Theory and Applications of Categories, Vol. 28, No. 12, 2013, pp. 304–331.

N-TUPLE GROUPOIDS AND OPTIMALLY COUPLED FACTORIZATIONS

DANY MAJARD

Abstract. In this paper, we prove that the category of vacant n-tuple groupoids is equivalent to the category of factorizations of groupoids by n factors that satisfy some Yang-Baxter type equation. Moreover we extend this equivalence to the category of maximally exclusive n-tuple groupoids, which we define, by dropping one assumption. The paper concludes by a note on how these results could tell us more about some Lie groups of interest.

Introduction

Extensions and factorizations of groups are important aspects of group theory as they allow us to understand a group by knowing some of its constituents and how they fit together. Agore and Militaru give a good overview of it in [1] but as a quick reminder, a factorization of a group G (by two factors) is given by two subgroups H and K such that the map of sets $H \times K \to G$ induced by multiplication is invertible. In other words, every element $g \in G$ can be uniquely written as the product hk for some elements $h \in H$ and $k \in K$. In general a factorization is explicitly written :

$$G = HK$$

The relationship between factorizations of groups and double groups has been known for quite some time and carries to smash products of Hopf algebras for the $\mathbb{K}$-linear case, as shown in Majid's book [13], Natale and Andruskiewitsch [2] or more recently in Mackenzie's article [11].
Our point of view being that groupoids are more fundamental than groups, we consider that factorizations of groupoids is what ought to be understood. Nonetheless, the article can safely be read replacing every "groupoid" by "group". Natale and Andruskiewitsch give a complete description of the structure of double groupoids and their relations to factorizations of groupoids by 2 factors in [3]. The aim of this paper is to extend this results to multiple factorizations and n-tuple groupoids, as much as possible. It is written as an answer to a note written by Brown [4] speculating on triple groups and matched triples of groups. Moreover following our previous article [12], we introduce new definitions,

Research supported by the Eduard Čech Institute for Algebra, Geometry and Physics.
Received by the editors 2012-05-01 and, in revised form, 2013-05-16.
Transmitted by Ronald Brown. Published on 2013-05-30.
2010 Mathematics Subject Classification: 18D05,18A32,18B40.
Key words and phrases: factorization, groupoid, cubical categories.
© Dany Majard, 2013. Permission to copy for private use granted.

that of maximal and exclusive n-tuple groupoids, which allow a wider class of n-tuple groupoids to be analyzed. We show that n-tuple groupoids belonging to both classes can represent further factorizations of groupoids and show how some semi-simple Lie groups may provide prime examples of these.
Since the first draft, it was brought to our attention that factorizations of categories into two factors may be understood as distributive laws, as shown in an article of Rosebrugh and Wood [15]. To our understanding the results are equivalent in the case of groupoids, though our approach is more straightforward and geometric. It would be interesting to investigate this connection for the case with three or more factors but it is out of the realm of this paper.

1. n-tuple categories

Let [n] be the set of integers from 1 to n, and for a nonempty subset I of $[n]$ define $\hat{I} := [n] \setminus I$. By abuse of notation braces will be omitted in subscripts, for example $n_{ij} := n_{\{ij\}}$ and $n_{\hat{i}} = n_{[n]\setminus\{i\}}$. Importantly **composition will be written in diagrammatic order**, i.e. fg will mean f first and then g, which is more classically written $g \circ f$.

1.1. Definition. *A n-**tuple category** $\mathfrak{C}$ is the collection of:*

Sets $\{\mathfrak{C}_I\}, \quad \forall I \subseteq [n]$

Maps $\{s_{J,I}\}, \{t_{J,I}\} \, \forall I \subseteq J \subseteq [n]$ *nonempty*

Maps $\{\circ_{I,i}\} \, \forall i \in I \subseteq [n]$

Maps $\{\imath_{J,I}\} \, \forall I, J \subseteq [n]$ *s.t.* $I \cap J = \emptyset$

where :

- *The following maps of sets are the source and target maps:*

$$s_{J,I}, t_{J,I} \, : \mathfrak{C}_J \to \mathfrak{C}_{J\setminus I} \quad \forall I \subseteq J \subseteq [n]$$

  The pullback of the pair $(t_{J,i},\, s_{J,i})$ is denoted by $\mathfrak{C}_J \underset{i}{\times} \mathfrak{C}_J$.

- *The following maps of sets are the identity maps:*

$$\imath_{J,I} \, : \mathfrak{C}_J \to \mathfrak{C}_{J\cup I} \quad \forall I, J \subseteq [n] \text{ s.t. } I \cap J = \emptyset$$

- *The following maps of sets are the composition maps:*

$$\circ_{I,i} : \, \mathfrak{C}_I \underset{i}{\times} \mathfrak{C}_I \to \mathfrak{C}_I \quad \forall I \text{ containing } i$$

- *Sources, targets and compositions are compatible the following way:*

$$s_{J,I}s_{J\setminus I,K} = s_{J,I\cup K} \qquad t_{J,I}t_{J\setminus I,K} = t_{J,I\cup K}$$
$$(s_{J,I} \underset{j}{\times} s_{J,I})\circ_{J\setminus I,j} = \circ_{J,j}s_{J,I} \qquad (t_{J,I} \underset{j}{\times} t_{J,I})\circ_{J\setminus I,j} = \circ_{J,j}t_{J,I}$$

- *Compositions are associative and satisfy the interchange laws:*

$$(\circ_{J,i} \underset{j}{\times} \circ_{J,i})\circ_{J,j} = (\circ_{J,j} \underset{i}{\times} \circ_{J,j})\circ_{J,i} \quad \forall i \neq j \in J \subseteq [n]$$

- *$\imath_{J,i}$ is an identity for the composition $\circ_{J,i}$ and $\circ_{J,j}(\imath_{J,I}) = (\imath_{J,I} \underset{j}{\times} \imath_{J,I})\circ_{J\cup I,j}$ for all $j \notin I$.*

The elements of $\mathfrak{C}_\emptyset$, $\{\mathfrak{C}_I\}$ for $I \subset [n]$ non empty and $\mathfrak{C}_{[n]}$ are respectively called objects, faces and n-cubes. The first subscript of the maps is generally omitted, so $\circ_{J,i}$ would just be written $\circ_i$, $s_{J,I}$ would be written s_I etc...

1.2. Lemma. *The interchange laws impose $\imath_i\imath_j = \imath_j\imath_i$, and therefore we can coherently define $\imath_{J\cup I} := \imath_I\imath_J$. Also for $i \neq j$, $\imath_i s_j = s_j\imath_i$.*

Proof. Using the fact that $\imath_i$ is an identity for $\circ_i$ we find that:

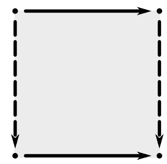
$$\begin{aligned}\Big(\big((\imath_i\imath_j) \underset{i}{\times} (\imath_j\imath_i)\big) \underset{j}{\times} \big((\imath_j\imath_i) \underset{i}{\times} (\imath_i\imath_j)\big)\Big)(\circ_i \underset{j}{\times} \circ_i)\circ_j &= \big((\imath_i\imath_j) \underset{j}{\times} (\imath_i\imath_j)\big)\circ_j \\ &= \imath_i\imath_j\end{aligned}$$

Yet using the interchange law and the fact that $\imath_j$ is an identity for $\circ_j$ we find that:

$$\begin{aligned}&\Big(\big((\imath_i\imath_j) \underset{i}{\times} (\imath_j\imath_i)\big) \underset{j}{\times} \big((\imath_j\imath_i) \underset{i}{\times} (\imath_i\imath_j)\big)\Big)(\circ_i \underset{j}{\times} \circ_i)\circ_j = \\ &\Big(\big((\imath_i\imath_j) \underset{j}{\times} (\imath_j\imath_i)\big) \underset{i}{\times} \big((\imath_j\imath_i) \underset{j}{\times} (\imath_i\imath_j)\big)\Big)(\circ_j \underset{i}{\times} \circ_j)\circ_i = \big((\imath_j\imath_i) \underset{i}{\times} (\imath_j\imath_i)\big)\circ_i \\ &\qquad = \imath_j\imath_i\end{aligned}$$

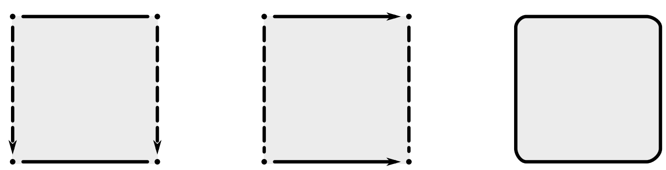
which shows that they are indeed equal. As for the second statement, we can use the uniqueness property of identities of associative binary operations. And for the last claim, since the source and composition maps intertwine, the j^{th} source of the i^{th} identity behaves like an i^{th} identity, uniqueness therefore shows that it is one. ∎

There exists a much more concise statement of these axioms that shrinks the whole definition to "An n-tuple category is an internal category in the category of $(n-1)$-tuple categories", cf [9]. To limit the amount of prerequisites, we preferred the direct approach. It should be noted also that for small integers, the denomination takes slightly different but obvious forms. For example a 2-tuple category is usually referred to as a double category, a 3-tuple category as a triple category etc...
Visually we can represent the elements of a n-tuple category as n-cubes with "oriented faces". The source and target maps encode their faces, which themselves form i-tuple categories. For example a general element of a double category is a square:

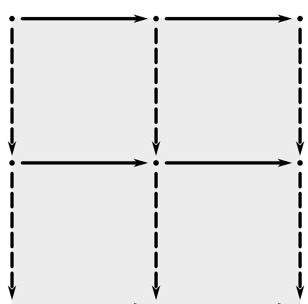


where arrows of the first type have been represented as plain, arrows of the second type in dashes and the objects have been represented by dots. From here on, **dots will represent possibly distinct objects** in the n-tuple category of choice. Note that there is no defined composition between arrows of different types, so there is generally no sense in talking of a square being commutative. Note as well that there can be many squares with the same boundary.

Graphically, two squares are composable in a given direction if they can be pasted next to each other in that direction. We will take the convention to **draw identities arrows as segments**, regardless of their type. Identity squares therefore look like:



The first two are identities on arrows, given respectively by $\imath_1$ and $\imath_2$, whereas the third is an identity on objects, given by $\imath_{12}$. The interchange law ensures that any assortment of the sort:



yields the same square regardless of the order in which it is composed. It is a good exercise to prove that $\imath_1\imath_2 = \imath_2\imath_1$ pictorially, using the interchange law. It is a good idea as well to read R.Paré and R.Dawson's article [14] on composability of plane diagrams in double categories.

General elements of a triple category are cubes that compose by pasting in one of the three directions. The generalization to higher dimension is straightforward, though representation on paper becomes challenging. While dealing with n-cubes, two notions are worth defining to make proofs more comprehensible. They are fairly transparent.

