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STAR-MULTIPLICATIVE GRAPHS IN POINTED PROTOMODULAR CATEGORIES

Dedicated to Dominique Bourn on the occasion of his sixtieth birthday

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ABSTRACT. Protomodularity, in the pointed case, is equivalent to the Split Short Five Lemma. It is also well known that this condition implies that every internal category is in fact an internal groupoid. In this work, this is condition (II) and we introduce two other conditions denoted (I) and (III). Under condition (I), every multiplicative graph is an internal category. Under condition (III), every star-multiplicative graph can be extended (uniquely) to a multiplicative graph, a problem raised by G. Janelidze in [10] in the semiabelian context.

When the three conditions hold, internal groupoids have a simple description, that, in the semiabelian context, correspond to the notion of internal crossed module, in the sense of [10].

1. Introduction

In a category $\mathbf{B}$, pointed with kernels of split epimorphisms, we consider the following three conditions:

(I) Split extensions are jointly epic;

(II) The Split Short Five Lemma holds;

(III) Admissibility is reflected by the Kernel Functor.

Condition (II) is well known (see [1] and references there, see also [9]). Condition (I) states that in every split extension

$$X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \ , \quad \alpha\beta = 1 \ , \quad k = \ker \alpha \ ,$$

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the pair (k,β) is jointly epic. This condition was originally one of the axioms considered by M. Gerstenhaber, in 1970, in the definition of Moore categories ([18]).

For a given jointly epic pair (k,β), a pair of morphisms (g,h), with common codomain,

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[dr, "g"'] & A \arrow[d, dashed] & B \arrow[l, "\beta"'] \arrow[dl, "h"] \\ & A' & \end{tikzcd}$$

is said to be admissible, w.r.t. (k,β), if there is a (necessarily unique) morphism $\varphi : A \longrightarrow A'$, such that $\varphi k = g$ and $\varphi\beta = h$. By condition (III) we mean that the kernel functor $Ker : \mathrm{Pt}_B(\mathbf{B}) \longrightarrow \mathbf{B}$, reflects admissibility in the following sense: a pair of morphisms in $\mathrm{Pt}_B(\mathbf{B})$ is admissible w.r.t. a jointly epic pair, provided this holds for their images by Ker. Details are presented in Section 2.

Condition (II) may also be interpreted as saying that the kernel functor reflects isomorphisms.

The reason for considering these particular three conditions, either individually or together, is the following.

Let the base category $\mathbf{B}$, be pointed with pullbacks along split epimorphisms and so, in particular, with products and kernels of split epimorphisms.

We consider the following chain of forgetful functors

$$\mathrm{Grpd}(\mathbf{B}) \xrightarrow{(4)} \mathrm{Cat}(\mathbf{B}) \xrightarrow{(3)} \mathrm{MG}(\mathbf{B}) \xrightarrow{(2)} \mathrm{SMG}(\mathbf{B}) \xrightarrow{(1)} \mathrm{RG}(\mathbf{B}),$$

from internal structures of groupoids to categories (forgets inverses), to multiplicative graphs (forgets associativity and codomains, but not domains!, see footnote at the end of page 174), to star-multiplicative graphs (keeps only composition around the origin), to reflexive graphs (forgets composition). We prove that

$$\begin{aligned} \text{(I)} &\implies \text{(3) is an isomorphism,} \\ \text{(II)} &\implies \text{(4) is an isomorphism,} \\ \text{(III)} &\implies \text{(2) is an isomorphism.} \end{aligned}$$

We also observe that the following implications hold

$$\begin{aligned} \text{(I)} &\implies \text{weakly Mal'cev (as defined in [17]),} \\ \text{(II) + equalizers} &\implies \text{(I),} \\ \text{(III) + products} &\implies \text{(I).} \end{aligned}$$

If $\mathbf{B}$ is abelian all categories in the above chain coincide.

If $\mathbf{B}$ is either the category of groups or the one of rings we have that $\mathrm{Grpd}(\mathbf{B}) \sim \mathrm{SMG}(\mathbf{B})$. This is not true in arbitrary protomodular categories. Indeed, in the category of digroups (sets with two group operations sharing the same unit), which is a semiabelian category, the functor (2) is not an isomorphism, as I learned from G. Janelidze. Hence, condition (III) arises as an attempt to fill in this gap.

Another approach to this problem, involving strong protomodularity, was presented in [15]. We remark that if in (I) the pair (k, β) is required to be jointly strongly epic, then we obtain (II) (see [1] and references there).

The classical notion of crossed module may be extended to the more general context of a category satisfying (I): it is a split extension

$$X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B\ , \quad \alpha\beta = 1\ , \quad k = \ker\alpha\ ,$$

together with a morphism

$$h : X \longrightarrow B\ ,$$

such that the dashed arrows in the following diagram can be inserted in order to make it commutative

$$\begin{tikzcd}
X \arrow[r, "\langle 1,0\rangle"] \arrow[d, equal] & X \times X \arrow[r, shift left, "\pi_2"] \arrow[d, dashed, "(a)"'] & X \arrow[l, shift left, "\langle 1,1\rangle"] \arrow[d, "h"] \\
X \arrow[r, "k"] \arrow[d, "h"'] & A \arrow[r, shift left, "\alpha"] \arrow[d, dashed, "(b)"'] & B \arrow[l, shift left, "\beta"] \arrow[d, equal] \\
B \arrow[r, "\langle 1,0\rangle"] & B \times B \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\langle 1,1\rangle"]
\end{tikzcd} \qquad (1.1)$$

Since the horizontal rows are split extensions, the morphisms represented by the dashed arrows are unique, provided they exist.

If $\mathbf{B}$ also has binary coproducts and coequalizers of reflexive pairs then we can define the category

$$\mathrm{Act}\,(\mathbf{B})$$

of internal actions in $\mathbf{B}$ (see [10] and [16]). If, as in a semi-abelian category [12], there is an equivalence

$$\mathrm{Act}\,(\mathbf{B}) \sim \mathrm{Pt}\,(\mathbf{B})$$

then the above definition coincides with the notion of internal crossed module, in the sense of Janelidze.

In this work we show that if our base category $\mathbf{B}$ (pointed with pullbacks along split epimorphisms) satisfies (I),(II) and (III), we have the following equivalences of categories

$$\mathrm{Grpd}\,(\mathbf{B}) \sim \mathrm{SMG}\,(\mathbf{B}) \sim \mathrm{Xmod}\,(\mathbf{B})\,.$$

In the case of groups (see Proposition 3.2), to give a internal groupoid is to give a split extension

$$X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

together with a morphism $h : X \longrightarrow B$ satisfying (with additive notation though the groups are not assumed to be commutative)

$$h\left(\beta\left(b\right)+k\left(x\right)-\beta\left(b\right)\right)=b+h\left(x\right)-b$$

which is equivalent to insert the arrow with label (b) in (1.1), and satisfying

$$k\left(\left(x',x'\right)+\left(x,0\right)-\left(x',x'\right)\right)=\beta h\left(x'\right)+k\left(x\right)-\beta h\left(x'\right)$$

which is equivalent to insert the arrow with label (a) in (1.1) and simplifies to

$$x'+x-x'=\beta h\left(x'\right)+k\left(x\right)-\beta h\left(x'\right)\ \in X\ .$$

The above equations look more familiar if we introduce the usual notation for actions, writing

$$\begin{aligned} b\cdot x &= \beta\left(b\right)+k\left(x\right)-\beta\left(b\right)\in X \\ h\left(x'\right)\cdot x &= \beta h\left(x'\right)+k\left(x\right)-\beta h\left(x'\right)\in X\ , \end{aligned}$$

so that the above conditions become the familiar axioms for a crossed module in Groups

$$\begin{aligned} h\left(b\cdot x\right) &= b+h\left(x\right)-b \\ h\left(x'\right)\cdot x &= x'+x-x'. \end{aligned}$$

For the case of rings (see Proposition 3.1) we have a split extension as above, together with a morphism $h : X \longrightarrow B$ satisfying, for every $x, y \in X \subseteq A$ and $b \in B$,

$$\begin{gathered} h(\beta(b)x)=bh(x)\quad,\ h(x\beta(b))=h(x)b \\ x\beta h(y)=xy=\beta h(x)y\ . \end{gathered}$$

In Section 4, assuming condition (I), we characterize, for a fixed split extension

$$X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B\ ,\quad \alpha\beta=1\ ,\quad k=\ker\alpha\ , \tag{1.2}$$

the morphisms

$$h : X \longrightarrow B$$

that give rise to a reflexive graph, a star-multiplicative graph, a multiplicative graph, an internal category, or an internal groupoid.

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2. Definitions and Technicalities

Let $\mathbf{B}$ be a category with pullbacks along split epis. Therefore, for every morphism $h : C \longrightarrow B$ and split epi $f : A \longrightarrow B$ with section $r : B \longrightarrow A$, the pullback of h along f exists

$$\begin{tikzcd} A \times_B C \arrow[r, shift left, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[l, shift left, "\langle rh,1\rangle"] \arrow[d, "h"] \\ A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "r"] \end{tikzcd} ;$$

and the second projection, π_2, is always a split epi with splitting $\langle rh, 1\rangle$.

