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INTERNAL PROFUNCTORS AND COMMUTATOR THEORY; APPLICATIONS TO EXTENSIONS CLASSIFICATION AND CATEGORICAL GALOIS THEORY

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Abstract. We clarify the relationship between internal profunctors and connectors on pairs (R, S) of equivalence relations which originally appeared in the new profunctorial approach of the Schreier-Mac Lane extension theorem [11]. This clarification allows us to extend this Schreier-Mac Lane theorem to any exact Mal'cev category with centralizers. On the other hand, still in the Mal'cev context and in respect to the categorical Galois theory associated with a reflection I, it allows us to produce the faithful action of a certain abelian group on the set of classes (up to isomorphism) of I-normal extensions having a given Galois groupoid.

Introduction

Any extension between the non-abelian groups K and Y can be canonically indexed by a group homomorphism $\phi : Y \to AutK/IntK$. The Schreier-Mac Lane extension theorem for groups [23] asserts that the class $Ext_\phi(Y, K)$ of non-abelian extensions between the groups K and Y indexed by ϕ is endowed with a simply transitive action of the abelian group $Ext_{\bar{\phi}}(Y, ZK)$, where ZK is the center of K and the index $\bar{\phi}$ is induced by ϕ. A recent extension [11] of this theorem to any action representative ([5], [4], [3]) category gave rise to an unexpected interpretation of this theorem in terms of internal profunctors which was closely related with the intrinsic commutator theory associated with the action representative category in question.
So that there was a need of clarification about the general nature of the relationship between profunctors and the main tool of the intrinsic commutator theory, namely the notion of connector on a pair (R, S) equivalence relations, see [12], and also [15], [21], [20], [16].
The first point is that many of the observations made in [11] in the exact Mal'cev and protomodular settings are actually valid in any exact category. The second point is a characterization of those profunctors $\underline{X}_1 \nrightarrow \underline{Y}_1$ which give rise to a connector on a pair of equivalence relations: they are exactly those profunctors whose associated discrete bifibration $(\underline{\phi}_1, \underline{\gamma}_1) : \underline{\Upsilon}_1 \to \underline{X}_1 \times \underline{Y}_1$ has its two legs $\underline{\phi}_1$ and $\underline{\gamma}_1$ internally fully faithful.

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The third point is the heart of Schreier-Mac Lane extension theorem, and deals with the notion of torsor. It is easy to define a torsor above a groupoid $\underline{Z}_1$ as a discrete fibration $\underline{\nabla}_1 X \to \underline{Z}_1$ from the indiscrete equivalence relation on an object X with global support, in a way which generalizes naturally the usual notion of G-torsor, when G is a group. The main point here is that when the groupoid $\underline{Z}_1$ is aspherical and abelian ([8]) with *direction* the abelian group A, there is a simply transitive action of the abelian group $TorsA$ on the set $Tors\underline{Z}_1$ of classes (up to isomorphism) of $\underline{Z}_1$-torsors.
This simply transitive action allows us on the one hand (fourth point) to extend now the Schreier-Mac Lane extension theorem to any exact Mal'cev category with centralizers, and on the other hand (fifth point) to produce, still in the exact Mal'cev context and in respect to the categorical Galois theory associated with a reflection I, the faithful action of a certain abelian group on the set of classes (up to isomorphism) of I-normal extensions having a fixed Galois groupoid. Incidentally we are also able to extend the notion of connector from a pair of equivalence relations to a pair of internal groupoids.
A last word: what is rather amazing here is that the notion of profunctor between groupoids which could rather seem, at first thought, as a vector of indistinction (everything being isomorphic) appears, on the contrary, as a important tool of discrimination.

The article is organized along the following lines:
Part 1) deals with some recall about the internal profunctors between internal categories and their composition, mainly from [18].
Part 2) specifies the notion of profunctors between internal groupoids and characterizes those profunctors $\underline{X}_1 \nrightarrow \underline{Y}_1$ whose associated discrete bifibration $(\underline{\phi}_1, \underline{\gamma}_1) : \underline{\Upsilon}_1 \to \underline{X}_1 \times \underline{Y}_1$ has its two legs $\underline{\phi}_1$ and $\underline{\gamma}_1$ internally fully faithful. It contains also our main theorem about the canonical simply transitive action on the $\underline{Z}_1$-torsors, when the groupoid $\underline{Z}_1$ is aspherical and abelian.
Part 3) deals with some recall about the notion of connector on a pair of equivalence relations, and extends it to a pair of groupoids.
Part 4) asserts the Schreier-Mac Lane extension theorem for exact Mal'cev categories with centralizers.
Part 5) describes the faithful action on the I-normal extensions having a fixed Galois groupoid.

1. Internal profunctors

In this section we shall recall the internal profunctors and their composition.

1.1. Discrete fibrations and cofibrations. We shall suppose that our ambient category $\mathbb{E}$ is a finitely complete category. We denote by $Cat\mathbb{E}$ the category of internal categories in $\mathbb{E}$, and by $()_0 : Cat\mathbb{E} \to \mathbb{E}$ the forgetful functor associating with any internal category $\underline{X}_1$ its "object of objects" X_0. This functor is a left exact fibration. Any fibre $Cat_X\mathbb{E}$ has the discrete equivalence relation $\Delta_1 X$ as initial object and the indiscrete equivalence relation $\nabla_1 X$ as terminal object.

An internal functor $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ is then $()_0$-cartesian if and only if the following square is a pullback in $\mathbb{E}$, in other words if and only if it is internally fully faithful:

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\
X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

Accordingly any internal functor $\underline{f}_1$ produces the following decomposition, where the lower quadrangle is a pullback:

$$\begin{tikzcd}
X_1 \arrow[rr, "f_1"] \arrow[dr, "\gamma_1"] \arrow[dd, "{(d_0,d_1)}"'] & & Y_1 \arrow[dd, "{(d_0,d_1)}"] \\
& Z_1 \arrow[ur, "\phi_1"'] \arrow[dl] & \\
X_0 \times X_0 \arrow[rr, "f_0 \times f_0"'] & & Y_0 \times Y_0
\end{tikzcd}$$

with the fully faithful functor $\underline{\phi}_1$ and the bijective on objects functor $\underline{\gamma}_1$. We need to recall the following pieces of definition:

1.2. Definition. *The internal functor $\underline{f}_1$ is said to be $()_0$-faithful when the previous factorization γ_1 is a monomorphism. It is said to be $()_0$-full when this same map γ_1 is a strong epimorphism. It is said to be a discrete cofibration when the following square with d_0 is a pullback:*

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, dotted, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, dotted, shift left=2, "d_1"] \\
X_0 \arrow[r, "f_0"'] \arrow[u, dotted] & Y_0 \arrow[u, dotted]
\end{tikzcd}$$

It is said to be a discrete fibration when the previous square with d_1 is a pullback.

Suppose $\underline{f}_1$ is a discrete cofibration; when the codomain $\underline{Y}_1$ is a groupoid, the domain $\underline{X}_1$ is a groupoid as well and the square with d_1 is a pullback as well; when the codomain $\underline{Y}_1$ is an equivalence relation, then the same holds for its domain $\underline{X}_1$. Accordingly, when the codomain $\underline{Y}_1$ of a functor $\underline{f}_1$ is a groupoid, it is a discrete cofibration if and only if it is a discrete fibration.

1.3. Lemma. *Any discrete fibration is $()_0$-faithful. A discrete fibration is $()_0$-cartesian if and only if it is monomorphic.*

Proof. Thanks to the Yoneda Lemma, it is sufficient to prove these assertions in *Set* which is straightforward. ∎

1.4. Profunctors. Let $(\underline{X}_1, \underline{Y}_1)$ be a pair of internal categories. Recall from [2] that an internal profunctor $\underline{X}_1 \nrightarrow \underline{Y}_1$ is given by a pair $X_0 \overset{f_0}{\leftarrow} U_0 \overset{g_0}{\rightarrow} Y_0$ of maps (i.e. a *span* in $\mathbb{E}$) together with a left action $d_1 : U_1^{Y_1} \to U_0$ of the category $\underline{Y}_1$ and a right action $d_0 : U_1^{X_1} \to U_0$ of the category $\underline{X}_1$ which commute with each others, namely which make commute the left hand side upper dotted square in the following diagram, where all those commutative squares that do not contain dotted arrows are pullbacks:

$$
\begin{tikzcd}
U_1 \arrow[r, shift left=2, "p_1"] \arrow[r, dotted, shift right=2, "\pi_0"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, dotted, shift left=2, "\pi_1"] & U_1^{Y_1} \arrow[l, "s_0"'] \arrow[r, "g_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, dotted, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "y_0"'] \arrow[d, dotted, shift left=2, "y_1"] \\
U_1^{X_1} \arrow[u] \arrow[r, shift left=2, "d_1"] \arrow[r, dotted, shift right=2, "d_0"'] \arrow[d, "f_1"'] & U_0 \arrow[u] \arrow[l, "s_0"'] \arrow[r, "g_0"] \arrow[d, "f_0"] & Y_0 \arrow[u] \\
X_1 \arrow[r, shift left=2, "x_1"] \arrow[r, dotted, shift right=2, "x_0"'] & X_0 \arrow[l, "s_0"'] &
\end{tikzcd}
$$

The middle horizontal reflexive graph is underlying a category $\underline{U}_1^{X_1}$, namely the category of "cartesian maps" above $\underline{X}_1$, while the middle vertical reflexive graph is underlying a category $\underline{U}_1^{Y_1}$, namely the category of "cocartesian maps" above $\underline{Y}_1$. The pairs (d_0, d_1) going out from $U_1^{Y_1}$ and $U_1^{X_1}$ are respectively coequalized by f_0 and g_0. A morphism of profunctors is a morphism of spans above $X_0 \times Y_0$ which commutes with the left and right actions. We define this way the category $\mathcal{P}rof(\underline{X}_1, \underline{Y}_1)$ of internal profunctors between $\underline{X}_1$ and $\underline{Y}_1$.

In the set theoretical context, a profunctor $\underline{X}_1 \nrightarrow \underline{Y}_1$ is explicitely given by a functor $U : \underline{X}_1^{op} \times \underline{Y}_1 \to Set$. An object of U_0 is then an element $\xi \in U(x, y)$ for any pair of object in $\underline{X}_1^{op} \times \underline{Y}_1$, in other words $U_0 = \Sigma_{(x,y)\in X_0 \times Y_0} U(x, y)$. Elements of U_0 can be figured as further maps "gluing" the groupoids $\underline{X}_1$ and $\underline{Y}_1$:

$$
\begin{tikzcd}
x \arrow[r, dashed, "\xi"] & y
\end{tikzcd}
$$

An object of $U_1^{X_1}$ is a pair $x' \overset{f}{\to} x \overset{\xi}{\dashrightarrow} y$ and we denote $d_0(f, \xi) = U(f, 1_y)(\xi)$ by $\xi.f$, while an object of $U_1^{Y_1}$ is a pair $x \overset{\xi}{\dashrightarrow} y \overset{g}{\to} y'$ and we denote $d_1(\xi, g) = U(1_x, g)(\xi)$ by $g.\xi$. An object of U_1 is thus a triple:

$$
\begin{tikzcd}
x' \arrow[d, "f"'] & y \arrow[d, "g"] \\
x \arrow[ur, dashed, "\xi"] & y'
\end{tikzcd}
$$

We have $\pi_0(g, \xi, f) = (\xi.f, g)$ and $\pi_1(g, \xi, f) = (f, g.\xi)$. The commutation of the two actions comes from the fact that we have: $(g.\xi).f = g.(\xi.f)$.

In this set theoretical context, the previous diagram can be understood as a map between $\phi = \xi.f$ and $\chi = g.\xi$, pictured this way:

$$\begin{tikzcd} x' \arrow[r, dotted, "\phi"] \arrow[d, "f"'] & y \arrow[d, "g"] \\ x \arrow[ur, dashed, "\xi"] \arrow[r, dotted, "\chi"'] & y' \end{tikzcd}$$

Accordingly this defines a reflexive graph given by the vertical central part of the following diagram in Set, with two morphisms of graphs:

$$\begin{tikzcd} X_1 \arrow[d, shift right=4, "x_0"'] \arrow[d, shift left=4, "x_1"] & U_1 \arrow[l, "f_1.p_0"'] \arrow[r, "g_1.p_1"] \arrow[d, shift right=4, "d_0.p_0"'] \arrow[d, shift left=4, "d_1.p_1"] & Y_1 \arrow[d, shift right=4, "y_0"'] \arrow[d, shift left=4, "y_1"] \\ X_0 \arrow[u] & U_0 \arrow[u] \arrow[l, "f_0"] \arrow[r, "g_0"'] & Y_0 \arrow[u] \end{tikzcd}$$

Actually this reflexive graph takes place in the more general scheme of a category we shall denote by $\underline{U}_1^{X_1} \sharp \underline{U}_1^{Y_1}$: its objects are the elements of U_0, a map between ξ and $\bar{\xi}$ being given by a pair (s,t) of map in $\underline{X}_1 \times \underline{Y}_1$ such that $\bar{\xi}.s = t.\xi$:

$$\begin{tikzcd} x \arrow[r, dashed, "\xi"] \arrow[d, "s"'] & y \arrow[d, "t"] \\ \bar{x} \arrow[r, dashed, "\bar{\xi}"'] & \bar{y} \end{tikzcd}$$

Clearly the categories $\underline{U}_1^{X_1}$ and $\underline{U}_1^{Y_1}$ are subcategories of $\underline{U}_1^{X_1} \sharp \underline{U}_1^{Y_1}$.
Actually, the same considerations hold in any internal context. The object of objects of the internal category $\underline{U}_1^{X_1} \sharp \underline{U}_1^{Y_1}$ is U_0, its object of morphisms Υ_1 is given by the pullback in $\mathbb{E}$ of the maps underlying the two actions:

$$\begin{tikzcd} \Upsilon_1 \arrow[r, "\upsilon_0"] \arrow[d, "\upsilon_1"'] & U_1^{Y_1} \arrow[d, "d_1"] \\ U_1^{X_1} \arrow[r, "d_0"'] & U_0 \end{tikzcd}$$

Moreover there is a pair $(\underline{\phi}_1 : \underline{\Upsilon_1} \to \underline{X}_1, \underline{\gamma}_1 : \underline{\Upsilon_1} \to \underline{Y}_1)$ of internal functors in $\mathbb{E}$ according to the following diagram:

$$\begin{tikzcd} X_1 \arrow[d, shift right=4, "x_0"'] \arrow[d, shift left=4, "x_1"] & \Upsilon_1 \arrow[l, "f_1.\upsilon_0"'] \arrow[r, "g_1.\upsilon_1"] \arrow[d, shift right=4, "d_0.\upsilon_0"'] \arrow[d, shift left=4, "d_1.\upsilon_1"] & Y_1 \arrow[d, shift right=4, "y_0"'] \arrow[d, shift left=4, "y_1"] \\ X_0 \arrow[u] & U_0 \arrow[u] \arrow[l, "f_0"] \arrow[r, "g_0"'] & Y_0 \arrow[u] \end{tikzcd}$$

Without entering into the details, let us say that this pair of functors is characteristic of the given profunctor under its equivalent definition of a *discrete bifibration* in the 2-category $Cat\mathbb{E}$. Moreover the commutation of the two actions induces a natural morphism of reflexive graph:

$$\begin{tikzcd} U_1 \arrow[r, "(\pi_0,\pi_1)"] \arrow[d, shift right=3, "d_0.p_0"'] \arrow[d, shift left=3, "d_1.p_1"] & \Upsilon_1 \arrow[d, shift right=3, "d_0.\upsilon_0"'] \arrow[d, shift left=3, "d_1.\upsilon_1"] \\ U_0 \arrow[u] \arrow[r, equal] & U_0 \arrow[u] \end{tikzcd}$$

The previous observations give rise to the following diagram in $Cat\mathbb{E}$:

$$\begin{tikzcd} & \underline{U}_1^{Y_1} \arrow[rr, "\underline{g}_1"] \arrow[dr] & & \underline{Y}_1 \\ \Delta_1 U_0 \arrow[ur] \arrow[dr] & & \underline{\Upsilon}_1 = \underline{U}_1^{X_1} \sharp \underline{U}_1^{Y_1} \arrow[ur, "\underline{\gamma}_1"] \arrow[dr, "\underline{\phi}_1"'] & \\ & \underline{U}_1^{X_1} \arrow[rr, "\underline{f}_1"'] \arrow[ur] & & \underline{X}_1 \end{tikzcd}$$

