow[d] & 2_Y \arrow[d] \\ \mathcal{A} \arrow[r] & \mathcal{B} \end{tikzcd}$$

the map $\mathcal{A} \to \mathcal{B}$ is a DK-equivalence. Let $S = Ob(\mathcal{A})$. This pushout can be calculated as the pushout

$$\begin{tikzcd} F_S U_S \mathcal{A} \arrow[r] \arrow[d, "\varepsilon^S_{\mathcal{A}}"] & F_S \mathcal{X} \arrow[d] \\ \mathcal{A} \arrow[r] & \mathcal{B} \end{tikzcd}$$

where $U_S\mathcal{A} \to \mathcal{X}$ is a certain map of $\mathcal{M}$-graphs with fixed set of objects S. But then the map $\mathcal{A} \to \mathcal{B}$ is known to be locally a weak equivalence of $\mathcal{M}$, see [15, Proof of Proposition 6.3(1)].

We now claim that if $\delta_y \colon \mathcal{I} \to \mathcal{H}$ is a map belonging to the set B2 from Theorem 3.4 and $\mathcal{A}$ is any $\mathcal{M}$-category, then in the pushout diagram

$$\begin{tikzcd} F'\mathcal{I} \arrow[r, "a"] \arrow[d, "F'\delta_y"'] & \mathcal{A} \arrow[d] \\ F'\mathcal{H} \arrow[r] & \mathcal{B} \end{tikzcd}$$

the map $\mathcal{A} \to \mathcal{B}$ is a DK-equivalence. We factorize the map δ_y as $\mathcal{I} \overset{\delta'_y}{\to} \mathcal{H}' \to \mathcal{H}$, where the simplicial category $\mathcal{H}'$ has $\{x\}$ as set of objects and $\mathcal{H}'(x,x) = \mathcal{H}(x,x)$, and then we take consecutive pushouts:

$$\begin{tikzcd} F'\mathcal{I} \arrow[r, "a"] \arrow[d, "F'\delta'_y"'] & \mathcal{A} \arrow[d, "j"] \\ F'\mathcal{H}' \arrow[r] \arrow[d] & \mathcal{A}' \arrow[d] \\ F'\mathcal{H} \arrow[r] & \mathcal{B} \end{tikzcd}$$

The map j can be obtained from the pushout diagram in $\mathcal{M}$-$\mathbf{Cat}(Ob(\mathcal{A}))$

$$\begin{tikzcd} a_!F'\mathcal{I} \arrow[r] \arrow[d, "a_!F'\delta'_y"'] & \mathcal{A} \arrow[d, "j"] \\ a_!F'\mathcal{H}' \arrow[r] & \mathcal{A}' \end{tikzcd}$$

where $a_! \colon Mon(\mathcal{M}) \to \mathcal{M}$-$\mathbf{Cat}(Ob(\mathcal{A}))$. By Lemma 3.6 the map δ'_y is a trivial cofibration in the category of simplicial monoids, therefore $F'\delta'_y$ is a trivial cofibration in the category of monoids in $\mathcal{M}$. Since $a_!$ is a left Quillen functor, j is a trivial cofibration in $\mathcal{M}$-$\mathbf{Cat}(Ob(\mathcal{A}))$.

The map $F'\mathcal{H}' \to F'\mathcal{H}$ is a full and faithful inclusion, so by Proposition 4.1 the map $\mathcal{A}' \to \mathcal{B}$ is a full and faithful inclusion. Therefore the map $\mathcal{A} \to \mathcal{B}$ is locally a weak equivalence of $\mathcal{M}$. Applying the functor $[\_]_{\mathcal{M}}$ to the diagram

$$\begin{tikzcd} F'\mathcal{I} \arrow[r, "a"] \arrow[d, "F'\delta_y"'] & \mathcal{A} \arrow[d] \\ F'\mathcal{H} \arrow[r] & \mathcal{B} \end{tikzcd}$$

and taking into account that F' preserves DK-equivalences and that $Ob(\mathcal{B}) = Ob(\mathcal{A}) \cup \{*\}$, it follows that $\mathcal{A} \to \mathcal{B}$ is a DK-equivalence as well. The claim is proved.