1.3. Definition. *Let X be an n-cube in a certain n-category. Then $s_{[n]}(X)$ is called the* ***source*** *of X and $t_{[n]}(X)$ its* ***sink****.*

The source is the only object in the cube that arrows only depart from and the sink is the only object that arrows only point towards.
The available notions of higher morphisms of n-tuple categories are many and we will not in this paper define all the higher menagerie. We will though define the corresponding notion of a functor.

1.4. Definition. *Let $\mathfrak{C}$ and $\mathfrak{C}'$ be n-tuple categories. An* ***n-tuple functor*** *$F : \mathfrak{C} \to \mathfrak{C}'$ is a collection $\{F_I : \mathfrak{C}_I \to \mathfrak{C}'_I\}$ of maps of sets such that:*

$$\begin{aligned}(F_I \underset{i}{\times} F_I)\circ'_i &= \circ_i F_I \\ F_I s'_i &= s_i F_{I\setminus\{i\}} \\ F_I t'_i &= t_i F_{I\setminus\{i\}} \\ F_I \imath'_i &= \imath_i F_{I\cup\{i\}}\end{aligned}$$

These compose associatively and have units, forming all together a category.

2. Vacant n-tuple groupoids

2.1. Definition. *An* ***n-tuple groupoid*** *is an n-tuple category whose n-cubes have inverses in every direction.*

The case $n = 1$, groupoids, should be familiar to the reader but we want to recall a few definitions that will be used throughout the paper.

2.2. Definition. *A groupoid (1-tuple) is* ***connected*** *if every homset is nonempty,* ***totally disconnected*** *if homsets between different objects are empty and* ***discrete*** *if all morphisms are identities.*

In other words, in an n-tuple groupoid, for every $a \in \mathfrak{C}_I$ and $i \in I$ there exists $b \in \mathfrak{C}_I$ such that:

$$a \circ_i b = \imath_i(s_i(a)) \qquad b \circ_i a = \imath_i(t_i(a))$$

It is a standard result that for an associative composition dual sided inverses are unique. These will be denoted as follows: suppose that X is an n-cube, then its inverse with respect to $\circ_i$ will be denoted X^{-i}. For the rest of the section, we will generically call τ the n-tuple groupoid of study, instead of $\mathfrak{C}$ and therefore τ_I will replace $\mathfrak{C}_I$.

Let's describe some basic properties of inverses.

2.3. LEMMA. *For $i \neq j \in [n]$ the source, target and identity maps commute with inverses, i.e.*

$$s_j(X^{-i}) = (s_j(X))^{-i}$$
$$t_j(X^{-i}) = (t_j(X))^{-i}$$
$$\imath_j(X^{-i}) = (\imath_j(X))^{-i}$$

PROOF. The proof for the source maps follows by uniqueness from

$$\begin{aligned} s_j(X) \circ_i s_j(X^{-i}) &= s_j(X \circ_i X^{-i}) = s_j(\imath_i(s_i(X))) = \imath_i(s_j(s_i(X))) \\ &= \imath_i(s_i(s_j(X))) \\ s_j(X^{-i}) \circ_i s_j(X) &= s_j(X^{-i} \circ_i X) = s_j(\imath_i(t_i(X))) = \imath_i(s_j(t_i(X))) \\ &= \imath_i(t_i(s_j(X))) \end{aligned}$$

The same proof works equally for target maps and identity maps. ■

2.4. LEMMA. *Let $X \in \tau$ be an n-cube, then $(X^{-i})^{-j} = (X^{-j})^{-i}$.*

PROOF. The proof follows from the previous lemma and the interchange law. Consider

$$\begin{aligned} \left(X \circ_j X^{-j}\right) \circ_i \left(X^{-i} \circ_j (X^{-j})^{-i}\right) &= \left(X \circ_i X^{-i}\right) \circ_j \left(X^{-j} \circ_i (X^{-j})^{-i}\right) \\ \imath_j(s_j(X)) \circ_i \left(X^{-i} \circ_j (X^{-j})^{-i}\right) &= \imath_i(s_i(X)) \circ_j \imath_i(s_i(X^{-j})) \\ &= \imath_i(s_i(X) \circ_j s_i(X^{-j})) \\ &= \imath_i(s_i(X \circ_j X^{-j})) \\ &= \imath_{\{ij\}}(s_{\{ij\}}(X)) \end{aligned}$$

and left compose by $\left(\imath_j(s_j(X))\right)^{-i} \circ_i$ to get:

$$\left(X^{-i} \circ_j (X^{-j})^{-i}\right) = \imath_j(s_j(X^{-i}))$$

It shows that $(X^{-j})^{-i}$ is the right j^{th} inverse of X^{-i}. The same argument shows that it is a left inverse also, which proves by uniqueness the lemma. ■

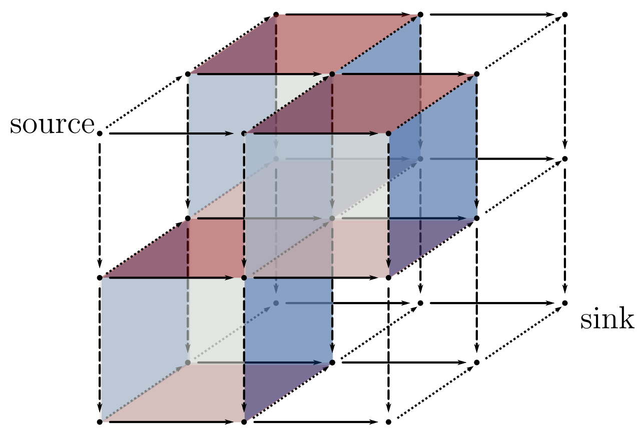
We can then define the unique inverse X^{-I} of X in the combined directions $i, j, k... \in I$.

2.5. DEFINITION. *Arrangements of n-cubes combinatorially equivalent to the one given by excluding the subspaces $x_i = \frac{1}{2}$ for all $i \in [n]$ from $[0,1]^n \subset \mathbb{R}^n$ are called* ***barycentric subdivisions*** *of the n-cube. Arrangements combinatorially equivalent to one found by excluding the subspaces $x_i = 1/3$ and $x_i = 2/3$ for all $i \in [n]$ are called* ***divisions in thirds****.*

With this definition in mind we can see the interchange law as ensuring that barycentric subdivisions have a uniquely defined composition.

2.6. Definition. *In a barycentric subdivision of the n-cube, the* ***partition*** *of a sub n-cube is an ordered pair (A, B) of complementary subsets of $[n]$ such that $i \in A$ if and only if the i^{th} source of the sub n-cube is part of the barycentric division of the i^{th} source of the original n-cube. The* ***depth*** *of a sub n-cube is the cardinality of B.*

By definition, the i^{th} target of a sub n-cube with partition (A, B) is internal to the subdivision if and only if $i \in A$. Moreover, the sub n-cube containing to the source and the one containing to the sink has depth 0 and n respectively. Here are for example the sub 3-cubes of depth 1:



2.7. Lemma. *A sub n-cube of depth i has a common boundary $(n-1)$-cube with i sub n-cubes of depth (i-1) and with $(n-i)$ sub n-cubes of depth $(i+1)$.*

Proof. Consider a sub n-cube Q with partition (A_Q, B_Q). Then $t_i(Q)$ is internal to the decomposition if and only if $i \in A_Q$. Similarly, for another sub n-cube Q', $s_i(Q')$ is internal to the subdivision if and only if $i \in B_Q$. Then defining $R := A_Q \cap B_{Q'}$ we can conclude that $s_R(Q') = t_R(Q)$ and that for $R \subset S \subset [n]$, $s_S(Q') \neq t_S(Q)$. Therefore a sub n-cube Q of depth $i-1$ shares a boundary $(n-1)$-cube with Q' of depth i if and only if $|A_Q \cap B_{Q'}| = 1$, or equivalently $B_Q \subset B_{Q'}$. Since $|B_{Q'}| = |B_Q| + 1 = i$ there are i such n-cubes Q for a given Q'. ∎

Every sub n-cube has n $(n-1)$-faces internal to the subdivision out of 2n such faces.

2.8. Lemma. *The intersection of a sub n-cube of depth i and the sub n-cube of depth 0 is an $(n-i)$-cube. Its intersection with the sub n-cube of depth n is an i-cube. Together, these two intersections contain edges of all n types.*

Proof. The sub n-cube α of depth 0 satisfies $A_\alpha = [n]$, so from the previous proof we can conclude that $Q \cap \alpha = s_{B_Q}$ and has codimension $|B_Q| = i$. Similarly if Ω is the sub n-cube of depth n, $B_\Omega = [n]$, so the intersection $Q \cap \Omega = t_{A_Q}$ and has dimension $|A_Q| = n - i$ and since $A_Q \cap B_Q = \emptyset$, the last claim is true. ∎

These little lemmas are helping us understand and formalize the adjacencies of sub-cubes inside a barycentric subdivision and will be helpful formulating some fairly intuitive proofs later on. Another very intuitive lemma that will be very helpful is the following:

2.9. Theorem. *Let $b(X)$ be a barycentric subdivision with one indeterminate sub n-cube X, then $b(X) = Y$ has a unique solution in τ.*

Proof. Thanks to the existence of inverses, we may assume w.l.o.g. that $b(X)$ has its indeterminate sub-cube X in depth 0. Define $b_0 := X$ and $b_i := \{\text{cubes in } b(X) \text{ whose } i^{th} \text{ source is the } i^{th} \text{ target of a cube in } b_{i-1}\}$. Then define $\overline{b}_i$ as the composition of the elements of b_i in $b(X)$. We then have

$$b(X) = (\cdots((X \circ_1 \overline{b}_1) \circ_2 \overline{b}_2) \cdots) \circ_n \overline{b}_n$$

so that if $b(X) = Y$ we can easily solve for X:

$$X = (\cdots((Y \circ_n \overline{b}_n^{\,-n}) \circ_{n-1} \overline{b}_{n-1}^{\,-(n-1)}) \cdots) \circ_1 \overline{b}_1^{\,-1}$$

which finishes the proof. ∎

Let X be an n-cube of τ and consider a barycentric subdivision where X has depth n. Place identities in all other positions of strictly positive depth. Then an n-cube u that can be placed with depth 0 must satisfy :

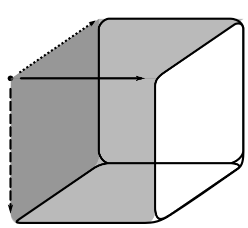
$$t_i(u) = \imath_{\hat{i}}(s_{[n]}(X))$$

i.e have all targets be identities on the source of X. We will now see what such n-cubes may tell us.