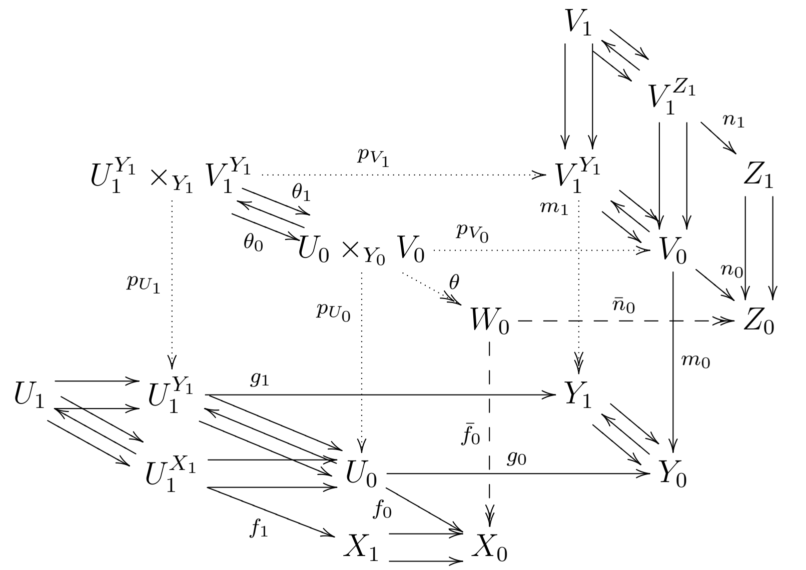
1.5. PROPOSITION. *Let be given a finitely complete category $\mathbb{E}$ and an internal profunctor $\underline{X}_1 \nrightarrow \underline{Y}_1$. Then, in the fibre $Cat_{U_0}\mathbb{E}$, the left hand side quadrangle above is a pullback such that: $\underline{U}_1^{X_1} \vee \underline{U}_1^{Y_1} = \underline{\Upsilon}_1 = \underline{U}_1^{X_1} \sharp \underline{U}_1^{Y_1}$.*

PROOF. Thanks to the Yoneda embedding, it is enough to check it in Set. A map between ξ and $\bar{\xi}$ in $\underline{\Upsilon}_1$, as above, is in $\underline{U}_1^{Y_1}$ (resp. in $\underline{U}_1^{X_1}$) if and only if $s = 1_x$ (resp. $t = 1_y$). Accordingly the intersection of these two subgroupoids is $\Delta_1 U_0$. The second point is a consequence of the fact that any map between ξ and $\bar{\xi}$ in $\underline{\Upsilon}_1$, as above, has a canonical decomposition through a map in $\underline{U}_1^{Y_1}$ and a map in $\underline{U}_1^{X_1}$:

$$\begin{tikzcd} x \arrow[r, dashed, "\xi"] \arrow[d, "1_x"'] & y \arrow[d, "t"] \\ x' \arrow[r, dashed, "\bar{\xi}.s=t.\xi"] \arrow[d, "s"'] & y \arrow[d, "1_{y'}"] \\ x' \arrow[r, dashed, "\bar{\xi}"'] & y' \end{tikzcd}$$

Accordingly any subcategory of $\underline{\Upsilon}_1$ which contains the two subcategories in question is equal to $\underline{\Upsilon}_1$. ∎

What is remarkable is that, in the set theoretical context, the profunctors can be composed on the model of the tensor product of modules, see [2]. The composition can be transposed in $\mathbb{E}$ as soon as *any internal category admits a* π_0 (i.e. a coequalizer of the domain and codomain maps) *which is universal* (i.e. stable under pullbacks), as it is the case when $\mathbb{E}$ is an elementary topos, see [18]. Let $Y_0 \overset{m_0}{\leftarrow} U_0 \overset{n_0}{\rightarrow} Z_0$ be the span underlying another profunctor $\underline{Y}_1 \nrightarrow \underline{Z}_1$ (and let $(\underline{\mu}_1, \underline{\nu}_1) : \underline{\Gamma}_1 \to \underline{Y}_1 \times \underline{Z}_1$ denote its associated discrete bifibration); then consider the following diagram induced by the dotted pullbacks:



It produces the following internal category $\underline{\Theta}_1$ and the following forgetful functor to $\underline{Y}_1$:

$$\begin{tikzcd}
\underline{\Theta}_1 : \arrow[d] & U_1^{Y_1} \times_{Y_1} V_1^{Y_1} \arrow[d, "g_1 \cdot p_{U_1}"'] \arrow[r, shift left=2, "\theta_1"] \arrow[r, shift right=2, "\theta_0"'] & U_0 \times_{Y_0} V_0 \arrow[l, "s_0"'] \arrow[d, "g_0 \cdot p_{U_0}"] \\
\underline{Y}_1 : & Y_1 \arrow[r, shift left=2, "y_1"] \arrow[r, shift right=2, "y_0"'] & Y_0 \arrow[l, "s_0"']
\end{tikzcd}$$

Then take the π_0 of the category $\underline{\Theta}_1$ (it is the coequalizer θ of the pair (θ_0, θ_1)) which produces the dashed span $X_0 \overset{\bar{f}_0}{\leftarrow} W_0 \overset{\bar{n}_0}{\to} Z_0$. The π_0 in question being stable under pullbacks, this span is endowed with a left action of $\underline{Z}_1$ and a right action of $\underline{X}_1$ and gives us the composite $(\underline{\mu}_1, \underline{\nu}_1) \otimes (\underline{\phi}_1, \underline{\gamma}_1)$.
Given an internal category $\underline{X}_1$, the unit profunctor is just given by the *Yoneda profunctor*, i.e. given by the following diagram, where X_1^{∇} is the object of the "commutative triangles" of the internal category $\underline{X}_1$ and $X_1^{\sqcap}$ is the object of the "triple of composable maps" of the internal category $\underline{X}_1$:

$$\begin{tikzcd}
X_1^{\sqcap} \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, dotted, "\pi_1"] \arrow[r, shift left=2, dotted, "\pi_0"] & X_1^{\Delta} \arrow[l, "s_0"'] \arrow[l, shift right=3, leftarrow, "p_1"'] \arrow[r, "p_2"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, dotted, "p_1"] & X_0 \arrow[d, shift right=2, "x_0"'] \arrow[d, shift left=2, dotted, "x_1"] \\
X_1^{\nabla} \arrow[u] \arrow[d, "p_0"'] \arrow[r, shift left=2, "p_2"] \arrow[r, shift right=2, dotted, "p_1"'] & X_1 \arrow[u] \arrow[l, "s_1"'] \arrow[r, "x_1"] \arrow[d, "x_0"] & X_0 \arrow[u] \\
X_1 & X_0 \arrow[l, "s_0"'] \arrow[l, shift right=3, leftarrow, "x_1"'] \arrow[l, shift left=3, leftarrow, dotted, "x_0"]
\end{tikzcd}$$

It is easy to check that $\underline{X}_1^{\Delta}\sharp\underline{X}_1^{\nabla}=\underline{X}_1^{2}$, namely the domain of the universal internal natural transformation with codomain $\underline{X}_1$.
This tensor product $\otimes : \mathcal{P}rof(\underline{X}_1,\underline{Y}_1)\times\mathcal{P}rof(\underline{Y}_1,\underline{Z}_1)\to\mathcal{P}rof(\underline{X}_1,\underline{Z}_1)$ is associative up to coherent isomorphism. By these units and the tensor composition, we get the bicategory $\mathcal{P}rof\mathbb{E}$ of profunctors in $\mathbb{E}$ [18].

1.6. $\underline{X}_1$-TORSOR. From now on, we shall be uniquely interested in the full subcategory $Grd\mathbb{E}$ of $Cat\mathbb{E}$ consisting of the internal groupoids in $\mathbb{E}$. Recall that any fibre $Grd_X\mathbb{E}$ is a protomodular category. We shall suppose moreover that the category $\mathbb{E}$ is at least regular.
We say that a groupoid $\underline{X}_1$ is *connected* when it has a global support in its fibre $Grd_{X_0}\mathbb{E}$, namely when the map $(d_0,d_1): X_1\to X_0\times X_0$ is a regular epimorphism. We say it is *aspherical* when, moreover, the object X_0 has a global support, namely when the terminal map $X_0\to 1$ is a regular epimorphism.

1.7. PROPOSITION. *Suppose $\mathbb{E}$ is a regular category. Let be given a discrete fibration $\underline{f}_1:\underline{X}_1\to\underline{Y}_1$ with $\underline{Y}_1$ an aspherical groupoid.*
1) If $\underline{f}_1$ is a monomorphism, then $\underline{X}_1$ is a connected groupoid.
2) If moreover X_0 has global support, then $\underline{f}_1$ is an isomorphism.
3) Any discrete fibration $\underline{f}_1$ between aspherical groupoids is a levelwise regular epimorphism.

PROOF. 1) Since $\underline{f}_1$ is a monomorphic discrete fibration it is $()_0$-cartesian according to Lemma 1.3 and the following commutative square is a pullback:

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\
X_0\times X_0 \arrow[r, "f_0\times f_0"'] & Y_0\times Y_0
\end{tikzcd}$$

Accordingly $\underline{X}_1$ is connected as soon as such is $\underline{Y}_1$.
2) Then, since $\underline{f}_1$ is discrete fibration, the following square that does not contain dotted arrows is still a pullback:

$$\begin{tikzcd}
X_0\times X_0 \arrow[r, "f_0\times f_0"] \arrow[d, dotted, shift right=2, "p_0"'] \arrow[d, shift left, "p_1"] & Y_0\times Y_0 \arrow[d, dotted, shift right=2, "p_0"'] \arrow[d, shift left, "p_1"] \\
X_0 \arrow[r, "f_0"'] \arrow[u, dotted] & Y_0 \arrow[u, dotted]
\end{tikzcd}$$

and, by the Barr-Kock Theorem, when X_0 has global support, it is the case also for the following one:

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d] & Y_0 \arrow[d] \\
1 \arrow[r, equal] & 1
\end{tikzcd}$$

Accordingly f_0 (and thus f_1) is an isomorphism.
3) Starting from any discrete fibration $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$, take the canonical reg-epi/mono decomposition of $\underline{f}_1$:

$$\begin{tikzcd}
X_1 \arrow[r, two heads, "q_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & U_1 \arrow[r, tail, "m_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
X_0 \arrow[r, two heads, "q_0"'] \arrow[u, dotted] & U_0 \arrow[r, tail, "m_0"'] \arrow[u, dotted] & Y_0 \arrow[u, dotted]
\end{tikzcd}$$

Since m_1 is a monomorphism, the left hand side squares are a pullback; since q_1 is a regular epimorphism the right hand side squares are still pullbacks. Then $\underline{m}_1$ is a monomorphic discrete fibration, and since X_0 has global support, so has U_0. Then, according to 2), the functor $\underline{m}_1$ is an isomorphism, and the discrete fibration $\underline{f}_1$ is a levelwise regular epimorphism. ■

1.8. Definition. *Let $\underline{X}_1$ be an aspherical groupoid in $\mathbb{E}$. A $\underline{X}_1$-torsor is a discrete fibration $\nabla_1 U \to \underline{X}_1$ where U is an object with global support:*

$$\begin{tikzcd}
U \times U \arrow[r, "\tau_1"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & X_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
U \arrow[r, "\tau"'] \arrow[u, dotted] & X_0 \arrow[u, dotted]
\end{tikzcd}$$

According to the previous proposition, the maps τ and τ_1 are necessarily regular epimorphisms. Moreover it is clear that this determines a profunctor: $1 \nrightarrow \underline{X}_1$. When $\underline{X}_1$ is a group G, we get back to the classical notion of G-torsor, if we consider G as a groupoid having the terminal object 1 as "object of objects". In order to emphazise clearly this groupoid structure, we shall denote it by $\underline{K}_1 G$.

2. Fully faithful profunctors

In this section we shall characterize those profunctors $\underline{X}_1 \nrightarrow \underline{Y}_1$ between groupoids whose associated discrete bifibration $(\underline{\phi}_1, \underline{\gamma}_1) : \underline{\Upsilon}_1 \to \underline{X}_1 \times \underline{Y}_1$ has its two legs $\underline{\phi}_1$ and $\underline{\gamma}_1$ internally fully faithful, namely those profunctors such that their associated discrete bifibration $(\underline{\phi}_1, \underline{\gamma}_1)$:

$$\begin{tikzcd}
 & \underline{U}_1^{Y_1} \arrow[rr, "\underline{g}_1"] \arrow[dr, tail] & & \underline{Y}_1 \\
\Delta_1 U_0 \arrow[ur, tail] \arrow[dr, tail] & & \underline{\Upsilon}_1 = \underline{U}_1^{X_1} \sharp \underline{U}_1^{Y_1} \arrow[ur, "\underline{\gamma}_1"] \arrow[dr, "\underline{\phi}_1"'] & \\
 & \underline{U}_1^{X_1} \arrow[rr, "\underline{f}_1"'] \arrow[ur, tail] & & \underline{X}_1
\end{tikzcd}$$

is obtained by the canonical decomposition of the discrete fibrations $\underline{f}_1$ and $\underline{g}_1$ through the $()_0$-cartesian maps.

When $\underline{X}_1$ and $\underline{Y}_1$ are internal groupoids, we have substantial simplifications in the presentation of profunctors $\underline{X}_1 \nrightarrow \underline{Y}_1$. First, since any discrete fibration between groupoids becomes a discrete cofibration, any commutative square in the definition diagram, even the dotted ones, becomes a pullback[1]. Accordingly *the objects* $\underline{U}_1$ *and* $\underline{\Upsilon}_1$ *defined above coincide*, so that *the reflexive graph* U_1 *is actually underlying the groupoid* $\underline{U}_1^{X_1} \sharp \underline{U}_1^{Y_1}$.
On the other hand, the new perfect symmetry of the definition diagram means that the pair $(\underline{\gamma}_1, \underline{\phi}_1) : \underline{U}_1 \to \underline{Y}_1 \times \underline{X}_1$ still determines a discrete bifibration, i.e. a profunctor $\underline{Y}_1 \nrightarrow \underline{X}_1$ in the opposite direction which we shall denote by $(\underline{\phi}_1, \underline{\gamma}_1)^*$.
When $\mathbb{E}$ is exact, any internal groupoid admits a π_0 which is stable under pullback; so that, in the context of exact categories, the profunctors between groupoids are composable. We shall be now interested in some special classes of profunctors between groupoids.