So far we have shown that the pushout of a map from J along any $\mathcal{M}$-functor is in $\mathrm{cof}(I) \cap \mathrm{W}$. Since a transfinite composition of weak equivalences of $\mathcal{M}$ is a weak equivalence, we readily obtain that $cell(J) \subset \mathrm{cof}(I) \cap \mathrm{W}$. Thus, condition C4 is checked.

Now, putting all the three steps together we obtain the desired model structure on $\mathcal{M}$-**Cat**.

Step 4. Suppose that $\mathcal{M}$ is right proper. Using the explicit construction of pullbacks in $\mathcal{M}$-**Cat**, the description of the fibrations between fibrant objects and [4, Lemma 9.4], we conclude that the model structure on $\mathcal{M}$-**Cat** is right proper. ∎

3.6. LEMMA. *Let $\mathcal{A}$ be a cofibrant simplicial category. Then for each $a \in Ob(\mathcal{A})$ the simplicial monoid $a^*\mathcal{A} = \mathcal{A}(a, a)$ is cofibrant.*

PROOF. Let $S = Ob(\mathcal{A})$. The simplicial category $\mathcal{A}$ is cofibrant in **S-Cat** if and only if it is cofibrant as an object of **S-Cat**(S). The cofibrant objects of **S-Cat**(S) are characterized in [7, 7.6]: they are the retracts of free simplicial categories. Therefore it suffices to prove that if $\mathcal{A}$ is a free simplicial category then $a^*\mathcal{A}$ is a free simplicial category for all $a \in S$. There is a full and faithful functor $\varphi\colon \mathbf{S}\text{-}\mathbf{Cat} \to \mathbf{Cat}^{\Delta^{op}}$ given by $Ob(\varphi(\mathcal{A})_n) = Ob(\mathcal{A})$ for all $n \geq 0$ and $\varphi(\mathcal{A})_n(a, a') = \mathcal{A}(a, a')_n$. Recall [7, 4.5] that $\mathcal{A}$ is a free simplicial category if and only if (i) for all $n \geq 0$ the category $\varphi(\mathcal{A})_n$ is a free category on a graph G_n, and (ii) for all epimorphisms $\alpha\colon [m] \to [n]$ of Δ, $\alpha^*\colon \varphi(\mathcal{A})_n \to \varphi(\mathcal{A})_m$ maps G_n into G_m.

Let $a \in S$. The category $\varphi(a^*\mathcal{A})_n$ is a full subcategory of $\varphi(\mathcal{A})_n$ with object set $\{a\}$, hence it is free as well. A set $G_n^{a^*\mathcal{A}}$ of generators can be described as follows. An element of $G_n^{a^*\mathcal{A}}$ is a path from a to a in $\varphi(\mathcal{A})_n$ such that every arrow in the path belongs to G_n and there is at most one arrow in the path with source and target a. The fact that $\varphi(a^*\mathcal{A})_n$ is indeed freely generated by $G_n^{a^*\mathcal{A}}$ follows from Lemma 3.7 and its proof. Since every epimorphism $\alpha\colon [m] \to [n]$ of Δ has a section, α^* maps $G_n^{a^*\mathcal{A}}$ into $G_m^{a^*\mathcal{A}}$. ∎