We assume that $\mathbf{B}$ is also pointed, so that, in particular, we have binary products and kernels of split epis. It is easy to see that the kernel of the split epi π_2 is given by the induced morphism $\langle k, 0\rangle$ into the pullback, where $k : X \longrightarrow A$ is the kernel of f:

$$\begin{tikzcd} X \arrow[r, "\langle k,0\rangle"] \arrow[d, equal] & A \times_B C \arrow[r, shift left, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[l, shift left, "\langle rh,1\rangle"] \arrow[d, "h"] \\ X \arrow[r, "k"] & A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "r"] \end{tikzcd} .$$

A reflexive graph in $\mathbf{B}$ is a diagram

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1_{C_0} = ce \ . \tag{2.1}$$

We recall that a multiplicative graph[1] (see [13], [8] and [11]) is a reflexive graph, as above, together with a multiplication $m : C_2 \longrightarrow C_1$ such that $me_1 = 1_{C_1} = me_2$, where C_2 is the pullback of c along the split epi d

$$\begin{tikzcd} C_2 \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\ C_1 \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"'] \end{tikzcd}$$

and

$$e_1 = \langle 1, ed\rangle$$
$$e_2 = \langle ec, 1\rangle$$

are the two canonical induced morphisms into the pullback.

[1] Instead of the original definition for a multiplicative graph, and for the purpose of this paper only, it is convenient to assume the extra condition $dm = d\pi_2$; otherwise the forgetful functor from multiplicative to star-multiplicative graphs would not be well defined. Also note that this extra condition is trivial once we assume (I).

An internal category is a multiplicative graph satisfying the additional requirements

$$dm = d\pi_2 \ , \ \ cm = c\pi_1$$

and the associativity condition (see also the note at the end of page 188).

An internal groupoid is an internal category where "every arrow is invertible" (see for example the Appendix of [1] for a precise definition).

A star-multiplicative graph is a reflexive graph, as in (2.1), together with a "star-multiplication", that is, a morphism

$$s : C_* \longrightarrow X$$

where C_* is obtained by pulling back ck along d, and $k : X \longrightarrow C_1$ is the kernel of the split epi d, as displayed below

$$\begin{tikzcd} X \arrow[r, "\langle k,0\rangle"] \arrow[d, equal] & C_* \arrow[r, shift left, "p_2"] \arrow[d, "p_1"'] & X \arrow[l, shift left, "\langle eck,1\rangle"] \arrow[d, "ck"] \\ X \arrow[r, "k"] & C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \end{tikzcd} ,$$

satisfying the following condition

$$s \langle k, 0 \rangle = 1_X = s \langle eck, 1 \rangle \, .$$

When $\mathbf{B}$ is the category of pointed sets, C_* is the set of composable pairs of arrows in C_1, starting at zero, in the following sense

$$C_* = \{(f, g) \in C_1 \times C_1 \mid d(f) = c(g) \text{ and } d(g) = 0\} \, ,$$

$$a \xleftarrow{f} b \xleftarrow{g} *$$

meaning that we can only compose two (appropriate) arrows $f \circ g$ in C_1, if the second one, g, starts at the origin (or zero, or star, hence the name star-multiplicative).

Every multiplicative graph (with $dm = d\pi_2$) is, in particular, a star-multiplicative graph, by restricting C_2 to C_* and C_1 to X:

$$\langle p_1, kp_2 \rangle : C_* \longrightarrow C_2 \ , \quad k : X \longrightarrow C_1$$

and defining s as the unique morphism such that $ks = m \langle p_1, kp_2 \rangle$ (which is well defined if $dm = d\pi_2$). Observe also that $\langle p_1, kp_2 \rangle$ is the kernel of the split epi $d\pi_2$.

This notion of star-multiplicative graph was introduced by G. Janelidze in [10] in order to describe internal crossed modules in semi-abelian categories. However it is not true that the star-multiplicative graphs are multiplicative in an arbitrary semi-abelian category. In [10], G. Janelidze asked for a description of semi-abelian categories with the property that every star-multiplication (uniquely) extends to an internal category structure.

Instead of semi-abelian categories we consider the problem in the context of pointed protomodular categories and conclude that under condition (III) we obtain the desired result.

Let $\mathbf{B}$ be a pointed category with kernels of split epis, and consider the category $\mathrm{Pt}\,(\mathbf{B})$ of points in $\mathbf{B}$, that is, objects are split epis

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd} \ , \ \alpha\beta = 1$$

and morphisms are pairs (f, g), making the obvious following squares commutative

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] \arrow[d, "f"'] & B \arrow[l, shift left, "\beta"] \arrow[d, "g"] \\ A' \arrow[r, shift left, "\alpha'"] & B' \arrow[l, shift left, "\beta'"] \end{tikzcd} . \tag{2.2}$$

For every $B \in \mathbf{B}$, since $\mathbf{B}$ has kernels of split epis, we may consider the category of points over B, denoted $\mathrm{Pt}_B\,(\mathbf{B})$ and the Kernel Functor, denoted by Ker

$$Ker : \mathrm{Pt}_B\,(\mathbf{B}) \longrightarrow \mathbf{B}$$

sending a morphism f in $\mathrm{Pt}_B\,(\mathbf{B})$

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "f_0"'] & A \arrow[r, shift left, "\alpha"] \arrow[d, "f"'] & B \arrow[l, shift left, "\beta"] \arrow[d, equal] \\ X' \arrow[r, "k'"] & A' \arrow[r, shift left, "\alpha'"] & B \arrow[l, shift left, "\beta'"] \end{tikzcd}$$

to the morphism $f_0 : X \longrightarrow X'$ such that $fk = k'f_0$.

A pair of morphisms (e_1, e_2) with a common codomain

$$e_1 : A \longrightarrow E \longleftarrow A' : e_2$$

is said to be *jointly epic* if, for every two parallel morphisms

$$f, g : E \longrightarrow D \ ,$$

we have

$$\left\{ \begin{array}{l} fe_1 = ge_1 \\ fe_2 = ge_2 \end{array} \right. \implies f = g \ .$$

Let (e_1, e_2) be a jointly epic pair

$$e_1 : A \longrightarrow E \longleftarrow A' : e_2 \ .$$

A pair (f,g) of morphisms with a common codomain

$$f : A \longrightarrow D \longleftarrow A' : g$$

is said to be *admissible* w.r.t. the pair (e_1, e_2), if there is a (necessarily unique) morphism

$$\varphi : E \longrightarrow D$$

such that

$$\varphi e_1 = f \ , \ \varphi e_2 = g \ .$$

In this case we write

$$[f \ g] : E \longrightarrow D$$

to denote the morphism φ.

In a pointed category, a diagram of the form

$$X \xrightarrow{k} A \underset{r}{\overset{f}{\rightleftarrows}} B \ \ , \ fr = 1 \ , \ fk = 0$$

will be called a *split chain* and, if k is a kernel of f, a *split extension*. If, furthermore the pair (k,r) is jointly epic we say that it is a *jointly epic split chain* or *jointly epic split extension*, respectively.

Let $\mathbf{A}$ and $\mathbf{B}$ be pointed categories. A functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ is said to *reflect admissibility* if for every split chain in $\mathbf{A}$

$$A \xrightarrow{e_1} E \underset{e_2}{\overset{\pi}{\rightleftarrows}} A' \ \ , \ \pi e_2 = 1 \ , \ \pi e_1 = 0_{\mathbf{A}} \ ,$$

whose image by F is a split extension in $\mathbf{B}$

$$FA \xrightarrow{F(e_1)} FE \underset{F(e_2)}{\overset{F(\pi)}{\rightleftarrows}} FA' \ ,$$

we have that:

- the pair $(F(e_1), F(e_2))$ is jointly epic in $\mathbf{B}$,
- the pair (e_1, e_2) is jointly epic in $\mathbf{A}$,
- and, a pair of morphisms (f,g) with common codomain

  $$f : A \longrightarrow D \longleftarrow A' : g$$

  in $\mathbf{A}$, is admissible w.r.t. (e_1, e_2) whenever $(F(f), F(g))$ is admissible with respect to $(F(e_1), F(e_2))$.

We are now in position to state the three properties (I), (II) and (III), mentioned in the Introduction.

Throughout $\mathbf{B}$ will denote a pointed category with kernels of split epis.

