

TRACED *-AUTONOMOUS CATEGORIES ARE COMPACT CLOSED

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ABSTRACT. We show that any traced *-autonomous category is compact closed.

1. Introduction

Suppose that $\mathbb{C} = (\mathbb{C}, I, \otimes, -\circ, \perp, Tr)$ is a traced *-autonomous category; here we understand that a *-autonomous category is a symmetric monoidal closed category $(\mathbb{C}, I, \otimes, -\circ)$ (we write $A -\circ B$ for the internal hom from A to B) equipped with a dualizing object \perp [Barr, 1979], and that the trace

$$Tr_{A,B}^X : \mathbb{C}(A \otimes X, B \otimes X) \longrightarrow \mathbb{C}(A, B)$$

is given on the symmetric monoidal structure in the sense of Joyal, Street and Verity [Joyal *et al.*, 1996] (rather than the trace for linearly distributive categories with MIX by Blute, Cockett and Seely [Blute *et al.*, 2000]).

In \mathbb{C} , we have the trace of the evaluation map

$$\frac{(X -\circ (\perp \otimes X)) \otimes X \xrightarrow{ev} \perp \otimes X}{X -\circ (\perp \otimes X) \xrightarrow{Tr^X ev} \perp}$$

which gives rise to a morphism $t_X : I \longrightarrow X \otimes (X -\circ I)$ via the isomorphism

$$(X -\circ (\perp \otimes X)) -\circ \perp \simeq X \otimes (X -\circ I).$$

It is then natural to ask if t_X satisfies the equations

$$X \xrightarrow{t_X \otimes X} X \otimes (X -\circ I) \otimes X \xrightarrow{X \otimes ev_{X,I}} X = id_X \tag{1}$$

and

$$X -\circ I \xrightarrow{(X -\circ I) \otimes t_X} (X -\circ I) \otimes X \otimes (X -\circ I) \xrightarrow{ev_{X,I} \otimes (X -\circ I)} X -\circ I = id_{X -\circ I} \tag{2}$$

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which mean that $X \multimap I$ is a (left) dual of X , hence \mathbb{C} is compact closed [Kelly and Laplaza, 1980]. Below we see that this is the case.

Before proceeding to the proof, let us explain how we came across this observation. In the sequel, we write $\widehat{f} : A \rightarrow B \multimap C$ for the transpose of $f : A \otimes B \rightarrow C$ in a symmetric monoidal closed category. In a traced symmetric monoidal closed category, we have a family of morphisms

$$\tau_B^X = \text{Tr}_{X \multimap (B \otimes X), B}^X(\text{ev}_{X, B \otimes X}) : X \multimap (B \otimes X) \rightarrow B.$$

It is easy to see that τ 's are sufficient to determine trace of any $f : A \otimes X \rightarrow B \otimes X$ as

$$\text{Tr}_{A, B}^X f = A \xrightarrow{\widehat{f}} X \multimap (B \otimes X) \xrightarrow{\tau_B^X} B.$$

In the case of traced *-autonomous categories, we can further restrict our attention to τ 's with $B = \perp$, from which τ_B^X for any B is recovered as

$$\begin{aligned} X \multimap (B \otimes X) &\simeq X \multimap ((B \wp \perp) \otimes X) \\ &\xrightarrow{X \multimap \delta} X \multimap (B \wp (\perp \otimes X)) \\ &\simeq B \wp (X \multimap (\perp \otimes X)) \\ &\xrightarrow{B \wp \tau_{\perp}^X} B \wp \perp \\ &\simeq B \end{aligned}$$

where δ denotes linear distributivity [Cockett and Seely, 1997].

Therefore, giving $\tau_{\perp}^X : (X \multimap (\perp \otimes X)) \rightarrow \perp$ for each X is enough to determine a trace. With some efforts of spelling out how the trace can be recovered directly from τ_{\perp}^X 's, we noticed that τ_{\perp}^X actually determines the unit map $t_X : I \rightarrow X \otimes (X \multimap I)$ of the duality between X and $X \multimap I$.