3.7. LEMMA. *A full subcategory of a free category is free.*

PROOF. Let $F(G)$ be a free category generated by a graph $G = (G_1 \rightrightarrows G_0)$. An arrow f of $F(G)$ is a generator if and only if f is *indecomposable* (f is not a unit and $f = vu$ implies v or u is a unit). Let C be a full subcategory of $F(G)$ with $Ob(C) = C_0 \subset G_0$. If $x, y \in C_0$, let us say that a path $(x_1, f_1, ..., f_{n-1}, x_n)\colon x \to y$ in the graph G is C_0-*free* if $target(f_i) \notin C_0$ for $1 \leq i < n$. Let G_1' be the set of C_0-free paths. It is easy to see that every arrow of C can be uniquely written as a finite composition of C_0-free paths, so that C is freely generated by the graph $(G_1' \rightrightarrows C_0)$. ∎

3.8. REMARK. The class of cofibrations of the model category constructed in Theorem 3.5 can be given an explicit description [17, Section 4.2].

3.9. REMARK. We noticed during the proof of Theorem 3.5 that our result is almost independent on Theorem 3.4, only a relaxed version of [3, Proposition 2.3] being needed. In particular, taking $\mathcal{M} = \mathbf{S}$ in Theorem 3.5 results in a weaker version of Bergner's result. However, using the fact that **S** has a monoidal fibrant replacement functor that preserves fibrations, the full Theorem 3.4 can be recovered.

3.10. Remark. One can change the assumptions of Theorem 3.5 and the recognition principle used in its proof to obtain a similar outcome. For example, let $\mathcal{M}$ be a cofibrantly generated monoidal **S**-model category having cofibrant unit and which satisfies the monoid axiom. Suppose furthermore that

(a) a transfinite composition of weak equivalences of $\mathcal{M}$ is a weak equivalence,

(b) $\mathcal{M}$ satisfies the technical condition of [11, Theorem 2.1], and

(c) in the Quillen pair $F\colon \mathbf{S} \rightleftarrows \mathcal{M}\colon G$ guaranteed by the definition, the functor G preserves weak equivalences.

Then [9, Theorem 11.3.1] can be used to show that $\mathcal{M}$-**Cat** admits a cofibrantly generated model category structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations. The proof proceeds in the same way as the proof of Theorem 3.5, Step 3 remains unchanged but Step 2 requires suitable modifications. Condition (b) can be relaxed, it was stated in this form in order to include examples such as compactly generated spaces [11].

4. Pushouts along full and faithful functors

A result of R. Fritsch and D.M. Latch [8, Proposition 5.2] says that the pushout of a full and faithful functor is full and faithful. The purpose of this section is to extend this result to categories enriched over a monoidal category.

Let $(\mathcal{V}, \otimes, I)$ be a cocomplete closed category. We denote by $\mathcal{V}$-**Cat** the category of small $\mathcal{V}$-categories and by $\mathcal{V}$-**Graph** that of small $\mathcal{V}$-graphs. A $\mathcal{V}$-functor, or a map of $\mathcal{V}$-graphs, that is locally an isomorphism (3.2) is said to be *full and faithful*. If S is a set, we denote by $\mathcal{V}$-**Cat**(S) (resp. $\mathcal{V}$-**Graph**(S)) the category of small $\mathcal{V}$-categories (resp. $\mathcal{V}$-graphs) with fixed set of objects S. The category $\mathcal{V}$-**Graph**(S) is a monoidal category with monoidal product $\square_S$ and unit which we denote by $\mathcal{I}_S$.

4.1. Proposition. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three small $\mathcal{V}$-categories and let $i\colon \mathcal{A} \to \mathcal{B}$ be a full and faithful inclusion. Then in the pushout diagram of $\mathcal{V}$-categories*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i"] \arrow[d, "f"'] & \mathcal{B} \arrow[d, "g"] \\ \mathcal{C} \arrow[r, "i'"] & \mathcal{D} \end{tikzcd}$$

the map $i'\colon \mathcal{C} \to \mathcal{D}$ is a full and faithful inclusion.