2.10. Definition. *Let τ be an n-tuple groupoid and define*

$$\begin{aligned} \tau_\lrcorner :&= \{n\text{-cubes whose recursive targets are identities}\} \\ &= \{X \in \tau | t_i(X) = \imath_{\hat{i}}(t_{[n]}(X))\} \end{aligned}$$

For $u \in \tau_\lrcorner$ and $X \in \tau$ such that $t_{[n]}(u) = s_{[n]}(X)$ define the ***transmutation of*** X ***by*** u*, denoted $u \cdot X$, to be the n-cube accepting a barycentric subdivision with u of depth 0, X of depth n and all others identities.*



For example, in dimension 3 an element of $\tau_\lrcorner$ looks like the above picture, where arrows without heads and non colored squares are identities. From it can be clearly deduced that the faces of an element of the core groupoid of a triple groupoid are themselves in the core groupoids of the boundary double groupoids. This is actually true for any n.

2.11. Lemma. *Let $u \in \tau_\lrcorner$, then $s_i(u) \in (\tau_{\hat{i}})_\lrcorner$*

Proof. All there is to prove is that for $j \neq i \in [n]$, $t_j(s_i(u)) = \imath_{\hat{i}\hat{j}}(t_{[n]}(u))$. But $t_j(s_i(u)) = s_i(t_j(u)) = s_i(\imath_{\hat{j}}(t_{[n]}(u))) = \imath_{\hat{i}\hat{j}}(t_{[n]}(u))$ as claimed. ∎

2.12. Lemma. *Let $u, v \in \tau_\lrcorner$, then $u \cdot v \in \tau_\lrcorner$. Moreover $(\tau_\lrcorner, \cdot, \imath)$ is a groupoid.*

Proof. Let the sources and targets of arrows in said groupoid be the sources and sinks of the n-cubes of $\tau_\lrcorner$. First note that $t_i(u \cdot X) = t_i(X)$, so that $v \in \tau_\lrcorner$ implies $u \cdot v \in \tau_\lrcorner$. This defines a composition on $\tau_\lrcorner$. The associativity of this composition is a direct consequence of the interchange law and the uniqueness of identities. Indeed, for $u, v, w, \in \tau_\lrcorner$, $(u \cdot v) \cdot w$ and $u \cdot (v \cdot w)$ are given by two different order of composition of the same division in thirds with u,v and w in this order on the diagonal and the rest filled by identities. But as the interchange laws ensures that any order of composition of a division in thirds yield the same result, associativity is proved. It is straightforward to see that the identities are given by identity n-cubes on objects. What remains to be proven is that inverses exist and uniqueness will follow.
To do so, first consider the equation $X \cdot u = id_{t(u)}$ in $\tau_\lrcorner$ and remark that, as all n-cubes of non zero depth in $X \cdot u$ are uniquely determined by u, it can be re-framed as an equation of the form $b(X) = Y$ in τ, as in Theorem 2.9. The theorem then proves the existence of left inverses in $\tau_\lrcorner$. This trick does not work for right inverses though as $u \cdot X$ has all but one sub-cube depending on the indeterminate X. So a little more work is needed. We will give a proof by induction.
Note that in dimension one $\tau_\lrcorner = \tau$, so the lemma is true. Now for any other dimension if a right inverse u^{-1} of u exists in $\tau_\lrcorner$, then $s_i(u^{-1}) = s_i(u^{-1}) \in (\tau_{\hat{i}})_\lrcorner$. That shows that the sub-cubes appearing in a solution of $u \cdot X = id_{s(u)}$ are fixed by the boundaries of u and exist provided the lemma is true for dimensions smaller than n. We can then rewrite the equation as $b(X) = Y$ in τ and use Theorem 2.9 to finish the proof. ∎

2.13. Definition. *The groupoid $\tau_\lrcorner$ is called the* ***core groupoid*** *of τ.*

Let $u \in \tau_\lrcorner$, then the assignment $u \to u\cdot$ defines an action of groupoids on $\tau_{[n]}$, the set of n-cubes of τ. The next lemma shows that it is transitive on n-cubes with common targets.

2.14. Lemma. *Let $X, Y \in \tau$ such that $t_i(X) = t_i(Y)$ $\forall i \in [n]$, then there exists a unique element $u_{XY} \in \tau_\lrcorner$ such that*

$$X = u_{XY} \cdot Y$$

Proof. We prove this as we did in 2.12. Indeed, the solution of $X = b(u)$, where the elements of the $b(u)$ are Y in depth n and identities on all other n-cubes of non zero depth, exists and is unique by Theorem 2.9. ∎

2.15. Definition. *Let $\tau_\bullet$ be the sub groupoid of $\tau_\lrcorner$ composed of n-cubes whose boundaries are all identities. It is called the* ***core bundle*** *of τ.*

2.16. Corollary. *Let $X, Y \in \tau$ such that $t_i(X) = t_i(Y)\, \forall i \in [n]$, then*

$$s_i(X) = s_i(Y)\, \forall i \in [n] \iff u_{XY} \in \tau_\bullet$$

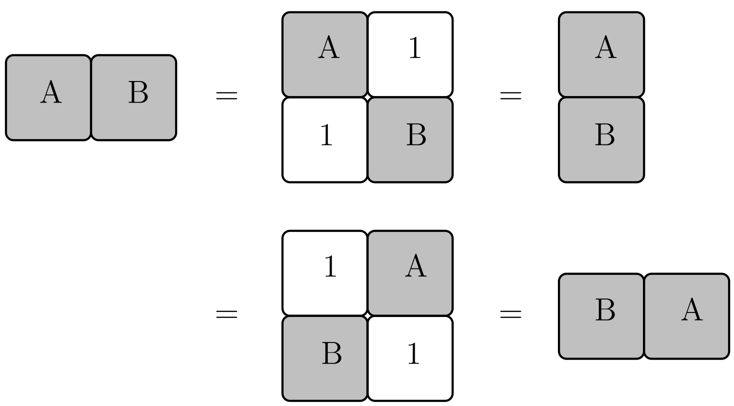
Proof. Let $i \in [n]$, then $s_i(X) = s_i(u_{XY} \cdot Y) = s_i(u_{XY}) \cdot s_i(Y)$, so that $s_i(u_{XY}) = s_i(X) \cdot s_i(Y)^{-1} = s_i(X) \cdot s_i(X)^{-1} = id_{s(u)}$, which proves the statement. ∎

2.17. Lemma. *The core bundle is an abelian group bundle over τ_0, the objects of τ.*

Proof. This is a generalized Eckmann-Hilton argument. N-cubes whose boundaries are identities "slide" along each other. First note that for $u, v \in \tau_\bullet$:

$$u \cdot v = u \circ_i v \qquad \forall i \in [n]$$

To see this, pick $i \in [n]$, $u, v \in \tau_\bullet$ and consider all sub n-cubes of the barycentric subdivision defining $u \cdot v$ whose i^{th} source contributes to the i^{th} source of $u \cdot v$. By definition all of them, except u, are identities on different faces of v, which are identities on $s_{[n]}(v) = t_{[n]}(u)$. Their composition is therefore u. The other sub n-cubes are those who contribute to the i^{th} target of $u \cdot v$. Once again they are all, except for v, identities. Their composition therefore yields v, which shows that $u \cdot v = u \circ_i v$.
Now pick two directions i, j and apply the following two dimensional argument in the plane ij :



It shows that $u \cdot v = u \circ_i v = v \circ_i u = v \cdot u$, and finishes the proof. Note that this proof is absolutely standard but we thought that it was worth repeating. ∎

2.18. Definition. *A n-tuple groupoid is* ***slim*** *if its core bundle is trivial.*

2.19. Corollary. *A n-tuple groupoid is slim iff there is at most one n-cube per boundary condition.*

2.20. Definition. *A n-tuple groupoid is* ***exclusive*** *if $\tau_\lrcorner = \tau_\bullet$.*

So in an exclusive n-tuple groupoid, an n-cube belongs to the core groupoid if and only if all its faces are identities.

2.21. Corollary. *A n-tuple groupoid is exclusive if and only if, for $X, Y \in \tau$, the following is true:*

$$\begin{aligned} &\forall i \in [n] \text{ either } s_i(X) = s_i(Y) \text{ or } t_i(X) = t_i(Y) \\ \iff\; & s_i(X) = s_i(Y) \text{ and } t_i(X) = t_i(Y) \qquad \forall i \in [n] \end{aligned}$$

Proof. Assume that τ is exclusive and the existence of two n-cubes that share either a source or a target in every direction. Define

$$I := \{i \in [n] | t_i(X) \neq t_i(Y)\}$$

Then $s_i(X) = s_i(Y)\, \forall i \in I$. But in this case $t_j(X^{-I}) = t_j(Y^{-I})\, \forall j \in [n]$. From 2.12 there exists $u \in \tau_\lrcorner$ such that $X^{-I} = u \cdot Y^{-I}$, which implies under the exclusivity assumption that $s_i(X^{-I}) = s_i(u) \cdot s_i(Y^{-I}) = s_i(Y^{-I})$. Therefore $s_i(X) = (s_i(X^{-I}))^{-I} = (s_i(Y^{-I}))^{-I} = s_i(Y)$.
Now assuming the second statement of the proposition, elements of the core groupoid always share all targets with an identity on their sink. All their boundaries are then forced to be identities as well, i.e. it is in the core bundle, showing that τ is exclusive. ■

We can summarize the previous result by saying that in an exclusive n-tuple groupoid, the boundary of its n-cubes are determined by one of their $(n-1)$-boundaries of each type.

2.22. Lemma. *Let τ be a n-tuple groupoid. If all boundary i-tuple groupoids are slim for $i > 1$ and all boundary double groupoids are exclusive then τ is exclusive.*
Moreover in this case the following is true :

$$t_{\hat{i}}(X) = t_{\hat{i}}(Y) \quad \forall i \in [n] \iff X = u \cdot Y \text{ for } u \in \tau_\bullet$$

Proof. Suppose that τ_{ij} is exclusive for all $i, j \in [n]$, then for $u \in \tau_\lrcorner$:

$$\begin{aligned} & t_{\hat{i}}(u) = \imath_i(s_{[n]}(u))\, \forall i \in [n] \\ \Rightarrow \quad & s_j t_{\hat{i}\hat{j}}(u) = \imath_i(s_{[n]}(u))\, \forall i \neq j \in [n] \\ \Rightarrow \quad & s_{jk} t_{\hat{i}\hat{j}\hat{k}}(u) = \imath_i(s_{[n]}(u))\, \forall i \neq j \neq k \in [n] \\ & \cdots \\ \Rightarrow \quad & s_{\hat{i}}(u) = \imath_i(s_{[n]}(u))\, \forall i \in [n] \end{aligned}$$

Which shows that all arrows of u are identities. Now since the boundary double groupoids of τ are slim, all boundary 2-cubes of u are identities. Since all boundary triple groupoids of τ are slim, all boundary 3-cubes of X are identities. Repeat the argument to dimension n-1 to prove that τ and all its boundary i-tuple groupoids are exclusive
Let $X, Y \in \tau$ be such that $t_{\hat{i}}(X) = t_{\hat{i}}(Y)\, \forall i \in [n]$, then a recursive argument on exclusivity shows that $t_i(X) = t_i(Y)\, \forall i \in [n]$. Hence by 2.14 $X = u \cdot Y$ for $u \in \tau_\lrcorner$, which by exclusivity reduces to $\tau_\bullet$. ■

This result shows that for a slim n-tuple groupoid satisfying the conditions of the above lemma following map is injective:

$$\Sigma : \tau_{[n]} \to \prod_{i=1}^{n} \tau_i$$
$$\Sigma(X) = (t_{\hat{1}}(X), \cdots, t_{\hat{n}}(X))$$

2.23. Definition. *An n-tuple groupoid τ is* ***maximal*** *if the above defined map Σ is surjective. It is* ***maximally exclusive*** *if*

- *it is maximal*
- *all boundary double groupoids are exclusive*
- *all boundary i-tuple groupoids are slim for $i > 1$*

It is ***vacant*** *if it is slim and maximally exclusive.*

We can then conclude, by Lemma 2.22, that a vacant n-tuple groupoid is one such that Σ is an isomorphism. In other words, for a vacant n-tuple groupoid all the information contained in an n-cube is reduced to all boundary arrows targeted at its sink. Moreover its existence for any such set of arrows is guaranteed.

