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EPIMORPHIC COVERS MAKE R_G^+ A SITE, FOR PROFINITE G

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ABSTRACT. Let G be a non-finite profinite group and let $G-\mathbf{Sets}_{df}$ be the canonical site of finite discrete G-sets. Then the category R_G^+, defined by Devinatz and Hopkins, is the category obtained by considering $G-\mathbf{Sets}_{df}$ together with the profinite G-space G itself, with morphisms being continuous G-equivariant maps. We show that R_G^+ is a site when equipped with the pretopology of epimorphic covers. We point out that presheaves of spectra on R_G^+ are an efficient way of organizing the data that is obtained by taking the homotopy fixed points of a continuous G-spectrum with respect to the open subgroups of G. Additionally, utilizing the result that R_G^+ is a site, we describe various model category structures on the category of presheaves of spectra on R_G^+ and make some observations about them.

1. Introduction

Let G be a profinite group that is not a finite group. Let R_G^+ be the category with objects all finite discrete left G-sets together with the left G-space G. The morphisms of R_G^+ are the continuous G-equivariant maps. Since G is not finite, the object G in R_G^+ is very different in character from all the other objects of R_G^+. In this paper, we show that R_G^+ is a site when equipped with the pretopology of epimorphic covers.

As far as the author knows, the category R_G^+ is first defined and used in the paper [DH04]. Let G_n be the profinite group $S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, where S_n is the nth Morava stabilizer group. In [DH04, Theorem 1], the authors construct a contravariant functor - that is, a presheaf -

$$\mathbf{F}\colon (R_{G_n}^+)^{\mathrm{op}} \to (\mathcal{E}_\infty)_{K(n)},$$

to the category $(\mathcal{E}_\infty)_{K(n)}$ of $K(n)$-local commutative S-algebras (see [EKMM97]), where $K(n)$ is the nth Morava K-theory (see [Rud98, Chapter 9] for an exposition of $K(n)$). The functor $\mathbf{F}$ has the properties that, if U is an open subgroup of G_n, then $\mathbf{F}(G_n/U) = E_n^{dhU}$, and $\mathbf{F}(G_n) = E_n$, where E_n is the nth Lubin-Tate spectrum (for salient facts about E_n and its importance in homotopy theory, see [DH95, Introduction]), and E_n^{dhU} is a spectrum that behaves like the U-homotopy fixed point spectrum of E_n with respect to the continuous U-action. Since $\mathrm{Hom}_{R_{G_n}^+}(G_n, G_n) \cong G_n$, functoriality implies that G_n acts on E_n by maps of commutative S-algebras. In Section 5, we will give several related examples of presheaves of spectra that illustrate the utility of the category R_G^+.

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The pretopology of epimorphic covers on a small category $\mathcal{C}$ is the pretopology $\mathcal{K}$ given by all covering families $\{f_i \colon C_i \to C \mid i \in I\}$ such that $\phi\colon \coprod_{i \in I} C_i \to C$ is onto, where $C_i, C \in \mathcal{C}$, $f_i \in \mathrm{Mor}_{\mathcal{C}}(C_i, C)$, and I is some indexing set. (Of course, one must prove that these covering families actually give a pretopology on $\mathcal{C}$.) We note that we do not require that ϕ be a morphism in $\mathcal{C}$; for our purposes, $\mathcal{C} = R_G^+$ and we only require that ϕ be an epimorphism in the category of all G-sets (so that ϕ does not have to be continuous). This assumption is important for our work, since, for example, $G \coprod G$ is not in R_G^+. Also, recall that the pretopology $\mathcal{K}$ is a familiar one. For example, for a profinite group G, $\mathcal{K}$ is the standard basis used for the site $G-\mathbf{Sets}_{df}$ of finite discrete G-sets ([Jar97, pg. 206]).

Note that R_G^+ is built out of the two subcategories $G-\mathbf{Sets}_{df}$ and the groupoid G. Since each of these categories is a site via $\mathcal{K}$ (for G, this is verified in Lemma 2.2 below), it is natural to wonder if R_G^+ is also a site via $\mathcal{K}$. Our main result (Theorem 3.1) verifies that this is indeed the case.

As discussed earlier, $\mathbf{F}$ is a presheaf of spectra on the site $R_{G_n}^+$. More generally, there is the category $\mathrm{PreSpt}(R_G^+)$ of presheaves of spectra on R_G^+. In [Dav06b], the author showed that, given a continuous G-spectrum Z, then, for any open subgroup U of G, there is a homotopy fixed point spectrum Z^{hU}, defined with respect to the continuous action of U on Z. In Examples 5.5 and 5.6, we see that there is a presheaf of spectra on R_G^+ that organizes in a functorial way the following data: Z, Z^{hU} for all U open in G, and the maps between these spectra that are induced by continuous G-equivariant maps between the G-spaces G and G/U. Thus, $\mathrm{PreSpt}(R_G^+)$ is a natural category within which to work with continuous G-spectra.