Proof. We shall construct $\mathcal{D}$ explicitly, as was done in the proof of [8, Proposition 5.2]. On objects we put $Ob(\mathcal{D}) = Ob(\mathcal{C}) \sqcup (Ob(\mathcal{B}) \setminus Ob(\mathcal{A}))$ and $\mathcal{D}(p,q) = \mathcal{C}(p,q)$ if $p, q \in Ob(\mathcal{C})$. For $p \in Ob(\mathcal{C})$ and $q \in Ob(\mathcal{B}) \setminus Ob(\mathcal{A})$ we define

$$\mathcal{D}(p,q) = \int^{x \in Ob(\mathcal{A})} \mathcal{B}(x,q) \otimes \mathcal{C}(p, f(x))$$

For $p \in Ob(\mathcal{B}) \setminus Ob(\mathcal{A})$ and $q \in Ob(\mathcal{C})$ we define

$$\mathcal{D}(p,q) = \int^{x \in Ob(\mathcal{A})} \mathcal{C}(f(x),q) \otimes \mathcal{B}(p,x)$$

For $p, q \in Ob(\mathcal{B}) \setminus Ob(\mathcal{A})$ we define $\mathcal{D}(p,q)$ to be the pushout

$$\begin{tikzcd}
\int^{x\in Ob(\mathcal{A})} \mathcal{B}(x,q)\otimes\mathcal{B}(p,x) \arrow[r] \arrow[d] & \int^{x\in Ob(\mathcal{A})}\int^{y\in Ob(\mathcal{A})} \mathcal{B}(x,q)\otimes\mathcal{C}(f(y),f(x))\otimes\mathcal{B}(p,y) \arrow[d] \\
\mathcal{B}(p,q) \arrow[r] & \mathcal{D}(p,q)
\end{tikzcd}$$

We shall describe a way to see that, with the above definition, $\mathcal{D}$ is indeed a $\mathcal{V}$-category.

Let $(\mathcal{B} \setminus \mathcal{A})^+$ be the preorder with objects all finite subsets $S \subset Ob(\mathcal{B}) \setminus Ob(\mathcal{A})$, ordered by inclusion. For $S \in (\mathcal{B} \setminus \mathcal{A})^+$, let $\mathcal{A}_S$ be the full sub-$\mathcal{V}$-category of $\mathcal{B}$ with objects $Ob(\mathcal{A}) \cup S$. Then $\mathcal{B} = \operatorname{colim}_{(\mathcal{B}\setminus\mathcal{A})^+} \mathcal{A}_S$. On the other hand, a filtered colimit of full and faithful inclusions of $\mathcal{V}$-categories is a full and faithful inclusion. This is because the forgetful functor from $\mathcal{V}$-**Cat** to $\mathcal{V}$-**Graph** preserves filtered colimits [13, Corollary 3.4] and a filtered colimit of full and faithful inclusions of $\mathcal{V}$-graphs is a full and faithful inclusion. Therefore one can assume from the beginning that $Ob(\mathcal{B}) = Ob(\mathcal{A}) \cup \{q\}$, where $q \notin Ob(\mathcal{A})$.

Case 1: f is full and faithful. In this case the pushout giving $\mathcal{D}(q,q)$ is simply $\mathcal{B}(q,q)$, all the other formulas remain unchanged. Then to show that $\mathcal{D}$ is a $\mathcal{V}$-category is straightforward.

Case 2: f is the identity on objects. The map i induces an adjoint pair

$$i_! \colon \mathcal{V}\text{-}\mathbf{Cat}(Ob(\mathcal{A})) \rightleftarrows \mathcal{V}\text{-}\mathbf{Cat}(Ob(\mathcal{B})) \colon i^*$$