2.24. Lemma. *Let τ be maximal, then so is τ_I for all $I \subset [n]$. Accordingly, if τ is maximally exclusive, then τ_I is vacant for all $I \subset [n]$ s.t. $|I| > 1$.*

Proof. Suppose that τ is maximal, then since for all $X \in \tau$ and all $i \in I$, $t_i(\imath_{\hat{I}}(t_I(X))) = t_i(X)$, the following map is surjective as well:

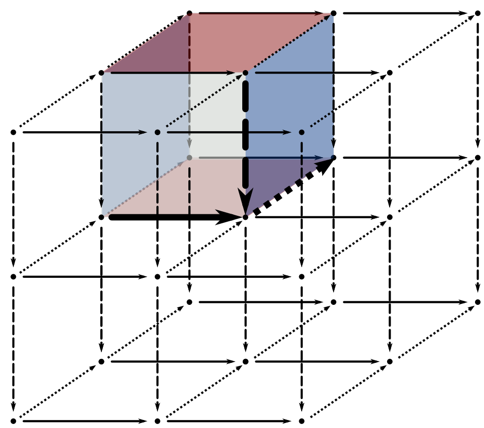
$$\Sigma|_I := \imath_{\hat{I}}\, \Sigma\, \pi : \tau_I \to \prod_{i \in I} \tau_i$$

where π is the projection onto the indices corresponding to I, proving the first part of the lemma.
Remark that we may now state that the three properties defining a maximally exclusive n-tuple groupoid are stable upon restriction to boundaries. So by proving the stability of maximality we actually proved stability of maximal exclusivity. Moreover, slimness of boundary i-tuple groupoids being part of the definition, we may conclude that the second statement of the lemma is also true. ∎

Now we want to look at a feature that is very specific to vacant n-tuple groupoids.

2.25. Lemma. *Let τ be a vacant n-tuple groupoid, $X, Y \in \tau$, then there exist a unique n-cube $X \cdot Y$ in τ that has a barycentric subdivision with X of depth 0 and Y of depth 1. Moreover $(\tau, \cdot)$ is a groupoid.*

Proof. According to Lemma 2.7, all sub-cubes of depth different from 0 or n have an internal edge of each kind shared with either X or Y. For example, in the case $n = 3$, a 3-cube of depth 1 shares internal edges of two kind with the depth 0 3-cube and of the third kind with the depth n 3-cube, as in the picture :



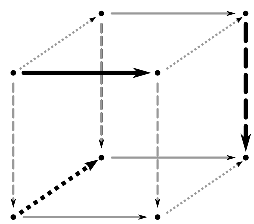
Since the n-tuple groupoid is vacant, these sub-cubes have a unique filler which proves that the composition is well defined. Its associativity is guaranteed by uniqueness of fillers and the interchange laws. Identities are identity n-cubes on source or sink.
Now to show that inverses exist, consider an n-cube X and place it in position 0. As depth 1 sub cubes' intersections with X are $(n-1)$-cubes, they only need one arrow in the direction not contained in the shared boundary to be determined. But for X to have an inverse, the n-cube Q of depth 1 whose yet undetermined arrows are in the i^{th} direction needs to satisfy $s_{\hat{i}}(Q) = s_{\hat{i}}(X)^{-1}$. That determines uniquely all depth 1 sub n-cubes. From Lemma 2.7 depth i cubes share at least two boundary $(n-1)$-cubes with depth $(i-1)$ sub n-cubes, for $i > 1$. This fact determines inductively all sub-cubes of depth greater than 1. Their composition is a n-cube with a boundary arrow of each type being an identity and is therefore $\imath_{[n]}(s_{[n]}(X))$, showing that X has a right inverse.
But since $(Y \cdot X)^{-[n]} = X^{-[n]} \cdot Y^{-[n]}$, a right inverse to $X^{-[n]}$ is a left inverse to X, proving that X is invertible. ■

2.26. Theorem. *Let τ be vacant, $X \in \tau$ and $f_i = t_{1\cdots(i-1)}s_{(i+1)\cdots n}(X)$. Then*

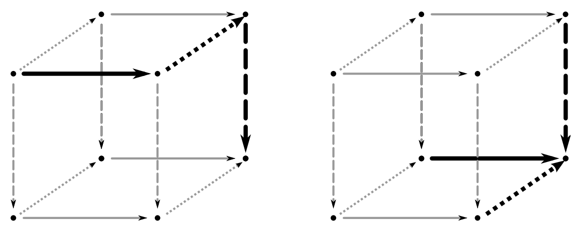
$$X = \imath_{\hat{1}}(f_1) \cdot \imath_{\hat{2}}(f_2) \cdots \imath_{\hat{n}}(f_n)$$

To prove this lemma, we need to see that there were many other acceptable definitions of maximality but that they mostly give the same notion of maximally exclusive n-tuple groupoids.
The idea behind maximality is that it should represent a positive answer to the question: given a certain set of boundary arrows in an abstract n-cube that contains exactly one arrow in each direction, can one find an actual n-cube with these boundary constrains?
The constrain is given by a graph Γ containing n arrows and an isomorphism between these and the set $[n]$ of numbers from 1 to n. Not all graphs are admissible though as

they need to be embeddable in the 1-skeleton of the n-cube. For example, in dimension 2 a non connected graph is not admissible, although it is admissible in dimension 3, as shows the following picture:



We will now show that the following two graphs Γ_1 and Γ_2 give the same notion of maximal exclusivity.



It will become clear that this should be true for any connected graph but we will not prove it as it is not entirely relevant to our goal. Remark that Γ_2 is the path we first used for maximality. Remark as well that for n=2, there is, up to symmetries only one path and therefore only one notion of maximality.

2.27. LEMMA. *Let τ be an n-tuple groupoid such that τ_{ij} is exclusive for all $i \neq j \in [n]$. Then τ is Γ_1-maximal if and only if it is Γ_2-maximal.*

PROOF. Each graph gives us a map $\gamma_i : \tau_{[n]} \to \prod_{j=1}^{n} \tau_j$ picking the boundary arrows of n-cube that are in Γ_i. Remark that γ_2 is just our previously defined Σ. Therefore

$$\gamma_2 = \Delta_n(t_{\hat{1}} \times \cdots \times t_{\hat{n}})$$

where Δ_n is the diagonal map $\tau_{[n]} \to \prod_{j=1}^{n} \tau_{[n]}$. Defining

$$c_{\hat{i}} := t_{1\cdots(i-1)} s_{(i+1)\cdots n}$$

we can rewrite γ_1 as

$$\gamma_1 = \Delta_n(c_{\hat{1}} \times \cdots \times c_{\hat{n}})$$

Now we need to show that γ_1 is surjective if and only if γ_2 is. Remark that the boundary double groupoids of τ being exclusive means that for each pair $i \neq j \in [n]$ there exists an isomorphism $\Phi_{i,j} : \tau_i\,{}_t\!\times_t \tau_j \to \tau_i\,{}_t\!\times_s \tau_j$ such that the following diagram commutes :

$$\begin{tikzcd}
& \tau_{ij} \arrow[dl, "\Delta"] \arrow[dr, "\Delta"'] & \\
\tau_{ij} \times \tau_{ij} \arrow[d, "t_j \times t_i"] & & \tau_{ij} \times \tau_{ij} \arrow[d, "s_j \times t_i"'] \\
\tau_i\,{}_t\!\times_t \tau_j \arrow[rr, "\Phi_{i,j}"'] & & \tau_i\,{}_t\!\times_s \tau_j
\end{tikzcd}$$

In other words, it is showing that in dimension two $\gamma_1 = \gamma_2\Phi_{1,2}$. We can now use this to generalize to any dimensions.
Considering the fact that

$$\gamma_2 = \Delta_{(n-1)}\big(t_{\hat{1}} \times \cdots \times t_{(\widehat{n-2})} \times t_{1\cdots(t-2)}\big)\Big(id \times \cdots \times id \times \big(\Delta(t_n \times t_{(n-1)})\big)\Big)$$

we can see that

$$\begin{aligned}
\gamma_2(id \times \cdots \times \Phi_{n,(n-1)}) &= \Delta_n(t_{\hat{1}} \times \cdots \times t_{(\widehat{n-2})} \times c_{(\widehat{n-1})} \times t_{\hat{n}}) \\
&= \Delta_n(t_{\hat{1}} \times \cdots \times t_{(\widehat{n-2})} \times c_{(\widehat{n-1})} \times c_{\hat{n}})
\end{aligned}$$

The right hand side corresponds to the graph obtained from Γ_2 where the arrow in the $(n-1)^{th}$ direction was sled back along the arrow in the n^{th} direction. We may keep using the same trick to slide the arrow in the $(n-2)^{th}$ direction back along the previously mentioned ones. It will be done by using $\Phi_{n,(n-2)}$ followed by $\Phi_{(n-1),(n-2)}$. At that point we will have showed that there exists a map ϕ such that

$$\gamma_2\phi = \Delta_n(t_{\hat{1}} \times \cdots \times t_{(\widehat{n-3})} \times c_{(\widehat{n-2})} \times c_{(\widehat{n-1})} \times c_{\hat{n}})$$

Repeating this argument shows that there exists an isomorphism Φ made out of $\Phi_{i,j}$'s such that $\gamma_2 = \gamma_1\Phi$. We can therefore conclude that γ_1 is surjective if and only if γ_2 is, proving the lemma. ∎

We are now ready to prove theorem 2.26.