2.1. Proposition. *Suppose* $\mathbb{E}$ *is a finitely complete category. Let be given a profunctor* $(\underline{\phi}_1, \underline{\gamma}_1) : \underline{U}_1 \to \underline{X}_1 \times \underline{Y}_1$ *between groupoids. Then its functorial leg* $\underline{\gamma}_1 : \underline{U}_1 \to \underline{Y}_1$ *is* $()_0$*-faithful if and only the groupoid* $\underline{U}_1^{X_1}$ *is an equivalence relation; it is* $()_0$*-cartesian if and only if we have* $\underline{U}_1^{X_1} = R[g_0]$*. By symmetry, the other leg* $\underline{\phi}_1 : \underline{U}_1 \to \underline{X}_1$ *is* $()_0$*-faithful if and only the groupoid* $\underline{U}_1^{Y_1}$ *is an equivalence relation; it is* $()_0$*-cartesian if and only if we have* $\underline{U}_1^{Y_1} = R[f_0]$

Proof. Thanks to the Yoneda embedding, it is enough to prove it in *Set*. Suppose that $\underline{\gamma}_1 : \underline{U}_1 \to \underline{Y}_1$ is faithful. Let be given two parallel arrows in $\underline{U}_1^{X_1} \subset \underline{U}_1$:

$$\begin{tikzcd} x \arrow[r, dashed, "\xi"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "f'"] & y \arrow[d, "1_y"] \\ x' \arrow[r, dashed, "\chi"'] & y \end{tikzcd}$$

The image of these two arrows of $\underline{U}_1$ by the functor $\underline{\gamma}_1$ is 1_y. Accordingly, since this functor is $()_0$-faithful, we get $f = f'$, and $\underline{U}_1^{X_1}$ is an equivalence relation. Conversely suppose $\underline{U}_1^{X_1}$ is an equivalence relation. Two maps in $\underline{U}_1$ having the same image g by $\underline{\gamma}_1$ determine a diagram:

$$\begin{tikzcd} x \arrow[r, dashed, "\xi"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "f'"] & y \arrow[d, "g"] \\ x' \arrow[r, dashed, "\chi"'] & y' \end{tikzcd}$$

which itself determines the following diagram:

$$\begin{tikzcd} x \arrow[r, dashed, "g.\xi"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "f'"] & y' \arrow[d, "1_{y'}"] \\ x' \arrow[r, dashed, "\chi"'] & y' \end{tikzcd}$$

[1] In this way, any profunctor between groupoids can be seen as a double augmented simplicial object in $\mathbb{E}$ such that any commutative square is a pullback

But since $\underline{U}_1^{X_1}$ is an equivalence relation, we get $f = f'$. Accordingly the functor $\underline{\gamma}_1 : \underline{U}_1 \to \underline{Y}_1$ is faithful.
Suppose we have moreover $\underline{U}_1^{X_1} = R[g_0]$. Any pair (ξ, χ) in U_0 with a map g between their respective codomains y and y', determines a pair $(g.\xi, \chi)$ in $R[g_0]$:

$$\begin{tikzcd} x \arrow[dr, dashed, "g.\xi"] \arrow[dd, dotted, "f"'] & \\ & y' \\ x' \arrow[ur, dashed, "\chi"'] & \end{tikzcd}$$

and since we have $\underline{U}_1^{X_1} = R[g_0]$, we get a map f such that $g.\xi = \chi.f$ which determines an arrow in $\underline{U}_1$ whose image by $\underline{\gamma}_1$ is g:

$$\begin{tikzcd} x \arrow[r, dashed, "\xi"] \arrow[d, "f"'] & y \arrow[d, "g"] \\ x' \arrow[r, dashed, "\chi"'] & y' \end{tikzcd}$$

Conversely suppose $\underline{\gamma}_1$ is fully faithful. Since it is faithful we observed that $\underline{U}_1^{X_1} \subset R[g_0]$. Now given any pair (ξ, χ) in $R[g_0]$, the fullness property of $\underline{\gamma}_1$ produces a map f which completes the following diagram:

$$\begin{tikzcd} x \arrow[r, dashed, "\xi"] \arrow[d, dotted, "f"'] & y \arrow[d, "1_y"] \\ x' \arrow[r, dashed, "\chi"'] & y \end{tikzcd}$$

and we get $R[g_0] \subset \underline{U}_1^{X_1}$. ∎

2.2. Definition. *An internal profunctor between groupoids is said to be faithful when its two legs $\underline{\phi}_1$ and $\underline{\gamma}_1$ are $()_0$-faithful. It is said to be fully faithful when its two legs $\underline{\phi}_1$ and $\underline{\gamma}_1$ are $()_0$-cartesian. It is said to be regularly fully faithful when moreover the maps f_0 and g_0 are regular epimorphisms.*

Examples. 1) Given any internal groupoid $\underline{X}_1$, its associated Yoneda profunctor is a regularly fully faithful profunctor:

$$\begin{tikzcd}[column sep=large, row sep=large]
R^2[x_0] \arrow[r, shift left=2, "p_3"] \arrow[r, dotted, shift right=2, "p_2"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, dotted, shift left=2, "p_1"] & R[x_0] \arrow[l, "s_1"'] \arrow[r, "p_2"] \arrow[d, shift right=2, "p_0"'] \arrow[d, dotted, shift left=2, "p_1"] & X_0 \arrow[d, shift right=2, "x_0"'] \arrow[d, dotted, shift left=2, "x_1"] \\
R[x_0] \arrow[u] \arrow[r, shift left=2, "p_2"] \arrow[r, dotted, shift right=2, "p_1"'] \arrow[d, "p_0"'] & X_1 \arrow[u] \arrow[l, "s_1"'] \arrow[r, two heads, "x_1"] \arrow[d, "x_0"] & X_0 \arrow[u] \\
X_1 \arrow[r, shift left=2, "x_1"] \arrow[r, dotted, shift right=2, "x_0"'] & X_0 \arrow[l, "s_0"'] &
\end{tikzcd}$$

It is this precise diagram which makes the internal groupoids monadic above the split epimorphisms, see [6].

2) Given an abelian group A in $\mathbb{E}$, any A-torsor determines a regularly fully faithful profunctor $\underline{K}_1 A \looparrowright \underline{K}_1 A$. For that, consider the following diagram where the upper right hand side squares are pullbacks by definition of an A-torsor:

$$
\begin{tikzcd}
R[h_1] \arrow[r, dotted, shift left=2, "p_1"] \arrow[r, dotted, shift right=2, "p_0"'] \arrow[d, shift right=2, "R(p_0)"'] \arrow[d, shift left=2, "R(p_1)"] & T \times T \arrow[l, dotted] \arrow[r, two heads, "h_1"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & A \arrow[d] \\
T \times T \arrow[r, dotted, shift left=2, "p_1"] \arrow[r, dotted, shift right=2, "p_0"'] \arrow[d, dashed, two heads, "h_1"'] & T \arrow[l, dotted] \arrow[u] \arrow[r, two heads, "\tau_T"'] \arrow[d, two heads, "\tau_T"] & 1 \arrow[u, shift right=2, "e"'] \\
A \arrow[r, dotted] & 1 \arrow[l, dotted, "e"] &
\end{tikzcd}
$$

and complete it by the horizontal kernel equivalence relations. Then, when A is abelian, the same map h_1 is the quotient of the vertical left hand side equivalence relation. Thanks to the Barr embedding, it is enough to prove it in *Set*, which is straighforward. Actually this is equivalent to saying that, when A is abelian, a principal left A-object becomes a principal symmetric two-sided object, according to the terminology of [1].

2.3. Proposition. *Suppose $\mathbb{E}$ is a regular category. A morphism $\tau : (\underline{\phi}_1, \underline{\gamma}_1) \to (\underline{\phi}'_1, \underline{\gamma}'_1)$ between profunctors above groupoids having their legs $\underline{\phi}_1$ and $\underline{\phi}'_1$ $()_0$-cartesian and regularly epimorphic is necessarily an isomorphism. In particular, a morphism between two regularly fully faithful profunctors is necessarily an isomorphism.*

Proof. Consider the following diagram which is part of the diagram induced by the morphism τ of profunctors:

$$
\begin{tikzcd}
R[f_0] \arrow[r, "R(\tau)"'] \arrow[rr, bend left, "g_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & R[f'_0] \arrow[r, "g'_1"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "y_0"'] \arrow[d, shift left=2, "y_1"] \\
U_0 \arrow[u] \arrow[r, "\tau"] \arrow[rr, bend right, "g_0"'] \arrow[d, "f_0"'] & U'_0 \arrow[u] \arrow[r, "g'_0"] \arrow[d, "f'_0"] & Y_0 \arrow[u] \\
X_0 \arrow[r, "1_{X_0}"'] & X_0 &
\end{tikzcd}
$$

The functor $\underline{g}_1$ and $\underline{g}'_1$ being discrete fibrations, the left hand side upper part of the diagram is a discrete fibration between equivalence relations. Since moreover f_0 is a regular epimorphism, the lower square is a pullback by the Barr-Kock theorem and τ an isomorphism. ∎

2.4. Proposition. *Suppose $\mathbb{E}$ is an efficiently regular category [9]. Let $\underline{X}_1 \nrightarrow \underline{Y}_1$ and $\underline{Y}_1 \nrightarrow \underline{Z}_1$ be two profunctors between groupoids whose first leg of their associated discrete bifibrations $(\underline{\phi}_1, \underline{\gamma}_1)$ and $(\underline{\mu}_1, \underline{\nu}_1)$ is $()_0$-cartesian. Then their profunctor composition does exist in $\mathbb{E}$ and has its first leg $()_0$-cartesian. When, moreover, their first legs are regularly epimorphic, their profunctor composition has its first leg regularly epimorphic. By symmetry, the same holds concerning the second legs. Accordingly regularly fully faithful profunctors are composable as profunctors, and are stable under this composition.*

Proof. Let us go back to the diagram defining the composition. When the first leg $\underline{\phi}_1$ is $()_0$-cartesian, we have $U_1^{Y_1} = R[f_0]$. By construction the groupoid $\underline{\Theta}_1$ determines a discrete fibration:

$$\begin{tikzcd}
U_1^{Y_1} \times_{Y_1} V_1^{Y_1} \arrow[r, shift left=2, "\theta_1"] \arrow[r, shift right=2, "\theta_0"'] \arrow[d, "p_{U_1}"'] & U_0 \times_{Y_0} V_0 \arrow[l, "s_0"' description] \arrow[r, dotted, two heads, "\theta"] \arrow[d, "p_{U_0}"'] & W_0 \arrow[d, dashed, "\bar{f}_0"] \\
R[f_0] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] & U_0 \arrow[l, "s_0"' description] \arrow[r, "f_0"'] & X_0
\end{tikzcd}$$

Since $\mathbb{E}$ is an efficiently regular category and the codomain of this discrete fibration is an effective equivalence relation, its domain is an effective equivalence relation as well. Thus, this domain admits a quotient θ, and a factorization $\bar{f}_0$ which makes the right hand side square a pullback. Moreover this quotient is stable under pullback, since $\mathbb{E}$ is regular. Accordingly the two profunctors can be composed. Let us notice immediately that when m_0 is a regular epimorphism, such is p_{U_0}. If moreover f_0 is a regular epimorphism, then $\bar{f}_0$ is a regular epimorphism as well, and we shall get the second point, once the first one is checked.
Suppose now the first leg $\underline{\phi}'_1$ is $()_0$-cartesian, i.e. $V_1^{Z_1} = R[m_0]$. We have to check that $W_1^{Z_1} = R[\bar{f}_0]$. Let us consider the diagram where $R[p_{U_0}]$ is the result of the pulling back along g_0 of $V_1^{Z_1} = R[m_0]$, and $R[p_{U_1}]$ is the result of the result of the pulling back along g_1 of $V_1 = R[m_1]$:

$$\begin{tikzcd}
R[p_{U_1}] \arrow[r, shift left=2, "R(\theta_1)"] \arrow[r, shift right=2, "R(\theta_0)"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & R[p_{U_0}] \arrow[l, "s_0"' description] \arrow[r, dotted, two heads, "\bar{\theta}"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & W_1^{Z_1} \arrow[d, dashed, shift right=2, "q_0"'] \arrow[d, dashed, shift left=2, "q_1"] \\
U_1^{Y_1} \times_{Y_1} V_1^{Y_1} \arrow[r, shift left=2, "\theta_1"] \arrow[r, shift right=2, "\theta_0"'] \arrow[d, "p_{U_1}"'] & U_0 \times_{Y_0} V_0 \arrow[l, "s_0"' description] \arrow[r, dotted, two heads, "\theta"] \arrow[d, "p_{U_0}"'] & W_0 \arrow[d, dashed, "\bar{f}_0"] \\
R[f_0] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] & U_0 \arrow[l, "s_0"' description] \arrow[r, "f_0"'] & X_0
\end{tikzcd}$$

Since any of the upper left hand side squares are pullbacks and $(\theta, \bar{\theta})$ is a pair of regular epimorphims, the upper right hand side squares are pullbacks. This implies that the right hand side vertical diagram is a kernel equivalence relation. Accordingly we have $W_1^{Z_1} = R[\bar{f}_0]$. ∎

It is clear that when $(\underline{\phi}_1, \underline{\gamma}_1) : \underline{X}_1 \nrightarrow \underline{Y}_1$ is a fully faithful (resp. regularly fully faithful) profunctor, the profunctor $(\underline{\phi}_1, \underline{\gamma}_1)^* : \underline{Y}_1 \nrightarrow \underline{X}_1$ is still fully faihtful (resp. regularly fully faithful).

2.5. Proposition. *Suppose $\mathbb{E}$ is an efficiently regular category. Given any regularly fully faithful profunctor $(\underline{\phi}_1, \underline{\gamma}_1) : \underline{X}_1 \nrightarrow \underline{Y}_1$, the profunctor $(\underline{\phi}_1, \underline{\gamma}_1)^* : \underline{Y}_1 \nrightarrow \underline{X}_1$ is its inverse with respect to the composition of profunctors.*

Proof. The heart of the composition $(\underline{\phi}_1, \underline{\gamma}_1)^* \otimes (\underline{\phi}_1, \underline{\gamma}_1)$ is given by the following dotted pullbacks where the unlabelled vertical dotted arrow is g_1:

$$\begin{tikzcd}
U_1 \arrow[rrr, dotted, "\pi_0"] \arrow[dr, shift left, "\pi_1"] \arrow[dr, shift right, "p_0"'] \arrow[ddd, dotted, "p_1"'] & & & R[f_0] \arrow[dr, shift left, "p_1"] \arrow[dr, shift right] \arrow[ddd, dotted] & & \\
& R[g_0] \arrow[rrr, dotted, "p_1"] \arrow[dr, dotted, "f_1"] \arrow[ddd, dotted, "p_0"'] & & & U_0 \arrow[dr, two heads, "f_0"] \arrow[ddd, "g_0"] & \\
& & X_1 \arrow[rrr, dashed, two heads, "x_1"] \arrow[ddd, dashed, "x_0"'] & & & X_0 \\
R[f_0] \arrow[rrr, "g_1"] \arrow[dr, shift left] \arrow[dr, shift right, "p_0"'] & & & Y_1 \arrow[dr, shift left, "y_1"] \arrow[dr, shift right] & & \\
& U_0 \arrow[rrr, "g_0"] \arrow[dr, two heads, "f_0"'] & & & Y_0 & \\
& & X_0 & & &
\end{tikzcd}$$

The commutations of this diagram shows that the maps θ_0 and θ_1 needed in the composition constuction are respectively p_0 and π_1. Accordingly their coequalizer is the regular epimorphism f_1, and the associated span is just (x_0, x_1). So that $(\underline{\phi}_1, \underline{\gamma}_1)^* \otimes (\underline{\phi}_1, \underline{\gamma}_1)$ is just the Yoneda profunctor associated with $\underline{X}_1$. By symmetry we obtain that $(\underline{\phi}_1, \underline{\gamma}_1) \otimes (\underline{\phi}_1, \underline{\gamma}_1)^*$ is the Yoneda profunctor associated with $\underline{Y}_1$. ∎

Accordingly, when $\mathbb{E}$ is a finitely complete efficiently regular category, we get the bigroupoid $\mathcal{R}f\mathbb{E}$ of the regularly fully faithful profunctors between internal groupoids in $\mathbb{E}$, and the associated groupoid $\mathbb{R}f\mathbb{E}$ whose morphisms are the isomorphic classes of regularly fully faithful profunctors.

Remark. By the specific (i.e the double choice of the map h_1 as a coequalizer) construction of the example 2 above, we associate with any A-torsor a regularly fully faithful profunctor $\underline{K}_1 A \nrightarrow \underline{K}_1 A$. What is very important is that the classical tensor product of A-torsors coincides with the composition of profunctors. Accordingly this construction defines the abelian group $TorsA$ as a subgroup of $\mathbb{R}f\mathbb{E}(\underline{K}_1 A, \underline{K}_1 A)$.

2.6. The canonical action on the $\underline{X}_1$-torsors when $\underline{X}_1$ is abelian. In this section, we shall show that when $\underline{X}_1$ is an aspherical abelian groupoid in $\mathbb{E}$, the set $Tors\underline{X}_1$ of isomorphic classes of $\underline{X}_1$-torsors is canonically endowed with a simply transitive action of an abelian group of the form $TorsA$, where A is an internal abelian group in $\mathbb{E}$.