We say that $\mathbf{B}$ satisfies:

(I) (the Jointly Epic Split Extension condition) if every split extension is jointly epic, that is, for every diagram of the form

$$X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B\ ,\quad \alpha\beta = 1\ ,\ \alpha k = 0\ ,$$

if $k = \ker\alpha$ then the pair (k, β) is jointly epic;

(II) (the Split Short Five Lemma) if given any diagram of the form

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "f"'] & A \arrow[r, shift left, "\alpha"] \arrow[d, "h"'] & B \arrow[l, shift left, "\beta"] \arrow[d, "g"] \\ X' \arrow[r, "k'"] & A' \arrow[r, shift left, "\alpha'"] & B' \arrow[l, shift left, "\beta'"] \end{tikzcd} , \tag{2.3}$$

where the rows are split extensions and the vertical arrows constitute a morphism of split extensions, if f and g are isomorphisms then h is an isomorphism as well;

(III) (the Reflected Admissibility property) if the Kernel Functor $Ker : \mathrm{Pt}_B(\mathbf{B}) \longrightarrow \mathbf{B}$ reflects admissibility.
In detail, given any diagram of the form

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "k"'] & Y \arrow[r, shift left, "p"] \arrow[d, "\bar{k}"'] & X' \arrow[l, shift left, "i_2"] \arrow[d, "k'"] \\ A \arrow[r, "e_1"] \arrow[d, shift left, "\alpha"] & E \arrow[r, shift left, "\pi"] \arrow[d, shift left, "\bar{\alpha}"] & A' \arrow[l, shift left, "e_2"] \arrow[d, shift left, "\alpha'"] \\ B \arrow[u, shift left, "\beta"] \arrow[r, equal] & B \arrow[u, shift left, "\bar{\beta}"] \arrow[r, equal] & B \arrow[u, shift left, "\beta'"] \end{tikzcd} \tag{2.4}$$

where the columns are split extensions and the appropriate horizontal arrows constitute morphisms of split extensions, with $\pi e_2 = 1_{A'}, \pi e_1 = \beta'\alpha$, if the top row is a split extension, then

$$\begin{array}{cc} (i_1, i_2) & \text{is jointly epic in } \mathbf{B} \\ (e_1, e_2) & \text{is jointly epic in } \mathrm{Pt}_B(\mathbf{B}) \end{array}$$

and furthermore, a pair (f, g) of morphisms in $\mathrm{Pt}_B(\mathbf{B})$

$$\begin{tikzcd}
X \arrow[r, "f_0"] \arrow[d, "k"'] & D_0 \arrow[d, "k_0"'] & X' \arrow[l, "g_0"'] \arrow[d, "k'"] \\
A \arrow[r, "f"] \arrow[d, shift left, "\alpha"] & D \arrow[d, shift left, "\alpha_0"] & A' \arrow[l, "g"'] \arrow[d, shift left, "\alpha'"] \\
B \arrow[u, shift left, "\beta"] \arrow[r, equal] & B \arrow[u, shift left, "\beta_0"] \arrow[r, equal] & B \arrow[u, shift left, "\beta'"]
\end{tikzcd}$$

is admissible (in $\mathrm{Pt}_B(\mathbf{B})$) w.r.t. (e_1, e_2) whenever the pair (f_0, g_0) is admissible w.r.t. (i_1, i_2).

2.1. Remark. In order to better understand condition (III) we should be able to characterize the split chains in $\mathrm{Pt}_B(\mathbf{B})$, whose image by *Ker* is a split extension in $\mathbf{B}$. Note that we are not assuming the existence of kernels in $\mathrm{Pt}_B(\mathbf{B})$; that would correspond to the existence of split pullbacks in the sense of [7], p. 43. Nevertheless, a split chain in $\mathrm{Pt}_B(\mathbf{B})$, say (e_1, π, e_2) as above, whose image by *Ker* is a split extension, always satisfies (upper left square in (2.4))

$$e_1 k = \ker \pi \ .$$

In fact, the square $\pi \bar{k} = k' p$ is a pullback because $1 : B \to B$ is a monomorphism and $\bar{k}, k'$ are kernels; hence if $i_1 = \ker p$ then $e_1 k = \ker \pi$.

Later on we will assume the existence of pullbacks along split epimorphisms so that the kernel functor will preserve split extensions. Also in this case, the split extensions in $\mathrm{Pt}_B(\mathbf{B})$ are precisely the split chains, say (e_1, π, e_2) as above, such that the square (see diagram (2.4))

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "e_1"'] & B \arrow[d, "\beta'"] \\
E \arrow[r, "\pi"] & A'
\end{tikzcd}$$

is a pullback.

Furthermore, if the category $\mathbf{B}$ is protomodular, then every split chain in $\mathrm{Pt}_B(\mathbf{B})$, whose image by *Ker* is a split extension in $\mathbf{B}$, is itself a split extension in $\mathrm{Pt}_B(\mathbf{B})$. As it is well known in this context the above square is a pullback if and only if $e_1 k = \ker \pi$ (with $k = \ker \alpha$ as in diagram (2.4)).

We deduce immediately:

2.2. Proposition. *Let* $\mathbf{B}$ *be a pointed category with kernels of split epis and equalizers. If* $\mathbf{B}$ *satisfies (II) then it also satisfies (I).*

Proof. For a given split extension

$$X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \ , \ \alpha\beta = 1 \ , \ k = \ker \alpha$$

the equalizer, q, of any two parallel morphisms

$$f, g : A \longrightarrow D$$

satisfying $fk = gk$ and $f\beta = g\beta$ is in fact an isomorphism, because it fits in the following diagram

$$\begin{tikzcd}
X \arrow[r, "k'"] \arrow[d, equal] & Y \arrow[r, shift left, "\alpha q"] \arrow[d, "q"'] & B \arrow[l, shift left, "\beta'"] \arrow[d, equal] \\
X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

where k' and β' are the unique morphisms such that $qk' = k$ and $q\beta' = \beta$. Obviously we have $\alpha q\beta' = 1$ and $k' = \ker(\alpha q)$ so by the Split Short Five Lemma we conclude that q is an isomorphism and hence that $f = g$. ∎

It is perhaps useful to remark that this result can also be deduced from the fact that property (II) is equivalent to split extensions being jointly strongly epic and the fact that, in a category with equalizers, any jointly strongly epic pair of arrows is jointly epic [1].

2.3. Proposition. *Let* $\mathbf{B}$ *be a pointed category with kernels of split epis and binary products. If* $\mathbf{B}$ *satisfies (III) then it also satisfies (I).*

Proof. We simply observe that any split extension is the image under Ker of some split chain in $\mathrm{Pt}_B(\mathbf{B})$, namely

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d] & Y \arrow[r, shift left, "p"] \arrow[d] & X' \arrow[l, shift left, "i_2"] \arrow[d] \\
X \times B \arrow[r, "i_1 \times 1"] \arrow[d, shift left] & Y \times B \arrow[r, shift left, "p \times 1"] \arrow[d, shift left] & X' \times B \arrow[l, shift left, "i_2 \times 1"] \arrow[d, shift left] \\
B \arrow[u, shift left] \arrow[r, equal] & B \arrow[u, shift left] \arrow[r, equal] & B \arrow[u, shift left]
\end{tikzcd} .$$

∎

2.4. Corollary. *Let* $\mathbf{B}$ *be a pointed category with finite limits. If* $\mathbf{B}$ *satisfies (II) or (III) then* $\mathbf{B}$ *also satisfies (I).*

2.5. Proposition. *Let* $\mathbf{B}$ *be pointed, with kernels of split epis and satisfying (I). Then* $\mathbf{B}$ *satisfies (II) if and only if given any diagram of the form* (2.3)*, with* f *and* g *isomorphisms, the pair* $(kf^{-1}, \beta g^{-1})$ *is admissible w.r.t.* (k', β')*.*

Proof. The fact that (k', β') and (k, β) are jointly epic forces the two compositions hh' and $h'h$ to be, respectively, the identity on A' and A (with $h' = [kf^{-1}, \beta g^{-1}]$). ∎

2.6. Proposition. *Let* **B** *be pointed, with kernels of split epis and satisfying (I). Then* **B** *satisfies (III) if and only if, given any diagram of the form*

$$
\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "k"'] & Y \arrow[r, shift left, "p"] \arrow[d, "\bar{k}"'] & X' \arrow[l, shift left, "i_2"] \arrow[d, "k'"] \\
A \arrow[r, "e_1"] \arrow[d, shift left, "\alpha"] & E \arrow[r, shift left, "\pi"] \arrow[d, shift left, "\bar{\alpha}"] & A' \arrow[l, shift left, "e_2"] \arrow[d, shift left, "\alpha'"] \\
B \arrow[u, shift left, "\beta"] \arrow[r, equal] & B \arrow[u, shift left, "\bar{\beta}"] \arrow[r, equal] & B \arrow[u, shift left, "\beta'"]
\end{tikzcd}
\tag{2.5}
$$

where the columns and the top row are split extensions and e_1, π, e_2 *are morphisms of split extensions, with* $\pi e_2 = 1, \pi e_1 = \beta'\alpha$*, i.e.,*

$$
\begin{array}{llll}
k = \ker \alpha \ , & \overline{k} = \ker \overline{\alpha} \ , & k' = \ker \alpha' \ , & i_1 = \ker p \ , \\
\alpha\beta = 1 \ , & \overline{\alpha}\overline{\beta} = 1 \ , & \alpha'\beta' = 1 \ , & p i_2 = 1 \ , \\
\overline{\alpha} e_1 = \alpha \ , & e_1 \beta = \overline{\beta} \ , & e_1 k = \overline{k} i_1 \ , & \\
\overline{\alpha} e_2 = \alpha' \ , & e_2 \beta' = \overline{\beta} \ , & e_2 k' = \overline{k} i_2 \ , & \\
\alpha' \pi = \overline{\alpha} \ , & \pi \overline{\beta} = \beta \ , & \pi \overline{k} = k' p \ , & \\
\pi e_2 = 1 \ , & \pi e_1 = \beta' \alpha \ , & &
\end{array}
$$

if the pair (f_0, g_0) *is admissible w.r.t* (i_1, i_2) *then the pair* (f, g) *is admissible w.r.t* (e_1, e_2) *where* f *and* g *, displayed as follows*

$$
\begin{tikzcd}
X \arrow[r, "f_0"] \arrow[d, "k"'] & D_0 \arrow[d, "k_0"'] & X' \arrow[l, "g_0"'] \arrow[d, "k'"] \\
A \arrow[r, "f"] \arrow[d, shift left, "\alpha"] & D \arrow[d, shift left, "\alpha_0"] & A' \arrow[l, "g"'] \arrow[d, shift left, "\alpha'"] \\
B \arrow[u, shift left, "\beta"] \arrow[r, equal] & B \arrow[u, shift left, "\beta_0"] \arrow[r, equal] & B \arrow[u, shift left, "\beta'"]
\end{tikzcd}
\ , \ k_0 = \ker \alpha_0 \ , \ \alpha_0 \beta_0 = 1 \ ,
$$

are morphisms of split extensions, that is, satisfying

$$
\begin{array}{lll}
f\beta = \beta_0 \ , & \alpha_0 f = \alpha \ , & fk = k_0 f_0 \ , \\
g\beta' = \beta_0 \ , & \alpha_0 g = \alpha' \ , & gk' = k_0 g_0 \ .
\end{array}
$$

Proof. Since **B** satisfies (I), we simply have to prove that the pair (e_1, e_2) is jointly epic.

