Theory and Applications of Categories, Vol. 25, No. 13, 2011, pp. 342–367.

COVARIANT PRESHEAVES AND SUBALGEBRAS

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Abstract. For small involutive and integral quantaloids $\mathcal{Q}$ it is shown that covariant presheaves on symmetric $\mathcal{Q}$-categories are equivalent to certain subalgebras of a specific monad on the category of symmetric $\mathcal{Q}$-categories. This construction is related to a weakening of the subobject classifier axiom which does not require the classification of all subalgebras, but only guarantees that classifiable subalgebras are uniquely classifiable. As an application the identification of closed left ideals of non-commutative C^*-algebras with certain «open» subalgebras of freely generated algebras is given.

Introduction

Let $\mathcal{Q}$ be a small involutive quantaloid. Then $\mathcal{Q}$ induces an involution on the quantaloid of symmetric $\mathcal{Q}$-categories and distributors. In particular the involute of every contravariant presheaf is covariant and vice versa. In this framework we ask the question whether there exists a concept of a weak subobject classifier in the sense that a subobject classified by a covariant presheaf is always uniquely classified. For this purpose we introduce a special kind of presheaves on symmetric $\mathcal{Q}$-categories which we call weak singletons. As a first property we note that weak singletons form a non-idempotent monad.

Weak singletons appear already in the theory of metric spaces. If symmetric $\mathcal{Q}$-categories are metric spaces, then maximal weak singletons coincide with extremal functions (cf. [11]). Further there is a close relationship between weak singletons and the type of singletons considered by H. Heymans (cf. [8]). In fact, if the Cauchy completion preserves the symmetry axiom (cf. [9], see also Proposition 3.1), then the monad associated with the Cauchy completion is a submonad of the weak singleton monad.

After this brief historical digression, we return to the problem of unique classification of subobjects. Let $\mathbf{W}$ denote the weak singleton monad. For every object a of the given involutive quantaloid $\mathcal{Q}$ there exists a $\mathbf{W}$-algebra structure on the free cocompletion of the trivial $\mathcal{Q}$-category $\underline{a}$. If $\mathcal{Q}$ is integral, this $\mathbf{W}$-algebra serves as a weak subobject classifier in the category of $\mathbf{W}$-algebras. Under the assumption of the integrality of $\mathcal{Q}$ we show that classifiable subalgebras are uniquely classifiable (cf. Theorem 4.3). Moreover, the classifiable hull of every subalgebra exists (cf. Section 5). Since in general there exist more subalgebras than «characteristic morphisms» , this property might be of some

Received by the editors 2010-04-23 and, in revised form, 2011-07-06.
Transmitted by Ieke Moerdijk. Published on 2011-07-14.
2000 Mathematics Subject Classification: 06F07, 18C15, 18D20, 18F20.
Key words and phrases: Involutive quantale, involutive quantaloid, symmetric $\mathcal{Q}$-category, covariant presheaf, monad of weak singletons, classifiable subalgebra, closed left ideal.

importance for a non-commutative model theory based on integral and involutive, small quantaloids.

The previous theory has an immediate application to $\mathcal{Q}$-enriched presheaves on symmetric $\mathcal{Q}$-categories. Because of the universal property of free $\mathbf{W}$-algebras every covariant presheaf on a symmetric $\mathcal{Q}$-category $\mathbb{X}$ has a unique extension to a $\mathbf{W}$-homomorphism on the *free* $\mathbf{W}$*-algebra* generated by $\mathbb{X}$. Then we conclude from Theorem 4.3 that covariant presheaves on symmetric $\mathcal{Q}$-categories are equivalent to classifiable subalgebras of free $\mathbf{W}$-algebras. Moreover there exists a functor from the category of $\mathbf{W}$-algebras to $\mathbf{Set}^{\mathcal{Q}}$ preserving monomorphisms. Hence covariant presheaves are not only subalgebras, but also traditional set-valued functors with domain $\mathcal{Q}$.

As a simple application of the previous results we make a contribution to the problem of constructing a non-commutative topological space for spectra of non-commutative C^*-algebras. Since Hilbert spaces occurring in the Gelfand-Naimark-Segal-construction depend usually on the respective pure states, it seems to be reasonable to choose an integral and involutive quantaloid as basis for the necessary considerations. Then on the set of all locally irreducible representations of a given unital, non-commutative C^*-algebra (cf. Section 6) we introduce an appropriate structure of a symmetric $\mathcal{Q}$-category. This approach leads to the $\mathcal{Q}$-category $\mathbb{I}_{loc}$. Then closed left ideals are identified with certain covariant presheaves on $\mathbb{I}_{loc}$. Subsequently, based on the equivalence between covariant presheaves and classifiable subalgebras of free algebras we obtain that closed left ideals can be interpreted as «open» subalgebras of the free algebra generated by $\mathbb{I}_{loc}$. In this sense free algebras of the weak singleton monad can be understood as non-commutative spaces of non-commutative C^*-algebras.

1. Quantaloids

In order to fix notation we begin with some basic definitions and properties from the theory of quantaloids (cf. [23]). A small quantaloid is a small category $\mathcal{Q}$ provided with the following additional properties:

- each hom-set is a complete lattice,
- the composition $\cdot$ of morphisms preserves arbitrary joins in both variables.

Objects (resp. morphisms) of $\mathcal{Q}$ are denoted by small Roman (resp. Greek) letters. Further, we write $\mathcal{Q}(a,b)$ for hom-sets of the form $\hom(a,b)$.

By the special adjoint functor theorem, for every morphism $a \xleftarrow{\lambda} b$ both maps

$$\lambda \cdot \_ \,:\, \mathcal{Q}(c,b) \to \mathcal{Q}(c,a) \quad \text{and} \quad \_ \cdot \lambda \,:\, \mathcal{Q}(a,c) \to \mathcal{Q}(b,c)$$

have right adjoints $\lambda \searrow \_$ and $\_ \swarrow \lambda$ which are determined by

$$\lambda \searrow \alpha \;=\; \bigvee \{\beta \in \mathcal{Q}(c,b) \,|\, \lambda \cdot \beta \;\le\; \alpha\}, \qquad \beta \swarrow \lambda \;=\; \bigvee \{\gamma \in \mathcal{Q}(a,c) \,|\, \gamma \cdot \lambda \;\le\; \beta\}.$$

It is easily seen that the following relations hold:

$$(\lambda \searrow \alpha) \cdot (\alpha \searrow \nu) \;\leq\; \lambda \searrow \nu \quad \text{and} \quad (\beta \swarrow \lambda) \cdot (\lambda \swarrow \nu) \;\leq\; \beta \swarrow \nu.$$

Further, $\mathcal{Q}(a,a)$ is always a unital quantale. A quantaloid $\mathcal{Q}$ is *integral* if for all objects a of $\mathcal{Q}$ the unit 1_a is the universal upper bound in $\mathcal{Q}(a,a)$.

A quantaloid $\mathcal{Q}$ is called *involutive* iff there exists a contravariant functor $j : \mathcal{Q} \to \mathcal{Q}$ satisfying the following conditions (cf. [6, 23]):

(q1) $j(a) = a$ (i.e. $j(1_a) = 1_a$) for all object a in $\mathcal{Q}$.

(q2) $j^2(\alpha) = \alpha$ for every morphism α in $\mathcal{Q}$.

(q3) If $\alpha,\, \beta \in \mathcal{Q}(a,b)$ with $\alpha \leq \beta$, then $j(\alpha) \leq j(\beta)$.

In order to shorten the notation we write α^o for $j(\alpha)$. In any involutive quantaloid the operations $\searrow$ and $\swarrow$ are mutually determined by each other (cf. Proposition 2.9 in [8]) — i.e.

$$(\alpha \searrow \beta)^o \;=\; \beta^o \swarrow \alpha^o. \tag{1.1}$$

With regard to the standing assumption in Section 4 we give here three examples of integral and involutive quantaloids which will be of some importance throughout this paper. We begin with a remark devoted to involutive quantales.

1.1. Remark. Let $Q = (Q, \leq, \&)$ be an arbitrary quantale according to the terminology in [22]. In this context quantales are not necessarily unital (i.e. quantales are not the same as one-object quantaloids (cf. Example 1.2)). The left- and right implication in Q are determined by:

$$\alpha \swarrow \beta \;=\; \bigvee \{\gamma \in Q \,|\, \gamma \,\&\, \beta \;\leq\; \alpha\}, \quad \alpha \searrow \beta \;=\; \bigvee \{\gamma \in Q \,|\, \alpha \,\&\, \gamma \;\leq\; \beta\}.$$

An element α is a left-divisor of an element β iff $\beta = \alpha \,\&\, (\alpha \searrow \beta)$; and β is a right-divisor of α iff $\alpha = (\alpha \swarrow \beta) \,\&\, \beta$.
Further, a unital quantale Q is *integral* if the unit coincides with the universal upper bound in Q. A quantale Q is *involutive* if Q is provided with an order preserving involution ^{o} s.t. (cf. [19, 21]):

$$(\alpha \,\&\, \beta)^o \;=\; \beta^o \,\&\, \alpha^o.$$

An element α of an involutive quantale is *symmetric* if $\alpha^o = \alpha$. Obviously, the universal lower (resp. upper) bound $\perp$ (resp. $\top$) is always symmetric. In the event that Q is unital, then also the unit is symmetric.
An important source of involutive and unital quantales are complete lattices L with a duality (resp. order reversing involution), that is, with a unary operation $'$ satisfying the following conditions:

$$\ell'' \;=\; \ell, \qquad \big(\bigvee_{i\in I} \ell_i\big)' \;=\; \bigwedge_{i\in I} \ell_i', \qquad \ell \in L, \quad \{\ell_i \mid i \in I\} \subseteq L.$$

The details of this situation are as follows (cf. Example 4 and 5 in [21]). On the set $Q(L)$ of all join preserving maps $L \xrightarrow{\sigma} L$ we define a partial ordering $\leq$ by:

$$\sigma_1 \leq \sigma_2 \iff \forall \ell \in L: \sigma_1(\ell) \leq \sigma_2(\ell).$$

Then $Q(L) = (Q(L), \leq, \&)$ is a unital quantale where $\&$ is determined by the usual composition of maps — i.e.

$$\sigma_1 \& \sigma_2(\ell) = \sigma_1(\sigma_2(\ell)), \qquad \ell \in L,$$

and the unit is given by the identity map id_L. Further, the order reversing involution on L induces a self-mapping $\sigma \longmapsto \sigma^o$ on $Q(L)$ by $\sigma^o(\ell) = \sigma_*(\ell')'$, $\ell \in L$ where σ_* is the right adjoint of σ — i.e.

$$\sigma_*(b) = \bigvee\{\ell \in L \mid s(\ell) \leq b\}, \qquad b \in L.$$

Obviously, $\sigma \longmapsto \sigma^o$ is an order preserving involution on $Q(L)$. Since adjoint situations can be composed, we obtain $(\sigma_1 \& \sigma_2)^o = \sigma_2^o \& \sigma_1^o$. Hence $Q(L) = (Q(L), \leq, \&, {}^o)$ is involutive and unital quantale.
If L has at least three elements, then $Q(L)$ is never integral. Therefore our interest goes to the involutive and integral subquantale $Q(L)_{\mathrm{id}}$ of $Q(L)$ whose support set is given by

$$Q(L)_{\mathrm{id}} = \{\sigma \in Q(L) \mid \sigma \leq \mathrm{id}_L\}.$$

The answer to the question, whether $Q(L)_{\mathrm{id}}$ is commutative or not, depends on the structure of the underlying lattice. If $(L,')$ is the ortho-lattice of all closed subspaces of a Hilbert space (see also Section 6), then $Q(L)_{\mathrm{id}}$ is commutative and is isomorphic to the power set of the set of all atoms of L. On the other hand, if $(L,')$ is given by the real unit interval $[0,1]$ provided with the order reversing involution $x \mapsto 1-x$, then it is easily seen that $Q([0,1])_{\mathrm{id}}$ is non-commutative.