Aspherical abelian groupoids

Recall from [8] the following:

2.7. DEFINITION. *An internal groupoid $\underline{Z}_1$ in $\mathbb{E}$ is said to be abelian when it is a commutative object in the protomodular fibre $Grd_{Z_0}\mathbb{E}$.*

We shall see that, in the Mal'cev context (see below), this is equivalent to saying that the map $(z_0, z_1) : Z_1 \to Z_0 \times Z_0$ is such that $[R[(z_0, z_1)], R[(z_0, z_1)]] = 0$.
When moreover the category $\mathbb{E}$ is efficiently regular, any aspherical abelian groupoid $\underline{Z}_1$ has admits *direction* [8], namely there exists an abelian group A in $\mathbb{E}$ which makes the following upper squares pullbacks squares:

$$\begin{tikzcd}
R[(z_0,z_1)] \arrow[r, two heads, "\nu_{\underline{Z}_1}"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & A \arrow[d, shift right] \\
Z_1 \arrow[r, two heads] \arrow[u] \arrow[d, "{(z_0,z_1)}"'] & 1 \arrow[u, shift right, "e"'] \\
Z_0 \times Z_0 &
\end{tikzcd}$$

Let us immediately notice that, in the Mal'cev context, this makes now *central* the kernel equivalence relation $R[(z_0, z_1)]$.

In the set theoretical context, a groupoid $\underline{Z}_1$ is abelian when, for any of its object z, the group Aut_z of endomaps at z is abelian. The groupoid $\underline{Z}_1$ is asherical, when it is non empty and connected. So, when the groupoid $\underline{Z}_1$ is aspherical and abelian, all the abelian groups Aut_z are isomorphic. Moreover, what is remarkable is that, given any map $\tau : z \to z'$ in $\underline{Z}_1$, the induced group homomorphism $Aut_z \to Aut_{z'}$ is *independent of the map* τ. The direction A of $\underline{Z}_1$ is then any of these abelian groups.

The previous construction provides a *direction functor* $d : AsGrd\mathbb{E} \to Ab\mathbb{C}$ from the category of aspherical groupoids in $\mathbb{E}$ to the category of abelian groups in $\mathbb{E}$. Suppose you have a $()_0$-cartesian functor $\underline{f}_1 : \underline{T}_1 \to \underline{Z}_1$ with $\underline{T}_1$ aspherical, then the following diagram shows that $d(\underline{f}_1)$ is a group isomorphism, since the lower square is a pullback (and consequently the upper left hand side ones):

$$\begin{tikzcd}
R[(t_0,t_1)] \arrow[r] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & R[(z_0,z_1)] \arrow[r, two heads, "\nu_{\underline{Z}_1}"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & d(\underline{Z}_1) \arrow[d, shift right] \\
T_1 \arrow[r, "f_1"] \arrow[d, "{(t_0,t_1)}"'] & Z_1 \arrow[r, two heads] \arrow[u] \arrow[d, "{(z_0,z_1)}"'] & 1 \arrow[u, shift right, "e"'] \\
T_0 \times T_0 \arrow[r, "f_0 \times f_0"'] & Z_0 \times Z_0 &
\end{tikzcd}$$

So, let $\underline{X}_1 \nrightarrow \underline{Y}_1$ be any fully faithful profunctor between aspherical groupoids whose associated discrete bifibration is given by the pair $(\underline{\phi}_1, \underline{\gamma}_1) : \underline{\Upsilon}_1 \to \underline{X}_1 \times \underline{Y}_1$. Then necessarily the groupoids $\underline{X}_1$ and $\underline{Y}_1$ have same direction A, and this fully faithful profunctor determines a group isomorphism $d(\underline{\gamma}_1).d(\underline{\phi}_1)^{-1} : A \to A$. We shall denote by $\mathbb{R}f_1\mathbb{E}(\underline{X}_1, \underline{Y}_1)$ the subset of $\mathbb{R}f\mathbb{E}(\underline{X}_1, \underline{Y}_1)$ consisting of those regularly faithful profunctors between aspherical groupoids such that $d(\underline{\gamma}_1).d(\underline{\phi}_1)^{-1} = 1_A$ (or equivalently $d(\underline{\gamma}_1) = d(\underline{\phi}_1)$).

2.8. PROPOSITION. *Let $\mathbb{E}$ be an efficiently regular category. Suppose the groupoid $\underline{Z}_1$ in $\mathbb{E}$ is aspherical with direction A and $\underline{g}_1 : \underline{\nabla}_1 X \twoheadrightarrow \underline{Z}_1$ is a $\underline{Z}_1$-torsor. Consider the following diagram with the horizontal kernel equivalence relations which make pullbacks the upper left hand side squares:*

$$\begin{tikzcd}
R[g_1] \arrow[rr, shift left=2, "p_1"] \arrow[rr, shift right=2, "p_0"'] \arrow[d, shift right, "R(p_0)"'] \arrow[d, shift left, "R(p_1)"] & & X \times X \arrow[ll] \arrow[r, two heads, "g_1"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & Z_1 \arrow[d, shift right=2, "z_0"'] \arrow[d, shift left=2, "z_1"] \\
R[g_0] \arrow[rr, shift left=2, "p_1"] \arrow[rr, shift right=2, "p_0"'] \arrow[d, dotted, "q"'] & & X \arrow[ll] \arrow[u] \arrow[r, two heads, "g_0"'] \arrow[d, "\tau_X"] & Z_0 \arrow[u] \\
A \arrow[rr, dotted, shift left] & & 1 \arrow[ll, dotted, shift left] &
\end{tikzcd}$$

Then there is a unique dotted arrow q which completes the previous diagram into a regularly fully faithful profunctor $\underline{K}_1 A \nrightarrow \underline{Z}_1$ whose legs of the associated discrete bifibration are such that $d(\underline{\gamma}_1) = d(\underline{\phi}_1)$.

PROOF. Since the category $\mathbb{E}$ is regular, it is sufficient to prove the result in Set, thanks to the Barr embedding [1]. For sake of simplicity, we denote by $\nu : R[(z_0, z_1)] \to A$ the mapping which defines the direction of $\underline{Z}_1$. The pullback defining A implies that, given a pair of parallel maps $(\phi, \psi) : u \rightrightarrows v$ in $\underline{Z}_1$, they are equal if and only if $\nu(\phi, \psi) = 0$. Let us denote by S the equivalence relation on $R[g_0]$ defined by the left hand side vertical diagram. We get $(x, x')S(y, y')$ if and only if $g_1(x, y) = g_1(x', y')$. On the other hand, we have necessarily: $\nu(1_{g_0(y)}, g_1(y, y')) + \nu(g_1(x, y), g_1(x, y)) = \nu(g_1(x, y), g_1(x, y')) = \nu(g_1(x, y), g_1(x', y')) + \nu(1_{g_0(x)}, g_1(x, x'))$, in other words we get:

$$\nu(1_{g_0(y)}, g_1(y, y')) = \nu(g_1(x, y), g_1(x', y')) + \nu(1_{g_0(x)}, g_1(x, x'))$$

So: $(x, x')S(y, y')$ if and only if $\nu(1_{g_0(y)}, g_1(y, y')) = \nu(1_{g_0(x)}, g_1(x, x'))$. Consequently the map $q : R[g_0] \to A$ defined by $q(x, x') = \nu(1_{g_0(x)}, g_1(x, x'))$ is such that $S = R[q]$ and, being surjective, it is the quotient map of this equivalence relation S. Conversely suppose you are given a regularly fully faithful profunctor:

$$\begin{tikzcd}
R[g_1] \arrow[rr, shift left=2, "p_1"] \arrow[rr, shift right=2, "p_0"'] \arrow[d, shift right, "R(p_0)"'] \arrow[d, shift left, "R(p_1)"] & & X \times X \arrow[ll] \arrow[r, two heads, "g_1"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & Z_1 \arrow[d, shift right=2, "z_0"'] \arrow[d, shift left=2, "z_1"] \\
R[g_0] \arrow[rr, shift left=2, "p_1"] \arrow[rr, shift right=2, "p_0"'] \arrow[d, "f_1"'] & & X \arrow[ll] \arrow[u] \arrow[r, two heads, "g_0"'] \arrow[d, "\tau_X"] & Z_0 \arrow[u] \\
A \arrow[rr, shift left] & & 1 \arrow[ll, shift left] &
\end{tikzcd}$$

such that $d(\underline{\gamma}_1) = d(\underline{\phi}_1)$. This means that for any pair $((x, y, z), (x, y', z))$ such that $g_0(y) = g_0(z)$ and $g_0(y') = g_0(z)$, we have $\nu(g_1(x, y'), g_1(x, y)) = f_1(y', z) - f_1(y, z)$. Whence: $f_1(y', z) = f_1(y', z) - f_1(z, z) = \nu(g_1(x, y'), g_1(x, z))$, for any x; in particular we have $f_1(y', z) = \nu(1_{g_0(y')}, g_1(y', z))$ and thus $f_1(y', z) = q(y', z)$. ∎

In other words there is a bijection between the sets $Tors\underline{Z}_1$ and $\mathbb{R}f_1\mathbb{E}(\underline{K}_1A, \underline{Z}_1)$.

THE CANONICAL ACTION

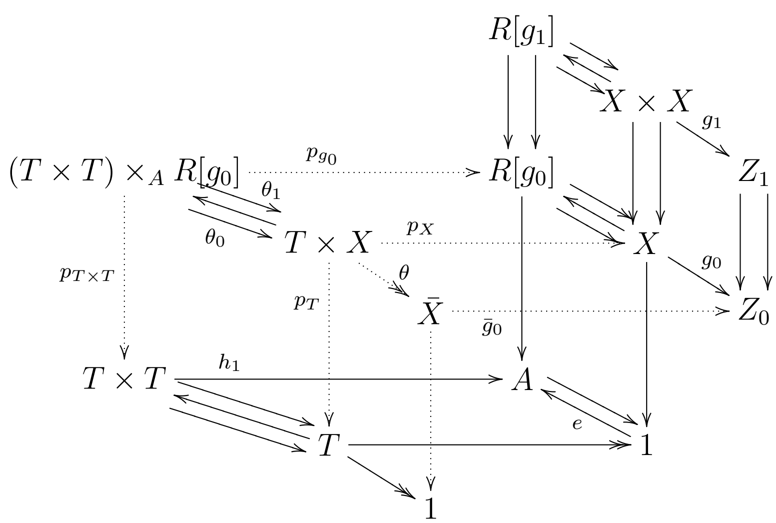
Suppose that our aspherical abelian groupoid $\underline{Z}_1$ is such that $Z_0 = 1$, namely that it is actually an abelian group A. We recalled that the set $TorsA$ is canonically endowed with an abelian group structure which is nothing but a subgroup of the group $\mathbb{R}f\mathbb{E}(\underline{K}_1A, \underline{K}_1A)$. Thanks to the previous proposition, we can now precise that this subgroup is $\mathbb{R}f_1\mathbb{E}(\underline{K}_1A, \underline{K}_1A)$.

2.9. THEOREM. *Let $\mathbb{E}$ be an efficiently regular category. Let $\underline{Z}_1$ be an aspherical abelian groupoid with direction A. Then there is a canonical simply transitive action of the abelian group $TorsA$ on the set $Tors\underline{Z}_1$ of isomorphic classes of $\underline{Z}_1$-torsors.*

PROOF. The action of the A-torsors on $\underline{Z}_1$-torsors will be naturally given by the composition (= tensor product) of profunctors $\underline{K}_1A \nrightarrow \underline{K}_1A \nrightarrow \underline{Z}_1$ whose image by the direction functor d will be 1_A since it is the case for both of them. The fact that this action is simply transitive comes from the fact that the regularly fully faithful profunctor $\underline{K}_1A \nrightarrow \underline{Z}_1$ arising from a $\underline{Z}_1$-torsor is an invertible profunctor, see Proposition 2.5. So that, starting from a pair $(\underline{g}_1, \underline{g}'_1)$ of $\underline{Z}_1$-torsors, the unique A-torsor relating them is necessarily given by the following composition:

$$\underline{K}_1A \overset{\underline{g}'_1}{\nrightarrow} \underline{Z}_1 \overset{(\underline{g}_1)^{-1}}{\nrightarrow} \underline{K}_1A$$

whose image by d is certainly 1_A. Accordingly this group action is nothing else but the simply transitive action of the sub-Hom-group $\mathbb{R}f_1\mathbb{E}(\underline{K}_1A, \underline{K}_1A)$ on the sub-Hom-set $\mathbb{R}f\mathbb{E}(\underline{K}_1A, \underline{Z}_1)$ inside the subgroupoid $\mathbb{R}f_1\mathbb{E}$. So, given a $\underline{Z}_1$-torsor $\underline{g}_1$ and an A-torsor $\underline{h}_1$, the tensor product $\underline{g}_1 \otimes \underline{h}_1$ will be given by the following diagram:



∎

2.10. The additive setting. Let us have a quick look at the translation of the previous theorem in the additive setting. So let $\mathbb{A}$ be an efficiently regular additive category, and C an object of $\mathbb{A}$. We are going to make explicit the previous simply transitive action in the slice category $\mathbb{A}/C$ (in the pointed category $\mathbb{A}$ any abelian group $TorsA$ is trivial). Giving an aspherical internal groupoid $\underline{A}_1$ in $\mathbb{A}/C$ is equivalent to giving an exact sequence:

$$\begin{tikzcd} A_1 \arrow[r, "\alpha"] & A_0 \arrow[r, two heads, "q"] & C \arrow[r] & 1 \end{tikzcd}$$

Let us denote by $\beta : B \rightarrowtail A_1$ the kernel of α. The direction of $\underline{A}_1$ is then nothing but $(p_C, \iota_C) : B \times C \leftrightarrows C$. It is well known that a torsor associated with this abelian group in $\mathbb{A}/C$ is nothing but an extension:

$$\begin{tikzcd} 1 \arrow[r] & B \arrow[r, tail, "m"] & H \arrow[r, two heads, "h"] & C \arrow[r] & 1 \end{tikzcd}$$

and the abelian group of torsors in $\mathbb{A}/C$ is nothing but the abelian group $Ext_{\mathbb{A}}(C, B)$. On the other hand, giving a $\underline{A}_1$-torsor is equivalent to giving an exact sequence together with a (regular epic) map $\tau : D \to A_0$ such that $q.\tau = f$ and $\tau.k = \alpha$:

$$\begin{tikzcd} 1 \arrow[r] & A_1 \arrow[r, tail, "k"] \arrow[d, equal] & D \arrow[r, two heads, "f"] \arrow[d, "\tau"'] & C \arrow[r] \arrow[d, equal] & 1 \\ & A_1 \arrow[r, "\alpha"'] & A_0 \arrow[r, two heads, "q"'] & C \arrow[r] & 1 \end{tikzcd}$$

The action of the abelian group $Ext_{\mathbb{A}}(C, B)$ can be described in the following way: starting with the previous "B-torsor" and "$\underline{A}_1$-torsor", take the pullback of f and h, then the result of the action is given by the following 3×3 construction:

$$\begin{tikzcd} & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\ 1 \arrow[r] & B \arrow[r, tail, "{(-1,\beta)}"] \arrow[d, "1_B"'] & B \times A_1 \arrow[r, two heads, "{(\beta,1_{A_1})}"] \arrow[d, "m\times k"] & A_1 \arrow[r] \arrow[d, dotted, "\bar{k}"] & 1 \\ 1 \arrow[r] & B \arrow[r, tail, "{(-m,k.\beta)}"'] \arrow[d] & H \times_C D \arrow[r, two heads, "w"'] \arrow[d, "h\times_C f"] & \bar{D} \arrow[r] \arrow[d, dotted, two heads, "\bar{f}"] & 1 \\ 1 \arrow[r] & 1 \arrow[r, tail] \arrow[d] & C \arrow[r, two heads, "1_C"'] \arrow[d] & C \arrow[r] \arrow[d] & 1 \\ & 1 & 1 & 1 & \end{tikzcd}$$