One has

$$i_!\mathcal{A}(a,a') = \begin{cases} \mathcal{A}(a,a') & \text{if } a, a' \in Ob(\mathcal{A}), \\ \emptyset & \text{otherwise}, \\ I & \text{if } a = a' = q \end{cases}$$

and i factors as $\mathcal{A} \to i_!\mathcal{A} \to \mathcal{B}$, where $i_!\mathcal{A} \to \mathcal{B}$ is the obvious map in $\mathcal{V}$-$\mathbf{Cat}(Ob(\mathcal{B})))$. Then the original pushout can be computed using the pushout diagram

$$\begin{tikzcd}
i_!\mathcal{A} \arrow[r] \arrow[d, "i_! f"'] & \mathcal{B} \arrow[d] \\
i_!\mathcal{C} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

in $\mathcal{V}$-$\mathbf{Cat}(Ob(\mathcal{B}))$. Next, we claim that $\mathcal{D}$ can be calculated as the pushout, in the category ${}_{\mathcal{B}}Mod_{\mathcal{B}}$ of $(\mathcal{B}, \mathcal{B})$-bimodules in

$$(\mathcal{V}\text{-}\mathbf{Cat}(Ob(\mathcal{B})), \Box_{Ob(\mathcal{B})}, \mathcal{I}_{Ob(\mathcal{B})})$$

of the diagram

$$\begin{tikzcd}
\mathcal{B}\Box_{i_!\mathcal{A}}\mathcal{B} \arrow[rr, "\mathcal{B}\Box_{i_!\mathcal{A}} i_! f \Box_{i_!\mathcal{A}}\mathcal{B}"] \arrow[d] & & \mathcal{B}\Box_{i_!\mathcal{A}} i_!\mathcal{C}\Box_{i_!\mathcal{A}}\mathcal{B} \arrow[d, "m"] \\
\mathcal{B} \arrow[rr] & & \mathcal{D}
\end{tikzcd}$$

For this we have to show that $\mathcal{D}$ is a monoid in ${}_{\mathcal{B}}Mod_{\mathcal{B}}$. We first show that $\mathcal{B}\Box_{i_!\mathcal{A}} i_!\mathcal{C}\Box_{i_!\mathcal{A}}\mathcal{B}$ is a monoid in ${}_{\mathcal{B}}Mod_{\mathcal{B}}$. There is a canonical isomorphism

$$i_!\mathcal{C}\Box_{i_!\mathcal{A}} i_!\mathcal{C} \cong i_!\mathcal{C}\Box_{i_!\mathcal{A}}\mathcal{B}\Box_{i_!\mathcal{A}} i_!\mathcal{C}$$

of $(i_!\mathcal{A}, i_!\mathcal{A})$-bimodules which is best seen pointwise, using coends. This provides a multiplication for $\mathcal{B}\Box_{i_!\mathcal{A}} i_!\mathcal{C}\Box_{i_!\mathcal{A}}\mathcal{B}$ which is again best seen to be associative by working pointwise, using coends. To define a multiplication for $\mathcal{D}$ consider the cube diagrams

$$\begin{tikzcd}[column sep=small]
\mathcal{B}\cdot i_!\mathcal{A}\cdot\mathcal{B}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{A}\cdot\mathcal{B} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{B}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{A}\cdot\mathcal{B} \arrow[dr] \arrow[dd] & \\
& \mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{A}\cdot\mathcal{B} \arrow[rr, crossing over] & & \mathcal{D}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{A}\cdot\mathcal{B} \arrow[dd] \\
\mathcal{B}\cdot i_!\mathcal{A}\cdot\mathcal{B}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B} \arrow[rr] \arrow[dr] & & \mathcal{B}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B} \arrow[dr] & \\
& \mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B} \arrow[rr] \arrow[from=uu, crossing over] & & \mathcal{D}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B}
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small]
\mathcal{B}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{A}\cdot\mathcal{B} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{B}\cdot_{\mathcal{B}}\mathcal{B} \arrow[dr] \arrow[dd] & \\
& \mathcal{D}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{A}\cdot\mathcal{B} \arrow[rr, crossing over] & & \mathcal{D}\cdot_{\mathcal{B}}\mathcal{B} \arrow[dd] \\
\mathcal{B}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B} \arrow[rr] \arrow[dr] & & \mathcal{B}\cdot_{\mathcal{B}}\mathcal{D} \arrow[dr] & \\
& \mathcal{D}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B} \arrow[rr] \arrow[from=uu, crossing over] & & \mathcal{D}\cdot_{\mathcal{B}}\mathcal{D}
\end{tikzcd}$$