Indeed, in diagram 2.5, the right hand upper square $k'p = \pi\bar{k}$ is a pullback because the lower horizontal morphism $1 : B \longrightarrow B$ is a monomorphism. Since the top row is a split extension it follows that $e_1 k = \ker \pi$. And since $e_1 k = \ker \pi$, we have that $e_1 k$ and e_2 are jointly epic by condition (I), and therefore also e_1 and e_2 are jointly epic. ∎

2.7. PROPOSITION. *In the context of Proposition 2.6, above, in order to have (III), it is sufficient to check that if the pair* (f_0, g_0) *is admissible w.r.t* (i_1, i_2) *then the pair* $(k_0 f_0, g)$ *is admissible w.r.t* $(e_1 k, e_2)$.

PROOF. Note that from 2.6 we know that $e_1 k = \ker \pi$. If $(k_0 f_0, g)$ is admissible w.r.t $(e_1 k, e_2)$, then we have a morphism (in $\mathrm{Pt}_B(\mathbf{B})$)

$$\gamma : E \longrightarrow D$$

such that $\gamma e_1 k = k_0 f_0$ and $\gamma e_2 = g$. We have to prove that $\gamma e_1 = f$, and in fact we have

$$\begin{cases} \gamma e_1 k = k_0 f_0 = f k \\ \gamma e_1 \beta = \gamma \bar{\beta} = \gamma e_2 \beta' = g \beta' = \beta_0 = f \beta \end{cases} \Longrightarrow \gamma e_1 = f$$

because (k, β) is jointly epic (since we have (I)). ∎

One last result relates condition (I) with weakly Mal'cev categories [17].

2.8. PROPOSITION. *Let* $\mathbf{B}$ *be a pointed category with pullbacks of split epis. If* $\mathbf{B}$ *satisfies (I), then* $\mathbf{B}$ *is a weakly Mal'cev category.*

PROOF. Given a diagram of the form

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & C \arrow[l, shift right, "g"'] \end{tikzcd} \quad , \quad fr = 1_B = gs \ ,$$

we may construct the pullback of g along the split epi f, and since g itself is a split epi, we obtain induced morphisms into the pullback $e_1 = \langle 1, sf \rangle, e_2 = \langle rg, 1 \rangle$, as follows

$$\begin{tikzcd} A \times_B C \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[r, shift left, "f"] \arrow[u, shift right, "e_1"'] & B \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd} .$$

We have to prove that the pair (e_1, e_2) is jointly epic.

Since we have kernels of split epis, and every split extension is jointly epic, the above diagram becomes

$$\begin{tikzcd} X \arrow[r, "\langle k,0 \rangle"] \arrow[d, equal] & A \times_B C \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ X \arrow[r, "k"] & A \arrow[r, shift left, "f"] \arrow[u, shift right, "e_1"'] & B \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

where $k = \ker f$, and hence the pair $(\langle k, 0 \rangle, e_2)$ is jointly epic (because $\langle k, 0 \rangle = \ker \pi_2$), but $\langle k, 0 \rangle$ is also equal to $e_1 k$,

$$e_1 k = \langle 1, sf \rangle k = \langle k, sfk \rangle = \langle k, 0 \rangle .$$

Now, $(e_1 k, e_2)$ is jointly epic, so (e_1, e_2) is also jointly epic. ∎

3. Examples

The following are some examples of categories satisfying one or more of the above properties (I),(II),(III).

The dual of the category of topological pointed spaces satisfies (I) but not (II). The following example, due to G. Gutierres, shows why condition (II) fails.

Consider the diagram of sets and maps

$$\begin{tikzcd} X \arrow[d, equal] & A \arrow[l, "k"'] \arrow[d, "f"'] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "p"'] \arrow[d, equal] \\ X & A' \arrow[l, "k'"'] \arrow[r, shift right, "s'"'] & B \arrow[l, shift right, "p'"'] \end{tikzcd}$$

where $X = B = \{0, 1\}$, $A = A' = \{0, 1, 2\}$, $k(0) = k(1) = 0$, $k(2) = 1$, $p(0) = 0$, $p(1) = 1$, $s(0) = 0$, $s(1) = s(2) = 1$, and similarly for k', p', s'. The map f is the identity map. Clearly, k, p, s and k', p', s' are continuous if considering the following topologies on the sets above: X is indiscrete; B has $\{1\}$ as the only non trivial open subset; A has $\{1, 2\}$ and $\{1\}$ as nontrivial open sets; while A' has $\{1, 2\}$ as the only non trivial open set. Now, it is also clear that f is continuous while its inverse is not.

By definition, all pointed protomodular categories satisfy (II).

In the case of syntactical examples we have that any quasivariety of universal algebras containing a unique constant 0 and a ternary term

$$p\left(x,y,z\right)$$

satisfying the following axioms

$$p\left(x,y,y\right) = p\left(p\left(x,y,0\right),0,y\right) = p\left(y,y,x\right) \tag{3.1}$$
$$p\left(0,y,y\right) = 0 \tag{3.2}$$
$$p\left(x,y,y\right) = p\left(x',y,y\right) \implies x = x' \tag{3.3}$$

also satisfy property I. In fact, given a split extension

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd} \tag{3.4}$$

and two morphisms

$$f : X \longrightarrow D \longleftarrow B : g$$

there is at most one morphism $\gamma : A \longrightarrow D$ with $\gamma k = f$ and $\gamma\beta = g$. The morphism γ can be defined as a function from the set A to the set D, if and only if the following equation has a (necessarily unique) solution γ_a (in D) for every a in A

$$p_D\left(fp_A\left(a,\beta\alpha\left(a\right),0\right),0,g\alpha\left(a\right)\right) = p_D\left(\gamma_a, g\alpha\left(a\right), g\alpha\left(a\right)\right).$$

We observe that $p_A(a, \beta\alpha(a), 0)$ is considered as an element in X, as $\alpha p_A(a, \beta\alpha(a), 0) = 0$; even if the assignment $a \longmapsto \gamma_a$ defines a map from the set A to the set D, some further restrictions are necessary in order to have a homomorphism of such structures.

If replacing the axiom $p(0, y, y) = 0$ by the stronger one $p(x, y, y) = x$ then the variety is protomodular (see [1], p.234) and we also have (II). An example as above, where $p(x, y, y) = x$ does not hold for every x, may be constructed in the set of natural numbers, with zero, as follows

$$
\begin{aligned}
&p(0,0,0) = 0 \\
&p(0,x,y) = \begin{cases} 0 & \text{if} \quad x = y \\ f_1(x,y) & \text{if} \quad x \neq y \end{cases} \\
&p(x,0,y) = \begin{cases} 0 & \text{if} \quad x = y \\ nx & \text{if} \quad x \neq y \end{cases} \\
&p(x,y,0) = \begin{cases} 0 & \text{if} \quad x = y \\ x & \text{if} \quad x \neq y \end{cases} \\
&p(x,0,0) = p(0,x,0) = p(0,0,x) = x \\
&p(x,y,z) = \begin{cases} x & \text{if} \quad x = y = z \\ nx & \text{if} \quad x \neq y = z \\ nz & \text{if} \quad x = y \neq z \\ f_2(x,y) & \text{if} \quad x = z \neq y \\ f_3(x,y,z) & \text{if} \quad x \neq y \neq z \neq x \end{cases}
\end{aligned}
$$

where x, y, z are non zero, n is any natural number other than 1 and 0, and f_1, f_2, f_3 are any maps on the natural numbers.

Another sort of examples, still in the varietal case, satisfying (I), are varieties of universal algebras containing a unique constant, 0, with n binary terms t_i and one $(n+1)$-ary term t, for some n, satisfying

$$
\begin{gathered}
t_i(x,x) = 0 \\
t(t_1(x,y), ..., t_n(x,y), y) = (t_1(x',y), ..., t_n(x',y), y) \implies x = x'.
\end{gathered}
$$

Again, to see that it satisfies (I), we introduce a derived binary operation

$$
e(x,y) = t(t_1(x,y), ..., t_n(x,y), y)
$$

and observe that for any split extension (3.4) and morphisms f and g as above, there is at most one morphism $\gamma : A \longrightarrow D$ with $\gamma k = f$ and $\gamma\beta = g$, and furthermore it is defined as a map if and only if the following equation has a (necessarily unique) solution γ_a in D for every a in A

$$
e(\gamma_a, g\alpha(a)) = t(ft_1(a, \beta\alpha(a)), ..., ft_n(a, \beta\alpha(a)), g\alpha(a)).
$$

Note that we consider $t_i(a, \beta\alpha(a))$ in X and not in A.