The kernel of the regular epimorphism $h \times_C f$ is the map $m \times k$; so that the 3×3 lemma produces the vertical right hand side exact sequence. The upper right hand side square is then certainly a pushout. Accordingly, the equality $\alpha.(\beta, 1_{A_1}) = \alpha.p_{A_1} = \tau.k.p_{A_1} = \tau.p_D.(m \times k)$ produces the required factorization $\bar{\tau} : \bar{D} \to A_0$ to have what is equivalent to a $\underline{A}_1$-torsor:

$$\begin{tikzcd}
B \times A_1 \arrow[r, two heads, "(\beta,1_{A_1})"] \arrow[d, tail, "m\times k"'] & A_1 \arrow[r, equal] \arrow[d, tail, dotted, "\bar{k}"] & A_1 \arrow[d, "\alpha"] \\
H \times_C D \arrow[r, two heads, "w"] \arrow[dr, "p_D"'] & \bar{D} \arrow[r, dashed, "\bar{\tau}"] & A_0 \\
& D \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Conversely, starting with what is equivalent to two $\underline{A}_1$-torsors, we get the unique B-torsor relating them by the vertical right hand side exact sequence given by the following 3×3 construction:

$$\begin{tikzcd}
& 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & B \arrow[r, tail, "(1,1)"] \arrow[d, "\beta"'] & B \times B \arrow[r, two heads, "(-1,1)"] \arrow[d, "(k.\beta)\times(k'.\beta)"] & B \arrow[r] \arrow[d, dotted] & 1 \\
1 \arrow[r] & A_1 \arrow[r, tail, "(k,k')"] \arrow[d] \arrow[dr, dotted, "\alpha"'] & D \times_{A_0} D' \arrow[r, two heads] \arrow[d, "\tau\times_{A_0}\tau'"] & H \arrow[r] \arrow[d, dotted] & 1 \\
1 \arrow[r] & Kerq \arrow[r, tail] \arrow[d] & A_0 \arrow[r, two heads, "q"'] \arrow[d] & C \arrow[r] \arrow[d] & 1 \\
& 1 & 1 & 1 &
\end{tikzcd}$$

3. Connector and centralizing double relation

We shall describe, here, the strong structural relationship between fully faithful profunctors and *connectors* between equivalence relations. Consider R and S two equivalence relations on an object X in any finitely complete category $\mathbb{E}$. Let us recall the following definition from [12], see also [24] and [15]:

3.1. Definition. *A connector for the pair (R, S) is a morphism*

$$p : S \times_X R \to X,\ (xSyRz) \mapsto p(x, y, z)$$

which satisfies the identities :

1) $xRp(x,y,z)$	*1') $zSp(x,y,z)$*
2) $p(x,y,y)=x$	*2') $p(y,y,z)=z$*
3) $p(x,y,p(y,u,v))=p(x,u,v)$	*3') $p(p(x,y,u),u,v)=p(x,y,v)$*

In set theoretical terms, Condition 1 means that with any triple $xSyRz$ we can associate a square:

$$\begin{tikzcd}
x \arrow[r, dotted, "R"] \arrow[d, "S"'] & p(x,y,z) \arrow[d, dotted, "S"] \\
y \arrow[r, "R"'] & z.
\end{tikzcd}$$

More acutely, any connected pair produces an equivalence relation $\underline{\Sigma}_1 \rightrightarrows \underline{R}_1$ in $Grd\mathbb{E}$ on the equivalence relation R whose two legs are discrete fibrations, in other words an equivalence relation in $DiF\mathbb{E}$:

$$\begin{tikzcd}[row sep=large, column sep=large]
S \times_X R \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "(d_0.p_0,p)"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "(p,d_1.p_0)"] & R \arrow[l] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
S \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[u] & X \arrow[l] \arrow[u]
\end{tikzcd}$$

It is called the *centralizing double relation* associated with the connector. It is clear that, conversely, any equivalence relation $\underline{\Sigma}_1 \rightrightarrows \underline{R}_1$ in $Grd\mathbb{E}$ whose two legs are discrete fibrations determines a connector between R and the image by the functor $()_0 : Grd\mathbb{E} \to \mathbb{E}$ of this equivalence relation $\underline{\Sigma}_1 \rightrightarrows \underline{R}_1$.

EXAMPLE 1) An emblematical example is produced by a given discrete fibration $\underline{f}_1 : \underline{R}_1 \to \underline{Z}_1$ whose domain R is an equivalence relation. For that consider the following diagram:

$$\begin{tikzcd}
R[f_1] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, shift right=2, "R(d_0)"'] \arrow[d, shift left=2, "R(d_1)"] & R \arrow[l] \arrow[r, "f_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Z_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
R[f_0] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[u] & X \arrow[l] \arrow[r, "f_0"'] \arrow[u] & Z_0 \arrow[u]
\end{tikzcd}$$

It is clear that $R[f_1]$ is isomorphic to $R[f_0] \times_X R$ and that the map

$$p : R[f_1] \stackrel{p_0}{\to} R \stackrel{d_1}{\to} X$$

determines a connector for the pair $(R, R[f_0])$.

2) For any groupoid $\underline{X}_1$, we have such a discrete fibration (which we shall denote by $\underline{\epsilon_1 X_1} : \underline{Dec_1 X_1} \to \underline{X}_1$ in $Grd\mathbb{E}$):

$$\begin{tikzcd}
R[d_0] \arrow[r, "d_2"] \arrow[d, shift right=2, "p_1"'] \arrow[d, shift left=2, "p_0"] & X_1 \arrow[d, shift right=2, "d_1"'] \arrow[d, shift left=2, "d_0"] \\
X_1 \arrow[r, "d_1"'] \arrow[u] & X_0 \arrow[u]
\end{tikzcd}$$

which implies the existence of a connector for the pair $(R[d_0], R[d_1])$. The converse is true as well, see [15] and [12]; given a reflexive graph :

$$\begin{tikzcd}
Z_1 \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "d_0"'] & Z_0 \arrow[l, "s_0"']
\end{tikzcd}$$

any connector for the pair $(R[d_0], R[d_1])$ determines a groupoid structure on this graph.
3) For any pair (X, Y) of objects, the pair $(R[p_X], R[p_Y])$ of effective equivalence relations is canonically connected.
4) The diagram defining any fully faithful profunctor $(\underline{\phi}_1, \underline{\gamma}_1) : \underline{X}_1 \nrightarrow \underline{Y}_1$ clearly determines a centralizing double relation, and thus a connector for the pair $(R[f_0], R[g_0])$.

Remark. The main point, here, is to emphasize that the diagram underlying the double centralizing relation associated with the existence of a connector is the core of the diagram defining a regularly fully faithful profunctor, and that, whenever the category $\mathbb{E}$ is exact, this core can be effectively completed into an actual regularly fully faithful profunctor by means of the quotients:

$$\begin{tikzcd}
S \times_X R \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "(d_0.p_0,p)"'] \arrow[d, shift right=3, "p_0"'] \arrow[d, shift left=3, "(p,d_1.p_0)"] & R \arrow[l] \arrow[r, dashed, two heads, "q_S^1"] \arrow[d, shift right=3, "d_0"'] \arrow[d, shift left=3, "d_1"] & Y_1 \arrow[d, shift right=3, "y_0"'] \arrow[d, shift left=3, "y_1"] \\
S \arrow[u] \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, dashed, two heads, "q_R^1"'] & X \arrow[u] \arrow[l] \arrow[r, dashed, two heads, "q_S"'] \arrow[d, dashed, two heads, "q_R"'] & Y_0 \arrow[u] \\
X_1 \arrow[r, shift left=3, "x_1"] \arrow[r, shift right=3, "x_0"'] & X_0 \arrow[l, "s_0"'] &
\end{tikzcd}$$

Now let us observe that:

3.2. Proposition. *Suppose p is a connector for the pair (R, S). Then the following reflexive graph is underlying a groupoid we shall denote by $R\sharp S$:*

$$\begin{tikzcd}
S \times_X R \arrow[r, shift left=2, "d_1.p_1"] \arrow[r, shift right=2, "d_0.p_0"'] & X \arrow[l]
\end{tikzcd}$$

Proof. Thank to the Yoneda embedding, it is enough to prove it in *Set*. This is straightforward just setting:

$$(zSuRv).(xSyRz) = xSp(u, z, y)Rv$$

The inverse of the arrow $xSyRz$ is $zSp(x, y, z)Rx$. ■

When $S \cap R = \Delta X$, the groupoid $S\sharp R$ is actually an equivalence relation.

3.3. Proposition. *Suppose p is a connector for the pair (R, S) and consider the following diagram in $Grd\mathbb{E}$:*

$$\begin{tikzcd}
\Delta X \arrow[r, tail] \arrow[d] & R \arrow[d, "i_R"] \\
S \arrow[r, tail, "i_S"'] & S\sharp R
\end{tikzcd}$$

It is a pullback such that the supremum of i_R and i_S is $1_{R\sharp S}$.

Proof. By the Yoneda embedding: the first point is straightforward; as for the second one: it is a direct consequence of the fact that any map $(xSyRz)$ in $S\sharp R$ is such that: $(xSyRz) = (ySyRz).(xSyRy) = i_R(yRz).i_S(xSy)$. ∎

So, when $S \cap R = \Delta X$, the equivalence relation $S\sharp R$ is nothing but $S \vee R$.

3.4. The Mal'cev context. Let $\mathbb{D}$ be now a Mal'cev category, i.e. a category in which any reflexive relation is an equivalence relation [14] [15].

Commutator theory

In a Mal'cev category, the previous conditions 2) on connectors imply the other ones, and moreover a connector is necessarily unique when it exists, and thus the existence of a connector becomes a property; we then write $[R, S] = 0$ when this property holds. From [12] recall that:

1) $R \wedge S = \Delta_X$ implies $[R, S] = 0$
2) $T \subset S$ and $[R, S] = 0$ imply $[R, T] = 0$
3) $[R, S] = 0$ and $[R', S'] = 0$ imply $[R \times R', S \times S'] = 0$
When $\mathbb{D}$ is a regular Mal'cev category, the direct image of an equivalence relation along a regular epimorphism is still an equivalence relation. In this case, we get moreover:
4) if $f : X \twoheadrightarrow Y$ is a regular epimorphism, $[R, S] = 0$ implies $[f(R), f(S)] = 0$
5) $[R, S_1] = 0$ and $[R, S_2] = 0$ imply $[R, S_1 \vee S_2] = 0$

As usual, an equivalence relation R is called *abelian* when we have $[R, R] = 0$, and central when we have $[R, \nabla_X] = 0$. An object X in $\mathbb{D}$ is called *commutative* when $[\nabla_X, \nabla_X] = 0$. We get also the following precision:

3.5. Proposition. *Let $\mathbb{D}$ be a Mal'cev category. Suppose p is a connector for the pair (R, S). The following diagram in $Grd\mathbb{D}$:*

$$\begin{tikzcd} \Delta X \arrow[r, tail] \arrow[d] & R \arrow[d, "i_R"] \\ S \arrow[r, tail, "i_S"'] & S\sharp R \end{tikzcd}$$

is a pushout in $Grd\mathbb{D}$.

Proof. Let us consider the following diagram in $Grd\mathbb{D}$:

$$\begin{tikzcd} \Delta X \arrow[r, tail] \arrow[d] & R \arrow[d, "f"] \\ S \arrow[r, "g"'] & \underline{X}_1 \end{tikzcd}$$

The previously observed decomposition:

$$(xSyRz) = (ySyRz).(xSyRy) = i_R(yRz).i_S(xSy)$$

in the groupoid $S\sharp R$ allows us to construct a unique morphism of reflexive graphs $\phi : S\sharp R \to \underline{X}_1$, just setting $\phi(xSyRz) = f(yRz).g(xSy)$. Now since $\mathbb{D}$ is a Mal'cev category, any morphism of reflexive graphs between the underlying graphs of groupoids is necessarily a functor [15]. ■

Finally, recall also the following from [13], see Proposition 3.2:

3.6. Proposition. *Let $\mathbb{C}$ be a regular Mal'cev category. Any decomposition in $Grd\mathbb{D}$ of a discrete fibration $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ through a regular epic functor:*

$$\underline{X}_1 \overset{q_1}{\twoheadrightarrow} \underline{Q}_1 \xrightarrow{\bar{f}_1} \underline{Y}_1$$

is necessarily made of discrete fibrations.

3.7. A glance at the notion of centralizing double groupoid. Making a further step, given any pair $(\underline{U}_1, \underline{V}_1)$ of internal groupoids in $\mathbb{E}$ having the same object of objects U_0, we shall say that they admit a *centralizing double groupoid*, if there is an internal groupoid in $DiF\mathbb{E}$ (objects: internal groupoids; maps: discrete fibrations):

$$\begin{tikzcd} \underline{W}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \underline{V}_1 \arrow[l] \end{tikzcd}$$

such that its image by the functor $()_0 : Grd\mathbb{E} \to \mathbb{E}$ is $\underline{U}_1$; namely if there is a diagram in $\mathbb{E}$ which reproduces the core of an internal profunctor between groupoids, where any commutative square in the following diagram is a pullback:

$$\begin{tikzcd} W_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & V_1 \arrow[l] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\ U_1 \arrow[u] \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & U_0 \arrow[l] \arrow[u] \end{tikzcd}$$

It is clear that in general this is a further structure. But in a Mal'cev category this becomes a property:

3.8. Proposition. *Let $\mathbb{D}$ be Mal'cev category, and $(\underline{U}_1, \underline{V}_1)$ a pair of internal groupoids having the same object of objects U_0. A centralizing double groupoid on this pair is unique if it exists. In this case we shall say that the groupoids $\underline{U}_1$ and $\underline{V}_1$ are connected.*

Proof. Consider the following pullback of split epimorphisms:

$$\begin{tikzcd} W_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, dotted, "d_0^{V_1}"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, dotted, "d_1^{U_1}"] & V_1 \arrow[l] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, dotted, "d_1^V"] \\ U_1 \arrow[u] \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, dotted, "d_0^U"'] & U_0 \arrow[l] \arrow[u] \end{tikzcd}$$

Since the category $\mathbb{D}$ is a Mal'cev category, in the pullback above, the pair $(s_0^{U_1} : U_1 \rightarrowtail W_1, s_0^{V_1} : V_1 \rightarrowtail W_1)$ is jointly strongly epic. Accordingly the unicity of the map $d_0^{V_1}$ is a consequence of the equations $d_0^{V_1}.s_0^{V_1} = 1_{V_1}$ and $d_0^{V_1}.s_0^{U_1} = s_0^V.d_0^U$, while the unicity of the map $d_1^{U_1}$ is a consequence of the equations $d_1^{U_1}.s_0^{U_1} = 1_{U_1}$ and $d_1^{U_1}.s_0^{V_1} = s_0^U.d_1^V$. ∎

Remark: Actually, in the Mal'cev context, a pair $(\underline{U}_1, \underline{V}_1)$ of groupoids is connected if and only if there is a pair of maps $(d_0^{V_1} : W_1 = U_1 \times_{U_0} V_1 \to V_1, d_1^{U_1} : W_1 = U_1 \times_{U_0} V_1 \to U_1)$ such that the four previous equations are satisfied with moreover the commutation equation $d_1^V.d_0^{V_1} = d_0^U.d_1^{U_1}$.

As previously for the connected equivalence relations, given any pair $(\underline{U}_1, \underline{V}_1)$ of groupoids which has a double centralizing groupoid, in a category $\mathbb{E}$, we can observe that the following reflexive graph is underlying a groupoid we shall denote by $\underline{U}_1\sharp\underline{V}_1$:

$$\begin{tikzcd} W_1 \arrow[r, shift left=2, "d_1.p_1"] \arrow[r, shift right=2, "d_0.p_0"'] & U_0 \arrow[l] \end{tikzcd}$$

In order to show this, the quickest argument is to consider the diagram defining the double centralizing groupoid as a double simplicial object where any commutative square is a pullback. The diagonal is then necessarily a simplicial object, while the fact that any of the structural commutative squares of this diagonal is a pullback is a straigthforward consequence of the fact that any of the commutative squares of the original double simplicial object is a pullback.

3.9. Proposition. *Let $\mathbb{D}$ be a Mal'cev category. Suppose the pair $(\underline{U}_1, \underline{V}_1)$ of internal groupoids is connected, then the following diagram in $Grd\mathbb{D}$:*

$$\begin{tikzcd} \Delta U_0 \arrow[r, tail] \arrow[d] & \underline{V}_1 \arrow[d, "i_{\underline{V}_1}"] \\ \underline{U}_1 \arrow[r, tail, "i_{\underline{U}_1}"'] & \underline{U}_1\sharp\underline{V}_1 \end{tikzcd}$$

is a pullback and a pushout.