The following list of examples shows that there exist at least three different procedures of constructing integral and involutive quantaloids from quantales.

1.2. Example. Let Q be an integral and involutive quantale. Then Q induces an integral and involutive quantaloid as follows: The set of objects is a singleton — i.e. $\mathrm{obj}(\mathcal{Q}) = \{*\}$, and the set of morphism is given by $\mathrm{mor}(\mathcal{Q}) = \mathcal{Q}(*,*) = Q$ where the composition, the involution and the lattice structure on $\mathcal{Q}(*,*)$ is determined by Q.

1.3. Example. Let Q be an arbitrary involutive quantale. Since Q is not necessarily unital, we need some more terminology. An element $\alpha \in Q$ is *stable*, if α is left- and right-divisor of α. Obviously, α is stable iff the relation

$$\alpha \& (\alpha \searrow \alpha) = \alpha = (\alpha \swarrow \alpha) \& \alpha \tag{1.2}$$

holds. Typical examples of stable elements are the following ones: every element in a unital quantale is stable. In non-unital quantales every idempotent element is stable.

After these preparations we can construct a small, involutive and integral quantaloid $\mathcal{Q}$ from Q as follows[1]:

- The set of *objects* of $\mathcal{Q}$ is the set $SH(Q)$ of all stable and symmetric elements of Q.
- For $a, b \in SH(Q)$ the hom-set $\mathcal{Q}(a,b)$ consists of all $\lambda \in Q$ satisfying the conditions:

$$\lambda \,\leq\, a \wedge b, \quad a \text{ is left divisor and } b \text{ is right divisor of } \lambda.$$

- For every triple (a,b,c) the composition law $\mathcal{Q}(a,b) \times \mathcal{Q}(b,c) \longrightarrow \mathcal{Q}(a,c)$ is determined by:

$$\lambda_2 \cdot \lambda_1 \,=\, \lambda_1 \,\&\, (b \searrow \lambda_2), \qquad \lambda_1 \in \mathcal{Q}(a,b),\ \lambda_2 \in \mathcal{Q}(b,c),$$

  where $\searrow$ denotes the right implication in Q (cf. Remark 1.1).
- The partial ordering on $\mathcal{Q}(a,b)$ is inherited from Q. In particular, the universal upper bound in $\mathcal{Q}(a,a)$ coincides with a.

Because of

$$a \,\&\, (\bigvee_{j \in J} (a \searrow \lambda_j) \,=\, \bigvee_{j \in J} \lambda_j \,=\, (\bigvee_{j \in J} (\lambda_j \swarrow b)) \,\&\, b$$

$\mathcal{Q}(a,b)$ is a complete lattice and joins are computed in Q. For $(\lambda_1, \lambda_2) \in \mathcal{Q}(a,b) \times \mathcal{Q}(b,c)$ the relation

$$(\lambda_1 \swarrow b) \,\&\, \lambda_2 \,=\, \lambda_1 \,\&\, (b \searrow \lambda_2) \tag{1.3}$$

holds. Thus $\lambda_2 \cdot \lambda_1 \in \mathcal{Q}(a,c)$ follows from (1.2), (1.3) and the stability of b. The associativity axiom is evident, and the universal upper bound in $\mathcal{Q}(a,a)$ is the unit in $\mathcal{Q}(a,a)$. Referring again to (1.3) it is easily seen that the composition preserves arbitrary joins in both variables. Hence $\mathcal{Q}$ is an integral quantaloid.

A contravariant endofunctor j on $\mathcal{Q}$ is determined by the involution ^{o} on Q — i.e. $j(\lambda) = \lambda^o,\ \lambda \in \mathrm{mor}(\mathcal{Q})$. Thus $\mathcal{Q}$ is also an involutive quantaloid.

Finally, if Q is integral, then the universal upper bound $\top$ of Q plays also a special role in $\mathcal{Q}$. For every object b of $\mathcal{Q}$ the hom-set $\mathcal{Q}(b,\top)$ is dominating — this means that the following relation holds:

$$\mathcal{Q}(b,\top) \,=\, \bigcup_{a \in SH(Q)} \mathcal{Q}(b,a). \tag{1.4}$$

[1] Here we make use of a more set-theoretical language in the presentation of categorical axioms (see e.g. p. 4 in [3]).

1.4. Example. We maintain the notation from Example 1.3. Further, let Q be an involutive quantale which is not integral. Typical examples are involutive and unital quantales having left-sided elements which are not right-sided. In this framework the formula (1.4) is in general not valid — e.g. $\bigcup\limits_{a\in SH(Q)} \mathcal{Q}(\top, a)$ is not always a hom-set. This observation is a motivation to enlarge the quantaloid $\mathcal{Q}$ constructed in Example 1.3. We add a further object ω to $\mathcal{Q}$ and introduce the following additional hom-sets:

- $\mathcal{Q}(\omega, a) \;=\; \{\lambda \in Q \mid \lambda \le a,\ a \text{ is a right divisor of } \lambda\}, \quad a \in SH(Q),$
- $\mathcal{Q}(a, \omega) \;=\; \{\lambda \in Q \mid \lambda \le a,\ a \text{ is a left divisor of } \lambda\}, \quad a \in SH(Q),$
- $\mathcal{Q}(\omega, \omega) \;=\; \{0, 1\}.$

The extension of the composition law is determined by:

- $\mathcal{Q}(\omega, b) \times \mathcal{Q}(b, c) \longrightarrow \mathcal{Q}(\omega, c)$ is defined by: $\lambda_2 \cdot \lambda_1 \;=\; \lambda_1 \,\&\, (b \searrow \lambda_2),$
- $\mathcal{Q}(a, b) \times \mathcal{Q}(b, \omega) \longrightarrow \mathcal{Q}(a, \omega)$ is defined by: $\lambda_2 \cdot \lambda_1 \;=\; \lambda_1 \,\&\, (b \searrow \lambda_2),$
- $\mathcal{Q}(a, \omega) \times \mathcal{Q}(\omega, c) \longrightarrow \mathcal{Q}(a, c)$ is defined by: $\lambda_2 \cdot \lambda_1 \;=\; \perp,$
- $\mathcal{Q}(\omega, b) \times \mathcal{Q}(b, \omega) \longrightarrow \mathcal{Q}(\omega, \omega)$ is defined by: $\lambda_2 \cdot \lambda_1 \;=\; 0,$
- 1 acts as unit in $\mathcal{Q}(\omega, \omega)$, and consequently 0 is the zero element in $\mathcal{Q}(\omega, \omega)$.

It is a matter of routine to check that the associativity and the identity axioms hold. Finally, the involution has the following extension:

$$\lambda \in \mathcal{Q}(\omega, b) \cup \mathcal{Q}(b, \omega),\ j(\lambda) = \lambda^o, \quad j(1) = 1,\ j(0) = 0.$$

Hence the enlargement of $\mathcal{Q}$ by the addition of ω leads again to an integral and involutive quantaloid satisfying the following additional property:

$$\bigcup_{a\in SH(Q)} \mathcal{Q}(b, a) \;\subseteq\; \mathcal{Q}(b, \omega), \qquad b \in SH(Q).$$

2. Presheaves on symmetric $\mathcal{Q}$-categories and the weak envelope

Let $\mathcal{Q}$ be a small quantaloid. A *$\mathcal{Q}$-category* is a triple $\mathbb{X} = (X, e_X, d_X)$ where X is a set, $X \xrightarrow{e_X} \mathrm{obj}(\mathcal{Q})$ and $X \times X \xrightarrow{d_X} \mathrm{mor}(\mathcal{Q})$ are maps subjected to the following axioms for all $x, y, z \in X$ (cf. [2, 26]):

(Q1) $d_X(x,y) \in \mathcal{Q}(e_X(y), e_X(x))$,

(Q2) $1_{e_X(x)} \le d_X(x,x)$,

(Q3) $d_X(x,y) \cdot d_X(y,z) \le d_X(x,z)$.

A $\mathcal{Q}$-category $\mathbb{X}$ is *skeletal* iff for all $x, y \in X$ the following implication holds:

$$e_X(x) = e_X(y) \text{ and } 1_{e_X(x)} \le d_X(x,y) \wedge d_X(y,x) \text{ imply } x = y.$$

A *$\mathcal{Q}$-functor* $\mathbb{X} \xrightarrow{\varphi} \mathbb{Y}$ between $\mathcal{Q}$-categories is a map $X \xrightarrow{\varphi} Y$ satisfying the following axioms for all $x, y \in X$:

(M1) $e_X(x) = e_Y\big(\varphi(x)\big)$,

(M2) $d_X(x,y) \le d_Y\big(\varphi(x), \varphi(y)\big)$.

$\mathcal{Q}$-categories and $\mathcal{Q}$-functors form a category $\mathbf{Cat}(\mathcal{Q})$ in an obvious way. It is not difficult to show that $\mathbf{Cat}(\mathcal{Q})$ is wellpowered, complete and cocomplete. Hence $\mathbf{Cat}(\mathcal{Q})$ is an (epi, extremal mono)-category. In particular, a $\mathcal{Q}$-functor $\mathbb{X} \xrightarrow{\varphi} \mathbb{Y}$ is an extremal monomorphism iff φ is an injective map provided with the following property for all $x_1, x_2 \in X$:

(EM) $d_X(x_1,x_2) = d_Y(\varphi(x_1), \varphi(x_2))$. (Fully Faithfulness)

The terminal object $\mathbf{1}$ in $\mathbf{Cat}(\mathcal{Q})$ has the form $(\mathrm{obj}.\mathcal{Q}, e_{\mathbf{1}}, d_{\mathbf{1}})$ where $e_{\mathbf{1}} = \mathrm{id}_{\mathrm{obj}\mathcal{Q}}$ and $d_{\mathbf{1}}(a,b) = \bigvee \mathcal{Q}(b,a)$.