For space considerations we have suppressed tensors (always over $i_!\mathcal{A}$, unless explicitly indicated) from notation. The right face of the first cube is the same as the left face of the latter cube. Let PO_1 (resp. PO_2) be the pushout of the left (resp. right) face of the first cube diagram. Let PO_3 be the pushout of the right face of the second cube diagram. We have pushout diagrams

$$\begin{tikzcd}
PO_1 \arrow[r] \arrow[d] & PO_2 \arrow[r] \arrow[d] & PO_3 \arrow[d] \\
\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B} \arrow[r] & \mathcal{D}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B} \arrow[r] & \mathcal{D}\cdot_{\mathcal{B}}\mathcal{D}
\end{tikzcd}$$

Using these pushouts and the fact that $\mathcal{B}\Box_{i_!\mathcal{A}} i_!\mathcal{C}\Box_{i_!\mathcal{A}}\mathcal{B}$ is a monoid one can define in a canonical way a map $\mu\colon \mathcal{D}\cdot_{\mathcal{B}}\mathcal{D}\to\mathcal{D}$. We omit the long verification that μ gives $\mathcal{D}$ the structure of a monoid. The map μ was constructed in such a way that m becomes a morphism of monoids. The fact that $\mathcal{D}$ has the universal property of the pushout in the category $\mathcal{V}$-$\mathbf{Cat}(Ob(\mathcal{B}))$ follows from its definition.

Case 3: f is arbitrary. Let $u = Ob(f)$. We factorize f as $\mathcal{A}\xrightarrow{f^u} u^*\mathcal{C}\to\mathcal{C}$, where $Ob(u^*\mathcal{C}) = Ob(\mathcal{A})$, $u^*\mathcal{C}(a,a') = \mathcal{C}(fa,fa')$ and f^u is the obvious map, and take consecutive pushouts:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "i"] \arrow[d, "f^u"'] & \mathcal{B} \arrow[d] \\
u^*\mathcal{C} \arrow[r] \arrow[d] & \mathcal{A}' \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

Now apply Case 2 to f^u and Case 1 to $u^*\mathcal{C}\to\mathcal{C}$. ■

5. Application: left Bousfield localizations of categories of monoids

This section was motivated by the paragraph 'As we mentioned above,...in general.' on page 111 of [12].

5.1. The problem. Let $\mathcal{M}$ be a (suitable) monoidal model category, $\mathrm{L}\mathcal{M}$ a left Bousfield localization of $\mathcal{M}$ which is itself a monoidal model category and $Mon(\mathcal{M})$ the category of monoids in $\mathcal{M}$. The problem is to induce on $Mon(\mathcal{M})$ a model category structure somehow related to $\mathrm{L}\mathcal{M}$. As pointed out in [12], such a model structure exists if, for example, (a) $\mathrm{L}\mathcal{M}$ satisfies the monoid axiom or (b) $Mon(\mathcal{M})$ has a suitable left proper model category structure. In order for (a) to be fulfilled one needs to know the (generating) trivial cofibrations of $\mathrm{L}\mathcal{M}$. However, it often happens that one does not have an explicit description of them. For (b), the category of monoids in a monoidal model category is rarely known to be left proper (it is left proper when the underlying model category has all objects cofibrant, for instance, which seems to us too restrictive to work with).

5.2. Our solution. We shall propose below a solution to the above problem. We shall reduce the verification of the monoid axiom for $\mathrm{L}\mathcal{M}$ to a smaller— and hopefully more tractable in practice, set of maps and we shall avoid left properness by using Theorem 1.1 via Proposition 2.3. The model category theoretical framework will be the 'combinatorial' counterpart of the one of [12, Section 8].