Of course that in the particular case where

$$e\,(x, y) = x$$

we have the characterization of pointed protomodular categories in the varietal case (see [6]).

Property (III) is still to be better understood. For the moment we just show that the categories of groups and rings do satisfy (III).

First we show that both categories satisfy (I) and give explicit formulas for a given pair of morphisms to be admissible w.r.t. a jointly epic pair obtained from a split extension. Then we use Proposition 2.7 to show that condition (III) is satisfied.

3.1. Proposition. *In rings every split extension is jointly epic, that is, property (I) holds in rings. Furthermore, given any split extension*

$$X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \tag{3.5}$$

the pair of morphisms (f, g)

$$f : X \longrightarrow D \longleftarrow B : g$$

is admissible w.r.t. (k, β) *if and only if for every* $x \in X$ *and* $b \in B$*,*

$$f\,(\beta\,(b)\,k\,(x)) = g\,(b)\,f\,(x) \quad , \quad f(k(x)\beta(b)) = f(x)g(b)$$

and in that case, the morphism $\gamma = [f, g] : A \longrightarrow D$ *is given by*

$$\gamma\,(a) = f\,(a - \beta\alpha\,(a)) + g\alpha\,(a)\,.$$

Proof. For every a in A we have $a = k\,(a - \beta\alpha\,(a)) + \beta\alpha\,(a)$ (because $a - \beta\alpha\,(a) \in X$) and since γ is such that $\gamma k = f$ and $\gamma\beta = g$ we obtain

$$\gamma\,(a) = f\,(a - \beta\alpha\,(a)) + g\alpha\,(a)\,.$$

This proves (k, β) to be jointly epic.

Now, since f and g are ring homomorphisms we have obviously

$$\gamma\,(a + a') = \gamma\,(a) + \gamma\,(a')\,.$$

Assuming $\gamma\,(aa') = \gamma\,(a)\,\gamma\,(a')$ in particular we have

$$\gamma\,(\beta\,(b)\,k\,(x)) = \gamma\,(\beta\,(b))\,\gamma\,(k\,(x))$$

but by definition of γ and because $\beta\,(b)\,k\,(x) \in X$ we have

$$f\,(\beta\,(b)\,k\,(x)) = g\,(b)\,f\,(x)\,.$$

A similar argument proves $f(k(x)\beta(b)) = f(x)g(b)$.

Conversely, assuming $f\,(\beta\,(b)\,k\,(x)) = g\,(b)\,f\,(x)$ and $f(k(x)\beta(b)) = f(x)g(b)$, some standard algebraic manipulation shows that $\gamma\,(aa') = \gamma\,(a)\,\gamma\,(a')$. ∎

3.2. Proposition. *In groups every split extension is jointly epic, and for every such*

$$X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \tag{3.6}$$

the pair of morphisms (f, g)

$$f : X \longrightarrow D \longleftarrow B : g$$

is admissible w.r.t. (k, β) *if and only if*

$$f\left(\beta\left(b\right)+k\left(x\right)-\beta\left(b\right)\right)=g\left(b\right)+f\left(x\right)-g\left(b\right)\ \ ,\quad \forall x\in X,\forall b\in B,$$

and in that case, the morphism $\gamma = [f, g] : A \longrightarrow D$ *is given by*

$$\gamma\left(a\right)=f\left(a-\beta\alpha\left(a\right)\right)+g\alpha\left(a\right).$$

Proof. This result is well known so we omit the proof. ∎

3.3. Proposition. *The condition (III) holds in the category of groups.*

Proof. Since groups satisfies (I), by Proposition 2.7, it is sufficient to prove that, in the notation of Proposition 2.6, if (f_0, g_0) is admissible w.r.t. (i_1, i_2) then $(k_0 f_0, g)$ is admissible w.r.t. $(e_1 k, e_2)$.

Assume all the notations of Proposition 2.6. Suppose (f_0, g_0) is admissible w.r.t. (i_1, i_2), that is (Proposition 3.2) we have

$$f_0\left(i_2\left(x'\right)+i_1\left(x\right)-i_2\left(x'\right)\right)=g_0\left(x'\right)+f_0\left(x\right)-g_0\left(x'\right)\ \ ,\quad \forall x\in X,\forall x'\in X'. \tag{3.7}$$

We will also need the fact that $f : A \longrightarrow D$ is a group homomorphism and because (k, β) is jointly epic we have that the pair $(fk, f\beta)$ is admissible w.r.t. (k, β), *i.e.*, (note $fk = k_0 f_0$)

$$k_0f_0\left(\beta\left(b\right)+k\left(x\right)-\beta\left(b\right)\right)=f\beta\left(b\right)+k_0f_0\left(x\right)-f\beta\left(b\right)\ \ ,\quad \forall b\in B,\forall x\in X. \tag{3.8}$$

We want to prove that $(k_0 f_0, g)$ is admissible w.r.t. $(e_1 k, e_2)$, that is,

$$k_0f_0\left(e_2\left(a'\right)+e_1k\left(x\right)-e_2\left(a'\right)\right)=g\left(a'\right)+k_0f_0\left(x\right)-g\left(a'\right)\ \ ,\quad \forall a'\in A',\forall x\in X.$$

First observe that $a' = k'(x') + \beta'(b')$, where $x' = a' - \beta'\alpha'(a') \in X'$ and $b' = \alpha'(a') \in B$, and then the above equation becomes (note that $e_2 k' = \bar{k} i_2, e_2 \beta' = \bar{\beta}, e_1 k = \bar{k} i_1, g k' = k_0 g_0, g\beta' = \beta_0$)

$$\begin{aligned} k_0f_0&\left(\bar{k}i_2\left(x'\right)+\left(\bar{\beta}\left(b'\right)+\bar{k}i_1\left(x\right)-\bar{\beta}\left(b'\right)\right)-\bar{k}i_2\left(x'\right)\right)\\ &\qquad\qquad\qquad=k_0g_0\left(x'\right)+\beta_0\left(b'\right)+k_0f_0\left(x\right)-\beta_0\left(b'\right)-k_0g_0\left(x'\right);\end{aligned}$$

the argument of $k_0 f_0$ on the left side above is considered as an element in X (because it is in the kernel of $\bar{\alpha}$) but to evaluate it we have to do computations in E; now, because

$\bar{k}$ is an inclusion, we may also evaluate as in Y and hence it becomes (note that i_1 is also an inclusion)

$$i_2(x') + i_1\left(\bar{\beta}(b') + \bar{k}i_1(x) - \bar{\beta}(b')\right) - i_2(x').$$

We may now use (3.7) to obtain

$$\begin{aligned} &k_0g_0(x') + k_0f_0\left(\bar{\beta}(b') + \bar{k}i_1(x) - \bar{\beta}(b')\right) - k_0g_0(x') \\ &\qquad = k_0g_0(x') + \beta_0(b') + k_0f_0(x) - \beta_0(b') - k_0g_0(x') \end{aligned}$$

which simplifies to

$$k_0f_0\left(\bar{\beta}(b') + \bar{k}i_1(x) - \bar{\beta}(b')\right) = \beta_0(b') + k_0f_0(x) - \beta_0(b').$$

Again we observe that $\bar{\beta}(b') + \bar{k}i_1(x) - \bar{\beta}(b')$ is an element in X (it is in the kernel of $\bar{\alpha}$) but it is evaluated as in E; now, because (in Groups) e_1 is a monomorphism and we have

$$\bar{\beta}(b') + \bar{k}i_1(x) - \bar{\beta}(b') = e_1\beta(b') + e_1k(x) - e_1\beta(b')$$

we may also consider

$$\beta(b') + k(x) - \beta(b'),$$

and using (3.8) we have the desired result since $f\beta = \beta_0$. ■

Of course that a simpler proof is obtained if (since we are in groups), instead of diagram (2.5), we consider the one obtained up to isomorphism as follows

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d] & X \rtimes X' \arrow[r, shift left, "{[0,1]}"] \arrow[d] & X' \arrow[l, shift left, "i_2"] \arrow[d] \\ X \rtimes B \arrow[r, "i_1 \rtimes 1"] \arrow[d, shift left] & (X \rtimes X') \rtimes B \arrow[r, shift left, "p \rtimes 1"] \arrow[d, shift left] & X' \rtimes B \arrow[l, shift left, "i_2 \rtimes 1"] \arrow[d, shift left] \\ B \arrow[u, shift left] \arrow[r, equal] & B \arrow[u, shift left] \arrow[r, equal] & B \arrow[u, shift left] \end{tikzcd}$$

where $i_1 = \binom{1}{0}, i_2 = \binom{0}{1}$ and it is completely determined by actions of B in X and X', and an action of X' in X, with the action of B on $X \rtimes X'$ given by $b \cdot (x, x') = (b \cdot x, b \cdot x')$. It is now routine calculations to check that

$$\left(\begin{bmatrix} f_0 \\ 0 \end{bmatrix}, \begin{bmatrix} g_0 & 0 \\ 0 & 1 \end{bmatrix}\right) = (k_0f_0, g)$$

is admissible w.r.t.