The homotopy fixed points referred to above use a certain model category structure on the category $\mathrm{PreSpt}(G-\mathbf{Sets}_{df})$ of presheaves of spectra on $G-\mathbf{Sets}_{df}$ (see Section 5 and [Dav06b, Section 3] for details). Thus, in Section 5, we describe various model category structures on $\mathrm{PreSpt}(R_G^+)$, and we make some observations about these model structures and the restriction functor $i_*\colon \mathrm{PreSpt}(R_G^+) \to \mathrm{PreSpt}(G-\mathbf{Sets}_{df})$ (see Definition 5.8), where the target category has the aforementioned model structure. It is our hope that one or some of these model structures on $\mathrm{PreSpt}(R_G^+)$ can be useful for the theory of homotopy fixed points for profinite groups, though we have not yet found any such applications.

The main result of this paper naturally motivates further work in exploring the site R_G^+. We give a brief discussion of some of the things that one might consider. The Grothendieck topos of sheaves on $G-\mathbf{Sets}_{df}$ is equivalent to the category of discrete G-sets, a category of continuous G-spaces. For some time, the author worked to show that the Grothendieck topos on the site R_G^+ was similarly equivalent to a category of topological G-spaces. (To be more precise, the author hoped to obtain some category related to pro-discrete G-spaces.) However, the author concluded that there was no such equivalence.

As noted above in several ways, the site R_G^+ is closely related to the site $G-\mathbf{Sets}_{df}$. In particular, each site has a topology given by the pretopology $\mathcal{K}$ of epimorphic covers. However, there is an important difference between R_G^+ and $G-\mathbf{Sets}_{df}$: the latter category is closed under pullbacks, but it is easy to see that R_G^+ does not have all pullbacks (see the discussion just after Theorem 3.1).

In a category with pullbacks, the canonical topology, the finest topology in which every representable presheaf is a sheaf, is given by all covering families of universal effective epimorphisms (see Expose IV, 4.3 of [Dem70]). This implies that $G-\mathbf{Sets}_{df}$ is a site with the canonical topology, when equipped with pretopology $\mathcal{K}$. However, due to the lack of sufficient pullbacks, we cannot conclude that $\mathcal{K}$ gives R_G^+ the canonical topology. Thus, it is natural to ask if the topology on R_G^+ is canonical. This is not an easy question to answer: a preliminary step is to determine whether or not the topology is subcanonical, and the author is currently working to solve this problem.

In Section 4, we exhibit a second pretopology on R_G^+ that generates a topology that is strictly coarser than that given by $\mathcal{K}$. We do not know if this second pretopology has any advantages over $\mathcal{K}$.

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2. Preliminaries

Before we prove our main result, we first collect some basic facts which will be helpful later. As stated in the Introduction, G always refers to an infinite profinite group. (If the profinite group G is finite, then $R_G^+ = G-\mathbf{Sets}_{df}$ and there is nothing to prove.)

2.1. Lemma. *Let $f\colon C \to G$ be any morphism in R_G^+ with $C \neq \varnothing$. Then $C = G$.*

Proof. Choose any $c \in C$ and let $f(c) = \gamma$. Choose any $\delta \in G$. Then the G-equivariance of f implies that $\delta = (\delta\gamma^{-1})\gamma = (\delta\gamma^{-1}) \cdot f(c) = f((\delta\gamma^{-1}) \cdot c)$. Thus, f is onto and $|\mathrm{im}(f)| = \infty$, so that C cannot be a finite set. ∎

2.2. Lemma. *For a topological group G, let $\mathbf{G}$ be the groupoid with the single object G and morphisms the G-equivariant maps $G \to G$ given by right multiplication by some element of G. Then $\mathbf{G}$ is a site with the pretopology $\mathcal{K}$ of epimorphic covers.*