It will be clear that the method could potentially be applied to other structures than monoids.

Recollections on enriched left Bousfield localization. We recall some facts from [1]. Let $\mathcal{V}$ be a monoidal model category and $\mathcal{M}$ a model $\mathcal{V}$-category with tensor, hom and cotensor denoted by

$$- * -\colon \mathcal{V} \times \mathcal{M} \to \mathcal{M}$$

$$Map(-,-)\colon \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{V}$$

$$(-)^{(-)}\colon \mathcal{V}^{op} \times \mathcal{M} \to \mathcal{M}$$

Let S be a set of maps of $\mathcal{M}$ between cofibrant objects.

5.3. Definition. A fibrant object W of $\mathcal{M}$ is *S-local* if for every $f \in S$ the map $Map(f, W)$ is a weak equivalence of $\mathcal{V}$. A map f of $\mathcal{M}$ is an *S-local equivalence* if for every S-local object W and for some (hence any) cofibrant approximation $\tilde{f}$ to f, the map $Map(\tilde{f}, W)$ is a weak equivalence of $\mathcal{V}$.

In the previous definition, if the map $Map(\tilde{f}, W)$ is a weak equivalence of $\mathcal{V}$, then for any other cofibrant approximation $\tilde{g}$ to f, the map $Map(\tilde{g}, W)$ is a weak equivalence of $\mathcal{V}$ [9, Proposition 14.6.6(1)]. Recall the following result of C. Barwick [1].

5.4. Theorem. *Let $\mathcal{V}$ be a combinatorial monoidal model category, $\mathcal{M}$ a left proper, combinatorial model $\mathcal{V}$-category and S a set of maps of $\mathcal{M}$ between cofibrant objects. Suppose that $\mathcal{V}$ has a set of generating cofibrations with cofibrant domains.*

Then the category $\mathcal{M}$ admits a left proper, combinatorial model category structure, denoted by $\mathrm{L}_S\mathcal{M}$, with the class of S-local equivalences as weak equivalences and the same cofibrations as the given ones. The fibrant objects of $\mathrm{L}_S\mathcal{M}$ are the S-local objects. $\mathrm{L}_S\mathcal{M}$ is a model $\mathcal{V}$-category.

Suppose that, moreover, $\mathcal{M}$ is a monoidal model $\mathcal{V}$-category which has a set of generating cofibrations with cofibrant domains. Let us denote by $\otimes$ the monoidal product on $\mathcal{M}$. If $X \otimes f$ is an S-local equivalence for every $f \in S$ and every X belonging to the domains and codomains of the generating cofibrations of $\mathcal{M}$, then $\mathrm{L}_S\mathcal{M}$ is a monoidal model $\mathcal{V}$-category.

The S-extended monoid axiom. Let $\mathcal{V}$ be a monoidal model category and $\mathcal{M}$ a monoidal model $\mathcal{V}$-category with monoidal product $\otimes$ and tensor, hom and cotensor denoted as in Section 5.2. If $i\colon K \to L$ is a map of $\mathcal{V}$ and $f\colon A \to B$ is a map of $\mathcal{M}$, we denote by $i *' f$ the canonical map

$$L * A \sqcup_{K*A} K * B \to L * B$$

Let S be a set of maps of $\mathcal{M}$ between cofibrant objects. For every $f \in S$, let $f = v_f u_f$ be a factorization of f as a cofibration u_f followed by a weak equivalence v_f; a concrete one is the mapping cylinder factorization.

5.5. Definition. We say that $\mathcal{M}$ satisfies the *S-extended monoid axiom* if, in the notation of [14, Section 3], every map in

$$(\{\text{trivial cofibrations of } \mathcal{M}\} \cup (\{\text{cofibrations of } \mathcal{V}\} *' u_f)_{f\in S}) \otimes \mathcal{M}\text{-cof}_{reg}$$

is an S-local equivalence.