From now on, let $\mathcal{Q}$ be an involutive small quantaloid. With the involution on $\mathcal{Q}$, it now makes sense to define a $\mathcal{Q}$-category $\mathbb{X}$ to be *symmetric* when the following axiom holds (cf. [8]):

(Q4) $d_X(x,y) = d_X(y,x)^o$ for all $x, y \in X$. (*Symmetry*)

The terminal object $\mathbf{1}$ in $\mathbf{Cat}(\mathcal{Q})$ is symmetric. Further, symmetric $\mathcal{Q}$-categories form a full subcategory $\mathbf{sCat}(\mathcal{Q})$ of $\mathbf{Cat}(\mathcal{Q})$. It is easily seen that $\mathbf{sCat}(\mathcal{Q})$ is also wellpowered, complete and cocomplete. Moreover, $\mathbf{sCat}(\mathcal{Q})$ is a coreflective subcategory of $\mathbf{Cat}(\mathcal{Q})$, and the corresponding coreflector is given by the symmetrization of $\mathcal{Q}$-categories (X, e_X, d_X) — i.e. d_X is replaced by d_X^s defined by (cf. [9]):

$$d_X^s(x,y) = d_X(x,y) \wedge d_X(y,x)^o, \quad x, y \in X.$$

Let a be an object of $\mathcal{Q}$. In order to recall the concept of covariant presheaves of type a we first have to specify the $\mathcal{Q}$-category $\mathbb{S}_a = (S_a, e_a, d_a)$ of «morphisms of type a» :

$$S_a = \{\lambda \in \mathrm{mor}(\mathcal{Q}) \mid \mathrm{codom}(\lambda) = a\}, \quad d_a(\lambda_1, \lambda_2) = \lambda_1 \searrow \lambda_2, \quad e_a(\lambda) = \mathrm{dom}(\lambda).$$

Obviously, $\mathbb{S}_a$ is the free cocompletion of the trivial $\mathcal{Q}$-category $\underline{a}$. Further, let $\mathbb{X}$ be an arbitrary $\mathcal{Q}$-category. A $\mathcal{Q}$-functor $\mathbb{X} \xrightarrow{\varphi} \mathbb{S}_a$ is called *a covariant presheaf on* $\mathbb{X}$ *of type* $a \in \mathcal{Q}$ (cf. [25]). A simple characterization of covariant presheaves can be given as follows: a map $X \xrightarrow{\varphi} \mathrm{mor}(\mathcal{Q})$ is a covariant presheaf on $\mathbb{X}$ of type a iff φ satisfies the following conditions:

(P1) $\varphi(x) \in \mathcal{Q}(e_X(x), a)$ for all $x \in X$.

(P2) $\varphi(x) \cdot d_X(x,y) \leq \varphi(y)$ for all $x, y \in X$. (*Right Extensionality*)

Hence covariant presheaves on $\mathbb{X}$ of type a and distributors from $\mathbb{X}$ to the trivial $\mathcal{Q}$-category $\underline{a}$ are equivalent concepts. It is not surprising that covariant presheaves on $\mathcal{Q}$-categories constitute again a $\mathcal{Q}$-category. Therefore let $\mathbb{X}$ be a $\mathcal{Q}$-category and $P_\ell(\mathbb{X})$ be the set of all pairs (a, φ) where a is an object of $\mathcal{Q}$ and φ is a covariant presheaf on $\mathbb{X}$ of type a. We specify the following maps $P_\ell(\mathbb{X}) \xrightarrow{e_\ell} \mathrm{obj}(\mathcal{Q})$ and $P_\ell(\mathbb{X}) \times P_\ell(\mathbb{X}) \xrightarrow{d_\ell} \mathrm{mor}(\mathcal{Q})$:

$$e_\ell(a, \varphi) = a, \quad d_\ell((a,\varphi),(b,\psi)) = \bigwedge_{x \in X} \varphi(x) \swarrow \psi(x).$$

Then it is easily seen that $\mathbb{P}_\ell(\mathbb{X}) = (P_\ell(\mathbb{X}), e_\ell, d_\ell)$ is a $\mathcal{Q}$-category and the (covariant) Yoneda embedding $\mathbb{X} \xrightarrow{\eta_{\mathbb{X}}} \mathbb{P}_\ell(\mathbb{X})$ (enriched in $\mathcal{Q}$) has the form

$$\eta_{\mathbb{X}}(x) = (e_X(x), \widetilde{x}), \qquad \text{where} \quad \widetilde{x}(z) = d_X(x,z), \quad x, z \in X.$$

It is well known that $(\eta_{\mathbb{X}}, \mathbb{P}_\ell(\mathbb{X}))$ is the *free completion* of $\mathbb{X}$ (cf. [25]) and the following important formula

$$d_\ell(a,\varphi), \eta_{\mathbb{X}}(x)) = \varphi(x), \qquad x \in X, \tag{2.1}$$

holds for all covariant presheaves on $\mathbb{X}$. In particular, $\mathbb{S}_a$ and $\mathbb{P}_\ell(\mathbb{X})$ are skeletal.

In the following considerations we will introduce weak singletons of symmetric $\mathcal{Q}$-categories as a special kind of covariant presheaves. For this purpose we choose a symmetric $\mathcal{Q}$-category and consider the symmetrization $\mathbb{P}_\ell^s(\mathbb{X})$ of $\mathbb{P}_\ell(\mathbb{X})$. In particular, d_ℓ^s is given by:

$$d_\ell^s((a,\varphi),(b,\psi)) = \bigwedge_{x \in X} (\varphi(x) \swarrow \psi(x)) \wedge (\varphi^0(x) \searrow \psi^0(x)), \qquad \varphi, \psi \in P_\ell(\mathbb{X}).$$

Since the relation (2.1) does not remain valid in $\mathbb{P}_\ell^s(\mathbb{X})$, we are interested in the largest full sub-category of $\mathbb{P}_\ell^s(\mathbb{X})$ satisfying (2.1). This approach leads to the full subcategory $\mathcal{W}(\mathbb{X})$ of $\mathbb{P}_\ell^s(\mathbb{X})$ whose objects are all those covariant presheaves σ of type a satisfying the following condition for all $x \in X$:

$$d_\ell^s((a,\sigma),(e_X(x), \widetilde{x})) = \sigma(x) \tag{2.2}$$

Since $\mathbb{X}$ is symmetric, (2.2) is equivalent to $\sigma(x) \leq \bigwedge_{z \in X} \sigma^o(z) \searrow d_X(z,x)$. Thus we introduce the following terminology.

2.1. Definition. Let $\mathbb{X}$ be a symmetric $\mathcal{Q}$-category. A covariant presheaf σ on $\mathbb{X}$ of type a is called a *weak singleton of* $\mathbb{X}$ *of type* a if σ satisfies the following condition for all $x, y \in X$:

(S1) $\sigma^o(x) \cdot \sigma(y) \ \leq \ d_X(x,y)$. (*Singleton Condition*)

A weak singleton σ of $\mathbb{X}$ of type a is a *singleton* of $\mathbb{X}$, if (σ, σ^o) is an adjoint pair of bimodules between $\underline{a}$ and $\mathbb{X}$ (cf. [8]) — this means in this context that σ fulfills the additional property :

(S2) $1_a \ \leq \ \bigvee\limits_{x \in X} \sigma(x) \cdot \sigma(x)^o$.

2.2. Remark. (a) In any symmetric $\mathcal{Q}$-category $\mathbb{X}$ for all $x \in X$ the covariant presheaf $\eta_{\mathbb{X}}(x)$ is always a singleton of $\mathbb{X}$.
(b) Let $\mathcal{Q}$ be an integral and involutive, small quantaloid. If σ is a weak singleton of $\mathbb{X}$ of type a, then for every $\varkappa \in \mathcal{Q}(a,b)$ the pair $\varkappa \cdot \sigma$ is again a weak singleton of $\mathbb{X}$ of type b.
(c) In general there exist weak singletons which are not singletons.

On the set $W_a(\mathbb{X})$ of all weak singletons on $\mathbb{X}$ of type a we introduce a partial ordering $\preceq$ by:

$$\sigma_1 \preceq \sigma_2 \quad \Longleftrightarrow \quad \forall\, x \in X : \ \sigma_1(x) \ \leq \ \sigma_2(x).$$

A maximal element in $(W_a(\mathbb{X}), \preceq)$ is called a *maximal weak singleton* of type a. The axiom of choice guarantees the existence of maximal weak singletons. In the case of integral quantaloids we can give a characterization.

2.3. Proposition. *Let $\mathcal{Q}$ be an integral quantaloid, and σ be a weak singleton of $\mathbb{X}$ of type a. Then the following assertions are equivalent:*

(i) *σ is maximal in $(W_a(\mathbb{X}), \preceq)$.*

(ii) *For all $y \in X$ the following relation holds:*

$$\sigma(y) \ = \ \bigwedge_{x \in X} \sigma^o(x) \searrow d_X(x,y).$$

Proof. The implication (ii) $\Longrightarrow$ (i) is obvious. Further, the singleton condition (S1) implies:

$$\sigma(y) \ \leq \ \bigwedge_{x \in X} \sigma^o(x) \searrow d_X(x,y).$$

In order to establish the inverse inequality, we fix $y_0 \in Y$ and consider the following weak singleton $\overline{\sigma}$ of type a defined by:

$$\overline{\sigma}(y) \ = \ \sigma(y) \vee \Big(\bigwedge_{X \in X} \sigma^o(x) \searrow d_X(x,y_0) \Big) \cdot d_X(y_0,y), \qquad y \in X.$$

In fact, because of the integrality of $\mathcal{Q}$ it is not difficult to show that (S1) holds. Then the relation $\bigwedge\limits_{X \in X} \sigma^o(x) \searrow d_X(x,y_0) \ \leq \ \sigma(y_0)$ follows from the maximality of σ. ∎

2.4. Example. (a) Let I be the real unit interval provided with the usual multiplication. Then I is an integral and commutative quantale with the identity map as involution. Further, let $\mathcal{Q}$ be the quantaloid induced by I in the sense of Example 1.2. Referring to F.W. Lawvere's fundamental paper [14] it is easily seen that symmetric $\mathcal{Q}$-categories $\mathbb{X} = (X, d_X, e_X)$ are equivalent to pseudo-metric spaces (X, ϱ_X). The correspondence between d_X and ϱ_X is given by $-\ln\big(d_X(x,y)\big) = \varrho_X(x,y)$, $x, y \in X$. Moreover, maximal weak singletons and *extremal functions* (cf. [11]) are equivalent concepts.
(b) Let Ω be a frame. Then the quantaloid $\mathcal{Q}_\Omega$ given by Ω in the sense of Example 1.3 is exactly the construction invented by R.F.C. Walters 1981 (cf. [26]). In particular the involution on $\mathcal{Q}_\Omega$ coincides with the identity functor. In this context, symmetric $\mathcal{Q}_\Omega$-categories and Ω-sets (cf. [7]) are equivalent concepts. A weak singleton σ of type a is a singleton iff a coincides with the *height of* σ — i.e $a = \bigvee_{x \in X} \sigma(x)$.