Proof. Any diagram $G \xrightarrow{f} G \xleftarrow{g} G$, where f and g are given by multiplication by γ and δ, respectively, can be completed to a commutative square

$$\begin{tikzcd} G \arrow[r, "f'"] \arrow[d, "g'"'] & G \arrow[d, "g"] \\ G \arrow[r, "f"] & G, \end{tikzcd}$$

where f' and g' are given by multiplication by δ^{-1} and γ^{-1}, respectively. This property suffices to show that G is a site with the atomic topology, in which every sieve is a covering sieve if and only if it is nonempty. It is easy to see that the only nonempty sieve of G is $\mathrm{Mor}_G(G, G)$ itself. Thus, the only covering sieve of G is the maximal sieve. Since every morphism of G is a homeomorphism, in the pretopology $\mathcal{K}$, the collection of covers is exactly the collection of all nonempty subsets of $\mathrm{Mor}_G(G, G)$. Then it is easy to check that $\mathcal{K}$ is the maximal basis that generates the atomic topology. ∎

Observe that if $f: G \to G$ is a morphism in R_G^+, then by G-equivariance, f is the map given by multiplication by $f(1)$ on the right. As mentioned earlier, we have

2.3. Lemma. *The category G–$\mathbf{Sets}_{df}$, a full subcategory of R_G^+, is closed under pullbacks.*

Proof. The pullback of a diagram in G–$\mathbf{Sets}_{df}$ is formed simply by regarding the diagram as being in the category T_G of discrete G-sets. The category T_G is closed under pullbacks, as explained in [MM94, pg. 31]. ∎

We recall the following useful result and its proof.

2.4. Lemma. *Let X be any finite set in R_G^+. We write $X = \coprod_{i=1}^n \overline{x_i}$, the disjoint union of all the distinct orbits $\overline{x_i}$, with each x_i a representative. Then X is homeomorphic to $\coprod_{i=1}^n G/U_i$, where $U_i = G_{x_i}$ is the stabilizer in G of x_i.*

Proof. Let $f: G/U_i \to \overline{x_i}$ be given by $f(\gamma U_i) = \gamma \cdot x_i$. Since X is a discrete G-set, the stabilizer U_i is an open subgroup of G with finite index, so that G/U_i is a finite set. Then f is open and continuous since it is a map between discrete spaces. Also, it is clear that f is onto. Now suppose $\gamma U_i = \delta U_i$. Then $\gamma^{-1}\delta \in U_i$, so that $(\gamma^{-1}\delta) \cdot x_i = (\gamma^{-1}) \cdot (\delta \cdot x_i) = x_i$. Thus, $\gamma \cdot x_i = \delta \cdot x_i$ and f is well-defined. Assume that $\gamma \cdot x_i = \delta \cdot x_i$. Then $\gamma^{-1}\delta \in G_{x_i}$ so that f is a monomorphism. ∎

2.5. Lemma. *Let X be a finite discrete G-set in R_G^+ and let $\psi: G \to X$ be any G-equivariant function. Then ψ is a morphism in R_G^+.*

Proof. As in Lemma 2.4, we identify X with $\coprod_{i=1}^n G/U_i$. Since ψ is G-equivariant and $\psi(\gamma) = \gamma \cdot \psi(1)$, ψ is determined by $\psi(1)$. Let $\psi(1) = \delta U_j$ for some $\delta \in G$ and some j. Then for any γ in G, $\gamma U_j = (\gamma\delta^{-1}\delta)U_j = (\gamma\delta^{-1}) \cdot \psi(1) = \psi(\gamma\delta^{-1})$, so that $\mathrm{im}(\psi) = G/U_j$. Since X is discrete, ψ is continuous, if, for any $x \in X$, $\psi^{-1}(x)$ is open in G. It suffices, by the identification, to let $x = \gamma U_j$, for any $\gamma \in G$. Then

$$\begin{aligned}\psi^{-1}(\gamma U_j) &= \{\zeta \in G \,|\, \psi(\zeta) = \gamma U_j\} = \{\zeta \in G \,|\, \zeta \cdot (\delta U_j) = \gamma U_j\} \\ &= \{\zeta \in G \,|\, \delta^{-1}\zeta^{-1}\gamma \in U_j\} = \gamma U_j \delta^{-1}.\end{aligned}$$

Since U_j is open and multiplication on the left or the right is always a homeomorphism in a topological group, we see that $\psi^{-1}(x)$ is an open set in G. ∎

3. The proof of the main result

With the lemmas of the previous section in hand, we are ready for

3.1. Theorem. *For any profinite group G, the category R_G^+, equipped with the pretopology $\mathcal{K}$ of epimorphic covers, is a small site.*

Before proving the theorem, we make some remarks about pullbacks in R_G^+ and how this affects our proof. In a category $\mathcal{C}$ with sufficient pullbacks, to prove that a pretopology is given by a function K, which assigns to each object C a collection $K(C)$ of families of morphisms with codomain C, one must prove the stability axiom, which says the following: if $\{f_i\colon C_i \to C | \, i \in I\} \in K(C)$, then for any morphism $g\colon D \to C$, the family of pullbacks

$$\{\pi_L\colon D \times_C C_i \to D | \, i \in I\} \in K(D).$$

Let us examine what this axiom would require of R_G^+.