Proof. Let $X, Y \in \tau$ such that $X = \imath_i(s_i(X))$ and $s_{[n]}(Y) = t_{[n]}(X)$. Let u be the sub n-cube of $X \cdot Y$ whose partition is $(i, [n] \setminus i)$. By definition u is of depth $(n-1)$ in the barycentric subdivision defining $X \cdot Y$ and $s_i(X \cdot Y) = s_i(X) \cdot s_i(u)$. Now since $X = \imath_i(s_i(X))$, $t_{\hat{i}}(X) = \imath_i(s_{[n]}(X)) = s_{\hat{i}}(u)$. But as τ is vacant, $u = \imath_i(s_i(u)) = \imath_i(t_i(u)) = \imath_i(s_i(Y))$. That shows that, for $\alpha \in \tau_{\hat{i}}$ and $Y \in \tau$

$$s_i(\imath_i(\alpha) \cdot Y) = \alpha \cdot s_i(Y)$$

Similarly one can show that

$$t_i(Y \cdot \imath_i(\alpha)) = t_i(Y) \cdot \alpha$$

From which we can conclude that :

$$\begin{aligned}
c_{\hat{i}}(\imath_{\hat{1}}(f_1)\cdot\imath_{\hat{2}}(f_2)\cdots\imath_{\hat{n}}(f_n)) &= t_{1\cdots(i-1)}s_{(i+1)\cdots(n-1)}s_n(\imath_{\hat{1}}(f_1)\cdot\imath_{\hat{2}}(f_2)\cdots\imath_{\hat{n}}(f_n)) \\
&= t_{1\cdots(i-1)}s_{(i+1)\cdots(n-1)}(\imath_{\hat{1}\hat{n}}(f_1)\cdot s_n(\imath_{\hat{2}}(f_2)\cdots\imath_{\hat{n}}(f_n))) \\
&= \cdots \\
&= t_{1\cdots(i-1)}s_{(i+1)\cdots(n-1)}(\imath_{\hat{1}\hat{n}}(f_1)\cdot\imath_{\hat{2}\hat{n}}(f_2)\cdots s_n(\imath_{\hat{n}}(f_n))) \\
&= t_{1\cdots(i-1)}s_{(i+1)\cdots(n-1)}(\imath_{\hat{1}\hat{n}}(f_1)\cdot\imath_{\hat{2}\hat{n}}(f_2)\cdots\imath_{\widehat{(n-1)}\hat{n}}(f_{n-1})) \\
&= \cdots \\
&= t_{1\cdots(i-1)}(\imath_{2\cdots i}(f_1)\cdot\imath_{13\cdots i}(f_2)\cdots\imath_{1\cdots(i-1)}(f_i)) \\
&= t_{2\cdots(i-1)}t_1(\imath_{2\cdots i}(f_1)\cdot\imath_{13\cdots i}(f_2)\cdots\imath_{1\cdots(i-1)}(f_i)) \\
&= t_{2\cdots(i-1)}\big(t_1(\imath_{2\cdots i}(f_1)\cdot\imath_{13\cdots i}(f_2)\cdots)\cdot\imath_{2\cdots(i-1)}(f_i)\big) \\
&= \cdots \\
&= t_{2\cdots(i-1)}(\imath_{3\cdots i}(f_2)\cdot\imath_{24\cdots i}(f_3)\cdot\imath_{2\cdots(i-1)}(f_i)\big) \\
&= t_{3\cdots(i-1)}t_2(\imath_{3\cdots i}(f_2)\cdot\imath_{24\cdots i}(f_3)\cdot\imath_{2\cdots(i-1)}(f_i)\big) \\
&= \cdots \\
&= t_{i-1}(\imath_{i-1}(f_i)) \\
&= f_i
\end{aligned}$$

Therefore X and $\imath_{\hat{1}}(f_1)\cdot\imath_{\hat{2}}(f_2)\cdots\imath_{\hat{n}}(f_n)$ have the same value along Γ_1, which by the previous lemma means that they have the same value along Γ_2 as well and therefore are equal. ∎

What this theorem says is that as a groupoid, $(G,\cdot)$ admits a factorization into n factors. We will describe the connection in further detail in the next section.

2.28. Lemma. *Let $X, Y \in \tau_{\hat{k}}$, then $\imath_k(X)\cdot\imath_k(Y) = \imath_k(X\cdot Y)$*

Proof. According to lemma 2.8, every sub-cube in the barycentric subdivision $\imath_k(X)\cdot\imath_k(Y)$ shares an edge in direction k with either $\imath_k(X)$ or $\imath_k(Y)$, and hence by vacancy is of the form $\imath_k()$. This shows that so is $\imath_k(X)\cdot\imath_k(Y)$. Now its k^{th} source has X in depth 0 and Y in depth $(n-1)$, which shows that it is indeed $X\cdot Y$, proving the lemma. ∎

2.29. Corollary. *Let $f\in\tau_i$ and $g\in\tau_j$ for $i<j\in[n]$ and $\imath_j(f)\cdot\imath_i(g)\in\tau_{ij}$ be defined as above in τ_{ij}. Then there exists a unique $f'\in\tau_i$ and $g'\in\tau_j$ such that*

$$\imath_{\hat{i}}(f)\cdot\imath_{\hat{j}}(g) = \imath_{\hat{i}\hat{j}}\big(\imath_j(f)\cdot\imath_i(g)\big) = \imath_{\hat{j}}(g')\cdot\imath_{\hat{i}}(f')$$

Proof. The fact that $\imath_{\hat{i}}(f)\cdot\imath_{\hat{j}}(g) = \imath_{\hat{i}\hat{j}}\big(\imath_j(f)\cdot\imath_i(g)\big)$ and that $\imath_{\hat{j}}(g')\cdot\imath_{\hat{i}}(f') = \imath_{\hat{i}\hat{j}}\big(\imath_i(g')\cdot\imath_j(f')\big)$ follows from the above lemma. Now define

$$\begin{aligned}
f' &:= t_j\big(\imath_j(f)\cdot\imath_i(g)\big) \\
g' &= s_i\big(\imath_j(f)\cdot\imath_i(g)\big)
\end{aligned}$$

and note that for $X \in \tau_{ij}$:

$$\begin{aligned} X &= \big(\imath_j(s_j(X)) \circ_i \imath_{ij}(s_j t_i(X))\big) \circ_j \big(X \circ_i \imath_i(t_i(X))\big) \\ &= \imath_j(s_j(X)) \cdot \imath_i(t_i(X)) \\ X &= \big(\imath_i(s_i(X)) \circ_i X\big) \circ_j \big(\imath_{ij}(s_i t_j(X)) \circ \imath_j(t_j(X))\big) \\ &= \imath_i(s_i(X)) \cdot \imath_j(t_j(X)) \end{aligned}$$

which proves the corollary. ∎

Let $\underline{\mathbf{nGpd}}$ be the category of n-tuple groupoids and n-tuple functors and let $\underline{\mathbf{nSub}}$ be the category defined by :

- Objects are $(n+1)$-tuples $(G, H_1, H_2, \cdots, H_n)$ where G is a groupoid, $\{H_i\}_{i\in[n]}$ are subgroupoids of G each containing all objects of G.[1]
- Arrows are functors $f : G \to G'$ such that $f(H_i) \subset H'_i \quad \forall i \in [n]$

In this section, we define the "commutative cubes" functor

$$\Gamma : \underline{\mathbf{nSub}} \to \underline{\mathbf{nGpd}}$$

and find a subcategory of $\underline{\mathbf{nGpd}}$ where it admits a left adjoint. We will then determine how to make this adjunction into an equivalence.
The "commutative cubes" functor takes a $(n+1)$-tuple $(G, H_1, H_2, \cdots, H_n)$ and sends it to the n-tuple groupoid $\Gamma(G, H_1, H_2, \cdots, H_n)$ defined by the criteria:

- Objects of $\Gamma(G, H_1, H_2, \cdots, H_n)$ are objects of G.
- $\Gamma(G, H_1, H_2, \cdots, H_n)_i = H_i$.
- $\Gamma(G, H_1, H_2, \cdots, H_n)_I$ is slim for any $I \subseteq [n]$ with more than one element.
- n-cubes are commutative cubes, to be defined below.

Let K be the set of all possible cubes that would comply with the first three conditions. Each path of length n from the source object to the sink object of an n-cube gives a sequence of composable arrows in G, with each arrow in a different subgroupoid. Each path then defines a map of sets $K \to G$ by composing the corresponding sequences. The n-cubes of K that have a constant value under all paths are the cubes of $\Gamma(G, H_1, H_2, \cdots, H_n)$.

2.30. Lemma. *$\Gamma(G, H_1, H_2, \cdots, H_n)$ is a slim n-tuple groupoid.*

Proof. As pasting commutative diagrams along common boundaries produces other commutative diagrams, all in an associative way, the compositions along faces are well defined. The rest of the proof is of no technical interest and very straightforward. Also, it is clear from the definition that the n-tuple groupoid defined is slim. ∎

[1] Any subgroupoid can be turned into one of this type by taking its union with the discrete groupoid on the objects of G.

Thus we showed that the map Γ is well defined on objects. For it to be a functor, we need to define it on arrows as well, so supposing that $F : (G, H_1, H_2, \cdots, H_n) \to (G', H'_1, H'_2, \cdots, H'_n)$ is a subgroupoid preserving functor, we can define $\Gamma(F)$ as the n-tuple functor from $\Gamma(G, H_1, H_2, \cdots, H_n)$ to $\Gamma(G', H'_1, H'_2, \cdots, H'_n)$ such that $\Gamma(F)_i = F|_{H_i}$. Since objects in the image of Γ are slim, the functor is uniquely defined by this criteria. The image of this functor is by definition in the category of slim n-tuple groupoids and in this category lie all the vacant n-tuple groupoids. But by Lemma 2.25 and Theorem 2.26, out of a vacant n-tuple groupoid one can canonically define a groupoid and n subgroupoids that satisfy the criteria for **nSub**. This defines a functor

$$\Lambda : \underline{\mathbf{nVacant}} \to \underline{\mathbf{nSub}}$$

by $\Lambda(\tau) = ((\tau, \cdot), \tau_1, \tau_2, \cdots, \tau_n)$, where we identified τ_i and $\imath_{\hat{i}}(\tau_i)$. The definition of the functor on morphisms is clear, $(\Lambda(F))(X) := F(X)$. If we can establish the preimage of **nVacant** under Γ, we may find an adjunction between Γ and Δ. Slimness being included in the definition of Γ, remains exclusivity and maximality to be characterized w.r.t. Γ. Out of the two, exclusivity is the easiest criteria to track.