Let $\mathbb{X}$ be a symmetric $\mathcal{Q}$-category. The full subcategory of $\mathbb{P}^s_\ell(\mathbb{X})$ whose objects are all weak singletons of $\mathbb{X}$ coincides with $\mathcal{W}(\mathbb{X})$. In particular, the underlying set $W(\mathbb{X})$ of *all* weak singletons of $\mathbb{X}$ is given by:

$$W(\mathbb{X}) = \bigcup_{a \in \mathrm{obj}(\mathcal{Q})} \{a\} \times W_a(\mathbb{X}). \tag{2.3}$$

In this context the symmetric $\mathcal{Q}$-category $\mathcal{W}(\mathbb{X})$ is called the *weak envelope* of $\mathbb{X}$.

Since $\mathbb{X}$ is symmetric, the Yoneda embedding $\eta_\mathbb{X}$ factors through $\mathcal{W}(\mathbb{X})$. Hence the formation of weak envelopes gives rise to a monad $\mathbf{W}$ on $\mathbf{sCat}(\mathcal{Q})$ whose *clone-composition function* $\circ$ is determined by:

$$\mathbb{X} \xrightarrow{\Phi} \mathcal{W}(\mathbb{Y}), \qquad \mathbb{Y} \xrightarrow{\Psi} \mathcal{W}(\mathbb{Z}), \qquad \mathbb{X} \xrightarrow{\Psi\circ\Phi} \mathcal{W}(\mathbb{Z}),$$
$$\Phi(x) = (e_X(x), \sigma_x), \quad \Psi(x) = (e_Y(y), \sigma_y), \quad \Psi \circ \Phi(x) = (e_X(x), \xi_x), \qquad x \in X, y \in Y,$$
$$\xi_x(z) = \bigvee_{y \in Y} \sigma_x(y) \cdot \sigma_y(z), \qquad z \in Z.$$

As usually the object function $\mathbb{X} \longmapsto \mathcal{W}(\mathbb{X})$ of $\mathbf{W}$ can be completed to an endofunctor $\mathcal{W}$ of $\mathbf{sCat}(\mathcal{Q})$. In particular, the action of $\mathcal{W}$ on $\mathcal{Q}$-functors is given as follows:

$$\mathbb{X} \xrightarrow{\Phi} \mathbb{Y}, \qquad \mathcal{W}(\Phi) = (\eta_\mathbb{Y} \cdot \Phi) \circ \mathrm{id}_{\mathcal{W}(\mathbb{X})}, \qquad \mathcal{W}(\Phi)(a, \sigma) = (a, \tau) \qquad \text{with}$$

$$\tau(y) = \bigvee_{x \in X} \sigma(x) \cdot d_Y(\varphi(x), y), \qquad (a, \sigma) \in W(\mathbb{X}),\ y \in Y. \tag{2.4}$$

Also for later purposes we quote the explicit form of the multiplication μ corresponding to $\mathbf{W}$

$$\mu_\mathbb{X} = \mathrm{id}_{\mathcal{W}(\mathbb{X})} \circ \mathrm{id}_{\mathcal{W}(\mathcal{W}(\mathbb{X}))}, \quad \mu_\mathbb{X}(b, \Sigma) = (b, \mathfrak{s}), \quad \mathfrak{s}(x) = \bigvee_{(a,\sigma) \in W(\mathbb{X})} \Sigma(a, \sigma) \cdot \sigma(x). \tag{2.5}$$

In the following considerations $\mathbf{W}$ is called the *weak singleton monad*.

3. Cauchy completion and the weak envelope of symmetric $\mathcal{Q}$-categories

It is well known that the Cauchy completion of a symmetric $\mathcal{Q}$-category is again symmetric iff for all objects a of $\mathcal{Q}$ every adjoint pair (τ, σ) between the trivial $\mathcal{Q}$-category $\underline{a}$ and $\mathbb{X}$ is a singleton of $\mathbb{X}$ — i.e. $\tau = \sigma^o$ (cf. [2]). Moreover, we can find in [9] the following result. If $\mathcal{Q}$ is an involutive and integral quantaloid provided with the property:

> for any object a_0 of $\mathcal{Q}$ and for any family $\big(a_i \xrightarrow{\alpha_i} a_0, \quad a_0 \xrightarrow{\beta_i} a_i\big)_{i \in I}$ of morphisms of $\mathcal{Q}$ the following implication holds:
>
> $$1_{a_0} \le \bigvee_{i \in I} \alpha_i \cdot \beta_i \quad \Longrightarrow \quad 1_{a_0} \le \bigvee_{i \in I} (\alpha_i \wedge \beta_i^0) \cdot (\alpha_i^0 \wedge \beta_i), \tag{3.1}$$

then the symmetric completion of any symmetric $\mathcal{Q}$-category coincides with its Cauchy completion (cf. Proof of Corollary 3.8 in [9]) — i.e. the Cauchy completion preserves the symmetry axiom. Even though (3.1) covers a large class of examples (cf. Section 4 in [9]), I conjecture that in the case of integral and involutive quantaloids (3.1) is not necessary for the preservation of symmetry under the Cauchy completion. In the next proposition we give now a necessary and sufficient condition for this property.

3.1. Proposition. *Let $\mathcal{Q}$ be an integral and involutive quantaloid. Then the following assertions are equivalent:*

(i) *The Cauchy completion preserves the symmetry axiom of $\mathcal{Q}$-categories.*

(ii) *For any family $\big(a_i \xrightarrow{\alpha_i} a_0, \quad a_0 \xrightarrow{\beta_i} a_i\big)_{i \in I}$ of morphisms in $\mathcal{Q}$ provided with the properties*

$$\forall\, i,j \in I: \quad \beta_i \cdot \alpha_j \cdot \alpha_j^o \le \alpha_i^o, \quad \beta_i^o \cdot \beta_i \cdot \alpha_j \le \beta_j^o, \qquad 1_{a_0} \le \bigvee_{i \in I} \alpha_i \cdot \beta_i \tag{3.2}$$

the following relations hold:

$$1_{a_0} \le \bigvee_{i \in I} \alpha_i \cdot \alpha_i^o \quad \text{and} \quad 1_{a_0} \le \bigvee_{i \in I} \beta_i^o \cdot \beta_i. \tag{3.3}$$

Proof. (a) ((i) $\Longrightarrow$ (ii)) Let $\big(a_i \xrightarrow{\alpha_i} a_0, \quad a_0 \xrightarrow{\beta_i} a_i\big)_{i \in I}$ be a family of morphisms in $\mathcal{Q}$ satisfying (3.2). First we construct a symmetric $\mathcal{Q}$-category as follows: $\mathbb{I} = (I, e_I, d_I)$ where

$$e_I(i) = a_i, \quad d_I(i,j) = (\alpha_i \searrow \alpha_j) \wedge (\beta_i^o \searrow \beta_j^o) \wedge (\alpha_i^o \swarrow \alpha_j^o) \wedge (\beta_i \swarrow \beta_j).$$

Secondly, we define a covariant (resp. contravariant) presheaf σ (resp. τ) on $\mathbb{I}$ of type a_0 by:

$$\sigma(i) = \alpha_i, \quad \tau(i) = \beta_i, \quad i \in I.$$

Since $\mathcal{Q}$ is integral, we conclude from (3.2) that (σ,τ) forms an adjoint pair of bimodules between $\underline{a_0}$ and $\mathbb{I}$. Since the Cauchy completion preserves the symmetry axiom, we obtain $\tau = \sigma^o$. Hence (3.3) holds.

(b) ((ii) $\Longrightarrow$ (i)) We choose an adjoint pair (σ,τ) of bimodules between $\underline{a_0}$ and a symmetric $\mathcal{Q}$-category $\mathbb{X}$. Then we conclude from the symmetry axiom and the extensionality axioms that

$$\tau(x)\cdot\sigma(y) \;\le\; (\tau^o(x)\searrow\tau^o(y))\wedge(\sigma^o(x)\swarrow\sigma^o(y))$$

holds for all $x,y\in X$. Hence the family $\big(e_X(x)\xrightarrow{\sigma(x)} a_0,\quad a_0\xrightarrow{\tau(x)} e_X(x)\big)_{x\in X}$ of morphisms in $\mathcal{Q}$ satisfies (3.2). Now we invoke (ii) and obtain:

$$1_{a_0}\le\bigvee_{y\in X}\sigma(y)\cdot\sigma^o(y)\;\le\;\bigwedge_{x\in X}\tau(x)\searrow\sigma^o(x),\quad 1_{a_0}\le\bigvee_{x\in X}\tau^o(x)\cdot\tau(x)\;\le\;\bigwedge_{y\in Y}\tau^o(y)\swarrow\sigma(y).$$

Hence $\tau = \sigma^0$ follows. ∎

Now we turn to the question under which condition the monad associated with the Cauchy completion is a submonad of the weak singleton monad. We begin with a technical lemma.

3.2. Lemma. *Let $\mathbb{X}$ be a symmetric $\mathcal{Q}$-category. If (a_1,σ_1) and (a_2,σ_2) are singletons of $\mathbb{X}$, then the following relation holds:*

$$d^s_\ell\big((a_1,\sigma_1),(a_2,\sigma_2)\big) \;=\; \bigvee_{x\in X}\sigma_1(x)\cdot\sigma_2^o(x).$$

Proof. Because of the singleton condition (S1) and the right extensionality of σ_1 the relation

$$\Big(\bigvee_{x\in X}\sigma_1(x)\cdot\sigma_2^o(x)\Big)\cdot\sigma_2(z)\;\le\;\sigma_1(z)$$

holds. Interchanging the role of σ_1 and σ_2 we obtain: $\Big(\bigvee\limits_{x\in X}\sigma_2(x)\cdot\sigma_1^o(x)\Big)\cdot\sigma_1(z)\;\le\;\sigma_2(z)$. Hence

$$\bigvee_{x\in X}\sigma_1(x)\cdot\sigma_2^o(x)\;\le\;d^s_\ell\big((a_1,\sigma_1),(a_2,\sigma_2)\big)$$

follows. Now we assume that σ_1 is singleton and not only a weak singleton. Then we conclude from (S2):

$$\begin{aligned} d^s_\ell\big((a_1,\sigma_1),(a_2,\sigma_2)\big) &= 1_{a_1}\cdot d^s_\ell\big((a_1,\sigma_1),(a_2,\sigma_2)\big)\\ &\le \bigvee_{x\in X}\sigma_1(x)\cdot\sigma_1^o(x)\cdot\big(\sigma_1^o(x)\searrow\sigma_2^o(x)\big)\\ &\le \bigvee_{x\in X}\sigma_1(x)\cdot\sigma_2^o(x). \end{aligned}$$

Hence the assertion is verified. ∎

If the Cauchy completion preserves symmetry, then Lemma 3.2 shows that the Cauchy completion of $\mathbb{X}$ is a full subcategory of the weak envelope $\mathcal{W}(\mathbb{X})$ of $\mathbb{X}$. In particular, the «clone-composition» function specified above factors through the Cauchy completion.