3.2. Example. The map $G \to *$ forms a covering family and so the stability axiom requires that $G \times_{\{*\}} G = G \times G$ be in R_G^+.

3.3. Example. Let C be any finite discrete G-set with more than one element and with trivial G-action, $g\colon G \to C$ any morphism, and consider the cover

$$\{f_i\colon C_i \to C | \, i \in I\} \in K(C),$$

where $C_j = C$ and $f_j\colon C \to C$ is the morphism mapping C to $g(1)$, for some $j \in I$. Because the action is trivial, f_j is G-equivariant. There certainly exist covers of C of this form, since one could let $f_k = \mathrm{id}_C$, for some $k \neq j$ in I, and then let the other f_i be any morphisms with codomain C. Hence, the stability axiom requires that

$$G \times_C C = \{(\gamma, c) | \, g(\gamma) = f_j(c)\} = \{(\gamma, c) | \, \gamma \cdot g(1) = g(1)\} = G_{g(1)} \times C = G \times C$$

exists in R_G^+, which is impossible.

Thus, the stability axiom for a pretopology must be altered so that one still obtains a topology. We list the correct axioms for our situation below. They are taken from [MM94, pg. 156].

1. If $f\colon C' \to C$ is an isomorphism, then $\{f\colon C' \to C\} \in K(C)$.
2. (stability axiom) If $\{f_i\colon C_i \to C | \, i \in I\} \in K(C)$, then for any morphism $g\colon D \to C$, there exists a cover $\{h_j\colon D_j \to D | \, j \in J\} \in K(D)$ such that for each j, $g \circ h_j$ factors through some f_i.
3. (transitivity axiom) If $\{f_i\colon C_i \to C | \, i \in I\} \in K(C)$, and if for each $i \in I$ there is a family $\{g_{ij}\colon D_{ij} \to C_i | \, j \in I_i\} \in K(C_i)$, then the family of composites

   $$\{f_i \circ g_{ij}\colon D_{ij} \to C | \, i \in I, j \in I_i\}$$

   is in $K(C)$.

Proof of Theorem 3.1. It is clear that the pretopology $\mathcal{K}$ satisfies axiom (1) above. Also, it is easy to see that axiom (3) holds. Indeed, using the above notation, choose any $c \in C$. Then there is some $c_i \in C_i$ for some i, such that $f_i(c_i) = c$. Similarly, there must be some $d_{ij} \in D_{ij}$ for some j, such that $g_{ij}(d_{ij}) = c_i$. Hence, $(f_i \circ g_{ij})(d_{ij}) = f_i(c_i) = c$, so that $\coprod_{i,j} D_{ij} \to C$ is onto. This verifies (3). We verify (2) by considering five cases.

Case (1): Suppose that D and each of the C_i are finite sets in R_G^+. By Lemma 2.1, C must be a finite set. Consider the cover $\{\pi_L(i)\colon D \times_C C_i \to D | \, i \in I\}$, where each $\pi_L(i)$ equals π_L and each $g \circ \pi_L(i)$ factors through f_i via the canonical map π_R. Now, for any $d \in D$, there exists some $l \in I$ such that $g(d) = f_l(c_l)$, for some $c_l \in C_l$. Thus, $(d, c_l) \in D \times_C C_l$, so that $\coprod_{i \in I} D \times_C C_i \to D$ maps (d, c_l) to d and is therefore an epimorphism. This shows that $\{\pi_L(i) | \, i \in I\}$ is in $K(D)$.

Case (2): Suppose that $D = G$ and that each C_i is a finite set in R_G^+. By Lemma 2.1, C is a finite set and we identify it with $\coprod_{i=1}^n G/U_i$, where $U_i = G_{x_i}$, the stabilizer in G of x_i. The map g is determined by $g(1) = \delta U_k$, for some $\delta \in G$ and some stabilizer U_k. Since $\coprod_{i \in I} C_i \to C$ is onto and $\mathrm{im}(g) = G/U_k$, there exists some $c_l \in C_l$, for some l, such that $f_l(c_l) = U_k$. Since C_l is a finite set, we can identify c_l with some μG_z, where $\mu \in G$ and G_z is the stabilizer of some element $z \in C_l$.