2.31. LEMMA. *Let $I \subseteq [n]$ have at least two elements, then $\Gamma(G, H_1, \cdots, H_n)_I$ is exclusive if and only if $\bigcap_{i \in I} H_i$ is discrete.*

PROOF. Remark that $\Gamma(G, H_1, \cdots, H_n)_I \simeq \Gamma(G, H_{i_1}, \cdots, H_{i_{|I|}})$ where $I = \{i_1, \cdots, i_{|I|}\}$ and that the only non identity arrows contained in an element X of its core groupoid are of the form $s_{\hat{i}}(X)$. Therefore:

$$\begin{aligned} X \in \big(\Gamma(G, H_1, \cdots, H_n)_I\big)_\lrcorner &\Rightarrow s_{\hat{i}}(X) = s_{\hat{j}}(X)\, \forall i, j \in I \\ &\Rightarrow s_{\hat{i}}(X) \in \cap_{j=1}^{|I|} H_{i_j}\ \forall i \in I \end{aligned}$$

Assuming that the above mentioned intersection is discrete, we can conclude that $s_{\hat{i}}(X) = \imath_i(s_{[n]}(X))\, \forall i \in I$, i.e. that the given boundary $|I|$-tuple groupoid is exclusive.
Now assume that there exists one arrow f in the above intersection that is not an identity. Then there exists $X \in \big(\Gamma(G, H_1, \cdots, H_n)_I\big)_\lrcorner$ such that $s_{\hat{i}}(X) = f\, \forall i \in I$. The fact that X is not in its core bundle finishes the proof of the statement. ■

As for maximality, we can start by this simple statement:

2.32. LEMMA. *If $\Gamma(G, H_1, \cdots, H_n)$ is Γ_2-maximal then $H_i H_j = H_j H_i\ \forall i, j \in [n]$. $\Gamma(G, H_1, H_2)$ is maximal if and only if $H_1 H_2 = H_2 H_1$.*

PROOF. The existence of identities show that if τ is Γ_2-maximal, so is τ_I, for all $I \subset [n]$. But if τ_{ij} is Γ_2 maximal, we can use the existence of inverses to show $H_i H_j \subset H_j H_i$ and $H_j H_i \subset H_i H_j$, and hence $H_i H_j = H_j H_i$. Now suppose that $H_1 H_2 = H_2 H_1$, then for a coloring (h_1, h_2) of the Γ_1 path we can find (h'_2, h'_1) in $H_2 \underset{1}{\times} H_1$ giving the same element of G under composition. This proves the existence of a commutative square and therefore of Γ_1-maximality, or in this case maximality in general. ■

It would be desirable to get a simple criteria for maximality. However, the difficulty to find such criteria makes such a quest out of the scope of this article. Luckily, the problem simplifies tremendously for maximal exclusivity. The case $n = 2$ suggests the following definition:

2.33. Definition. *Let G be a groupoid and H_1, H_2 be subgroupoids such that*

$$H_1 H_2 = H_2 H_1$$
$$H_1 \cap H_2 \text{ is discrete.}$$

Then the subgroupoids are said to be ***coupled****.*

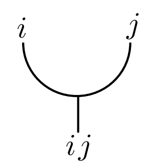
2.34. Corollary. *Let $(G, H_1, H_2) \in \underline{\mathbf{2Sub}}$. Then H_1 and H_2 are coupled if and only if $\Gamma(G, H_1, H_2)$ is vacant.*

It would be convenient if pairwise coupling was the criteria for vacancy. The reality is a little more complex, but pairwise coupling provides some sort of "braiding" that will prove itself crucial.

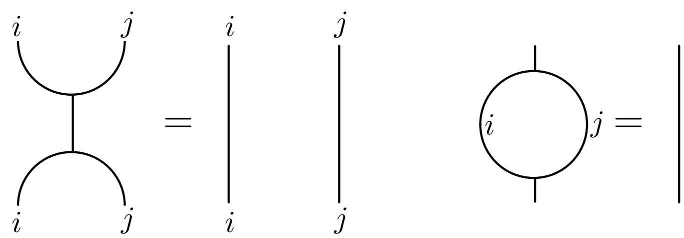
2.35. Lemma. *Let H_1 and H_2 be coupled and μ be the groupoid composition. Then there exists an isomorphism $\Phi_{12} : H_1 \underset{1}{\times} H_2 \to H_2 \underset{1}{\times} H_1$ in* $\underline{\mathbf{Set}}$ *such that $\mu = \Phi_{12}\mu$*

Proof. The proof follows the same pattern as that of Lemma 2.27. ■

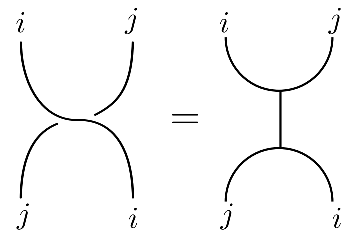
The fact that $H_i H_j = H_j H_i$ means that $H_i H_j$ is a subgroupoid of G. For the graphically minded, we may represent $\mu : H_i \underset{1}{\times} H_j \to H_i H_j$ by : When $H_i \cap H_i$ is discrete, this



map is invertible:



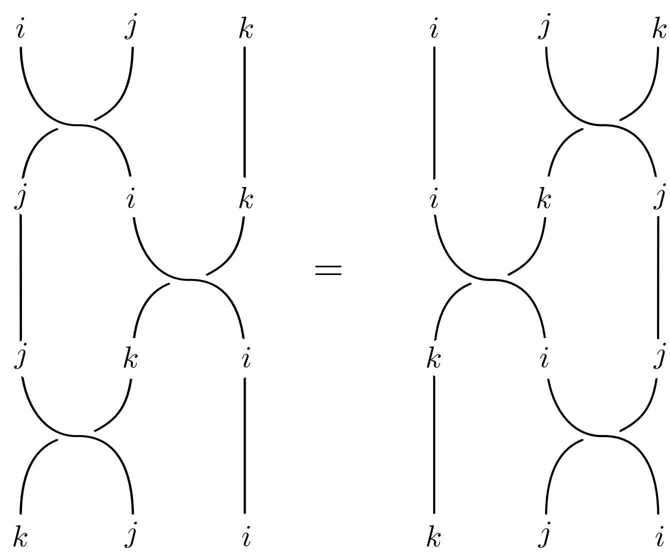
and the "braiding" Φ_{ij} is given by:



2.36. DEFINITION. *Let G be a groupoid and $\{H_i\}_{[n]}$ a set of pairwise coupled subgroupoids. This set is said to be* ***optimally coupled*** *if, for any $i \neq j \neq k \in [n]$ the following is true:*

$$(\Phi_{ij} \times id_{H_k})(id_{H_j} \times \Phi_{ik})(\Phi_{jk} \times id_{H_i}) = (id_{H_i} \times \Phi_{jk})(\Phi_{ik} \times id_{H_j})(id_{H_k} \times \Phi_{ij})$$

One can readily recognize the Yang-Baxter equation, and hence understand our use of the word "braiding". Graphically, it is the usual braiding condition :



which can be drawn using multiplication branchings as well. We encourage the reader to play with these diagrams. We are now ready to formulate the theorem.

2.37. THEOREM. *Let $(G, H_1, \cdots, H_n) \in$* ***nSub****, then the subgroupoids are optimally coupled if and only if $\Gamma(G, H_1, \cdots, H_n)$ is vacant.*

PROOF. Let us start with $\Rightarrow$, i.e. with a optimally coupled set of subgoupoids. The fact that the subgoupoids are pairwise coupled shows that all boundary double groupoids of $\Gamma(G, H_1, \cdots, H_n)$ are exclusive. Moreover by definition it is slim and all its boundary i-tuple groupoids are slim for $i > 1$. The only criteria left to check is therefore maximality. Suppose that you color the Γ_1 path, the one corresponding to the usual ordering of $[n]$, with elements of the appropriate subgroupoids. For any other path, or equivalently

ordering of $[n]$, there exists a sequence of transposition taking one ordering to the other. Geometrically, each transposition moves the path along a 2-dimensional face of the n-cube. To this sequence corresponds, through the "braiding" isomorphisms, a coloring of the chosen path by elements of the corresponding subgroups that yield the same morphism under composition in G as Γ_1.
We can use this trick to color every paths but for it to correspond to an actual cube, we need to show that the coloring was independent of the sequence of path chosen. This corresponds to requiring that the isomorphism corresponding to two different sequences of transpositions, or tangles, are equal. It is a well known result that this coherence condition is satisfied as long as the Yang-Baxter equation is satisfied. Therefore by definition, there exists a unique cube in $\Gamma(G, H_1, \cdots, H_n)$ for any coloring of the Γ_1 path, which proves vacancy.
Now we can show $\Leftarrow$. Assume that $\Gamma(G, H_1, \cdots, H_n)$ is vacant. Then so are its boundary i-tuple groupoids. For $i = 2$, Lemma 2.34 tells us that the subgroupoids are pairwise coupled. As for $i = 3$, there exists a sequence of paths source-sink differing by only two edges at every step that goes around the 3-cube and comes back to the initial path. the corresponding isomorphism is then forced to be an identity, which is equivalent to the Yang-Baxter condition. ∎

Now that we characterized the preimage of **nVacant** under Γ, we need to find the image of Λ. It is already known that in the case $n = 2$ the equivalence of categories is between vacant double groupoids and factorizations of a groupoid by two subgroupoids [3]. Let's show light on the relationship between optimal coupling and factorizations.

2.38. Definition. *Let G be a groupoid and $(H_1, \cdots, H_n)$ an ordered n-tuple of subgroupoids. The n-tuple is a **factorization** of G provided that every arrow $g \in G$ can be uniquely written as:*

$$g = h_1 h_2 \cdots h_n$$

for $h_i \in H_i$.

We can see factorization as the requirement that the composition $\mu : H_1 \underset{1}{\times} \cdots \underset{1}{\times} H_n \to G$ in **Set** is iso. We will use this criteria in the following proofs.

2.39. Lemma. *Let $(H_1, \cdots, H_n)$ be a factorization of G and (H_i, H_j) be coupled for all $i \neq j \in [n]$. Then the subgroupoids are optimally coupled and for any $\sigma \in S_n$, the set $(H_{\sigma(1)}, \cdots, H_{\sigma(n)})$ is a factorization.*

Proof. Let's start by seeing that, following Lemma 2.35 and thanks to associativity of composition, two coupled subgroups H_i and H_{i+1} give an isomorphism $id \times \cdots \times \Phi_{i,i+1} \times \cdots \times id$:

$$H_1 \underset{1}{\times} \cdots \underset{1}{\times} H_i \underset{1}{\times} H_{i+1} \underset{1}{\times} \cdots \underset{1}{\times} H_n \to H_1 \underset{1}{\times} \cdots \underset{1}{\times} H_{i+1} \underset{1}{\times} H_i \underset{1}{\times} \cdots \underset{1}{\times} H_n$$

such that $\mu = (id \times \cdots \times \Phi_{i,i+1} \times \cdots \times id)\mu$. But since the composition on the left side is an isomorphism, so is the composition on the right side. We have therefore shown that $(H_1, \cdots, H_{i+1}, H_i, \cdots, H_n)$ is a factorization of G.
Since S_n is generated by the transpositions, we have proved the second statement of the lemma. But since all of the orderings of the subgroupoids are factorizations, the two isomorphisms present in the Yang-Baxter equation have to be equal, by uniqueness, proving the optimal coupling. ∎

The above proof has a very nice graphic counterpart. We will leave to the reader the pleasure of discovering it.