To sum up we have the following result.

3.3. PROPOSITION. *Let $\mathcal{Q}$ be an integral and involutive quantaloid. If the Cauchy completion preserves symmetry (i.e. the assertion* (ii) *in Proposition 3.1 holds), then the Cauchy completion can be restricted to* $\mathbf{sCat}(\mathcal{Q})$ *and the monad associated with the Cauchy completion is a submonad of the weak singleton monad.*

4. Presheaves on symmetric $\mathcal{Q}$-categories and $\mathbf{W}$-subalgebras

We begin with the classification problem of presheaves on symmetric $\mathcal{Q}$-categories.

CLASSIFICATION PROBLEM. For every object $a \in \mathcal{Q}$ let $\mathbf{1} \xrightarrow{t_a} \mathbb{S}_a$ be the $\mathcal{Q}$-functor defined by

$$t_a(b) \;=\; \bigvee \mathcal{Q}(b,a), \qquad b \in \mathrm{obj}(\mathcal{Q}). \tag{4.1}$$

Since $\mathbf{1}$ is symmetric t_a can also be viewed as a $\mathcal{Q}$-functor from $\mathbf{1}$ to the symmetrization $\mathbb{S}_a^s$ of $\mathbb{S}_a$. Further, let $\mathbb{X}$ be a symmetric $\mathcal{Q}$-category and φ be a covariant presheaf on $\mathbb{X}$ of type a . Then φ can also be considered as $\mathcal{Q}$-functor from $\mathbb{X}$ to $\mathbb{S}_a^s$. Since $\mathbf{sCat}(\mathcal{Q})$ is complete, we can form the pullback of t_a along φ in $\mathbf{sCat}(\mathcal{Q})$ — i.e.

$$\begin{tikzcd} \mathbb{U} \arrow[r, "!"] \arrow[d, "\iota"'] & \mathbf{1} \arrow[d, "t_a"] \\ \mathbb{X} \arrow[r, "\varphi"'] & \mathbb{S}_a^s \end{tikzcd} \tag{4.2}$$

Hence the carrier set U of the subobject $\mathbb{U} \xrightarrow{\iota} \mathbb{X}$ classified by φ is given by:

$$U \;=\; \{x \in X \mid \varphi(x) \;=\; t_a(e_X(x))\}$$

where ι denotes the inclusion map from U to X. On the other hand, the covariant presheaf ψ on $\mathbb{X}$ of type a defined by

$$g(x) \;=\; \bigvee_{u \in U} t_a(e_X(u)) \cdot d_X(u,x), \quad x \in X$$

classifies also $\mathbb{U} \xrightarrow{\iota} \mathbb{X}$ in the sense of (4.2). But, unfortunately this classification is *not* unique. In general, the presheaves φ and ψ are different. Simple counterexamples can be given among other things in the context of pseudo-metric spaces (cf. Example 2.4(a)).

The aim of the following considerations is to overcome the non-uniqueness of the classification in $\mathbf{sCat}(\mathcal{Q})$ where we might pay the price that not every subobject is classifiable.

We will show that in the case of integral and involutive, small quantaloids there exists a bijection between covariant presheaves on symmetric $\mathcal{Q}$-categories and special subalgebras (cf. [15, 17]) in the sense of the weak singleton monad (cf. Section 2). For this purpose we exhibit a $\mathbf{W}$-algebra structure on the symmetric $\mathcal{Q}$-category $\mathbb{S}_a^s$.

We fix an object a of $\mathcal{Q}$ and define a map ξ from $W(\mathbb{S}_a^s)$ to S_a as follows:

$$\xi(b,\sigma) \;=\; \bigvee \big\{ \varkappa \in \mathcal{Q}(b,a) \;\mid\; \forall\, \lambda \in S_a :\; \varkappa \cdot \sigma(\lambda) \;\le\; \lambda \big\}. \tag{4.3}$$

Because of

$$\xi(b_1,\sigma_1) \cdot d_\ell^s\big((b_1,\sigma_1),(b_2,\sigma_2)\big) \cdot \sigma_2(\lambda) \;\le\; \xi(b_1,\sigma_1)\cdot\sigma_1(\lambda) \;\le\; \lambda$$

the map ξ is a $\mathcal{Q}$-functor $\mathcal{W}(\mathbb{S}_a^s) \xrightarrow{\xi} \mathbb{S}_a^s$.

In the next remark we explain the relationship between ξ and the free completion of the trivial $\mathcal{Q}$-category $\underline{a}$.

4.1. Remark. First we notice that $\mathbb{S}_a^s$ is not necessarily complete. But there exists a $\mathcal{Q}$-functor $\mathbb{S}_a^s \xrightarrow{\varphi} \mathbb{P}_\ell(\underline{a})$ induced by the involution of the underlying quantaloid — i.e. $\varphi(\lambda) = \lambda^o$, $\lambda \in S_a$. Moreover, the evaluation of covariant presheaves determines a distributor Θ from $\mathbb{S}_a^s$ to $\mathcal{W}(\mathbb{S}_a^s)$ by:

$$\Theta\big((b,\sigma),\lambda\big) \;=\; \sigma(\lambda), \qquad \big((b,\sigma),\lambda\big) \in W(\mathbb{S}_a^s) \times S_a.$$

Then the Θ-weighted limit $\mathcal{W}(\mathbb{S}_a^s) \xrightarrow{\psi} \mathbb{P}_\ell(\underline{a})$ of φ has the following form:

$$\psi(b,\sigma) \;=\; \bigvee_{\lambda \in S_a} \sigma(\lambda)\cdot\lambda^0, \quad (b,\sigma) \in W(\mathbb{S}_a^s).$$

In particular, ψ factors through φ. Hence we conclude from (S1) that ξ is the unique $\mathcal{Q}$-functor satisfying the condition that $\varphi \cdot \xi$ coincides with the Θ-weighted limit of φ.

4.2. Proposition. *The pair* $(\mathbb{S}_a^s, \xi)$ *is a* $\mathbf{W}$*-algebra.*

Proof. The relation $\eta_{\mathbb{X}} \cdot \xi = \mathrm{id}_{\mathbb{X}}$ follows immediately from the definition of $\eta_{\mathbb{X}}$ and ξ. In order to verify the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{W}(\mathcal{W}(\mathbb{S}_a^s)) \arrow[r, "\mathcal{W}(\xi)"] \arrow[d, "\mu_{\mathbb{S}_a^s}"'] & \mathcal{W}(\mathbb{S}_a^s) \arrow[d, "\xi"] \\
\mathcal{W}(\mathbb{S}_a^s) \arrow[r, "\xi"'] & \mathbb{S}_a^s
\end{tikzcd}$$

we choose a weak singleton (b,Σ) of $\mathcal{W}(\mathbb{S}_a^s)$ and conclude from (2.2), (2.4) and (2.5):

$$\mu_{\mathbb{S}_a^s}(b,\Sigma)(z) \;\le\; [\mathcal{W}(\xi)(b,\Sigma)](z), \quad z \in X. \tag{4.4}$$

Then the commutativity of the previous diagram follows from (4.4) and the following implication:

$$\Big(\forall\,\lambda \in S_a:\ \varkappa \cdot \mu_{\mathbb{S}_a^s}(b,\Sigma)(\lambda)\ \le\ \lambda\Big) \Longrightarrow \Big(\forall\,\lambda \in S_a:\ \varkappa \cdot [\mathcal{W}(\xi)(b,\Sigma)](\lambda)\ \le \lambda\Big)$$

where $\varkappa\in\mathcal{Q}(b,a)$. In fact, if $\varkappa\cdot\mu_{\mathbb{S}_a^s}(b,\Sigma)(\lambda)\ \le\ \lambda$, then the relations (2.5) and (4.3) imply: $\varkappa\cdot\Sigma(c,\sigma)\ \le\ \xi(c,\sigma)$. Now we obtain:

$$\varkappa\cdot[\mathcal{W}(\xi)(b,\Sigma)](\lambda)\ \le \bigvee_{(c,\sigma)\in W(\mathbb{S}_a^s)} \xi(c,\sigma)\cdot d_a^s(\xi(c,\sigma),\lambda)\ \le\ \lambda.$$

Hence the previous implication has been established. ■

Further, the $\mathcal{Q}$-functor $\mathbf{1}\xrightarrow{t_a}\mathbb{S}_a^s$ turns out to be $\mathbf{W}$-homomorphism

$$(\mathbf{1},!_{\mathcal{W}(\mathbf{1})})\xrightarrow{t_a}(\mathbb{S}_a^s,\xi).$$

In fact, because of (4.1) the relation $\varkappa\cdot\sigma(c)\cdot d_a^s(t_a(c),\lambda)\ \le\ \lambda$ holds for all $(b,\sigma)\in W(\mathbf{1})$ and for all $\varkappa\in\mathcal{Q}(b,a)$.

Now we make the following

Standing Assumption. $\mathcal{Q}$ is always an integral and involutive, small quantaloid.