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = (e_1k, e_2),$$

whenever (f_0, g_0) is admissible w.r.t.

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = (i_1, i_2).$$

The reason why we choose to present the proof stated above, is because it may shed some light in the process of obtaining more general results than groups. For example rings.

3.4. Proposition. *The category of rings satisfies (III), the Kernel Reflected Admissibility Property.*

Proof. We will follow the argument used to prove the result in Groups.

Assume we have (f_0, g_0) admissible w.r.t. (i_1, i_2), that is (see Proposition 3.1)

$$f_0\left(i_2\left(x'\right) i_1\left(x\right)\right) = g_0\left(x'\right) f\left(x\right) \quad , \quad \forall x \in X, \forall x' \in X'. \tag{3.9}$$

We will also need the fact that (k, β) is jointly epic and that $(fk, f\beta)$ is admissible w.r.t. (k, β), *i.e.*, (note $fk = k_0 f_0$)

$$k_0 f_0\left(\beta\left(b\right) k\left(x\right)\right) = f\beta\left(b\right) k_0 f_0\left(x\right) \quad , \quad \forall b \in B, \forall x \in X. \tag{3.10}$$

We have to prove that $(k_0 f_0, g)$ is admissible w.r.t. $(e_1 k, e_2)$, *i.e.*,

$$k_0 f_0\left(e_2\left(a'\right) e_1 k\left(x\right)\right) = g\left(a'\right) k_0 f_0\left(x\right) \quad , \quad \forall a' \in A', \forall x \in X.$$

We observe that

$$e_2\left(a'\right) = e_2 k'\left(x'\right) + e_2 \beta'\left(b'\right)$$
$$g\left(a'\right) = g k'\left(x'\right) + g \beta'\left(b'\right)$$

with $x' = a' - \beta' \alpha'\left(a'\right)$, $b' = \alpha'\left(a'\right)$ and so we have

$$k_0 f_0\left(e_2 k'\left(x'\right) e_1 k\left(x\right)\right) + k_0 f_0\left(e_2 \beta'\left(b'\right) e_1 k\left(x\right)\right) = g k'\left(x'\right) k_0 f_0\left(x\right) + g \beta'\left(b'\right) k_0 f_0\left(x\right)$$

or equivalently (since $e_2 k' = \bar{k} i_2, e_2 \beta' = e_1 \beta, e_1 k = \bar{k} i_1, g k' = k_0 g_0, g \beta' = \beta_0$)

$$k_0 f_0\left(\bar{k} i_2\left(x'\right) \bar{k} i_1\left(x\right)\right) + k_0 f_0\left(e_1 \beta\left(b'\right) e_1 k\left(x\right)\right) = k_0 g_0\left(x'\right) k_0 f_0\left(x\right) + \beta_0\left(b'\right) k_0 f_0\left(x\right),$$

again, we have that the elements $\bar{k} i_2\left(x'\right) \bar{k} i_1\left(x\right)$ and $i_2\left(x'\right) i_1\left(x\right)$ are the same when considered in X and also the elements $e_1 \beta\left(b'\right) e_1 k\left(x\right)$ and $\beta\left(b'\right) k\left(x\right)$ are the same if considered in X, simply because $\bar{k}$ is a monomorphism and $e_1 \beta\left(b'\right) e_1 k\left(x\right) = e_1 k\left(\beta\left(b'\right) k\left(x\right)\right) = \bar{k} i_1\left(\beta\left(b'\right) k\left(x\right)\right)$, the resulting equation from above is

$$k_0 f_0\left(i_2\left(x'\right) i_1\left(x\right)\right) + k_0 f_0\left(\beta\left(b'\right) k\left(x\right)\right) = k_0 g_0\left(x'\right) k_0 f_0\left(x\right) + f\beta\left(b'\right) k_0 f_0\left(x\right),$$

and it follows directly from (3.9) and (3.10). ■

4. Results

From now on we assume that $\mathbf{B}$ is a pointed category with pullbacks along split epis, and so, in particular, it has binary products and kernels of split epis. Denoting by $\mathrm{RG}(\mathbf{B})$, $\mathrm{SMG}(\mathbf{B})$, $\mathrm{MG}(\mathbf{B})$, $\mathrm{Cat}(\mathbf{B})$, $\mathrm{Grpd}(\mathbf{B})$, respectively the categories of reflexive graphs, star-multiplicative graphs, multiplicative graphs, categories and groupoids, internal to $\mathbf{B}$, we have the following obvious[2] chain of forgetful functors

$$\mathrm{Grpd}(\mathbf{B}) \xrightarrow{(4)} \mathrm{Cat}(\mathbf{B}) \xrightarrow{(3)} \mathrm{MG}(\mathbf{B}) \xrightarrow{(2)} \mathrm{SMG}(\mathbf{B}) \xrightarrow{(1)} \mathrm{RG}(\mathbf{B}).$$

[2]Note that the forgetful functor (2) is only defined for the multiplicative graphs satisfying $dm = d\pi_2$, and so we restrict only to those; but please note that under condition (I), the equation $dm = d\pi_2$ is trivially satisfied.

4.1. Proposition. *If* **B** *satisfies (II), the Split Short Five Lemma, then the forgetful functor (4) is an isomorphism.*

Proof. We have to prove that an internal category in **B** is always an internal groupoid.

Let be given an internal category in **B**, that is, a diagram

$$\begin{tikzcd} C_2 \arrow[r, shift left=4, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=4, "\pi_1"'] & C_1 \arrow[l, shift right=2, "e_2"'] \arrow[l, shift left=2, "e_1"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

satisfying

$$de = 1 = ce$$
$$me_1 = 1 = me_2$$
$$dm = d\pi_2\ ,\ \ cm = c\pi_1$$
$$\text{associativity}$$

where C_2 is the object in the following (split) pullback diagram with projections π_1, π_2 and $e_1 = \langle 1, ed\rangle\, , e_2 = \langle ec, 1\rangle$ the induced morphisms into the pullback

$$\begin{tikzcd} C_2 \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\ C_1 \arrow[r, shift left, "d"] \arrow[u, shift right, "e_1"'] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"'] \end{tikzcd} .$$

It is a fact (see [3],[1]) that the above structure is an internal groupoid if and only if the following morphism

$$\langle m, \pi_2\rangle : C_2 \longrightarrow C_d$$

is an isomorphism, with C_d obtained by pulling back d along the split epi d, as follows

$$\begin{tikzcd} C_2 \arrow[dr, dashed] \arrow[drr, bend left, "\pi_2"] \arrow[ddr, bend right, "m"'] & & \\ & C_d \arrow[r, "\pi_2'"] \arrow[d, "\pi_1'"'] & C_1 \arrow[d, "d"] \\ & C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \end{tikzcd} .$$

We simply observe that $\langle m, \pi_2\rangle$ fits in the following diagram and hence it is a morphism of split extensions

$$\begin{tikzcd} X \arrow[r, "\langle k,0\rangle"] \arrow[d, equal] & C_2 \arrow[r, shift left, "\pi_2"] \arrow[d, "\langle m,\pi_2\rangle"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[d, equal] \\ X \arrow[r, "\langle k,0\rangle"'] & C_d \arrow[r, shift left, "\pi_2'"] & C_1 \arrow[l, shift left, "\langle 1,1\rangle"] \end{tikzcd} ,\quad k = \ker d\ .$$

By the Split Short Five Lemma, the morphism $\langle m, \pi_2\rangle$ is in fact an isomorphism, and the given internal category is in fact an internal groupoid. ∎

4.2. Proposition. *If* $\mathbf{B}$ *satisfies (I), the Jointly Epic Split Extension condition, then the forgetful functor (3) is an isomorphism.*

Proof. From Proposition 2.8 we conclude that $\mathbf{B}$ is a weakly Mal'cev category. In [17] it is proved that in a weakly Mal'cev category, every multiplicative graph is in fact an internal category. ■

4.3. Proposition. *If* $\mathbf{B}$ *satisfies (III), the Reflected Admissibility property, then the forgetful functor (2) is an isomorphism.*