2.40. Corollary. *Let τ be a vacant n-tuple groupoid, then $\Lambda(\tau)$ is an optimally coupled factorization.*

Proof. This follows from Theorem 2.26 and Corollary 2.29 ∎

We therefore know that pairwise coupled factorizations give vacant n-tuple groupoids under Γ, since they are optimally coupled but we wish to provide an example showing that optimal coupling doesn't always come from a factorization. Consider a situation where (H_1, H_2, H_3) are optimally coupled but $H_1H_2 \cap H_3$ is not discrete. Then composition is not an isomorphism and the triple does not form a factorization of the groupoid G. Yet they provide a vacant triple groupoid.

We may now show that if we restrict ourselves to optimally coupled sets of subgroupoids, the pair (Λ, Γ) is an adjunction and that this adjunction is an equivalence once restricted to optimally coupled factorizations.

2.41. Definition. *Let $\underline{\textbf{\textit{nCpl}}}$ be the full subcategory of $\underline{\textbf{\textit{nSub}}}$ whose objects are n-tuples $(G, H_1, H_2, \cdots, H_n)$ such that $\{H_i\}_{[n]}$ is optimally coupled, and $\underline{\textbf{\textit{nCplFct}}}$ the full subcategory of $\underline{\textbf{\textit{nCpl}}}$ whose objects are factorizations of G.*

2.42. Theorem. *$\Gamma : \underline{\textbf{\textit{nCpl}}} \to \underline{\textbf{\textit{nVacant}}}$ has Λ for left adjoint. Moreover $(\Lambda, \Gamma|_{\underline{\textbf{\textit{nCplFct}}}})$ is an equivalence of categories.*

Proof. Let $(G, H_1, \cdots, H_n) \in \underline{\mathbf{nCpl}}$ and $\tau \in \underline{\mathbf{nVacant}}$. Then from Lemma 2.26, which states that $\{\tau_i\}_{[n]}$ provides a factorization of $(\tau, \cdot)$, we can build an isomorphism

$$\begin{aligned}&\underline{\mathbf{nCpl}}\big(\Lambda(\tau),\, (G, H_1, \cdots, H_2)\big) \simeq \\ &\{(f_1, \cdots, f_n) | f_i : \tau_i \to H_i \text{ and } f_i = f_j \text{ on objects}\}\end{aligned}$$

moreover τ has at most one n-cube per acceptable 1-boundary, so an n-tuple functor is fixed by its values on arrows, i.e.

$$\begin{aligned}&\underline{\mathbf{nVacant}}\big(\tau,\, \Gamma(G, H_1, \cdots, H_n)\big) \simeq \\ &\{(f_1, \cdots, f_n) | f_i : \tau_i \to H_i \text{ and } f_i = f_j \text{ on objects}\}\end{aligned}$$

proving the adjunction.
Now if $(G, H_1, H_2, \cdots, H_n) \in \underline{\mathbf{nCplFct}}$, every arrow of G can be written uniquely as a composition of arrows of $\{H_i\}$, hence :

$$\underline{\mathbf{nCplFct}}\big((G, H_1, \cdots, H_2), \Lambda(\tau)\big) \simeq \\ \{(f_1, \cdots, f_n) | f_i : H_i \to \tau_i \text{ and } f_i = f_j \text{ on objects}\}$$

and since $\Gamma(G, H_1, \cdots, H_n)$ is vacant, we have an isomorphism:

$$\underline{\mathbf{nVacant}}\big(\Gamma(G, H_1, \cdots, H_n), \tau\big) \simeq \\ \{(f_1, \cdots, f_n) | f_i : H_i \to \tau_i \text{ and } f_i = f_j \text{ on objects}\}$$

which proves the equivalence of categories between $\underline{\mathbf{nCplFct}}$ and $\underline{\mathbf{nVacant}}$. ∎

This theorem allows us to see some decompositions of groups as higher dimensional groups, where dimension is taken in a very categorical sense.

3. Maximally exclusive n-tuple groupoids

In this section we will expand our results to those n-tuple groupoids that would be vacant, would they be slim. We have to add a few technicalities to get an adjunction again.

3.1. Definition. *Let $(\tau_{\hat{1}}, \cdots, \tau_{\hat{n}})$ be the boundary $(n-1)$-tuple groupoids of some n-tuple groupoid τ. Then the **coarse n-tuple groupoid** $\square(\tau_{\hat{1}}, \cdots, \tau_{\hat{n}})$ is the slim n-tuple groupoid such that $\square(\tau_1, \cdots, \tau_n)_i = \tau_i$ and an n-cube exists iff its boundary is admissible. The **frame** $\blacksquare\tau$ of τ is then the image of the functor :*

$$\Pi : \tau \to \square(\tau_{\hat{1}}, \cdots, \tau_{\hat{i}}, \cdots, \tau_{\hat{n}})$$

such that $\Pi s_i = s_i \Pi$ and $\Pi t_i = t_i \Pi$ for all $i \in [n]$.

In this light, a maximally exclusive n-tuple groupoid is one whose frame is vacant. Instrumental to our defining a diagonal composition for n-cubes was the uniqueness provided by vacancy to determine fillers for sub cubes of intermediary depth in the barycentric subdivision . But in the present case this uniqueness is no longer available, though the boundaries of such cubes are fixed. We therefore need to make a consistent choice of fillers to define a diagonal groupoid out of a maximally exclusive n-tuple groupoid.
Let $! : \blacksquare\tau \to \tau$ be a section of Π as n-tuple graphs and $X, Y \in \tau$, then one can use the section to fill the barycentric subdivision of the n-cube with X of depth 0 and Y of depth n. Denote the composite of the subdivision by $X \cdot_! Y$, and the graph defined by $(\tau_{[n]}, s_{[n]}, t_{[n]})$ with the above product by $(\tau, \cdot_!)$.

3.2. Lemma. *Let τ be an n-tuple groupoid. Then if the section $! : \blacksquare\tau \to \tau$ is an n-tuple functor, then $(\tau, \cdot_!)$ is a groupoid.*

PROOF. Let $X, Y, Z \in \tau$ such that $t_{[n]}(X) = s_{[n]}(Y)$ and $t_{[n]}(Y) = s_{[n]}(Z)$. Then $(X \cdot_! Y) \cdot_! Z$ and $X \cdot_! (Y \cdot_! Z)$ are both equal to the cube obtained by composing the subdivision in thirds of the n-cube where X, Y, Z are placed on the diagonal source-sink and all other sub n-cubes are filled with elements given by the section. Since the frame is vacant, such a filling exists and since the interchange laws hold, the composition is associative. The identities of the groupoid are given by the identity n-cubes on objects and inverses are given by the following argument:
Let X be placed in position of depth 0. From a Lemma 2.12, there exists a filling of the barycentric subdivision with section n-cubes such that all boundaries of the composition are identities on $s_{[n]}(X)$. In other words, $\exists Y \in \blacksquare\tau$ such that $X \cdot_! !(Y) = u_X \in \tau_\bullet$. Then:

$$\begin{aligned} X \cdot_! !(Y) \cdot (u_X)^{-i} &= u_X \cdot_! (u_X)^{-i} \\ &= u_X \circ_i (u_X)^{-i} \\ &= \imath_{[n]}(s_{[n]}(X)) \end{aligned}$$

Which shows that X has a right inverse. The same procedure shows that it has a left inverse and therefore that $(\tau, \cdot_!)$ is a groupoid. ■

As promised in the introduction, we can extend Theorem 2.42 to further decompositions of groupoids. Let **nSemiCpl** be the category defined by :

- Objects are $(n+2)$-tuples $(G, A, H_1, H_2, \cdots, H_n)$ where $(G, H_1, \cdots, H_n) \in$ **nCpl**, $A \in G$ is an abelian group bundle on the objects of G such that $ha(h)^{-1} \in A$ for all $h \in H_i$ and all $i \in [n]$.
- Arrows are functors $f : G \to G'$ such that $f(H_i) \subset H'_i$ for all $i \in [n]$ and $f(A) \in A'$.

Let **nSemi** be the full subcategory of **nSemiCpl** where objects $(G, A, H_1, \cdots, H_n)$ are factorizations. Let **nMaxExcl** be the category whose objects are sections of frame maps of n-tuple groupoids and arrows are section preserving n-tuple functors. Then we can build a functor

$$\tilde{\Gamma} : \mathbf{nSemi} \to \mathbf{nMaxExcl}$$

where the n-tuple groupoid of $\tilde{\Gamma}(G, A, H_1, \cdots, H_n)$ has for n-cubes pairs (X, a) with $X \in \Gamma(G, H_1, \cdots, H_n)$ and $a \in A$ and for compositions:

$$(X, a) \circ_i (Y, b) = \big(X \circ_i Y,\, a s_{\hat{i}}(X) b (s_{\hat{i}}(X)^{-1})\big)$$

and has for section $! : \Gamma(G, H_1 \cdots, H_n) \to \tilde{\Gamma}(G, A, H_1 \cdots, H_n)$ given by $!(X) = \big(X, \imath(s_{[n]}(X))\big)$. A direct computation shows that these compositions define an n-tuple groupoid. On arrows of **nSemi**, $\tilde{\Gamma}$ is given by:

$$\tilde{\Gamma}(F)(X, a) = \big(\Gamma(F)(X), F(a)\big)$$

Once again a direct computation shows that this defines a functor. We are now ready to state the theorem.