Define the cover to be $\{\lambda\colon G \to G\}$, where $\lambda(\gamma) = \gamma\delta^{-1}$. Define $\alpha_l\colon G \to C_l$ to be the G-equivariant map given by $1 \mapsto \mu G_z$. By Lemma 2.5, α_l is continuous and is a morphism in R_G^+. Since λ is a homeomorphism, the cover $\{\lambda\}$ is in $K(D)$. Now,

$$(g \circ \lambda)(1) = g(\delta^{-1}) = \delta^{-1} \cdot g(1) = U_k = \mu \cdot f_l(G_z) = \mu \cdot f_l(\mu^{-1} \cdot \alpha_l(1)) = (f_l \circ \alpha_l)(1).$$

This shows that $g \circ \lambda$ factors through f_l via α_l.

Case (3): Suppose that not all the C_i are finite sets and that $D = G$. Also, assume that $C = G$, so that $C_i = G$, for all $i \in I$. Choose any $k \in I$, let $\alpha_k = \mathrm{id}_G$, and define $\lambda\colon G \to G$ to be multiplication on the right by $f_k(1)g(1)^{-1}$. Then $g \circ \lambda = f_k \circ \mathrm{id}_G$, since

$$(g \circ \lambda)(1) = g(f_k(1)g(1)^{-1}) = f_k(1)g(1)^{-1} \cdot g(1) = f_k(1) = (f_k \circ \alpha_k)(1),$$

so that the stability axiom is satisfied by letting the covering family be $\{\lambda\}$.

Case (4): Suppose that not all the C_i are finite sets, $D = G$, and C is a finite set. With C as in Lemma 2.4, let $g(1) = \delta U_k \in C$, as in Case (2). Then there exists some l such that $f_l(c_l) = U_k$, for some $c_l \in C_l$. Now we consider two subcases.

Case (4a): Suppose that C_l is a finite set in R_G^+. Just as in Case (2), we construct maps λ and α_l, so that $g \circ \lambda$ factors through f_l via α_l and $\{\lambda\} \in K(D)$.

Case (4b): Suppose that $C_l = G$. By G-equivariance, $f_l(1) = c_l^{-1} U_k$. Then define $\lambda\colon G \to G$ by $1 \mapsto \delta^{-1}$ and $\alpha_l\colon G \to G$ by $1 \mapsto c_l$. Then $g \circ \lambda$ factors through f_l via α_l, since

$$(g \circ \lambda)(1) = g(\delta^{-1}) = \delta^{-1} \cdot g(1) = U_k = f_l(c_l) = (f_l \circ \alpha_l)(1).$$

Thus, $\{\lambda\} \in K(D)$, since λ is a homeomorphism. This completes Case (4).

Now we consider the final possibility, *Case (5)*: suppose that not all of the C_i are finite sets and suppose that D is a finite set. This implies that C is a finite set. This case is more difficult than the others because it combines several of the previous constructions and the desired cover consists of more than one morphism. For each $d \in D$, we make a choice of some $i(d) \in I$ and some $c_d \in C_{i(d)}$, such that c_d is in the preimage of $g(d)$ under $\coprod_{i \in I} C_i \to C$. Then write $D = D_{df} \coprod D_G$, where D_{df} is the set of all d such that $C_{i(d)}$ is a finite set, and D_G is the set of all d such that $C_{i(d)} = G$. Now consider the cover $\{h_d \colon D_d \to D | \, d \in D\}$, where

$$D_d = \begin{cases} D \times_C C_{i(d)} & \text{if } d \in D_{df}, \\ G & \text{if } d \in D_G \end{cases}$$

and h_d is defined in the following paragraph.

If $d \in D_{df}$, then set $h_d = \pi_L$ and let $\alpha_d \colon D \times_C C_{i(d)} \to C_{i(d)}$ be the canonical map π_R; it is clear that the required square commutes. Now suppose $d \in D_G$. Then there exists $c_d \in C_{i(d)} = G$, such that $g(d) = f_{i(d)}(c_d)$. We write $f_{i(d)}(1) = \theta U_k \in C$, for some $\theta \in G$ and for some stabilizer U_k. Then we define $\alpha_d \colon G \to C_{i(d)} = G$ by $1 \mapsto \theta^{-1}$. Also, we define $h_d \colon G \to D$ by $1 \mapsto (\theta^{-1} c_d^{-1}) \cdot d$. Lemma 2.5 shows that h_d is a morphism in R_G^+. Then $g \circ h_d = f_{i(d)} \circ \alpha_d$, as desired, since

$$\begin{aligned} (g \circ h_d)(1) &= g((\theta^{-1} c_d^{-1}) \cdot d) = (\theta^{-1} c_d^{-1}) \cdot g(d) \\ &= (\theta^{-1} c_d^{-1}) \cdot f_{i(d)}(c_d) = f_{i(d)}(\theta^{-1}) = (f_{i(d)} \circ \alpha_d)(1). \end{aligned}$$