4.3. Theorem. *Let $(\mathbb{X},\vartheta)$ be a $\mathbf{T}$-algebra and $(\mathbb{X},\vartheta)\xrightarrow{\varphi}(\mathbb{S}_a^s,\xi)$ be a $\mathbf{W}$-homomorphism. Further, let $\big((\mathbb{U},\zeta),\iota\big)$ be the subalgebra of $(\mathbb{X},\vartheta)$ determined by the following pullback diagram*

$$\begin{tikzcd}
(\mathbb{U},\zeta) \arrow[r, "!_{\mathbb{U}}"] \arrow[d, "\iota"] & (\mathbf{1},!_{\mathcal{W}(\mathbf{1})}) \arrow[d, "t_a"] \\
(\mathbb{X},\vartheta) \arrow[r, "\varphi"'] & (\mathbb{S}_a^s,\xi)
\end{tikzcd}$$

in the category of $\mathbf{W}$-algebras. Then the following relation holds:

$$\varphi(x)\ =\ \bigvee_{u\in U} t_a(e_X(u))\cdot d_X\big(\iota(u),x\big),\qquad x\in X. \tag{4.5}$$

Proof. Since in the category of $\mathbf{W}$-algebras pullbacks are computed at the level of $\mathbf{sCat}(\mathcal{Q})$ (cf. [17]), the carrier set U of the subalgebra $(\mathbb{U},\zeta)$ together with its embedding $\mathbf{W}$-homomorphism ι is given by:

$$U\ =\ \Big\{x\in X\ \Big|\ \varphi(x)\ =\ t_a(e_X(x))\Big\},\qquad \iota\ =\ \text{inclusion map}.$$

Then for all $x\in X$ the inequality

$$\bigvee_{u\in U} t_a(e_X(u))\cdot d_X(u,x)\ \le\ \varphi(x) \tag{4.6}$$

follows from the right extensionality of φ. On the other hand, if we fix $x_0 \in X$, then we conclude from the standing assumption that σ with

$$\sigma(x) \;=\; \varphi(x_0) \cdot d_X(x_0, x), \qquad x \in X,$$

is a weak singleton of $\mathbb{X}$ of type a (cf. Remark 2.2(b)). Referring again to the right extensionality axiom we obtain:

$$[\mathcal{W}(\varphi)(a,\sigma)](\lambda) \;=\; \bigvee_{x \in X} \varphi(x_0) \cdot d_X(x_0, x) \cdot d_a^s(\varphi(x), \lambda) \;\leq\; \lambda.$$

Since 1_a is the universal upper bound in $\mathcal{Q}(a,a)$, the relation $\xi(\mathcal{W}(\varphi)(a,\sigma)) \;=\; t_a(a)$ follows immediately from the definition of ξ. Now we invoke the property that φ is a $\mathbf{W}$-homomorphism and obtain:

$$\varphi(\vartheta(a,\sigma)) \;=\; t_a(a), \quad e_X(\vartheta(a,\sigma)) \;=\; a. \tag{4.7}$$

Hence $\vartheta(a,\sigma) \in U$. Finally, we conclude from (2.2):

$$\varphi(x_0) \;=\; \sigma(x_0) \;\leq\; t_a(a) \cdot d_X(\vartheta(a,\sigma), x_0). \tag{4.8}$$

Thus (4.5) follows from (4.6), (4.7) and (4.8). ∎

Theorem 4.3 suggests the following terminology: A subalgebra $(\mathbb{U},\zeta) \xrightarrow{\iota} (\mathbb{X},\vartheta)$ is called t_a*-classifiable* iff there exists a $\mathbf{W}$-homomorphism $(\mathbb{X},\vartheta) \xrightarrow{\varphi} (\mathbb{S}_a^s,\xi)$ s.t. the diagram

$$\begin{array}{ccc} (\mathbb{U},\zeta) & \xrightarrow{!_{\mathbb{U}}} & (\mathbf{1}, !_{\mathcal{W}(\mathbf{1})}) \\ \big\downarrow{\scriptstyle \iota} & & \big\downarrow{\scriptstyle t_a} \\ (\mathbb{X},\vartheta) & \xrightarrow[\varphi]{} & (\mathbb{S}_a^s,\xi) \end{array}$$

is a pullback square. Then Theorem 4.3 implies that every t_a-classifiable subalgebra is *uniquely* t_a-classifiable.

As an application of the previous construction we describe those classifiable subalgebras which are classified by covariant presheaves. Because of the universal property of *free* $\mathbf{W}$*-algebras* every covariant presheaf φ on a symmetric $\mathcal{Q}$-category $\mathbb{X}$ of type a can uniquely be extended to a $\mathbf{W}$-homomorphism

$$(\mathcal{W}(\mathbb{X}), \mu_{\mathbb{X}}) \xrightarrow{\varphi^\sharp} (\mathbb{S}_a^s, \xi).$$

It is well known that $\varphi^\sharp$ is given by (cf. I.4.12 in [17]): $\varphi^\sharp \;=\; \xi \cdot \mathcal{W}(\varphi)$. Then the carrier set U of the t_a-classifiable subalgebra of $(\mathcal{W}(\mathbb{X}), \mu_{\mathbb{X}})$ classified by $\varphi^\sharp$ contains all weak singletons σ of $\mathbb{X}$ of type b with

$$t_a(b) \cdot \sigma(x) \cdot d_a^s\big(\varphi(x), \lambda\big) \;\leq\; \lambda.$$

In particular, U has the following form:

$$U \;=\; \Big\{(b,\sigma) \in W(\mathbb{X}) \;\Big|\; \forall\, x \in X:\; t_a(b) \cdot \sigma(x) \;\leq\; \varphi(x)\Big\}. \tag{4.9}$$

4.4. REMARK. (a) In the case of pseudo-metric spaces (X, ϱ_X) (cf. Example 2.4(a)) it follows immediately from Theorem 4.3 and the subsequent considerations that every non-expansive map $X \xrightarrow{f} [0,+\infty]$ can be identified with the metric space of all non-expansive functions $X \xrightarrow{s} [0,+\infty]$ provided with the following additional properties:

$$f(x) \le s(x), \qquad \varrho_X(x,y) \le s(x)+s(y), \quad x,y \in X.$$

(b) Let Ω be a frame, and $\mathcal{Q}_\Omega$ be the quantaloid induced by Ω in the sense of Example 1.3. If $\top$ denotes the universal upper bound in Ω, then $(t_\top, \mathbb{S}^s_\top)$ is the subobject classifier in the category **CΩ-Set** of complete Ω-sets (cf. [7]). Since in the case of $\mathcal{Q}_\Omega$ the Cauchy completion preserves the symmetry axiom (cf. [26] or Proposition 3.1), the monad associated with the Cauchy completion is a submonad of **W** (cf. Proposition 3.3). Hence the **W**-algebra $(\mathbb{S}^s_\top, \xi)$ and the **W**-homomorphism $t_\top$ form a modification of $(t_\top, \mathbb{S}^s_\top)$. Moreover, the formula (4.5) in Theorem 4.3 can be seen as an enlargement of the construction provided by the subobject classifier axiom in **CΩ-Set** to the more general setting determined by the category of **W**-algebras.

The next proposition shows that every **W**-algebra gives rise to a set-valued functor with domain $\mathcal{Q}$.

4.5. PROPOSITION. *Every* **W**-*algebra* $(\mathbb{X}, \vartheta)$ *induces a functor* $\mathcal{Q} \xrightarrow{\mathcal{F}} \mathbf{Set}$ *by:*

$$\mathcal{F}(a) = X_a = \{x \in X \mid e_X(x) = a\}, \quad a \in \mathrm{obj}(\mathcal{Q}),$$
$$a \xrightarrow{\lambda} b \ \ (\text{in } \mathcal{Q}), \qquad X_a \xrightarrow{\mathcal{F}(\lambda)} X_b, \qquad \mathcal{F}(\lambda)(x) = \vartheta(b, \lambda \cdot \widetilde{x}).$$

PROOF. It is easily seen that $\mathcal{F}$ preserves the respective identities. In order to show that $\mathcal{F}$ preserves also the composition, we fix $x \in X_a$ and choose arrows $a \xrightarrow{\lambda_1} b \xrightarrow{\lambda_2} c$ in $\mathcal{Q}$. Then we introduce a weak singleton Σ of $\mathcal{W}(\mathbb{X})$ of type c as follows:

$$\Sigma(k,\sigma) = \lambda_2 \cdot d^s_\ell\big((b, \lambda_1 \cdot \widetilde{x}), (k,\sigma)\big), \qquad (k,\sigma) \in W(\mathbb{X}).$$

Obviously, $\mathcal{W}(\vartheta)(c, \Sigma) = (c, \sigma_0)$ has the form

$$\begin{aligned}\sigma_0(z) &= \bigvee_{(k,\sigma)\in S(\mathbb{X})} \lambda_2 \cdot d^s_\ell\big((b,\lambda_1\cdot\widetilde{x}),(k,\sigma)\big) \cdot d_X\big(\vartheta(k,\sigma), z\big),\\ &= \lambda_2 \cdot d_X\big(\vartheta(b, \lambda_1 \cdot \widetilde{x}), z\big).\end{aligned}$$

Further, we obtain from (2.2):

$$\bigvee_{(k,\sigma)\in S(\mathbb{X})} \Sigma(k,\sigma)\cdot\sigma(z) = \lambda_2 \cdot \lambda_1 \cdot d_X(x,z) = (\lambda_2 \cdot \lambda_1) \cdot \widetilde{x}(z), \quad z \in X.$$

Then $\mathcal{F}(\lambda_2 \cdot \lambda_1) = \mathcal{F}(\lambda_2) \circ \mathcal{F}(\lambda_1)$ follows from $\vartheta \cdot \mu_{\mathbb{X}} = \vartheta \cdot \mathcal{W}(\vartheta)$. ∎

It is not difficult to see that the object function specified in Proposition 4.5 can be completed to a functor from the category of all $\mathbf{W}$-algebras to $\mathbf{Set}^{\mathcal{Q}}$ which obviously preserves monomorphisms. Hence covariant presheaves are not only subalgebras, but also set-valued functors on $\mathcal{Q}$ from which the given covariant presheaf can be reconstructed. In fact, if $\mathcal{F}$ is the set-valued functor on $\mathcal{Q}$ induced by the free $\mathbf{W}$-algebra $\big(\mathcal{W}(\mathbb{X}), \mu_{\mathbb{X}}\big)$ — i.e.

$$\mathcal{F}(b) \;=\; \{\sigma \mid (b,\sigma) \in W(\mathbb{X})\}, \qquad b \xrightarrow{\;\lambda\;} c \;, \qquad \mathcal{F}(\lambda)(\sigma) \;=\; \lambda \cdot \sigma,$$

and $\mathcal{S}$ be the subfunctor of $\mathcal{F}$ induced by the subalgebra $(\mathbb{U}, \zeta) \xrightarrow{\;\iota\;} (\mathcal{W}(\mathbb{X}), \mu_{\mathbb{X}})$ corresponding to a covariant presheaf φ on $\mathbb{X}$ of type a, then the $\mathbf{W}$-homomorphism $\varphi^\sharp$ can be reconstructed from $\mathcal{S}$ in the following way. Since $\mathcal{S}$ is given by (cf. (4.9))

$$\mathcal{S}(b) \;=\; \Big\{\sigma \in \mathcal{F}(b) \,\Big|\, t_a(b) \cdot \sigma \;\leq\; \varphi\Big\}, \quad b \in \mathrm{obj}(\mathcal{Q}),$$

the corresponding characteristic morphism $\mathcal{F} \xrightarrow{\;\chi\;} \Omega$ has the form (cf. pp. 38-39 in [16]):

$$\mathcal{F}(b) \xrightarrow{\;\chi_b\;} \Omega(b), \quad \chi_b(\sigma) = \{\lambda \in \mathrm{mor}(\mathcal{Q}) \mid \mathrm{dom}(\lambda) = b,\ \lambda \cdot \sigma \in \mathcal{S}(\mathrm{codom}(\lambda)\},\ \sigma \in \mathcal{F}(b)$$

where Ω is the subobject classifier in $\mathbf{Set}^{\mathcal{Q}}$. Then we reconstruct $\varphi^\sharp$ from χ as follows:

$$\varphi^\sharp(b,\sigma) \;=\; \xi(\mathcal{W}(\varphi))(b,\sigma) \;=\; \bigvee_{\lambda \in \chi_b(s)} t_a(\mathrm{codom}(\lambda)) \cdot \lambda, \qquad (b,\sigma) \in W(\mathbb{X}).$$

5. Classifiable subalgebras

Let a be an object of the underlying quantaloid $\mathcal{Q}$. Since not every subalgebra is t_a-classifiable (i.e. there exist more subalgebras than «characteristic morphisms»), it is important to show that the t_a-classifiable hull of subalgebras exists.