Proof. We have to prove that, given a reflexive graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1 = ce$$

and a morphism

$$s : A \times_B X \longrightarrow X$$

where $k : X \longrightarrow A$ is the kernel of the the split epi d , and $A \times_B X$ is the object in the following pullback diagram (with kernels)

$$\begin{tikzcd} X \arrow[r, "{\langle k,0\rangle}"] \arrow[d, equal] & A \times_B X \arrow[r, shift left, "p_2"] \arrow[d, "p_1"'] & X \arrow[l, shift left, "{\langle eck,1\rangle}"] \arrow[d, "ck"] \\ X \arrow[r, "k"] & A \arrow[r, shift left, "d"] & B \arrow[l, shift left, "e"] \end{tikzcd}$$

satisfying

$$s\,\langle k, 0\rangle = 1_X = s\,\langle eck, 1\rangle\,,$$

it is always possible to define (and in a unique way) a morphism

$$m : A \times_B A \longrightarrow A$$

with $A \times_B A$ as in the following (split) pullback diagram

$$\begin{tikzcd} A \times_B A \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & A \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "d"] & B \arrow[l, shift left, "e"] \arrow[u, shift right, "e"'] \end{tikzcd}$$

satisfying

$$me_1 = 1_A = me_2\ .$$

Consider the following diagram

$$\begin{tikzcd} X \arrow[r, "{\langle k,0\rangle}"] \arrow[d, "k"'] & A \times_B X \arrow[r, shift left, "p_2"] \arrow[d, "1\times_B k"'] & X \arrow[l, shift left, "{\langle eck,1\rangle}"] \arrow[d, "k"] \\ A \arrow[r, "e_1"] \arrow[d, shift left, "d"] & A \times_B A \arrow[r, shift left, "\pi_2"] \arrow[d, shift left, "d\pi_2"] & A \arrow[l, shift left, "e_2"] \arrow[d, shift left, "d"] \\ B \arrow[u, shift left, "e"] \arrow[r, equal] & B \arrow[u, shift left, "{\langle e,e\rangle}"] \arrow[r, equal] & B \arrow[u, shift left, "e"] \end{tikzcd} \tag{4.1}$$

Its columns are split extensions: to see that $1 \times_B k$ is the kernel of $d\pi_2$ simply observe the following diagram

$$\begin{tikzcd}
 & & Y \arrow[d, dashed, "<f{,}kg'>"] \arrow[ddll, dashed, "<f{,}g'>"'] \arrow[ddr, dashed, "g'"] \arrow[drr, bend left, "kg'"] & & & \\
 & & A \times_B A \arrow[rr, "\pi_2"] \arrow[dd, "\pi_1"] & & A \arrow[r, "d"] \arrow[dd, "c"] & B \\
A \times_B X \arrow[urr] \arrow[rrr, "p_2"] \arrow[drr, "p_1"'] & & & X \arrow[ur, "k"] \arrow[dr] & & \\
 & & A \arrow[rr, "d"] & & B . &
\end{tikzcd}$$

Since all the appropriate squares in (4.1) commute, the top horizontal arrows are the image under the Kernel Functor of e_1, e_2 and π_2, considered as morphisms of spit extensions.

Now, the top row is a split extension, so that by the Kernel Reflected Admissibility Property (III) we may conclude the following assertions[3]

- $(\langle k, 0\rangle, \langle eck, 1\rangle)$ is jointly epic;
- (e_1, e_2) is jointly epic;
- for every pair of morphisms (f, g) in $\mathrm{Pt}_B(\mathbf{B})$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, shift left, "d"] & D \arrow[d, shift left, "\alpha"] & A \arrow[l, "g"'] \arrow[d, shift left, "d"] \\
B \arrow[u, shift left, "e"] \arrow[r, equal] & B \arrow[u, shift left, "\beta"] \arrow[r, equal] & B \arrow[u, shift left, "e"]
\end{tikzcd}$$

  if $(Ker(f), Ker(g))$ is admissible w.r.t. $(\langle k, 0\rangle, \langle eck, 1\rangle)$ then (f, g) is admissible w.r.t. (e_1, e_2).

In particular the pair of morphisms $(1_A, 1_A)$ in $\mathrm{Pt}_B(\mathbf{B})$

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "k"'] & X \arrow[d, "k"'] & X \arrow[l, "1_X"'] \arrow[d, "k"] \\
A \arrow[r, "1_A"] \arrow[d, shift left, "d"] & A \arrow[d, shift left, "d"] & A \arrow[l, "1_A"'] \arrow[d, shift left, "d"] \\
B \arrow[u, shift left, "e"] \arrow[r, equal] & B \arrow[u, shift left, "e"] \arrow[r, equal] & B \arrow[u, shift left, "e"]
\end{tikzcd}$$

is such that the pair $(1_X, 1_X)$ is admissible w.r.t. $(\langle k, 0\rangle, \langle eck, 1\rangle)$, because we have given $s : A \times_B X \longrightarrow X$ with $s\langle k, 0\rangle = 1_X$ and $s\langle eck, 1\rangle = 1_X$, and hence we may conclude

[3]In fact the pair (e_1, e_2) is jointly epic in $\mathrm{Pt}_B(\mathbf{B})$, but in the presence of binary products it is also jointly epic in the ground category $\mathbf{B}$.

that the pair $(1_A, 1_A)$ is admissible w.r.t. (e_1, e_2), which gives us the desired (unique) morphism

$$m : A \times_B A \longrightarrow A$$

with $me_1 = 1_A$ and $me_2 = 1_A$. ∎

This does not seem to be a necessary condition for (2) to be an isomorphism. Indeed, from the proof above, it is clear that in the presence of pullbacks along split epimorphisms, we could restrict condition (III) to those split extensions in $\mathrm{Pt}_B(\mathbf{B})$, that are induced by a reflexive graph in $\mathbf{B}$; rather than asking the reflected admissibility condition for an arbitrary split chain in $\mathrm{Pt}_B(\mathbf{B})$, whose image by Ker is a split extension in $\mathbf{B}$.

Next we show that under (I), the Jointly Epic Split Extension condition, all the forgetful functors in the chain above are injective on objects. This means that for a reflexive graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1 = ce$$

it is a property whether or not it admits a star-multiplication, a multiplication (which makes it automatically an internal category, by Proposition 4.2), a multiplication with inverses (making it a groupoid).

It is remarkable that property (I) provides a tool to characterize the different levels at which a given reflexive graph is.

Similar situations were already observed for example in the case of pointed protomodular categories where in fact every split extension is jointly strongly epic, and the notions of central morphism, connector, and cooperator are introduced, with very strong classifying properties (see for example [1], [4], [2], [5] and references there).

Under (I) the notion of reflexive graph itself may be decomposed into a split extension

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "d"] & B \arrow[l, shift left, "e"] \end{tikzcd} \quad , \quad de = 1, k = \ker d$$

together with a morphism

$$h : X \longrightarrow B$$

having the property that the pair $(h, 1_B)$ is admissible w.r.t. (k, e), thus giving $c : A \longrightarrow B$ as the (unique) morphism with

$$ck = h \ , \quad ce = 1 \ .$$

We will start with a very general setting and then restrict to the present case.

Let $\mathbf{B}$ be any category and consider the category $\mathbf{B}^D$ where objects are diagrams in $\mathbf{B}$ of the form

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[rr, bend right, "h"'] & A & B \arrow[l, "e"'] \end{tikzcd}$$

with the only requirement that the pair (k, e) is jointly epic.

Now let $\mathbf{B}$ be pointed with kernels of split epis, and restrict $\mathbf{B}^D$ to the category of diagrams as above such that (k, e) is jointly epic and the pairs $(0_{X,B}, 1_B)$ and $(h, 1_B)$ are admissible w.r.t. (k, e) and, furthermore, the morphism k is a kernel for

$$[0, 1] : A \longrightarrow B.$$

We have thus the following proposition.

4.4. Proposition. *If $\mathbf{B}$ satisfies (I), the Jointly Epic Split Extension property, then the functor*

$$\mathbf{B}^D \longrightarrow \mathrm{RG}\,(\mathbf{B})$$

which assigns to each object

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[rr, bend right, "h"'] & A & B \arrow[l, "e"'] \end{tikzcd}$$

the reflexive graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "{[0,1]}"] \arrow[r, shift right=2, "{[h,1]}"'] & B \arrow[l, "e" description] \end{tikzcd}$$

is an equivalence.

Proof. Given a reflexive graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1 = ce,$$

if considering k the kernel of the split epi d, we have that (k, e) is jointly epic and hence

$$\begin{aligned} d &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\ c &= \begin{bmatrix} ck & 1 \end{bmatrix}. \end{aligned}$$

■

Let again $\mathbf{B}$ be a pointed category with pullbacks along split epis satisfying property (I).

We consider a fixed jointly epic pair

$$X \xrightarrow{k} A \xleftarrow{e} B \tag{4.2}$$

such that the pair $(0_{X,B}, 1_B)$ is admissible w.r.t. (k, e), and $k = \ker([0, 1])$, where we write $[0, 1] : A \longrightarrow B$ for the induced morphism with $[0, 1]k = 0_{X,B}$ and $[0, 1]e = 1_B$.