The only remaining detail is to show that $\{h_d | \, d \in D\} \in K(D)$; that is, we must show that $\phi \colon \coprod_{d \in D} D_d \to D$ is an epimorphism. Let d be any element in D. Suppose $d \in D_{df}$, with $c_d \in C_{i(d)}$, such that $f_{i(d)}(c_d) = g(d)$. Then $(d, c_d) \in D \times_C C_{i(d)}$ and $\phi(d, c_d) = \pi_L(d, c_d) = d$. Now suppose $d \in D_G$. With $c_d \in G$ and θ as above, $c_d\theta \in D_d = G$ and $\phi(c_d\theta) = h_d(c_d\theta) = (c_d\theta) \cdot h_d(1) = d$. Therefore, ϕ is an epimorphism. ∎

4. A second pretopology on R_G^+

In this section, we obtain a second pretopology on R_G^+ that generates a strictly coarser topology than that given by $\mathcal{K}$. We use $\mathcal{L}$ to denote the pretopology of epimorphic covers for the site $G-\mathbf{Sets}_{df}$.

Let $\mathcal{M}$ be the function on R_G^+ defined by

$$\mathcal{M}(C) = \begin{cases} \mathcal{L}(C) & \text{if } C \in G-\mathbf{Sets}_{df}, \\ \{R \,|\, R \subset \mathrm{Hom}_{R_G^+}(G, G); R \neq \varnothing\} & \text{if } C = G. \end{cases}$$

Thus, $\mathcal{M}$ consists exactly of the covers from the sites G (the groupoid) and $G-\mathbf{Sets}_{df}$.

Now we show that $\mathcal{M}$ makes R_G^+ a site.

4.1. Theorem. *For any profinite group G, the category R_G^+ is a site with pretopology $\mathcal{M}$.*

Proof. As in the proof of Theorem 3.1, since R_G^+ does not have enough pullbacks, we use the alternative set of axioms for a pretopology stated just before the proof of Theorem 3.1.

Now suppose that $f\colon C' \to C$ is an isomorphism in R_G^+. If C is in $G-\mathbf{Sets}_{df}$, then C' is too. Since f is a morphism in the category $G-\mathbf{Sets}_{df}$, $\{f\} \in \mathcal{L}(C) = \mathcal{M}(C)$. If $C = G$, then $C' = G$ and it is clear that $\{f\} \in \mathcal{M}(G)$. This verifies the first axiom.

Axiom three is also true, since, using the notation of its statement above (just before the proof of Theorem 3.1), all the C_i and D_{ij} are in only $G-\mathbf{Sets}_{df}$ or in only groupoid G, according to whether C is in $G-\mathbf{Sets}_{df}$ or G, respectively.

In the notation of Axiom 2 (as stated in Section 3), suppose that $C = G$, so that each C_i is equal to G. If $D = \varnothing$, then it is easy to see that the stability axiom is satisfied by using the cover $\{\varnothing \to \varnothing\} \in \mathcal{M}(\varnothing)$. Now suppose that $D = G$. Then case (3) in the proof of Theorem 3.1 shows that Axiom 2 holds.

Now suppose $C \in G-\mathbf{Sets}_{df}$ and consider any $\{f_i\colon C_i \to C \mid i \in I\} \in \mathcal{L}(C)$. All the C_i must be finite sets, and, if $g\colon D \to C$ is any morphism in R_G^+, then D is either a finite set or G. If D is a finite set, then case (1) of the proof of Theorem 3.1 gives the required cover, since $G-\mathbf{Sets}_{df}$ is closed under pullbacks. Finally, if $D = G$, then case (2) of the proof of Theorem 3.1 gives a satisfactory cover of G. ∎

As stated at the beginning of this section, it is not hard to see that $\mathcal{M}$ generates a strictly coarser topology than $\mathcal{K}$. For example, for $\mathcal{K}$, the trivial one-element G-set $\{*\}$ has a covering sieve of the form

$$(\{G \to \{*\}\}) = \{G \to \{*\}, \varnothing \to \{*\}\},$$

which is clearly not a covering sieve for $\mathcal{M}$.

5. Model category structures on presheaves of spectra on the site R_G^+

By Theorems 3.1 and 4.1, we know how to make the category R_G^+ into a site via pretopologies $\mathcal{K}$ and $\mathcal{M}$, respectively. Thus, in this section, we consider model category structures on the category of presheaves of spectra on the site (R_G^+, K), where K is any pretopology on R_G^+.