To make the proof short and readable, we use some basic results on (enriched) extraordinary natural transformations of Eilenberg and Kelly [Eilenberg and Kelly, 1966, Kelly, 1982], though it is also possible to derive the result by direct calculation from scratch.

2. A Characterization of Compact Closedness

There are many ways of characterizing compact closed categories as special symmetric monoidal closed categories [Day, 1977, Kelly and Laplaza, 1980]. In our development, the following characterization turns out to be useful:

2.1. PROPOSITION. *Suppose that \mathbb{C} is a symmetric monoidal closed category such that there is an extraordinary \mathbb{C} -natural transformation $t_X : I \rightarrow X \otimes (X \multimap I)$ with t_I invertible. Then \mathbb{C} is compact closed.*

PROOF. Recall that, for a symmetric monoidal closed category \mathbb{V} , \mathbb{V} -categories \mathbb{A} , \mathbb{B} , a \mathbb{V} -functor $F : \mathbb{A}^{\text{op}} \otimes \mathbb{A} \rightarrow \mathbb{B}$ and an object B of \mathbb{B} , a family of morphisms $\alpha_X : B \rightarrow F(X, X)$ is said to be an extraordinary \mathbb{V} -natural transformation [Kelly, 1982] when the following diagram commutes for all X, X' .

$$\begin{array}{ccc}
 & \mathbb{A}(X, X') & \\
 F(X, -) \swarrow & & \searrow F(-, X') \\
 \mathbb{B}(F(X, X), F(X, X')) & & \mathbb{B}(F(X', X'), F(X, X')) \\
 \mathbb{B}(\alpha_X, id) \searrow & & \swarrow \mathbb{B}(\alpha_{X'}, id) \\
 & \mathbb{B}(B, F(X, X')) &
 \end{array}$$

In the proposition, the assumption that $t_X : I \rightarrow X \otimes (X \multimap I)$ is extraordinarily \mathbb{C} -natural means that the following diagram commutes for all X and X' :

$$\begin{array}{ccc}
 & X \multimap X' & \\
 (X \multimap X') \otimes t_X \swarrow & & \searrow t_{X'} \otimes (X \multimap X') \\
 (X \multimap X') \otimes X \otimes (X \multimap I) & & X' \otimes (X' \multimap I) \otimes (X \multimap X') \\
 ev \otimes (X \multimap I) \searrow & & \swarrow X' \otimes comp \\
 & X' \otimes (X \multimap I) &
 \end{array}$$

By letting X be I in the diagram above, we see that (modulo some obvious simplifications)

$$X' \xrightarrow{t_{X'} \otimes X'} X' \otimes (X' \multimap I) \otimes X' \xrightarrow{X' \otimes ev} X'$$

agrees with

$$X' \xrightarrow{X' \otimes t_I} X' \otimes I \otimes (I \multimap I) \xrightarrow{\simeq} X'.$$

Similarly, by letting X' be I in the diagram, we have that

$$X \multimap I \xrightarrow{(X \multimap I) \otimes t_X} (X \multimap I) \otimes X \otimes (X \multimap I) \xrightarrow{ev \otimes (X \multimap I)} X \multimap I$$

agrees with

$$X \multimap I \xrightarrow{t_I \otimes (X \multimap I)} I \otimes (I \multimap I) \otimes (X \multimap I) \xrightarrow{\simeq} X \multimap I.$$

Hence

$$t'_X = I \xrightarrow{\simeq} I \otimes (I \multimap I) \xrightarrow{t_I^{-1}} I \xrightarrow{t_X} X \otimes (X \multimap I)$$

satisfies the equations (1) and (2) for making $X \multimap I$ a left dual of X . ■

Note that, in the proof above, t'_X agrees with t_X if $t_I : I \rightarrow I \otimes (I \multimap I)$ itself is the canonical isomorphism from I to $I \otimes (I \multimap I)$.