3.3. Theorem. *The functor $\tilde{\Gamma} : \underline{\textbf{\textit{nSemiSub}}} \to \underline{\textbf{\textit{nMaxExcl}}}$ has a left adjoint $\tilde{\Lambda}$ defined by:*

$$\begin{aligned}\tilde{\Lambda}(\tau, !) &= \big((\tau, \cdot_!), \tau_\bullet, \tau_1, \cdots, \tau_n\big)\\ \tilde{\Lambda}(F) &= F\end{aligned}$$

Moreover $(\tilde{\Gamma}|_{\underline{\textbf{\textit{nSemi}}}}, \tilde{\Lambda})$ is an equivalence of categories

Proof. Let τ, ω be maximally exclusive, then :

$$\underline{\textbf{nMaxExcl}}(\omega, \tau) \simeq \{(F_0, F_1, \cdots, F_n) | F_0 : \omega_\bullet \to \tau_\bullet \text{ and } F_i : \omega_i \to \tau_i\}$$

Moreover for $(G, A, H_1, \cdots, H_n)$ and $(K, B, L_1, \cdots, L_n) \in \underline{\textbf{nSemi}}$, we have:

$$\begin{gathered}\underline{\textbf{nSemi}}\big((G, H_1, \cdots, H_n), (K, B, L_1, \cdots, L_n)\big) \simeq\\ \{(F_0, F_1, \cdots, F_n) | F_0 : A \to B \text{ and } F_i : H_i \to L_i\}\end{gathered}$$

Considering that the image of $\tilde{\Lambda}$ is by definition in $\underline{\textbf{nSemi}}$, it is enough, following the argument of Theorem 2.42 to prove the two statements. ■

3.4. Examples. Following the work on Lie double groupoids and algebroids of MacKenzie [10], we can define the notions of Lie n-tuple groupoid. Then we may use decompositions such as the Iwasawa decomposition to present certain Lie groups as n-tuple groups. In our previous paper we presented the Poincaré group as a maximal exclusive double group and hinted towards a possible corresponding triple group. We can now state what needs to be checked to validate this claim.

3.5. Theorem. *Every Iwasawa decomposition $G = KAN$ of a semi-simple Lie group corresponds to a vacant double group with boundary groups K and AN. Moreover there exists a corresponding vacant triple group with boundary groups K, A and N if and only if $KA = AK$ and $KN = NK$.*

Proof. An Iwasawa decomposition of a group G is a factorization (G, K, A, N) stemming from a decomposition of its Lie algebra $\mathfrak{g}$ as a direct sum $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, see [8] for details. Since $\mathfrak{a} \oplus \mathfrak{n}$ is a subalgebra, AN is a subgroup of G. Therefore, according to Lemma 2.39 and Theorem 2.37, the triple (G, K, AN) gives a vacant double group under Γ.
From the direct sum, it is clear that $K \cap A$, $A \cap N$ and $K \cap N$ are trivial. According to lemma 2.39, it suffices for the other two pairs of factors to be coupled to give a vacant triple group. The only remaining criteria are then the ones listed in the lemma. ■

Our main interest was in the possible new ways of seeing the Poincaré group and we had hoped for it to be giving a triple group. Unfortunately it is not the case, or at least not as originally conceived.

3.6. COROLLARY. *The Poincaré group has a decomposition of the form :*

$$Poinc \simeq (KAN) \ltimes \mathbb{R}^4_+$$

where K, A, and N are an Iwasawa decomposition of SO(3,1) given by:

$$K := exp\left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & -a & 0 & c \\ 0 & -b & -c & 0 \end{bmatrix} \quad |a, b, c \in \mathbb{R} \right\} \simeq SO(3)$$

$$A := exp\left\{ \begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad |a \in \mathbb{R} \right\} \simeq SO(1,1)$$

$$N := exp\left\{ \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & a & b \\ a & -a & 0 & 0 \\ b & -b & 0 & 0 \end{bmatrix} \quad |a, b \in \mathbb{R} \right\}$$

The double group $\tilde{\Gamma}(Poinc, \mathbb{R}^4_+, SO(3), AN)$ is a maximal exclusive double group. However, $\tilde{\Gamma}(Poinc, \mathbb{R}^4_+, SO(3), SO(1,1), N)$ is not maximal.

PROOF. The Iwasawa decomposition of the Lorentz group $SO(3,1)$ is a standard computation, see [7], and the semi-direct product with the translation group is a well known feature of Euclidian and Minkowskian isometry groups. From Theorems 3.3 and 3.5, $\tilde{\Gamma}(Poinc, \mathbb{R}^4_+, SO(3), AN)$ gives a maximal exclusive double group. A quick computation shows that:

$$[\mathfrak{a}, \mathfrak{k}] = \left\{ \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 \\ -b & 0 & 0 & 0 \end{bmatrix} \quad |a \in \mathbb{R} \right\} \subset \mathfrak{g}$$

does not belong to $\mathfrak{k} \oplus \mathfrak{a}$. Therefore $\mathfrak{k} \oplus \mathfrak{a}$ is not a Lie subalgebra of $\mathfrak{g}$, and KA not a subgroup of G. From theorem 3.5 we know that there does not exist a corresponding maximally exclusive triple group. ∎

There exists of course a way to get a vacant triple group out of the Poincaré group. We may consider the factorization $(Poinc, \mathbb{R}^4_+, SO(3), AN) \in$ **3Sub** and note that as $\mathbb{R}^4_+$ is normal in $Poinc$, so that it is coupled with $SO(3)$ and AN. Since one of these subgroups is normal, the coupling is optimal (this is very easy to prove) and we can then conclude that $(Poinc, \mathbb{R}^4_+, SO(3), AN)$ is in **3CplFct** and has an associated vacant triple group.

Conclusion

We have in these pages completely characterized maximally exclusive n-tuple groupoids, generalizing the results on vacant double groupoids obtained in the past by various authors. It remains that these cases are some of the most simple cases available. We found that dropping the exclusivity assumption results in a much more complicated structure. Even within these simple cases some questions remain unanswered. A complete and precise definition of core diagram [12] has yet to be given for the cases $n > 2$ and it seems that it would be a weaker invariant than in the two dimensional case. The classification of the classes of double groupoids that have isomorphic core diagrams has not been found either. More importantly the proper representation theory of these entities has not been discussed anywhere, to our knowledge. Considering that a group as important as the Poincaré group is an example of double and triple group, as other Lie groups, it seems urgent to take a look at these questions. It is our hope that this quick exposition of the subject matter will encourage further development of higher dimensional group theory and representation theory, beyond the globular approach discussed in [5, 6].

Acknowledgements: The author would like to thank Louis Crane and David Yetter for their help and support, as well as Victor Turchin for his critical reading.

References

[1] A. L. Agore and G. Militaru. Extending structures i: the level of groups. November 2010, 1011.1633.

[2] Nicolas Andruskiewitsch and Sonia Natale. Double categories and quantum groupoids. *Publ. Mat. Urug.*, 10:11–51., 2005, math/0308228.

[3] Nicolas Andruskiewitsch and Sonia Natale. The structure of double groupoids. *J. Pure Appl. Algebra Volume 1031-1045*, J. 1031-1045:Pages 1031–1045, 2009, math/0602497.

[4] Ronald Brown. Double groupoids, matched pairs and then matched triples. April 2011, arXiv:1104.1644.

[5] L. Crane and M. D. Sheppeard. 2-categorical poincare representations and state sum applications. 2007, math/0306440.

[6] Louis Crane and David Yetter. Measurable categories and 2-groups. *Applied Categorical Structures*, 13:501–516, 2005, math/0305176. 10.1007/s10485-005-9004-5.

[7] Jean Gallier. Notes on differential geometry and Lie groups. *University of Pennsylvannia*, 2012.

[8] A.W. Knapp. *Lie Groups Beyond an Introduction.* Progress in Mathematics. Birkhäuser Boston, 2002.

[9] T. Leinster. *Higher operads, higher categories*, volume 298. Cambridge Univ Pr, 2004.

[10] Kirill C. H. Mackenzie. Double Lie algebroids and second-order geometry, ii. *Adv. Math.*, 154(1):46–75, 1997.

[11] Kirill C. H. Mackenzie. Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids. *J. Reine Angew. Math.*, 658:193–245, 2011.

[12] Dany Majard. On double groups and the Poincaré group. December 2011, 1112.6208.

[13] S. Majid. *Foundations of Quantum Group Theory*. Cambridge University Press, 2000.

[14] R.Dawson-R.Pare. General associativity and general composition for double categories. *Cahiers de topologie et geometrie differentielle categoriques*, 33(1):57–79, 1993.

[15] R. Rosebrugh and R.J. Wood. Distributive laws and factorization. *Journal of Pure and Applied Algebra*, 175(1):327–353, 2002.

Masaryk University Faculty of Science
Department of Mathematics and Statistics
Kotlářská 2
611 37 Brno, Czech Republic
Email: majardd@math.muni.cz

This article may be accessed at http://www.tac.mta.ca/tac/ or by anonymous ftp at ftp://ftp.tac.mta.ca/pub/tac/html/volumes/28/12/28-12.{dvi,ps,pdf}

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.
Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at `http://www.tac.mta.ca/tac/` and by ftp. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

INFORMATION FOR AUTHORS The typesetting language of the journal is TEX, and LATEX2e strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at `http://www.tac.mta.ca/tac/`.

MANAGING EDITOR. Robert Rosebrugh, Mount Allison University: `rrosebrugh@mta.ca`

TEXNICAL EDITOR. Michael Barr, McGill University: `barr@math.mcgill.ca`

ASSISTANT TEX EDITOR. Gavin Seal, Ecole Polytechnique Fédérale de Lausanne: `gavin_seal@fastmail.fm`

TRANSMITTING EDITORS.
Clemens Berger, Université de Nice-Sophia Antipolis: `cberger@math.unice.fr`
Richard Blute, Université d' Ottawa: `rblute@uottawa.ca`
Lawrence Breen, Université de Paris 13: `breen@math.univ-paris13.fr`
Ronald Brown, University of North Wales: `ronnie.profbrown(at)btinternet.com`
Valeria de Paiva: `valeria.depaiva@gmail.com`
Ezra Getzler, Northwestern University: `getzler(at)northwestern(dot)edu`
Kathryn Hess, Ecole Polytechnique Fédérale de Lausanne: `kathryn.hess@epfl.ch`
Martin Hyland, University of Cambridge: `M.Hyland@dpmms.cam.ac.uk`
Anders Kock, University of Aarhus: `kock@imf.au.dk`
Stephen Lack, Macquarie University: `steve.lack@mq.edu.au`
F. William Lawvere, State University of New York at Buffalo: `wlawvere@buffalo.edu`
Tom Leinster, University of Edinburgh: `Tom.Leinster@ed.ac.uk`
Ieke Moerdijk, Radboud University Nijmegen: `i.moerdijk@math.ru.nl`
Susan Niefield, Union College: `niefiels@union.edu`
Robert Paré, Dalhousie University: `pare@mathstat.dal.ca`
Jiri Rosicky, Masaryk University: `rosicky@math.muni.cz`
Giuseppe Rosolini, Università di Genova: `rosolini@disi.unige.it`
Alex Simpson, University of Edinburgh: `Alex.Simpson@ed.ac.uk`
James Stasheff, University of North Carolina: `jds@math.upenn.edu`
Ross Street, Macquarie University: `street@math.mq.edu.au`
Walter Tholen, York University: `tholen@mathstat.yorku.ca`
Myles Tierney, Rutgers University: `tierney@math.rutgers.edu`
Robert F. C. Walters, University of Insubria: `robert.walters@uninsubria.it`
R. J. Wood, Dalhousie University: `rjwood@mathstat.dal.ca`