Let $\mathbf{Ab}$ be the category of abelian groups, and let Spt denote the model category of Bousfield-Friedlander spectra of pointed simplicial sets. We refer to the objects of Spt as simply "spectra." Also, we let $\mathrm{PreSpt}(R_G^+, K)$ be the category of presheaves of spectra on the site (R_G^+, K).

Now we give some interesting examples of objects in $\mathrm{PreSpt}(R_G^+, K)$.

5.1. Example. In the Introduction, we saw that the Devinatz-Hopkins functor $\mathbf{F}$ is an object in $\mathrm{PreSpt}(R_{G_n}^+, K)$.

For the next example, if X is a spectrum, then, for each $k \geq 0$, we let X_k be the kth pointed simplicial set constituting X, and, for each $l \geq 0$, $X_{k,l}$ is the pointed set of l-simplices of X_k.

5.2. Example. Let X be a discrete G-spectrum (see [Dav06b] for a definition of this term), so that each $X_{k,l}$ is a pointed discrete G-set. If $C \in R_G^+$, then let $\mathrm{Hom}_G(C, X)$ be the spectrum, such that

$$\mathrm{Hom}_G(C, X)_k = \mathrm{Hom}_G(C, X_k),$$

where

$$\mathrm{Hom}_G(C, X)_{k,l} = \mathrm{Hom}_G(C, X_k)_l = \mathrm{Hom}_G(C, X_{k,l}).$$

Above, the set $X_{k,l}$ is given the discrete topology, since it is naturally a discrete G-set. Then $\mathrm{Hom}_G(-, X)$ is an object in $\mathrm{PreSpt}(R_G^+, K)$. It is easy to see that if U is an open subgroup of G, then $\mathrm{Hom}_G(G/U, X) \cong X^U$, the U-fixed point spectrum of X. Also, $\mathrm{Hom}_G(G, X) \cong X$.

Now we recall [Dav06a, Lemma 3.1], since this result (and its corollary below (see [Dav06a, (3.3)])) will be helpful in our next example. We note that this result is only a slight extension of [Jar97, Remark 6.26]: if U is normal in G, then the lemma below is an immediate consequence of Jardine's remark.

5.3. Lemma. *Let X be a discrete G-spectrum. Also, let $f\colon X \to X_{f,G}$ be a trivial cofibration, such that $X_{f,G}$ is fibrant, where all this takes place in the model category of discrete G-spectra* (*see* [Dav06b]). *If U is an open subgroup of G, then $X_{f,G}$ is fibrant in the model category of discrete U-spectra.*

5.4. Corollary. *Let X and U be as in the preceding lemma. Then $X^{hU} = (X_{f,G})^U$.*

Proof. Let f be as in the above lemma. Since f is G-equivariant, it is U-equivariant. Also, since f is a trivial cofibration in the model category of discrete G-spectra, it is a trivial cofibration in the model category of spectra. The preceding two facts imply that f is a trivial cofibration in the model category of discrete U-spectra. By the lemma, $X_{f,G}$ is fibrant in this model category. Thus, $X^{hU} = (X_{f,G})^U$. ∎

5.5. Example. Let X be a discrete G-spectrum. Then $\mathrm{Hom}_G(-, X_{f,G})$ is a presheaf in $\mathrm{PreSpt}(R_G^+, K)$. In particular, notice that

$$\mathrm{Hom}_G(G/U, X_{f,G}) \cong (X_{f,G})^U = X^{hU}$$

and

$$\mathrm{Hom}_G(G, X_{f,G}) \cong X_{f,G} \simeq X.$$

5.6. Example. For any unfamiliar concepts in this example, we refer the reader to [Dav06b]. Let $Z = \mathrm{holim}_i Z_i$ be a continuous G-spectrum, so that $\{Z_i\}_{i \geq 0}$ is a tower of discrete G-spectra, such that each Z_i is a fibrant spectrum. Then

$$P(-) = \operatorname*{holim}_i \mathrm{Hom}_G(-, (Z_i)_{f,G}) \in \mathrm{PreSpt}(R_G^+, K),$$

where

$$P(G/U) \cong \operatorname*{holim}_i ((Z_i)_{f,G})^U = \operatorname*{holim}_i (Z_i)^{hU} = Z^{hU}$$

and

$$P(G) \cong \operatorname*{holim}_i (Z_i)_{f,G} \simeq Z.$$

The above examples show that the category $\mathrm{PreSpt}(R_G^+, K)$ gives an efficient way of organizing homotopy fixed point data for discrete and continuous G-spectra. Also, notice that when X is a discrete G-spectrum, Example 5.5 shows that the presheaf $\mathrm{Hom}_G(-, X_{f,G})$ has X as its G-sections (up to weak equivalence), whereas a theorem is required to show if and when the presheaf of spectra $\mathrm{Hom}_G(-, X_{f,G})$ on the site G–$\mathbf{Sets}_{df}$ has X as sections (up to weak equivalence). This observation is a minor organizational advantage of the category $\mathrm{PreSpt}(R_G^+, K)$ over the category of presheaves of spectra on $G-\mathbf{Sets}_{df}$.