2.2. REMARK. In Proposition 2.1, the assumption that t_I is invertible cannot be dropped. For instance, consider the category $\omega\mathbf{Cppo}_\perp$ of pointed ω -complete partial orders and strict ω -continuous functions. $\omega\mathbf{Cppo}_\perp$ is symmetric monoidal closed, with Sierpinski space as the unit object, smash products as tensor and strict function spaces as internal hom. In $\omega\mathbf{Cppo}_\perp$, there is an extraordinary $\omega\mathbf{Cppo}_\perp$ -natural transformation $t_X : I \rightarrow X \otimes (X \multimap I)$ given by the constant functions returning the least element. However, t_I is not invertible, and $\omega\mathbf{Cppo}_\perp$ is not compact closed.

3. Proof of the Main Result

As in the introduction, let us define

$$\tau_B^X = \text{Tr}_{X \multimap (B \otimes X), B}^X(\text{ev}_{X, B \otimes X}) : X \multimap (B \otimes X) \rightarrow B$$

in traced symmetric monoidal closed categories.

3.1. LEMMA. *In a traced symmetric monoidal closed category \mathbb{C} with an object B ,*

$$\widehat{\tau}_B^X : I \rightarrow (X \multimap (B \otimes X)) \multimap B$$

is extraordinary \mathbb{C} -natural in X .

PROOF. The extranaturality amounts to the commutativity of

$$\begin{array}{ccc}
 & (X \multimap X') \otimes (X' \multimap (B \otimes X)) & \\
 \text{id} \otimes \widehat{\tau}_B^X \swarrow & & \searrow \text{id} \otimes \widehat{\tau}_B^{X'} \\
 (X \multimap X') \otimes (X' \multimap (B \otimes X)) \otimes ((X \multimap (B \otimes X)) \multimap B) & & (X \multimap X') \otimes (X' \multimap (B \otimes X)) \otimes ((X' \multimap (B \otimes X')) \multimap B) \\
 \text{comp} \otimes \text{id} \downarrow & & \downarrow \text{comp} \otimes \text{id} \\
 (X \multimap (B \otimes X)) \otimes ((X \multimap (B \otimes X)) \multimap B) & & (X' \multimap (B \otimes X')) \otimes ((X' \multimap (B \otimes X')) \multimap B) \\
 \text{ev} \searrow & & \swarrow \text{ev} \\
 & B &
 \end{array}$$

which is a consequence of the sliding property (dinaturality) of trace. ■

3.2. LEMMA. *In a $*$ -autonomous category \mathbb{C} , the isomorphism*

$$\varphi_{X,Y} : (X \multimap (\perp \otimes Y)) \multimap \perp \xrightarrow{\cong} X \otimes (Y \multimap I)$$

given by

$$\begin{aligned} (X \multimap (\perp \otimes Y)) \multimap \perp &\simeq (X \multimap (((\perp \otimes Y) \multimap \perp) \multimap \perp)) \multimap \perp \\ &\simeq (X \multimap ((Y \multimap (\perp \multimap \perp)) \multimap \perp)) \multimap \perp \\ &\simeq (X \multimap ((Y \multimap I) \multimap \perp)) \multimap \perp \\ &\simeq ((X \otimes (Y \multimap I)) \multimap \perp) \multimap \perp \\ &\simeq X \otimes (Y \multimap I) \end{aligned}$$

is \mathbb{C} -natural in X and Y .

PROOF. Each isomorphism involved in φ is \mathbb{C} -natural. ■

3.3. LEMMA. [Eilenberg and Kelly, 1966, Kelly, 1982] *Assume that \mathbb{V} is a symmetric monoidal closed category. Let \mathbb{A}, \mathbb{B} be \mathbb{V} -categories and G, H be \mathbb{V} -functors of the form $\mathbb{A}^{\text{op}} \otimes \mathbb{A} \rightarrow \mathbb{B}$ and suppose that B is an object of \mathbb{B} . If $\alpha_X : B \rightarrow G(X, X)$ is extraordinary \mathbb{V} -natural in X and $\beta_{X,Y} : G(X, Y) \rightarrow H(X, Y)$ is \mathbb{V} -natural in X and Y , then*

$$\beta_{X,X} \circ \alpha_X : B \rightarrow H(X, X)$$

is extraordinary \mathbb{V} -natural in X .