Proof. The proof is the same as for the equivalence relations in the Mal'cev context. The first point is straightforward. As for the second one, consider any commutative diagram in $Grd\mathbb{D}$:

$$\begin{tikzcd} \Delta U_0 \arrow[r, tail] \arrow[d] & \underline{V}_1 \arrow[d, "\underline{f}_1"] \\ \underline{U}_1 \arrow[r, "\underline{g}_1"'] & \underline{X}_1 \end{tikzcd}$$

Any map in $\underline{U}_1\sharp\underline{V}_1$ is an object of the pullback $W_1 = U_1 \times_{U_0} V_1$, namely a pair of a map in $\underline{U}_1$ and of a map in $\underline{V}_1$. Accordingly it is possible to construct a unique morphism of reflexive graphs $\phi : W_1 \to X_1$. Now, since $\mathbb{D}$ is a Mal'cev category, it is necessarily underlying an internal functor, since $\underline{U}_1\sharp\underline{V}_1$ and $\underline{X}_1$ are actual groupoids, again see [15]. ∎

3.10. Corollary. *Let $\mathbb{D}$ be a Mal'cev category and $\underline{X}_1 \nrightarrow \underline{Y}_1$ an internal profunctor. In the following diagram, the left hand side quadrangle is a pushout in $Grd\mathbb{D}$:*

$$\begin{tikzcd}
 & \underline{U}_1^{Y_1} \arrow[rr, "\underline{g}_1"] \arrow[r, hook] & \Upsilon_1 = \underline{U}_1^{X_1} \sharp \underline{U}_1^{Y_1} \arrow[r, "\underline{\gamma}_1"] \arrow[dr, "\underline{\phi}_1"'] & \underline{Y}_1 \\
\Delta_1 U_0 \arrow[ur, hook] \arrow[dr, hook] & & & \\
 & \underline{U}_1^{X_1} \arrow[uur, hook] \arrow[rr, "\underline{f}_1"'] & & \underline{X}_1
\end{tikzcd}$$

Example: It is well known that, in the category Gp of groups, an internal groupoid is the same thing as a *crossed module*. It is easy to check that a pair of crossed modules $H \overset{h}{\to} G \overset{h'}{\leftarrow} H'$ corresponds to a pair of groupoids $\underline{H}_1$ and $\underline{H}'_1$ having a centralizing double groupoid if and only if the restriction of the action of the group G on H to the subgroup $h'(H')$ is trivial (which implies $[h(H), h'(H')] = 0$), and symmetrically the restriction of the action of the group G on H' to the subgroup $h(H)$ is trivial. When this is the case, the crossed module corresponding to the groupoid $\underline{H}_1 \sharp \underline{H}'_1$ is nothing but the factorization $\phi : H \times H' \to G$ induced by the equality $[h(H), h'(H')] = 0$, and the following diagram becomes a pushout inside the category X-Mod of crossed modules:

$$\begin{tikzcd}
 & H' \arrow[r] \arrow[dd, "h'"'] & H \times H' \arrow[ddl, "\phi"] \\
1 \arrow[ur] \arrow[d] & & \\
H \arrow[urr] \arrow[r, "h"'] & G &
\end{tikzcd}$$

Remark. Again, the main point, here, is as above to emphasize that the diagram underlying a double centralizing groupoid is the core of the diagram defining a profunctor; when the category Mal'cev $\mathbb{D}$ is exact, this core can be effectively completed into an actual profunctor by means of the quotients, thanks to Proposition 3.6:

$$\begin{tikzcd}
W_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & V_1 \arrow[l, "s_0"'] \arrow[r, dotted, two heads] \arrow[d, shift left=2, "d_1"] \arrow[d, shift right=2, "d_0"'] & Y_1 \arrow[d, shift right=2, "y_0"'] \arrow[d, shift left=2, "y_1"] \\
U_1 \arrow[u] \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, dotted, two heads] & U_0 \arrow[l, "s_0"'] \arrow[u] \arrow[r, dotted, two heads] \arrow[d, dotted, two heads] & Y_0 \arrow[u] \\
X_1 \arrow[r, shift left=2, "x_1"] \arrow[r, shift right=2, "x_0"'] & X_0 \arrow[l, "s_0"'] &
\end{tikzcd}$$

4. Schreier-Mac Lane extension theorem

A first application of the existence of the canonical action on $\underline{Z}_1$-torsors deals with the classification of extensions in $\mathbb{D}$, when the category $\mathbb{D}$ is an exact regular Mal'cev category which admits centralizers.

It was already observed in [7] that, in any finitely complete exact category $\mathbb{E}$, any commutative object X (i.e. such that $[\nabla X, \nabla X] = 0$, with a given commutative connector π) with global support has a *direction*, namely: there exists an abelian group A in $\mathbb{E}$ such that the following squares are pullbacks, where $\pi_1(x, y, z) = (x, \pi(x, y, z))$:

$$\begin{tikzcd}
X \times X \times X \arrow[r, shift left=2, "p_2"] \arrow[r, shift right=2, "\pi_1"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "\bar{\pi}_1"] & X \times X \arrow[l] \arrow[r, dotted, two heads, "\nu_X"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] \arrow[u] & A \arrow[d, shift right=1] \\
X \times X \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[u] & X \arrow[l] \arrow[r, two heads] \arrow[u] & 1 \arrow[u, shift right=1, "\omega"']
\end{tikzcd}$$

which is given by the quotient of the upper left hand side horizontal equivalence relation; and consequently any commutative object with global support gave rise to an A-torsor. Accordingly the set of isomorphic classes of commutative objects with global support and direction A has the abelian group structure of $TorsA$.

Now let $\mathbb{D}$ be an exact Mal'cev category. The previous result applied to the slice category $\mathbb{D}/Y$ says that:

1) any extension $f : X \twoheadrightarrow Y$ which has an abelian kernel equivalence relation produces, as its direction, an abelian group structure $E \leftrightarrows Y$ in $\mathbb{D}/Y$

2) the set $Ext_E Y$ of extensions above Y having an abelian kernel equivalence relation and $E \leftrightarrows Y$ as direction is nothing but the abelian group $TorsE$.

The aim of this section is to show that when, moreover, the Mal'cev category $\mathbb{D}$ has centralizers of equivalence relations, there is a way of producing an index ϕ for any extension $f : X \twoheadrightarrow Y$ which determines, on the set $Ext_\phi Y$ of extensions with this given index ϕ, the simply transitive action of an abelian group of type $Ext_E Y$. This observation is a generalization of the Schreier-Mac Lane extension theorem for groups; it was originated from a first generalization of this Schreier-Mac Lane extension theorem to any action representative category, see [11].

4.1. Mal'cev categories with centralizers. Let now $\mathbb{D}$ be a Mal'cev category.

4.2. Definition. *When R is a equivalence relation on an object X, we define $Z(R)$, and call* centralizer *of R, the largest equivalence relation on X connected with R, i.e. such that $[R, S] = 0$. We shall say that the Mal'cev category $\mathbb{D}$ has centralizers, when any equivalence relation R has a centralizer $Z(R)$.*

It is clear, for instance, that a naturally Mal'cev category in the sense of [19] is nothing but a Mal'cev category with centralizers, such that, for any equivalence relation R on

X, we have $Z(R) = \nabla_X$. More generally, recall that, in a Mal'cev category, an internal reflexive graph $(d_0, d_1) : X_1 \rightrightarrows X_0$ is a groupoid if and only if we have $[R[d_0], R[d_1]] = 0$, namely $R[d_1] \subset Z(R[d_0])$. Of course, there is an extremal situation:

4.3. Definition. *Let $\mathbb{D}$ be a Mal'cev category. A groupoid $\underline{X}_1$ in $\mathbb{D}$ is said to be eccentral when we have $Z(R[d_0]) = R[d_1]$.*

In other words a groupoid $\underline{X}_1$ in $\mathbb{D}$ is eccentral if and only if, given any reflexive relation $\underline{\Sigma}_1$ on $\underline{Dec}_1\underline{X}_1$ (see example 2) of connector) in $DiF\mathbb{D}$, its legs (p_0, p_1):

$$\begin{tikzcd} \underline{\Sigma}_1 \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & \underline{Dec}_1\underline{X}_1 \arrow[l] \arrow[r, dotted, two heads, "\epsilon_1\underline{X}_1"] & \underline{X}_1 \end{tikzcd}$$

are coequalized by $\epsilon_1\underline{X}_1$.

In the regular context, one important point is that eccentral groupoids are strongly related to centralizers. Before going any further, we have to recall [6] that the functor $\underline{Dec}_1 : Grd\mathbb{D} \to Grd\mathbb{D}$ is underlying a comonad such that the following diagram, in the category $Grd\mathbb{D}$, is a kernel equivalence relation with its quotient:

$$\begin{tikzcd} \underline{Dec}_1^2\underline{X}_1 \arrow[r, shift left, "\epsilon_1\underline{Dec}_1\underline{X}_1"] \arrow[r, shift right, "\underline{Dec}_1\epsilon_1\underline{X}_1"'] & \underline{Dec}_1\underline{X}_1 \arrow[r, two heads, "\epsilon_1\underline{X}_1"] & \underline{X}_1 \end{tikzcd}$$

4.4. Proposition. *Suppose the Mal'cev category $\mathbb{D}$ is regular and $\underline{X}_1$ is an eccentral groupoid. Then any regular epimorphic discrete fibration $\underline{j}_1 : \underline{R}_1 \twoheadrightarrow \underline{X}_1$ where R is an equivalence relation on an object X, is such that $R[j_0]$ is the centralizer $Z(R)$.*

Proof. Consider the following diagram in $Grd\mathbb{D}$ where $\underline{\Sigma}_1$ is the double centralizing relation associated with a connected pair $[R, S] = 0$. We shall show that $\underline{\Sigma}_1$ factorizes through $R[\underline{j}_1]$ or, equivalently, that $\underline{j}_1$ coequalizes p_0 and p_1. For that take the direct image along the regular epimorphic discrete fibration $\underline{Dec}_1\underline{j}_1$ of the equivalence relation $\underline{Dec}_1\underline{\Sigma}_1$:

$$\begin{tikzcd} \underline{Dec}_1\underline{\Sigma}_1 \arrow[r, dotted, two heads] \arrow[dr, shift left, "\underline{Dec}_1p_0"] \arrow[dr, shift right, "\underline{Dec}_1p_1"'] \arrow[dd, "\epsilon_1\Sigma"'] & \underline{Dec}_1\underline{j}_1(\underline{\Sigma}_1) \arrow[r, dashed] \arrow[dr, shift left, "\pi_0"] \arrow[dr, shift right, "\pi_1"'] & \underline{Dec}_1^2\underline{X}_1 \arrow[d, shift left] \arrow[d, shift right] \\ & \underline{Dec}_1\underline{R}_1 \arrow[ul] \arrow[r, dotted, two heads, "\underline{Dec}_1\underline{j}_1"'] \arrow[d, "\epsilon_1\underline{R}_1"] & \underline{Dec}_1\underline{X}_1 \arrow[ul] \arrow[d, "\epsilon_1\underline{X}_1"] \\ \underline{\Sigma}_1 \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & \underline{R}_1 \arrow[l] \arrow[r, dotted, two heads, "\underline{j}_1"'] & \underline{X}_1 \end{tikzcd}$$

The maps π_0 and π_1 are discrete fibrations since they come from a decomposition of a discrete fibration through a regular epic functor. Accordingly $\underline{Dec}_1\underline{j}_1(\underline{\Sigma}_1)$ is a reflexive relation in $DiF\mathbb{D}$, which factorizes through $\underline{Dec}_1^2\underline{X}_1$ since $\underline{X}_1$ is an eccentral groupoid. So $\epsilon_1\underline{X}_1.\underline{Dec}_1\underline{j}_1$ coequalizes $\underline{Dec}_1p_0$ and $\underline{Dec}_1p_1$, and since $\epsilon_1\underline{\Sigma}_1$ is a regular epic functor, the functor $\underline{j}_1$ coequalizes p_0 and p_1. ∎

4.5. Exact Mal'cev setting. In this section we shall show that when $\mathbb{D}$ is an exact Mal'cev category, the existence of centralizers is characterized by the existence of "enough" eccentral groupoids.

4.6. Proposition. *Let $\mathbb{D}$ be an exact Mal'cev category with centralizers. Given any equivalence relation R, there is a, unique up to isomorphism, regular epic discrete fibration $\underline{j}_1 : \underline{R}_1 \twoheadrightarrow \underline{X}_1$ towards an eccentral groupoid.*

Proof. Let $\underline{\Sigma}_1$ be the double centralizing relation associated with the connected pair $[R, Z(R)] = 0$. Since $\mathbb{D}$ is exact, we can take a levelwise quotient of this double relation which produces a regular epic discrete fibration:

$$\begin{tikzcd} \underline{\Sigma}_1 \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & \underline{R}_1 \arrow[r, two heads, "\underline{j}_1"] & \underline{X}_1 \end{tikzcd}$$

We have to show now that the groupoid $\underline{X}_1$ is eccentral. Suppose given an equivalence relation $\underline{\Lambda}_1$ on $\underline{Dec_1}\underline{X}_1$ which is in DiF. Then consider the inverse image of $\underline{\Lambda}_1$ along the regular epic discrete fibration $\underline{Dec_1 j}_1$ in the following diagram:

$$\begin{tikzcd} \underline{Dec_1}\underline{j}_1^{-1}(\underline{\Lambda}_1) \arrow[r, two heads, "h_1"] \arrow[d] \arrow[dr, shift left=2, "\bar{\pi}_0"] \arrow[dr, shift right=2, "\bar{\pi}_1"'] & \underline{\Lambda}_1 \arrow[r, dashed] \arrow[dr, shift left=2, "\pi_0"] \arrow[dr, shift right=2, "\pi_1"'] & \underline{Dec_1^2}\underline{X}_1 \arrow[d, shift left] \arrow[d, shift right] \\ \underline{\Gamma}_1 \arrow[d, dotted] \arrow[dr, shift left=2, "\check{\pi}_0"] \arrow[dr, shift right=2, "\check{\pi}_1"'] & \underline{Dec_1}\underline{R}_1 \arrow[r, two heads, "\underline{Dec_1 j}_1"'] \arrow[d, "\underline{\epsilon_1 R}_1"] \arrow[ul] & \underline{Dec_1}\underline{X}_1 \arrow[d, "\underline{\epsilon_1 X}_1"] \arrow[ul] \\ \underline{\Sigma}_1 \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & \underline{R}_1 \arrow[r, two heads, "\underline{j}_1"'] \arrow[l] \arrow[ul] & \underline{X}_1 \end{tikzcd}$$

Its direct image $\underline{\Gamma}_1$ along the regular epic discrete fibration $\underline{\epsilon_1 R}_1$ is an equivalence relation in DiF which is a double centralizing relation associated with R. Accordingly this direct image factorizes through $\underline{\Sigma}_1$ according to Proposition 4.4, and produces the left hand side vertical dotted factorization. Accordingly the pair $(\bar{\pi}_0, \bar{\pi}_1)$ is coequalized by $\underline{\epsilon_1 X}_1.\underline{Dec_1 j}_1$. And since h_1 is an epimorphism, the pair (π_0, π_1) is coequalized by $\underline{\epsilon_1 X}_1$. Accordingly $\underline{\Lambda}_1$ factorizes through $\underline{Dec_1^2}\underline{X}_1$, and $\underline{X}_1$ is eccentral.