A given morphism $h : X \longrightarrow B$ is said to be of

type 1 if the pair $(h, 1_B)$ is admissible w.r.t. (k, e).
In this case $[h, 1] : A \longrightarrow B$ will denote the unique morphism with the property that $[h, 1]k = h$, $[h, 1]e = 1$;

type 2 if the pair $(1_X, 1_X)$ is admissible w.r.t. $(\langle k,0\rangle, \langle eh,1\rangle)$.
Here $A \times_B X$ is obtained by pulling back h along the split epi $[0,1] = d$, that is

$$\begin{tikzcd}
X \arrow[r, "\langle k{,}0\rangle"] \arrow[d, equal] & A \times_B X \arrow[r, shift left, "p_2"] \arrow[d, "p_1"'] & X \arrow[l, shift left, "\langle eh{,}1\rangle"] \arrow[d, "h"] \\
X \arrow[r, "k"] & A \arrow[r, shift left, "d"] & B \arrow[l, shift left, "e"] .
\end{tikzcd}$$

We write $[1_X, 1_X] : A \times_B X \longrightarrow X$ for the unique morphism satisfying

$$[1,1]\langle k,0\rangle = 1 = [1,1]\langle eh,1\rangle .$$

type 3 if it is of type 1 and, in addition, the pair $(k, 1_A)$ is admissible w.r.t. $(e_1 k, e_2)$

$$X \xrightarrow{e_1 k} A_2 \xleftarrow{e_2} A .$$

Here A_2 is the pullback of $[h\ 1]$ along $[0\ 1]$, and $e_1 = \langle 1, ed\rangle, e_2 = \langle ec, 1\rangle$ are the induced morphisms into the pullback, as displayed in the following diagram

$$\begin{tikzcd}
X \arrow[r, "\langle k{,}0\rangle"] \arrow[d, equal] & A_2 \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & A \arrow[l, shift left, "e_2"] \arrow[d, shift right, "{[h\,1]}"'] \\
X \arrow[r, "k"] & A \arrow[r, shift left, "{[0\,1]}"] \arrow[u, shift right, "e_1"'] & B \arrow[l, shift left, "e"] \arrow[u, shift right, "e"'] .
\end{tikzcd} \tag{4.3}$$

We also write

$$m = \begin{bmatrix} k & 1_A \end{bmatrix} : A_2 \longrightarrow A$$

and observe that

$$\begin{aligned}
\pi_2 &= \begin{bmatrix} 0_{X,A} & 1_A \end{bmatrix} \\
\pi_1 &= \begin{bmatrix} k & \begin{bmatrix} eh & e \end{bmatrix} \end{bmatrix} \\
e_1 k &= \langle k, 0_{X,A}\rangle = \ker \pi_2 \ .
\end{aligned}$$

type 4 if it is of type 1 and, in addition, the pair $(\langle k, 0_{X,A}\rangle, e_2)$ is admissible with respect to $(\langle k, 0_{X,A}\rangle, \langle 1_A, 1_A\rangle)$

$$\langle k, 0_{X,A}\rangle : X \longrightarrow A_d \longleftarrow A : \langle 1_A, 1_A\rangle .$$

That is, we may insert the dashed arrow, denoted by $\begin{bmatrix} \langle k, 0_{X,A}\rangle & e_2 \end{bmatrix}$, in the following diagram of split extensions

$$\begin{tikzcd}
X \arrow[r, "\langle k{,}0\rangle"] \arrow[d, equal] & A_d \arrow[r, shift left, "\pi_2'"] \arrow[d, dashed] & A \arrow[l, shift left, "\langle 1{,}1\rangle"] \arrow[d, equal] \\
X \arrow[r, "\langle k{,}0\rangle"] & A_2 \arrow[r, shift left, "\pi_2"] & A \arrow[l, shift left, "e_2"] ;
\end{tikzcd}$$

where A_2, π_2, e_2 are as in (4.3) and A_d is the pullback of $d = [0\ 1]$ along itself, as displayed in the following diagram

$$\begin{tikzcd} X \arrow[r, "\langle k,0\rangle"] \arrow[d, equal] & A_d \arrow[r, shift left, "\pi_2'"] \arrow[d, "\pi_1'"'] & A \arrow[l, shift left, "<ed{,}1>"] \arrow[d, "d"] \\ X \arrow[r, "k"] & A \arrow[r, shift left, "{[0\,1]}"] & B \arrow[l, shift left, "e"] \end{tikzcd} .$$

Note that we are replacing the induced section of of π_2', $\langle ed, 1\rangle$, by the morphism $\langle 1, 1\rangle$.

4.5. Proposition. *Let* $\mathbf{B}$ *be a pointed category with pullbacks of split epis and satisfying (I). To give a reflexive graph in* $\mathbf{B}$ *is to give a split extension*

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "{[0\,1]}"] & B \arrow[l, shift left, "e"] \end{tikzcd}$$

together with a morphism

$$h : X \longrightarrow B$$

of type 1.
Furthermore:

- *the reflexive graph is a star-multiplicative graph if and only if* h *is also of type 2 w.r.t.* (k, e)*;*
- *the reflexive graph is a multiplicative graph (and hence an internal category) if and only if* h *is of type 3 (w.r.t.* (k, e)*);*
- *the reflexive graph is an internal groupoid if and only if* h *is of type 3 and type 4.*

Proof. That type 1 corresponds to reflexive graphs follows from Proposition 4.4.

Type 1 and type 2, when combined, give

$$c = \begin{bmatrix} h & 1 \end{bmatrix} : A \longrightarrow B$$

and

$$s = \begin{bmatrix} 1_X & 1_X \end{bmatrix} : A \times_B X \longrightarrow X$$

satisfying the required conditions for a star-multiplicative graph. Of course, since (k, e) and $(\langle k, 0\rangle, \langle eh, 1\rangle)$ are jointly epic, every star-multiplicative graph is obtained in this way.

Type 3 is the same with multiplicative graphs.

Type 3 and type 4 give in fact an internal groupoid, since we can fill in the following diagram

$$\begin{tikzcd}
X \arrow[r, "\langle k,0\rangle"] \arrow[d, equal] & A_d \arrow[r, shift left, "\pi_2'"] \arrow[d, shift right, "{[<k,0>\, e_2]}"'] & A \arrow[l, shift left, "<1,1>"] \arrow[d, equal] \\
X \arrow[r, "\langle k,0\rangle"] & A_2 \arrow[r, shift left, "\pi_2"] \arrow[u, shift right, "<m,\pi_2>"'] & A \arrow[l, shift left, "e_2"]
\end{tikzcd}$$

with the two morphisms in the middle, with opposite directions. Since $(\langle k,0\rangle, \langle 1,1\rangle)$ and $(\langle k,0\rangle, e_2)$ are jointly epic it follows that they are inverse to each other. ∎

Finally we conclude by presenting a simple characterization of internal groupoids (a similar result, but in a slightly different context, was presented in [15]).

4.6. Proposition. *Let* $\mathbf{B}$ *be a pointed category with pullbacks of split epis and satisfying (I),(II) and (III). Giving an internal groupoid in* $\mathbf{B}$ *is to give a split extension*

$$\begin{tikzcd}
X \arrow[r, "k"] & A \arrow[r, shift left, "{[0\ 1]}"] & B \arrow[l, shift left, "e"]
\end{tikzcd}$$

together with a morphism

$$h : X \longrightarrow B$$

such that the dashed arrows in the following diagram can be inserted in order to make it commutative

$$\begin{tikzcd}
X \arrow[r, "\langle 1,0\rangle"] \arrow[d, equal] & X \times X \arrow[r, shift left, "\pi_2"] \arrow[d, dashed, "(a)"'] & X \arrow[l, shift left, "\langle 1,1\rangle"] \arrow[d, "h"] \\
X \arrow[r, "k"] \arrow[d, "h"'] & A \arrow[r, shift left, "{[0\ 1]}"] \arrow[d, dashed, "(b)"'] & B \arrow[l, shift left, "e"] \arrow[d, equal] \\
B \arrow[r, "\langle 1,0\rangle"] & B \times B \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\langle 1,1\rangle"]
\end{tikzcd} .$$

Proof. It is sufficient to prove that every star-multiplicative graph is of this form (since we are assuming (I), (II) and (III) and we have Propositions 4.3, 4.2 and 4.1).

Inserting (b) is equivalent to the fact that $(h,1)$ is admissible w.r.t. (k,e) (*i.e.*, h is of type 1 above), and we have

$$\text{(b)} = \left\langle \begin{bmatrix} h & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right\rangle .$$

Inserting (a) is equivalent to say that the diagram

$$\begin{tikzcd}
X \arrow[r, "\langle 1,0\rangle"] \arrow[d, equal] & X \times X \arrow[r, shift left, "\pi_2"] \arrow[d, dashed] & X \arrow[l, shift left, "\langle 1,1\rangle"] \arrow[d, equal] \\
X \arrow[r, "\langle k,0\rangle"] & A \times_B X \arrow[r, shift left, "p_2"] & X \arrow[l, shift left, "\langle eh,1\rangle"]
\end{tikzcd}$$

can be completed, *i.e.*, the dashed arrow can be inserted; because of (II), the dashed arrow, existing is an isomorphism, and hence it is also equivalent to say that h is of type 2, with the morphism $[1_X, 1_X]$ given by

$$A \times_B X \xrightarrow{\cong} X \times X \xrightarrow{\pi_1} X \ .$$

∎

5. Conclusion

In [10] it is said that in a semi-abelian category in general, not every internal groupoid is obtained from the description above, this means that we have to impose condition (III) in a semi-abelian category in order to have the desired result. This is a new condition (at least for the knowledge of the author) and some future work is needed in order to better understand it. We have only checked that it holds in the semi-abelian categories of groups and rings. It is also clear that it holds in every additive category, with kernels of split epimorphisms, since in that case every pair is admissible.

As for the future development of this topics we observe that two extreme cases may occur: either (III) is too restrictive and only a few semi-abelian categories (of interest) satisfy it, or the opposite; in the second case, the results here presented have a wide range of application, and certainly deserve to be further investigated; if, instead, the first is the case, then this condition may be a good approximation for an axiomatic treatment of the category of groups and rings [9].

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