3.4. COROLLARY. *In a traced $*$ -autonomous category \mathbb{C} , $t_X = \varphi_{X,X} \circ \widehat{\tau}_\perp^X : I \rightarrow X \otimes (X \multimap I)$ is extraordinary \mathbb{C} -natural in X . \square*

3.5. LEMMA. $t_I : I \rightarrow I \otimes (I \multimap I)$ agrees with the canonical isomorphism from I to $I \otimes (I \multimap I)$.

PROOF. A consequence of the vanishing property of trace. ■

Putting Proposition 2.1, Corollary 3.4 and Lemma 3.5 together, we obtain our main result.

3.6. THEOREM. *Any traced $*$ -autonomous category is compact closed.*

It is possible that a compact closed category is equipped with a dualizing object which is not isomorphic to the unit object (and par not isomorphic to tensor). For instance, the linearly ordered set of integers \mathbb{Z} is compact closed with unit $I = 0$ and tensor $X \otimes Y = X + Y$ and duality $X^* = -X$, while any element of \mathbb{Z} serves as a dualizing object. (The same can be done for any partially ordered Abelian group regarded as a compact closed poset.)

Since a compact closed category has a unique trace (cf. [Hasegawa, 2009]), we have:

3.7. THEOREM. *To give a traced $*$ -autonomous category is to give a compact closed category with a dualizing object.*

Note that a dualizing object in a compact closed category is just an object \perp such that the unit morphism $I \rightarrow \perp \otimes \perp^*$ is invertible, cf. the Abelian group example above.

4. On Linear Distributivity

In a compact closed category with a dualizing object \perp , linear distributivity [Cockett and Seely, 1997] on the $*$ -autonomous structure is invertible. To see this, recall that the linear distributivity $\delta : (A \wp B) \otimes C \longrightarrow A \wp (B \otimes C)$ in a $*$ -autonomous category (regarded as a symmetric linearly distributive category with negation) amounts to the canonical morphism $(A^\perp \multimap B) \otimes C \longrightarrow A^\perp \multimap (B \otimes C)$ which is just the associativity isomorphism $((A^\perp)^* \otimes B) \otimes C \simeq (A^\perp)^* \otimes (B \otimes C)$ in a compact closed category.

Conversely, a $*$ -autonomous category with invertible linear distributivity is compact closed. We have

$$A \multimap B \simeq A^\perp \wp B \simeq A^\perp \wp (I \otimes B) \stackrel{\delta^{-1}}{\simeq} (A^\perp \wp I) \otimes B \simeq (A \multimap I) \otimes B$$

In particular, the canonical map $(A \multimap I) \otimes A \longrightarrow A \multimap A$ is invertible, and it follows that the category is compact closed (cf. [Day, 1977]).

Together with Theorem 3.7, we have that the following three structures are essentially the same:

- a traced $*$ -autonomous category,
- a compact closed category equipped with a dualizing object, and
- a $*$ -autonomous category with invertible linear distributivity.

4.1. REMARK. As noted in [Cockett and Seely, 1997], in a symmetric linearly distributive category with invertible linear distributivity and also equipped with a tensor-inverse of \perp (an object \perp^* such that there is an isomorphism $I \simeq \perp \otimes \perp^*$ subject to a coherence axiom), the par $A \wp B$ is isomorphic to the “ \perp -shifted tensor” $A \otimes \perp^* \otimes B$. This is the case for $*$ -autonomous categories with invertible linear distributivity (equivalently: traced $*$ -autonomous categories, or compact closed categories with a dualizing object), in which $\perp^* = \perp \multimap I$ serves as a tensor-inverse of \perp and we have $A \wp B \simeq A \otimes (\perp \multimap I) \otimes B$.

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