Suppose now there are two regular epic discrete fibrations $\underline{j}_1$ and $\underline{j}'_1$ to eccentral groupoids $\underline{X}_1$ and $\underline{X}'_1$. Then $R[j_0] = Z(R) = R[j'_0]$. Accordingly X_0 is isomorphic to X'_0. Since $\underline{j}_1$ and $\underline{j}'_1$ are discrete fibrations, the two equivalence relations $R[j_1]$ and $R[j'_1]$ are part of the double centralizing relation associated with the pair $(R, Z(R))$. Since $\mathbb{D}$ is a Mal'cev category, this double centralizing relation is unique (up to isomorphism), and consequently we get $R[j_1] = R[j'_1]$; so that X_1 is isomorphic to X'_1. ■

4.7. Theorem. *Let $\mathbb{D}$ be an exact Mal'cev category. Then $\mathbb{D}$ has centralizers if and only if $\mathbb{D}$ has "enough" eccentral groupoids with respect to $DiF\mathbb{D}$: namely, from any groupoid $\underline{T}_1$ there is a regular epic discrete fibration $\underline{\phi}_1 : \underline{T}_1 \twoheadrightarrow \underline{X}_1$ with $\underline{X}_1$ an eccentral groupoid. In this case the eccentral groupoid is unique up to isomorphism and we call it the index-groupoid of the groupoid $\underline{T}_1$.*

PROOF. If $\mathbb{D}$ has enough groupoids, $\mathbb{D}$ has centralizers according to Proposition 4.4. Conversely suppose $\mathbb{D}$ has centralizers. Let $\underline{j}_1 : \underline{Dec}_1\underline{T}_1 \twoheadrightarrow \underline{X}_1$ be the regular epic discrete fibration, with $\underline{X}_1$ eccentral, given by the previous proposition. Since $\underline{X}_1$ is eccentral, the functor $\underline{j}_1$ trivializes the equivalence relation $\underline{Dec}_1^2\underline{T}_1$ since it is in DiF, according to Proposition 4.4.

$$\begin{tikzcd}
Dec^2\underline{T}_1 \arrow[r, shift left, "\underline{\epsilon}_1 Dec\underline{T}_1"] \arrow[r, shift right, "Dec\underline{\epsilon}_1\underline{T}_1"'] & Dec\underline{T}_1 \arrow[r, two heads, "\underline{\epsilon}_1\underline{T}_1"] \arrow[dr, "\underline{j}_1"'] & \underline{T}_1 \arrow[d, dashed, "\underline{\phi}_1"] \\
 & & \underline{X}_1
\end{tikzcd}$$

Accordingly there is a factorization $\underline{\phi}_1$ which is regular epic and a discrete fibration, since so is $\underline{j}_1$. The eccentral codomain $\underline{X}_1$ of this factorization $\underline{\phi}_1$ is unique up to isomorphism since, by the previous proposition, it was already the case for the codomain of $\underline{j}_1$. ∎

4.8. THE SCHREIER-MAC LANE THEOREM. In this section we shall suppose $\mathbb{D}$ is an exact Mal'cev category with centralizers. Let us start with any extension $f : X \twoheadrightarrow Y$. Consider its kernel equivalence relation:

$$\begin{tikzcd}
R[f] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & X \arrow[l] \arrow[r, dotted, two heads, "f"] & Y
\end{tikzcd}$$

and then take its index $\underline{q}_1 : \underline{R}_1[f] \twoheadrightarrow \underline{Q}_1$ to the eccentral groupoid $\underline{Q}_1$. Since $\mathbb{D}$ is exact, then the groupoid $\underline{Q}_1$ admits a $\pi_0(\underline{Q}_1)$, namely the coequalizer of the pair (d_0, d_1) below. Whence a factorization $\phi : Y \to \pi_0(\underline{Q}_1)$ which is necessarily a regular epimorphism:

$$\begin{tikzcd}
R[f] \arrow[rr, two heads, "q_1"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & & Q_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
X \arrow[u] \arrow[rr, two heads, "q_0"'] \arrow[d, "f"'] & & Q_0 \arrow[u] \arrow[d, "q"] \\
Y \arrow[rr, two heads, "\phi"'] & & \pi_0(\underline{Q}_1)
\end{tikzcd}$$

A morphism between two extensions above Y having the same index-groupoid $\underline{Q}_1$ and the same index ϕ is necessarily an isomorphism. We shall denote by $Ext_\phi Y$ the set of all isomorphic classes of extensions with index-groupoid $\underline{Q}_1$ and index ϕ. Now, consider the following diagram where the right hand side part is made of pullbacks:

$$\begin{tikzcd}
R[f] \arrow[r, dotted, "f_{1\phi}"] \arrow[rr, bend left, "q_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & D_1\phi \arrow[r, two heads, "d_{1\phi}"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & Q_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
X \arrow[r, dotted, "f_\phi"] \arrow[rr, bend left, "q_0"] \arrow[dr, "f"'] & D_0\phi \arrow[r, two heads, "d_{0\phi}"] \arrow[d, "q_\phi"] & Q_0 \arrow[d, "q"] \\
& Y \arrow[r, two heads, "\phi"'] & \pi_0(\underline{Q}_1)
\end{tikzcd}$$

This produces a groupoid $\underline{D}_{1\phi}$ such that $\pi_0(\underline{D}_{1\phi}) = Y$, and the two upper internal functors are discrete fibrations since so is the functor $\underline{q}_1$. The groupoid $\underline{D}_{1\phi}$ is then aspherical in the slice category $\mathbb{D}/Y$. Accordingly, with the discrete fibration $\underline{f}_{1\phi}$, we get a $\underline{D}_{1\phi}$-torsor in $\mathbb{D}/Y$.

4.9. Proposition. *The morphism f_ϕ is an epimorphism.*

Proof. It is a consequence of Proposition 1.7. ∎

This construction, associating the $\underline{D}_{1\phi}$-torsor $\underline{f}_{1\phi}$ with the extension f, produces a mapping which is clearly injective:

$$\Theta : Ext_\phi Y \to Tors\underline{D}_{1\phi}$$

4.10. Theorem. *The mapping Θ is bijective.*

Proof. Let $\underline{g}_1 : \underline{R}_1[f'] \twoheadrightarrow \underline{D}_{1\phi}$ be a $\underline{D}_{1\phi}$-torsor. Then the following diagram:

$$\begin{tikzcd}
R[f'] \arrow[r, two heads, "g_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & D_1\phi \arrow[r, two heads, "d_{1\phi}"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & Q_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
X' \arrow[r, two heads, "g_0"] \arrow[dr, "f'"'] & D_0\phi \arrow[r, two heads, "d_{0\phi}"] \arrow[d, "q_\phi"] & Q_0 \arrow[d, "q"] \\
& Y \arrow[r, two heads, "\phi"'] & \pi_0(\underline{Q}_1)
\end{tikzcd}$$

shows that the regular epic discrete fibration $\underline{d}_{1\phi}.\underline{g}_1$ is necessarily the index of $\underline{R}_1[f']$. Accordingly the extension $f' : X' \twoheadrightarrow Y$ is such that its index is the factorization ϕ and consequently belongs to $Ext_\phi Y$. ∎

We shall work now in the slice category $\mathbb{D}/Y$, where the groupoid $\underline{D}_{1\phi}$ is aspherical. It is abelian, since the category $\mathbb{D}/Y$ is Mal'cev. Let us denote its direction, which is an abelian group in $\mathbb{D}/Y$, in the following way:

$$\begin{tikzcd} E\phi \arrow[r, shift left, "e_\phi"] & Y \arrow[l, shift left, tail, "s_\phi"] \end{tikzcd}$$

Accordingly the map e_ϕ has an abelian kernel equivalence relation. We recalled above that a $E\phi$-torsor in $\mathbb{D}/Y$ is nothing but an extension $e : E \twoheadrightarrow Y$ in $\mathbb{D}$ having an abelian kernel relation $R[e]$ and the abelian group $E\phi$ in $\mathbb{D}/Y$ as *direction*, and that the abelian group $TorsE\phi$ is nothing but the group $Ext_{E\phi}Y$ of the extensions with abelian kernel equivalence relation having the abelian group $E\phi$ in $\mathbb{D}/Y$ as direction (see also Section *Baer sums* in [9]). Finally, according to the simply transitive action given by our Theorem 2.9, we get what we were aiming to:

4.11. THEOREM. *Suppose $\mathbb{D}$ is an exact Mal'cev category with centralizers. Let $f : X \twoheadrightarrow Y$ be any extension with index ϕ. There is on the set $Ext_\phi Y$ a canonical simply transitive action of the abelian group $Ext_{E\phi}Y$.*

5. Reg-epi and Birkhoff reflections

A second application of the canonical action on $\underline{Z}_1$-torsors will deal with the categorical Galois theory. We suppose $j : \mathbb{C} \rightarrowtail \mathbb{D}$ is a full replete inclusion and $\mathbb{D}$ is regular. Recall the following:

5.1. DEFINITION. *A reflection $I : \mathbb{D} \to \mathbb{C}$ of the inclusion j is said to be a reg-epi reflection when any projection $\eta_X : X \twoheadrightarrow IX$ is a regular epimorphism. It is said to be a Birkhoff reflection [13] when moreover for any regular epimorphism $f : X \twoheadrightarrow Y$ the factorization $R(f)$ is a regular epimorphism:*

$$\begin{tikzcd} R[\eta_X] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "R(f)"'] & X \arrow[r, two heads, "\eta_X"] \arrow[d, two heads, "f"'] & IX \arrow[d, two heads, "If"] \\ R[\eta_Y] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & Y \arrow[r, two heads, "\eta_Y"'] & IY \end{tikzcd}$$

A reflection I is a reg-epi reflection if and only if the subcategory $\mathbb{C}$ is stable under subobjects. When I is a Birkhoff reflection, the right hand square above is a pushout. Accordingly $\mathbb{C}$ is stable under regular epimorphism and is certainly a regular category. Since $\mathbb{C}$ is also stable under monomorphism, we conclude that $\mathbb{C}$ is a Birkhoff subcategory of $\mathbb{D}$ in the sense of [17]. When $\mathbb{D}$ *is an exact Mal'cev category*, we have the converse: *if $\mathbb{C}$ is stable under regular epimorphism, then any reg-epi reflection is a Birkhoff reflection.* So any reflection to a Birkhoff subcategory (subvariety) $\mathbb{C}$ of an exact Mal'cev category (variety) $\mathbb{D}$ determines a Birkhoff reflection. For instance, when the exact Mal'cev category $\mathbb{D}$ is pointed and finitely cocomplete, the inclusion $Ab(\mathbb{D}) \rightarrowtail \mathbb{D}$ of the subcategory

of abelian objects in $\mathbb{D}$ has a reflection $A : \mathbb{D} \to Ab(\mathbb{D})$; accordingly, this is a Birkhoff reflection.
Now, let $(\alpha, o) : A \leftrightarrows I(Y)$ be an abelian group in the slice category $\mathbb{C}/I(Y)$. When $\mathbb{D}$ is efficiently regular (we need that to have a group structure on the extensions), pulling back along $\eta_Y : Y \twoheadrightarrow I(Y)$ produces a group homomorphism $\eta^*_{Y,A} : Ext^{\mathbb{C}}_A I(Y) \to Ext^{\mathbb{D}}_{\eta^*_Y(A)} Y$:

5.2. Proposition. *Suppose $\mathbb{D}$ is an efficiently regular category and I a reg-epi reflection. Then the group homomorphism $\eta^*_{Y,A}$ is a monomorphism.*

Proof. Let $\psi : C \twoheadrightarrow I(Y)$ be a regular epimorphism in $\mathbb{C}$ with abelian kernel equivalence relation and direction α. Suppose is image by $\eta^*_{Y,A}$ is 0. This means that its pullback $\bar{\psi}$ along η_Y is split:

$$\begin{tikzcd} W \arrow[r, two heads, "\pi"] \arrow[d, two heads, shift right, "\bar{\psi}"'] & C \arrow[d, two heads, "\psi"] \\ I \arrow[u, shift right, "\sigma"'] \arrow[r, two heads, "\eta_Y"'] & I(Y) \end{tikzcd}$$

Since C is in $\mathbb{C}$, the map $\pi.\sigma : Y \to C$ produces a splitting of ψ which makes it 0 in the abelian group $Ext^{\mathbb{C}}_A I(Y)$. ∎

5.3. I-normal maps and Galois groupoids. When $\mathbb{D}$ is a regular Mal'cev category, recall from [13] that any reg-epi reflection I preserves the pullbacks of any pair of split epimorphisms and consequently preserves internal groupoids. This implies in particular that the image $I(R[f])$ of the kernel equivalence relation of any map f is a groupoid and the upper part of the following diagram produces an internal functor we shall denote by $\underline{\eta}_1 f : R[f] \to I(R[f])$:

$$\begin{tikzcd} R[f] \arrow[r, "\eta_{R[f]}"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & I(R[f]) \arrow[d, shift right=2, "I(p_0)"'] \arrow[d, shift left=2, "I(p_1)"] \\ X \arrow[u, dotted] \arrow[r, "\eta_X"'] \arrow[d, "f"'] & I(X) \arrow[u, dotted] \arrow[d, "I(f)"] \\ Y \arrow[r, "\eta_Y"'] & I(Y) \end{tikzcd}$$

Following [17], we shall be now interested in certain classes of maps with respect to the reg-epi reflection I:

5.4. Definition. *Given a reg-epi reflection I, a map $f : X \to Y$ in $\mathbb{D}$ is said to be I-trivial when the following square is a pullback:*

$$\begin{tikzcd} X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"'] & IX \arrow[d, "If"] \\ Y \arrow[r, two heads, "\eta_Y"'] & IY \end{tikzcd}$$

A map f is said to be I-normal when the projection $p_0 : R[f] \to X$ is I-trivial.

Accordingly, in the Mal'cev context, when the map f is I-normal, the functor $\underline{\eta}_1 f$: $R[f] \to I(R[f])$ becomes a *discrete fibration.* According to [17], when f is moreover a regular epimorphism (namely, *an I-normal extension*), the groupoid $I(R[f])$ is called the *Galois groupoid* of the I-normal extension f. It is then an aspherical groupoid in the category $\mathbb{C}/I(Y)$. Consider now the following diagram where any of the right hand side square is a pullback:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "\check{f}_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & G_1^f \arrow[r, two heads, "\bar{\eta}_{G_1^f}"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & I(R[f]) \arrow[d, shift right=2, "I(p_0)"'] \arrow[d, shift left=2, "I(p_1)"] \\
X \arrow[r, two heads, "\check{f}"'] \arrow[d, two heads, "f"'] & G_0^f \arrow[u, dotted] \arrow[r, two heads, "\bar{\eta}_{G_0^f}"'] \arrow[d, "\phi_f"'] & I(X) \arrow[u, dotted] \arrow[d, "I(f)"] \\
Y \arrow[r, "="'] & Y \arrow[r, two heads, "\eta_Y"'] & I(Y)
\end{tikzcd}$$

The upper vertical central part of this diagram is underlying an internal groupoid in $\mathbb{D}/Y$ which is nothing but $\eta_Y^*(I(R[f]))$ and will be denoted by $\underline{G}_1^f$. So, when f is an I-normal extension, the groupoid $\underline{G}_1^f$ is aspherical in the slice category $\mathbb{D}/Y$ and the discrete fibration $\underline{\check{f}}_1$ determines a $\underline{G}_1^f$-torsor in this category. The image by I of this groupoid $\underline{G}_1^f$ is not necessarily $I(R[f])$ unless the reflection I is admissible [17], namely unless I-trivial extensions are stable under pullback. If it is the case, the map $I(\check{f})$ is then necessarily an isomorphism. Now, according to [13], any Birkhoff reflection is admissible.