Since the maximal covariant presheaf φ_a on $\mathbb{X}$ of type a is the composition of e_X with t_a, it is easily seen that φ_a is always a $\mathbf{W}$-homomorphism for any structure morphism $\mathcal{W}(\mathbb{X}) \xrightarrow{\;\vartheta\;} \mathbb{X}$. Hence every $\mathbf{W}$-algebra $(\mathbb{X}, \vartheta)$ viewed as subalgebra of itself is trivially t_a-classifiable.

In the next step we show that the non empty intersection of t_a-classifiable subalgebras is again t_a-classifiable.

We begin with the internalization of the *conjunction of type a*. The product

$$(\mathbb{S}_a^s, \xi) \times (\mathbb{S}_a^s, \xi) \;=\; (\mathbb{T}_a, \zeta)$$

of the $\mathbf{W}$-algebra $(\mathbb{S}_a^s, \xi)$ with itself can be specified as follows. First we compute the product $\mathbb{S}_a^s \times \mathbb{S}_a^s = \mathbb{T}_a$ in $\mathbf{sCat}(\mathcal{Q})$:

$$T_a \;=\; \{(\lambda_1, \lambda_2) \in S_a \times S_a \mid e_a(\lambda_1) \;=\; e_a(\lambda_2\},$$

$$d_{T_a}\big((\lambda_1,\lambda_2), (\widehat{\lambda}_1, \widehat{\lambda}_2)\big) \;=\; d_a(\lambda_1, \widehat{\lambda}_1) \wedge d_a(\lambda_2, \widehat{\lambda}_2), \quad e_{T_a}(\lambda_1,\lambda_2) \;=\; e_a(\lambda_1) \;(=\; e_a(\lambda_2)).$$

Then the structure morphism ζ is given by $\langle \xi \cdot \mathcal{W}(\pi_1), \xi \cdot \mathcal{W}(\pi_2)\rangle$ where $T_a \xrightarrow{\pi_i} S_a$ is the projection onto the i-th coordinate $(i = 1, 2)$. Finally, we define a covariant presheaf $f_\wedge$ on $\mathbb{S}_a^s \times \mathbb{S}_a^s$ of type a as follows:

$$f_\wedge(\lambda_1, \lambda_2) \;=\; \lambda_1 \wedge \lambda_2, \qquad (\lambda_1, \lambda_2) \in T_a.$$

Since for any weak singleton (b, σ) of $\mathbb{T}_a$ the implication

$$\varkappa_1 \cdot \sigma(\lambda_1, \lambda_2) \;\le\; \lambda_1, \quad \varkappa_2 \cdot \sigma(\lambda_1, \lambda_2) \;\le\; \lambda_2 \;\Longrightarrow\; (\varkappa_1 \wedge \varkappa_2) \cdot \sigma(\lambda_1, \lambda_2) \;\le\; \lambda_1 \wedge \lambda_2.$$

is obvious, it is easily seen that $f_\wedge$ is a $\mathbf{W}$-homomorphism from $(\mathbb{S}_a^s, \xi) \times (\mathbb{S}_a^s, \xi)$ to $(\mathbb{S}_a^s, \xi)$. In particular, the diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "t_a"] & (\mathbb{S}_a^s, \xi) \\
\mathbf{1} \arrow[u] \arrow[r, "\langle t_a, t_a\rangle"'] & (\mathbb{S}_a^s \xi) \times (\mathbb{S}_a^s, \xi) \arrow[u, "f_\wedge"']
\end{tikzcd}$$

is a pullback square. Hence, by analogy to topos theory we call $f_\wedge$ *the conjunction of type* a in the category of $\mathbf{W}$-algebras.

5.1. THEOREM. *The binary intersection of* t_a*-classifiable* $\mathbf{W}$*-subalgebras is again* t_a*-classifiable.*

PROOF. Let $(\mathbb{U}_i, \zeta_i) \xrightarrow{\iota_i} (\mathbb{X}, \vartheta)$ be a t_a-classifiable $\mathbf{W}$-subalgebra classified by a $\mathbf{W}$-homomorphism $(\mathbb{X}, \vartheta) \xrightarrow{\varphi_i} (\mathbb{S}_a^s, \xi)$ $(i = 1, 2)$. Then we form the pullback square

$$\begin{tikzcd}
(\mathbb{U}_0, \zeta_0) \arrow[r] \arrow[d] \arrow[dr, dotted, "\iota_0"'] & (\mathbb{U}_2, \zeta_2) \arrow[d, "\iota_2"] \\
(\mathbb{U}_1, \zeta_1) \arrow[r, "\iota_1"'] & (\mathbb{X}, \vartheta)
\end{tikzcd}$$

and observe that the outer rectangle of the following diagram

$$\begin{tikzcd}
(\mathbb{U}_0, \zeta_0) \arrow[r, "\iota_0"] \arrow[d, "!"'] & (\mathbb{X}, \vartheta) \arrow[d, "\langle \varphi_1, \varphi_2\rangle"] \\
\mathbf{1} \arrow[r, "\langle t_a, t_a\rangle"] \arrow[d] & (\mathbb{S}_a^s, \xi) \times (\mathbb{S}_a^s, \xi) \arrow[d, "f_\wedge"] \\
\mathbf{1} \arrow[r, "t_a"] & (\mathbb{S}_a^s, \xi)
\end{tikzcd}$$

is also a pullback square. ∎

Since the category of $\mathbf{W}$-algebras is complete, the previous proof holds also for any non empty family of t_a-classifiable $\mathbf{W}$-subalgebras. Just for the record we state:

5.2. COROLLARY. *The non empty intersection of t_a-classifiable $\mathbf{W}$-subalgebras is again t_a-classifiable.*

We can summarize the previous results in the following statement:

> For any subalgebra $(\mathbb{V}, \zeta)$ of $(\mathbb{X}, \vartheta)$ the *t_a-classifiable hull of* $(\mathbb{V}, \zeta)$ exists and coincides with the intersection of all t_a-classifiable subalgebras of $(\mathbb{X}, \vartheta)$ containing $(\mathbb{V}, \zeta)$.

We hope that the previous result might play a significant role in the development of a non-commutative model theory.

6. An application to the spectrum of C^*-algebras

We begin with a concretization of Example 1.4. For this purpose let $\mathcal{H}$ be a Hilbert space, and $M_{\mathcal{H}}$ be the complete lattice of all closed, linear subspaces ℓ of $\mathcal{H}$ provided with the ortho-complement $^{\perp}$ as order reversing involution. In particular 1 (0) denotes the universal upper (lower) bound in $M_{\mathcal{H}}$. Now we consider the involutive and unital quantale $Q(M_{\mathcal{H}})$ induced by $(M_{\mathcal{H}}, {}^{\perp})$ in the sense of Remark 1.1. Since the universal upper bound in $Q(M_{\mathcal{H}})$ has the form:

$$\top(\ell) \;=\; \begin{cases} 1 & : \quad \ell \neq 0, \\ 0 & : \quad \ell = 0, \end{cases}$$

we can give a simple description of left- and right-sided elements in $Q(M_{\mathcal{H}})$. An element $\sigma \in Q(M_{\mathcal{H}})$ is left-sided iff there exists $b \in M_{\mathcal{H}}$ s.t. for all $\ell \in M_{\mathcal{H}}$:

$$\sigma(\ell) \;=\; \begin{cases} 1 & : \quad \ell \not\leq b, \\ 0 & : \quad \ell \leq b. \end{cases}$$

An element $\sigma \in Q(M_{\mathcal{H}})$ is right-sided iff there exists $a \in M_{\mathcal{H}}$ s.t. for all $\ell \in M_{\mathcal{H}}$:

$$\sigma(\ell) \;=\; \begin{cases} a & : \quad \ell \neq \perp, \\ \perp & : \quad \ell = 0. \end{cases}$$

Obviously, the subquantale of all two-sided elements coincides with $\{\perp, \top\}$ — i.e. $Q(M_{\mathcal{H}})$ is simple.

In order to avoid superfluous information we wish that the semigroup operation is only determined by the multiplication of right-sided elements with left-sided ones. Hence we replace $\&$ by the semigroup operation $\odot$ defined by

$$\sigma_1 \odot \sigma_2 \;=\; \sigma_1 \,\&\, \top \,\&\, \sigma_2$$

Again we obtain an involutive and simple quantale $Q(\mathcal{H}) = (Q(M_{\mathcal{H}}), \odot, \leq, {}^{o})$. Even though $Q(\mathcal{H})$ is not unital, the universal upper bound is still *extensive* — i.e.

$$\sigma \leq (\top \odot \sigma) \wedge (\sigma \odot \top).$$

Moreover, stable and symmetric elements in $Q(\mathcal{H})$ have a simple form. The details are as follows: since $\odot$ coincides with $\&$ on all left-sided (resp. right-sided) elements, the subquantale of all left-sided (resp. right-sided) elements is idempotent. Hence $\odot$ fulfills the following properties:

$$(\sigma_1 \odot \sigma_2) \odot (\sigma_3 \odot \sigma_4) = (\sigma_1 \odot \sigma_3) \odot (\sigma_2 \odot \sigma_4), \tag{6.1}$$

$$\sigma_1 \odot \sigma_1 \odot \sigma_2 = \sigma_1 \odot \sigma_2 = \sigma_1 \odot \sigma_2 \odot \sigma_2. \tag{6.2}$$

Property (6.1) means bi-symmetry, while (6.2) is a kind of generalized idempotency of $\odot$. An important corollary from these properties is the fact that $\sigma \in Q(\mathcal{H})$ is stable (cf. Example 1.3) iff σ is idempotent w.r.t. $\odot$. Hence σ is *stable and symmetric* in $Q(\mathcal{H})$ iff there exists $a \in M_{\mathcal{H}}$ s.t. σ has the following form for all $\ell \in M_{\mathcal{H}}$:

$$\sigma(\ell) = \sigma_a(\ell) = \begin{cases} a & : \quad \ell \not\leq a^{\perp}, \\ 0 & : \quad \ell \leq a^{\perp}. \end{cases} \tag{6.3}$$

Finally, with every pair $(a, b) \in M_{\mathcal{H}} \times M_{\mathcal{H}}$ we associate an element σ_b^a of $Q(M_{\mathcal{H}})$ defined by:

$$\sigma_b^a(\ell) = \begin{cases} a & : \quad \ell \not\leq b, \\ 0 & : \quad \ell \leq b. \end{cases} \tag{6.4}$$