As usual [14, Lemma 3.5(2)], if $\mathcal{V}$ and $\mathcal{M}$ are cofibrantly generated and every map in

$$(\{\text{generating trivial cofibrations of } \mathcal{M}\} \cup (\{\text{generating cofibrations of } \mathcal{V}\} *' u_f)_{f\in S}) \otimes \mathcal{M}\text{-cof}_{reg}$$

is an S-local equivalence, then the S-extended monoid axiom holds.

Let $Mon(\mathcal{M})$ be the category of monoids in $\mathcal{M}$ and let

$$T\colon \mathcal{M} \rightleftarrows Mon(\mathcal{M})\colon U$$

be the free-forgetful adjunction.

5.6. Definition. A monoid M in $\mathcal{M}$ is *TS-local* if $U(M)$ is S-local. A map f of monoids in $\mathcal{M}$ is a *TS-local equivalence* if $U(f)$ is an S-local equivalence.

5.7. Theorem. *Let $\mathcal{V}$ be a combinatorial monoidal model category having a set of generating cofibrations with cofibrant domains. Let $\mathcal{M}$ be a left proper, combinatorial monoidal model $\mathcal{V}$-category which has a set of generating cofibrations with cofibrant domains. Let us denote by $\otimes$ the monoidal product on $\mathcal{M}$. Let S be a set of maps of $\mathcal{M}$ between cofibrant objects. Suppose that $X \otimes f$ is an S-local equivalence for every $f \in S$ and every X belonging to the domains and codomains of the generating cofibrations of $\mathcal{M}$ and that $\mathcal{M}$ satisfies the S-extended monoid axiom.*

Then the category $Mon(\mathcal{M})$ admits a combinatorial model category structure with TS-local equivalences as weak equivalences and with $T(\{\text{cofibrations of } \mathcal{M}\})$ *as cofibrations.The fibrant objects are the TS-local objects.*

Proof. We shall apply Proposition 2.3. We take $\mathcal{E}$ to be $Mon(\mathcal{M})$, W to be the class of TS-local equivalences, I to be the set $T(\{\text{generating cofibrations of } \mathcal{M}\})$ and J to be

$$T(\{\text{generating trivial cofibrations of } \mathcal{M}\} \cup (\{\text{generating cofibrations of } \mathcal{V}\} *' u_f)_{f\in S})$$

Notice that a map g of monoids in $\mathcal{M}$ belongs to $\text{inj}(T(\{\text{generating cofibrations of } \mathcal{M}\}))$ if and only if $U(g)$ belongs to $\text{inj}(\{\text{generating cofibrations of } \mathcal{M}\})$ if and only if $U(g)$ is a trivial fibration of $\mathcal{M}$. Therefore condition C2 from Proposition 2.3 holds.

We claim that a monoid M in $\mathcal{M}$ is naively fibrant if and only if M is TS-local. We may assume without loss of generality that $U(M)$ is fibrant. We observe that if i is any map of $\mathcal{V}$ and $f \in S$, then M has the right lifting property with respect to $T(i *' u_f)$ if and only if $Map(u_f, U(M))$ has the right lifting property with respect to i. Since $Map(v_f, U(M))$ is a weak equivalence of $\mathcal{V}$ and $Map(u_f, U(M))$ is a fibration of $\mathcal{V}$, the claim follows from this observation.

Let now g be a map of monoids in $\mathcal{M}$ between TS-local monoids such that g is both a TS-local equivalence and a naive fibration. Then $U(g)$ is an S-local equivalence between S-local objects, so $U(g)$ is a weak equivalence. $U(g)$ is also a fibration, therefore condition C5 from Proposition 2.3 holds.

Condition C4 from Proposition 2.3 is guaranteed by the S-extended monoid axiom and [14, Proof of Lemma 6.2]. ∎

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