5.5. The faithful action on $Ext_Y/\underline{C}_1$. We shall suppose now I is a Birkhoff reflection on an efficiently regular Mal'cev category $\mathbb{D}$. Let $\underline{C}_1$ be an aspherical groupoid in the slice category $\mathbb{C}/I(Y)$. In this section, we shall be interested in those I-normal extensions $f : X \twoheadrightarrow Y$ which have (up to isomorphims) $\underline{C}_1$ as Galois groupoid.
It is clear that any morphism $\gamma : H' \to H$ in $\mathbb{D}/Y$ between such I-normal extensions is an isomorphism, since this morphism determines a morphism of $\eta_Y^*(\underline{C}_1)$-torsors in $\mathbb{D}/Y$. We shall denote by $Ext_Y/\underline{C}_1$ the set of isomorphic classes of the I-normal extensions $h : H \twoheadrightarrow Y$ which have $\underline{C}_1$ as Galois groupoid. The last diagram of the previous section described an inclusion: $Ext_Y/\underline{C}_1 \subset Tors_Y \eta_Y^*(\underline{C}_1)$. We are now in position to assert:

5.6. Theorem. *Let $\mathbb{D}$ be an efficiently regular Mal'cev category and I a Birkhoff reflection. Let $\underline{C}_1$ be an aspherical groupoid in the slice category $\mathbb{C}/I(Y)$ and $(\alpha, o) : A \leftrightarrows I(Y)$ its direction. If the set $Ext_Y/\underline{C}_1$ of I-normal extensions $f : X \twoheadrightarrow Y$ having $\underline{C}_1$ as Galois groupoid is non-empty, there is on $Ext_Y/\underline{C}_1$ a canonical faithful action of the abelian group $Ext_A^{\mathbb{C}} I(Y)$.*

Proof. Let us set $\underline{\bar{C}}_1 = \eta_Y^*(\underline{C}_1)$; it is an apherical groupoid in $\mathbb{D}/Y$ whose direction is the abelian group $\eta_Y^*(A)$ in $\mathbb{D}/Y$ which we shall denote by $(\beta, \omega) : B \leftrightarrows Y$. Since $Ext_Y/\underline{C}_1$ is a subset of $Tors_Y \underline{\bar{C}}_1$, it is enough to check that the restriction of the canonical simply transitive action of the group $Ext_B^{\mathbb{D}} Y$ to the subgroup $Ext_A^{\mathbb{C}} I(Y)$ (see Proposition 5.2) is stable on this subset $Ext_Y/\underline{C}_1$. So, let $\psi : C \twoheadrightarrow I(Y)$ be a regular epimorphism in $\mathbb{C}$ with abelian kernel equivalence relation and direction $(\alpha, o) : A \leftrightarrows I(Y)$. Then its pullback $\bar{\psi}$:

$$\begin{tikzcd}
W \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\bar{\psi}"'] & C \arrow[d, two heads, "\psi"] \\
Y \arrow[r, two heads, "\eta_Y"'] & I(Y)
\end{tikzcd}$$

determines an I-trivial $(\eta_Y^*(A) = B)$-torsor in $\mathbb{D}/Y$. Let us check that the action of this B-torsor on the $\underline{\bar{C}}_1$-torsor $\underline{\check{f}}_1$ associated with f actually determines an element of $Ext_Y/\underline{C}_1$. For that, let us consider the following diagram:

$$\begin{tikzcd}
 & & & & & R[\check{f}_1] \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr, shift left] & & \\
 & & & & & & R[f] \arrow[ul, shift left] \arrow[dr, "\check{f}_1"] \arrow[dd, shift left] \arrow[dd, shift right] & \\
R[\bar{\psi}] \times_{B_f} R[\check{f}] \arrow[rrrrr, dotted, "p_{\check{f}}"] \arrow[ddd, dotted, "p_{\bar{\psi}}"'] \arrow[drr, shift left, "\theta_1"] \arrow[drr, shift right, "\theta_0"'] & & & & & R[\check{f}] \arrow[ddd, dotted] \arrow[dr, shift left] & & \bar{C}_1 \arrow[dd, shift left] \arrow[dd, shift right] \\
 & & W \times_Y X \arrow[rrrr, dotted, "p_X"] \arrow[dr, dotted, "\theta"] \arrow[ddd, dotted, "p_W"'] & & & & X \arrow[ul, shift left] \arrow[dr, "\check{f}"] \arrow[ddd, "f"] & \\
 & & & \bar{X} \arrow[rrrr, dashed, two heads, "\check{\check{f}}"] \arrow[ddd, dashed, "\bar{f}"'] & & & & \bar{C}_0 \arrow[ddd, dashed, "\phi_f"] \\
R[\bar{\psi}] \arrow[rrrrr] \arrow[drr, shift left] \arrow[drr, shift right] & & & & & B \arrow[dr, shift left, "\beta"] & & \\
 & & W \arrow[rrrr, two heads, "\bar{\psi}"] \arrow[dr, two heads, "\bar{\psi}"'] & & & & Y \arrow[ul, shift left] \arrow[dr, equal] & \\
 & & & Y \arrow[rrrr, equal] & & & & Y
\end{tikzcd}$$

The result of this action is the $\underline{\bar{C}}_1$-torsor $\check{\check{f}}$. So, certainly $\bar{f} = \phi_f.\check{\check{f}}$ is an I-normal regular epimorphism, since the following upper squares are pullbacks:

$$\begin{tikzcd}
R[\bar{f}] \arrow[r, two heads, "\check{\check{f}}_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & \bar{C}_1 \arrow[r, two heads, "\eta_{\bar{C}_1}"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & C_1 \arrow[d, shift right] \arrow[d, shift left] \\
\bar{X} \arrow[r, two heads, "\check{\check{f}}"'] \arrow[d, "\bar{f}"'] & \bar{C}_0 \arrow[r, two heads, "\eta_{\bar{C}_0}"'] \arrow[d, "\phi_f"'] \arrow[u, dotted] & C_0 \arrow[d] \arrow[u, dotted] \\
Y \arrow[r, "="'] & Y \arrow[r, two heads, "\eta_Y"'] & I(Y)
\end{tikzcd}$$

In order to check that this I-normal extension $\bar{f}$ has $\underline{C}_1$ as Galois groupoid, it remains to show that $I(\check{\check{f}})$ is an isomorphism. The two following squares are pullbacks:

$$\begin{tikzcd}
\bar{X} \arrow[d, "\bar{f}"'] & W \times_Y X \arrow[l, two heads, "\theta"'] \arrow[r, two heads, "p_X"] \arrow[d, "p_W"'] & X \arrow[d, "f"] \\
Y & W \arrow[l, two heads, "\bar{\psi}"] \arrow[r, two heads, "\bar{\psi}"'] & Y
\end{tikzcd}$$

the right hand side one by definition, the left hand side one since the pair of maps $(p_W, p_{\bar{\psi}})$ determines a discrete fibration between equivalence relations. Moreover the map $\bar{\psi}$ is I-trivial by construction, and the reflection I, being admissible, preserves the pullback of I-trivial regular epimorphisms. Accordingly their images by I are pullbacks in $\mathbb{C}$:

$$
\begin{tikzcd}
I(\bar{X}) \arrow[d, "I(\bar{f})"'] & I(W \times_Y X) \arrow[l, two heads, "I(\theta)"'] \arrow[r, two heads, "I(p_X)"] \arrow[d, "I(p_W)"'] & I(X) \arrow[d, "I(f)"] \\
I(Y) & C \arrow[l, two heads, "\psi"] \arrow[r, two heads, "\psi"'] & I(Y)
\end{tikzcd}
$$

We noticed that $I(\check{f})$ is an isomorphism; this implies that the following dashed square is a pullback:

$$
\begin{tikzcd}
I(W \times_Y X) \arrow[rr, two heads, "I(p_X)"] \arrow[dr, "I(\theta)"'] \arrow[dd, "I(p_W)"'] & & I(X) \arrow[dr, "I(\check{f})"] \arrow[dd, "I(f)"] & \\
& I(\bar{X}) \arrow[rr, dashed, "I(\bar{\check{f}})"] \arrow[dd, dashed, "I(\bar{f})"'] & & I(G_0^f) \arrow[dd, "I(\phi_f)"] \\
C \arrow[rr, two heads, "\psi"] \arrow[dr, "\psi"'] & & I(Y) \arrow[dr, equal] & \\
& I(Y) \arrow[rr, equal] & & I(Y)
\end{tikzcd}
$$

and that consequently the map $I(\bar{\check{f}})$ is an isomorphism. ∎

Remark. It seems that there is no reason in general for which this action would be transitive. Certainly any pair (f, f') of I-normal extensions having $\underline{C}_1$ as Galois groupoid produces a B-torsor $\phi : T \twoheadrightarrow \bar{C}_0$ in $\mathbb{D}/Y$ according to Theorem 2.9. It is determined by the following diagram where the two squares are pullbacks:

$$
\begin{tikzcd}
X' \arrow[d, "\check{f}'"'] & X' \times_Y X \arrow[l, two heads, "p_{X'}"'] \arrow[r, two heads, "p_X"] \arrow[d, "\theta"'] & X \arrow[d, "\check{f}"] \\
\bar{C}_0 & T \arrow[l, two heads, "\phi"] \arrow[r, two heads, "\phi"'] & \bar{C}_0
\end{tikzcd}
$$

This B-torsor comes from an A-torsor in $\mathbb{C}/I(Y)$ if and only if the map ϕ is I-trivial, which would imply that the projections p_X and $p_{X'}$ would be themselves I-trivial:

$$
\begin{tikzcd}
X' \times_Y X \arrow[r, two heads, "p_X"] \arrow[d, "p_{X'}"'] & X \arrow[d, "f"] \\
X' \arrow[r, two heads, "f'"'] & Y
\end{tikzcd}
$$

5.7. Central extensions in the pointed setting. We shall suppose now the category $\mathbb{D}$ is a pointed finitely cocomplete exact protomodular category. We already recalled that the inclusion $Ab(\mathbb{D}) \to \mathbb{D}$ of the abelian object admits a reflection $A : \mathbb{D} \to$

$Ab(\mathbb{D})$ which is a Birkhoff reflection. It is known that the A-normal extensions are nothing but the usual central extensions, see [16], [25] and [13]. We shall have a quick look at the previous theorem in this setting. Again, since the category $Ab(\mathbb{D})$ is additive, giving an internal groupoid in $Ab(\mathbb{D})/A(Y)$ is equivalent to giving an exact sequence in $Ab(\mathbb{D})$:

$$A_1 \xrightarrow{\alpha} A_0 \overset{q}{\twoheadrightarrow} A(Y) \longrightarrow 1 \quad (*)$$

Let us denote by $\beta : B \rightarrowtail A_1$ the kernel of α. The direction of $\underline{A}_1$ is then nothing but $(p_Y, \iota_Y) : B \times A(Y) \leftrightarrows A(Y)$. A torsor associated with this abelian group in $Ab(\mathbb{D})/A(Y)$ is nothing but an extension in $Ab(\mathbb{D})$:

$$1 \longrightarrow B \overset{m}{\rightarrowtail} H \overset{h}{\twoheadrightarrow} A(Y) \longrightarrow 1$$

and the associated abelian group of torsors is nothing but the group $Ext_{Ab(\mathbb{D})}(A(Y), B)$. Now let $f : X \twoheadrightarrow Y$ be a central (=A-normal) extension whose image by the reflection A is the internal groupoid corresponding to the previous exact sequence $(*)$. This means that $A(f) = q$ (up to isomorphism) and that the central kernel of f is A_1:

$$\begin{tikzcd}
1 \arrow[r] & A_1 \arrow[r, tail, "k"] \arrow[d, equal] & X \arrow[r, two heads, "f"] \arrow[d, "\eta_X"'] & Y \arrow[r] \arrow[d, "\eta_Y"] & 1 \\
 & A_1 \arrow[r, "\alpha"'] & A_0 \arrow[r, two heads, "q"'] & A(Y) \arrow[r] & 1
\end{tikzcd}$$

Theorem 5.6 says that the set $ZExt_{(q,\alpha)}(Y, A_1)$ of central extensions whose image by the reflection A is the exact sequence $(*)$ admits, when it is non-empty, a faithful action of the abelian group $Ext_{Ab(\mathbb{D})}(A(Y), B)$. This action can be described in the following way: starting with the previous B-torsor in $Ab(\mathbb{D})$ and the extension f, take the pullback of f along $\eta_Y^*(h) = \bar{h}$, then the result of the action is given by the following construction:

$$\begin{tikzcd}
B \times A_1 \arrow[r, two heads, "{(\beta, 1_{A_1})}"] \arrow[d, tail, "\bar{m} \times k"'] & A_1 \arrow[r, equal] \arrow[d, tail, dotted, "\bar{k}"] & A_1 \arrow[d, "\alpha"] \\
\bar{H} \times_Y X \arrow[r, two heads, dotted] \arrow[d, "p_{\bar{H}}"'] \arrow[dr] & \bar{X} \arrow[r, dashed, "\eta_{\bar{X}}"] \arrow[d, two heads, dotted, "\bar{f}"] & A_0 \arrow[d, "q"] \\
\bar{H} \arrow[r, "\bar{h}"'] & Y \arrow[r, "\eta_Y"'] & A(Y)
\end{tikzcd}$$

The kernel of the regular epimorphism $\bar{h}.p_{\bar{H}}$ is the map $\bar{m} \times k$, where $\bar{m}$ is the kernel of $\bar{h}$ induced by m; so that the pushout along the regular epimorphism $(\beta, 1_{A_1}) : B \times A_1 \twoheadrightarrow A_1$ produces an exact sequence in $\mathbb{D}$:

$$1 \longrightarrow A_1 \overset{\bar{k}}{\rightarrowtail} \bar{X} \overset{\bar{f}}{\twoheadrightarrow} Y \longrightarrow 1$$

Theorem 5.6 asserts that its image by the reflection A is the original exact sequence $(*)$.

References

[1] M. Barr, *Exact categories*, Lecture Notes in Math., **236**, Springer, 1971, 1-120.

[2] J. Benabou, *Les distributeurs*, Rapport **33**, 1973, Inst. de Math. Pure et Appl. Univ. Cath. Louvain la Neuve.

[3] F. Borceux and D. Bourn, Split extension classifier and centrality, in *Categories in Algebra, Geometry and Math. Physics*, Contemporary Math., **431**, 2007, 85-14.

[4] F. Borceux, G. Janelidze and G.M. Kelly, *Internal object actions*, Commentationes Mathematicae Universitatis Carolinae, **46**, 2005, 235-255.

[5] F. Borceux, G. Janelidze and G.M. Kelly, *On the representability of actions in a semi-abelian category*, Th. and Applications of Categories, **14**, 2005, 244-286.

[6] D. Bourn, The shift functor and the comprehensive factorization for internal groupoids, Cahiers de Top. et Géom. Diff. Catégoriques, **28**, 1987, 197-226.

[7] D. Bourn, Baer sums and fibered aspects of Mal'cev operations, Cahiers de Top. et Géom. Diff. Catégoriques, **40**, 1999, 297-316.

[8] D. Bourn, Aspherical abelian groupoids and their directions, J. Pure Appl. Algebra, **168**, 2002, 133-146.

[9] D. Bourn, Baer sums in homological categories, Journal of Algebra, **308**, 2007, 414-443.

[10] D. Bourn, Abelian groupoids and non pointed additive categories, Theory and Appl. of categories, **20**, 2008, 48-73.

[11] D. Bourn, Commutator theory, action groupoids, and an intrinsic Schreier-Mac Lane theorem, Advances in Math., **217**, 2008, 2700-2735.

[12] D. Bourn and M. Gran, Centrality and connectors in Maltsev categories, Algebra Universalis, **48**, 2002, 309-331.

[13] D. Bourn and D. Rodelo, Comprehensive factorization and universal I-central extension in the Mal'cev context, Preprint CMUC, Universidade de Coimbra, 10-02, 2010.

[14] A. Carboni, J. Lambek and M.C. Pedicchio, Diagram chasing in Mal'cev categories, J. Pure Appl. Algebra, **69**, 1991, 271-284.

[15] A. Carboni, M.C. Pedicchio and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, CMS Conference Proceedings, **13**, 1992, 97-109.

[16] M. Gran, Central extensions and internal groupoids in Maltsev categories, J. Pure Appl. Algebra, **155**, 2001, 139-156.

[17] G. Janelidze and G.M. Kelly, Galois theory and a general notion of central extension, J. Pure Appl. Algebra, **97**, 1994, 135-161.

[18] P.T. Johnstone, Topos theory, Academic Press, London, 1977.

[19] P.T. Johnstone, Affine categories and naturally Mal'cev categories, J. Pure Appl. Algebra, **61**, 1989, 251-256.

[20] P.T. Johnstone, *The closed subgroup theorem for localic herds and pregroupoids*, J. Pure Appl. Algebra, **70**, 1989, 97-106.

[21] A. Kock, *Generalized fibre bundles*, in *Categorical Algebra and its Applications*, Lecture Notes in Math., **1348**, Springer, 1989, 194-207.

[22] A. Kock, Pregroupoids and their enveloping groupoids, Preprint Univ. of Aarhus, 2005.

[23] S. Mac Lane, Homology, Springer, 1963.

[24] M.C. Pedicchio, A categorical approach to commutator theory, Journal of Algebra, **177**, 1995, 647-657.

[25] V. Rossi, Admissible Galois structures and coverings in regular Mal'cev categories, Appl. Cat. Structures, **14**, 2006, 291-311.

[26] J.D.H. Smith, *Mal'cev varieties*, Lecture Notes in Math., **554**, Springer, 1976.

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