The product w.r.t. $\odot$ has always the form described in (6.4). In fact, if $\sigma_1(\top) = a$ and $b = \bigvee\{\ell \in M_{\mathcal{H}} \mid \sigma_2(\ell) = 0\}$, then $\sigma_1 \odot \sigma_2 = \sigma_b^a$. Also the following relations hold:

$$\sigma_a = \sigma_{a^{\perp}}^a, \qquad \left(\sigma_b^a\right)^o = \sigma_{a^{\perp}}^{b^{\perp}}, \qquad \top = \sigma_0^1.$$

After these preparations we consider the involutive and integral quantaloid $\mathcal{Q}_{\mathcal{H}}$ induced by $Q(M_{\mathcal{H}})$ in the sense of Example 1.4. It is not difficult to see that the set of objects and hom-sets have the form:

- $\mathrm{obj}(\mathcal{Q}_{\mathcal{H}}) = \{\sigma_a \mid a \in M_{\mathcal{H}}\} \dot{\bigcup} \{\omega\}$.
- $\mathcal{Q}_{\mathcal{H}}(\sigma_a, \sigma_b) = \begin{cases} \{\perp, \sigma_a\}, & a = b, \\ \{\perp\}, & a \neq b. \end{cases}$
- $\mathcal{Q}_{\mathcal{H}}(\omega, \sigma_a) = \{\sigma_{a^{\perp}}^b \mid b \leq a\}$, $\quad \mathcal{Q}_{\mathcal{H}}(\sigma_a, \omega) = \{\sigma_b^a \mid a^{\perp} \leq b\}$, $\quad \mathcal{Q}_{\mathcal{H}}(\omega, \omega) = \{0, 1\}$.
- The partial ordering on the hom-sets is inherited from $Q(M_{\mathcal{H}})$.

As special properties of the composition in $\mathcal{Q}_{\mathcal{H}}$ we first note that all hom-sets $\mathcal{Q}_{\mathcal{H}}(\alpha,\alpha)$ are isomorphic to the Boolean algebra **2**. Hence $\mathcal{Q}_{\mathcal{H}}$ can be viewed as the *quantisation* of **2**. Furthermore, in the case of $b \neq c$ it is remarkable to see that $\mathcal{Q}_{\mathcal{H}}(b,c)$ has the structure of an idempotent quantale. Of course, the multiplication in $\mathcal{Q}_{\mathcal{H}}(b,c)$ is induced by $\odot$ and is compatible with the composition in the following sense:

$$(\sigma_1 \odot \sigma_2)\cdot\tau \;=\; (\sigma_1\cdot\tau)\odot(\sigma_2\cdot\tau), \quad (\sigma_1,\sigma_2)\in\mathcal{Q}_{\mathcal{H}}(\beta,\gamma),\ \tau\in\mathcal{Q}_{\mathcal{H}}(\alpha,\beta),\ \beta\neq\gamma \qquad (6.5)$$

where we have made use of (6.1) and (6.2).

Now we turn to spectra of unital C^*-algebras which are understood as the quantale of all *closed left-ideals* (see also [22]). Further, we agree with the conception that irreducible representations play the role of «points» for non-commutative C^*-algebras (cf. pp. 14 in [1]). By means of the Gelfand-Neumark-Segal-construction every pure state ϱ of a C^*-algebra A induces an irreducible representation $(\mathcal{H}_\varrho,\vartheta_\varrho)$ of A (cf. [12]). In contrast to the commutative setting it is interesting to note that in the case of non-commutative C^*-algebras the Hilbert space dimension of the underlying Hilbert space $\mathcal{H}_\varrho$ depends on the pure state ϱ and might possibly vary. In order to overcome this obstacle and to choose an underlying Hilbert space which is independent from the respective pure states, we first recall some terminology from [10].

6.1. Remark. Let $\mathcal{P}(A)$ be the set of all pure states of a unital C^*-algebra A. With every pure state ϱ we associated the irreducible representation $(\mathcal{H}_\varrho,\vartheta_\varrho)$ given by the Gelfand-Naimark-Segal-construction.
(a) A Hilbert space $\mathcal{H}$ is *admissible for* A iff for every pure state $\varrho\in\mathcal{P}(A)$ there exists an isometry $\mathcal{H}_\varrho \xrightarrow{\Phi_\varrho} \mathcal{H}$. It is easily seen that for every C^*-algebra an admissible Hilbert space *exists.*
(b) A representation $(\mathcal{H},\vartheta)$ of A is called *locally irreducible* iff $\mathcal{H}$ is admissible for A and there exists a pure state ϱ of A and an isometry $\mathcal{H}_\varrho \xrightarrow{\Phi_\varrho} \mathcal{H}$ s.t. for all $a\in A$ the following relation holds:

$$\pi(a) \;=\; \Phi_\varrho\circ\vartheta_\varrho(a)\circ\Phi_\varrho^*, \qquad a\in A \qquad (6.6)$$

where $\mathcal{H} \xrightarrow{\Phi_\varrho^*} \mathcal{H}_\varrho$ denotes the adjoint operator corresponding to Φ_ϱ.

In the following considerations we fix a unital C^*-algebra A. Then we choose an admissible Hilbert space $\mathcal{H}$ for A and consider the involutive and unital quantale $Q(M_{\mathcal{H}})$ — the Hilbert quantale associated with $\mathcal{H}$ (cf. Remark 1.1, [20]). Further, let $\mathrm{sp}(A)$ be the spectrum of A, and $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$. From [10] we quote the following result.

6.2. Theorem. *Let $A \xrightarrow{\vartheta} \mathcal{L}(\mathcal{H})$ be a $*$-homomorphism s.t. $(\mathcal{H},\vartheta)$ is a locally irreducible representation of A. Then ϑ induces a quantale homomorphism $\sigma(A) \xrightarrow{h_\vartheta} Q(M_{\mathcal{H}})$*

defined by

$$[h_\vartheta(I)](\ell) = \text{top. closure}(\text{lin. hull}\{\vartheta(a)(x) \mid a \in I,\, x \in \ell\}), \quad I \in \mathrm{sp}(A),\, \ell \in M_{\mathcal{H}}. \quad (6.7)$$

Moreover h_ϑ fulfills the following properties:

(i) $h_\vartheta(A)$ *is an idempotent and symmetric element of* $Q(M_{\mathcal{H}})$.

(ii) $h_\vartheta(I_1 \cdot I_2) = h_\vartheta(I_1) \,\&\, \top \,\&\, h_\vartheta(I_2)$ *where* $I_1, I_2 \in \mathrm{sp}(A)$.

Further, let X_A be the set of all $*$-homomorphisms $A \xrightarrow{\vartheta} \mathcal{L}(\mathcal{H})$ s.t. $(\mathcal{H}, \vartheta)$ is a locally irreducible representation of A. We conclude from Theorem 6.2 that every element $\vartheta \in X_A$ induces a quantale homomorphism h_ϑ from $\mathrm{sp}(A)$ to $Q(\mathcal{H})$ s.t. $h_\vartheta(A)$ is symmetric and stable in the sense of $Q(\mathcal{H})$. Hence there exists a map $X_A \xrightarrow{e} \mathrm{obj}(\mathcal{Q}_{\mathcal{H}})$ determined by

$$e(\vartheta) = h_\vartheta(A), \qquad \vartheta \in X_A. \tag{6.8}$$

Moreover, every closed left-ideal I of A induces a map $X_A \xrightarrow{f_I} \mathrm{mor}(\mathcal{Q}_{\mathcal{H}})$ satisfying the following property

$$f_I(\vartheta) = h_\vartheta(I) \in \mathcal{Q}_{\mathcal{H}}(e(\vartheta), \omega), \qquad \vartheta \in X_A. \tag{6.9}$$

Now we provide (X_A, e) with the structure of a symmetric $\mathcal{Q}_{\mathcal{H}}$-category and define a map $X_A \times X_A \xrightarrow{d} \mathrm{mor}(\mathcal{Q}_{\mathcal{H}})$ by:

$$d(\vartheta_1, \vartheta_2) = \bigvee \Big\{ \sigma \in \mathcal{Q}_{\mathcal{H}}\big(e_0(\vartheta_2), e_0(\vartheta_1)\big) \,\Big|\, \forall I \in \mathrm{sp}(A) : \\ f_I(\vartheta_1) \cdot \sigma \le f_I(\vartheta_2),\ \sigma \cdot f_I(\vartheta_2)^o \le f_I(\vartheta_1)^o \Big\}$$

Then $\mathbb{I}_{loc} = (X_A, e, d)$ is a symmetric $\mathcal{Q}_{\mathcal{H}}$-category, and every map f_I is a covariant presheaf on $\mathbb{I}_{loc}$ of type ω.

Further, the correspondence $I \longmapsto f_I$ is injective, because every closed left-ideal is an intersection of maximal left-ideals and every maximal left-ideal can be identified with a pure state (cf. 10.2.10 in [13]). If we make use of the quantale structure on $\mathcal{Q}_{\mathcal{H}}(e(\vartheta), \omega)$ induced by $\odot$ (see above), then the property (ii) of Theorem 6.2 means: $f_{I_1 \cdot I_2} = f_{I_1} \odot f_{I_2}$. Hence the spectrum of A is isomorphic to $\{f_I \mid I \in \mathrm{sp}(A)\}$ — a result which is closely related to Proposition 4 in [4] (see also Remark (ii) in [24]).

If we now speculate and view closed left-ideals I (resp. f_I) as «open subsets» of a non-commutative topological space (cf. [5, 18]), then the *problem* arises that all f_I are covariant presheaves (in particular $\mathcal{Q}_{\mathcal{H}}$-functors) and not $\mathcal{Q}_{\mathcal{H}}$-subcategories. Because of the methods in Section 4 we can solve this problem. First, we extend the covariant presheaf f_I to a $\mathbf{W}$-homomorphism $f_I^\sharp$ on the free $\mathbf{W}$-algebra $(\mathcal{W}(\mathbb{I}_{loc}), \mu_{\mathbb{I}_{loc}})$. Secondly, we identify $f_I^\sharp$ with a specific subalgebra of $(\mathcal{W}(\mathbb{I}_{loc}), \mu_{\mathbb{I}_{loc}})$ whose carrier set is determined by formula (4.9). Hence $(\mathcal{W}(\mathbb{I}_{loc}), \mu_{\mathbb{I}_{loc}})$ is the *non-commutative space* of the C^*-algebra A. In particular, «open subsets» corresponding to closed left-ideals of A are certain *subalgebras* of $(\mathcal{W}(\mathbb{I}_{loc}), \mu_{\mathbb{I}_{loc}})$.

Acknowledgement. I am very grateful to the referee for his helpful suggestions and his reference to [9].

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