Now we equip $\mathrm{PreSpt}(R_G^+, K)$ with a model category structure M_c; let $\mathrm{PreSpt}^c(R_G^+, K)$ be the resulting model category. Notice that since Examples 5.1, 5.2, 5.5, and 5.6 are independent of the model category structure, $\mathrm{PreSpt}^c(R_G^+, K)$ is always useful for organizing homotopy fixed points "in the G-world." However, it is natural to wonder if there is a particular model category $\mathrm{PreSpt}^c(R_G^+, K)$ that can provide some additional utility with regard to homotopy fixed points. Though we have not succeeded in finding a good answer to this question, we are able to make a few observations in the following discussion.

First, we briefly review the more important model structures M_c that are available. In making the recollections that follow, we were aided by the helpful exposition in [CHSW08, pp. 560–561], and any facts or results in the remainder of this section that are stated without citation or justification can be found in these pages.

Our review begins with some definitions. Let $f\colon P \to Q$ be a morphism of presheaves of spectra on R_G^+. Then f is an *objectwise weak equivalence* (*objectwise cofibration, objectwise fibration*) if $f(C)$ is a weak equivalence (cofibration, fibration) of spectra, for all $C \in R_G^+$. Then M_{in}, the injective model structure, is the model category structure where the weak equivalences and cofibrations are the objectwise weak equivalences and the objectwise cofibrations, respectively (see [Jar87], [Jar97, Remark 2.36]).

5.7. DEFINITION. *Let $P\colon (R_G^+)^{\mathrm{op}} \to \mathrm{Spt}$ be a presheaf of spectra. Then, for each $n \in \mathbb{Z}$,*

$$\pi_n(P)\colon (R_G^+)^{\mathrm{op}} \to \mathbf{Ab}, \quad C \mapsto \pi_n(P(C)),$$

is a presheaf of abelian groups. Then the associated sheaf $\tilde{\pi}_n(P)$ of abelian groups is the sheafification of $\pi_n(P)$.

We say that the morphism f of presheaves is a *local stable equivalence* if the induced map $\tilde{\pi}_n(P) \to \tilde{\pi}_n(Q)$ of sheaves is an isomorphism, for all $n \in \mathbb{Z}$. Then $\mathrm{M}_{L(in)}$, the local injective model structure, is defined by taking the weak equivalences and cofibrations to be the local stable equivalences and objectwise cofibrations, respectively (see [Jar87], [Jar97, Theorem 2.34]). In $\mathrm{M}_{L(in)}$, a fibration is referred to as a *global fibration.*

Additionally, M_{pr}, the projective model structure, is given by taking the weak equivalences and fibrations to be the objectwise weak equivalences and objectwise fibrations, respectively. In M_{pr}, each cofibration (that is, each *projective cofibration*) is an objectwise cofibration. Also, there is $\mathrm{M}_{L(pr)}$, the local projective model structure, in which the weak equivalences are the local stable equivalences and the cofibrations are exactly the projective cofibrations (see [Bla01, Section 1] for an explanation of the terminology).

As alluded to in Lemma 5.3, homotopy fixed point spectra for profinite group actions are defined by using the model category structure on the category of discrete G-spectra that is specified by defining a morphism f to be a weak equivalence (cofibration) if it is a weak equivalence (cofibration) of spectra. As explained in [Dav06b, Section 3], this model category structure is obtained from the local injective model structure on the category $\mathrm{PreSpt}(G\text{–}\mathbf{Sets}_{df})$ (the category of presheaves of spectra on the canonical site $G\text{–}\mathbf{Sets}_{df}$), in which the weak equivalences are the local stable equivalences and the cofibrations are defined objectwise.

Now, since $G\text{–}\mathbf{Sets}_{df}$ is a full subcategory of R_G^+, we can make the following definition.

5.8. Definition. *If P is a presheaf of spectra in $\mathrm{PreSpt}(R_G^+, K)$, let P_{df} be the presheaf in $\mathrm{PreSpt}(G-\mathbf{Sets}_{df})$ that is obtained by the composition*

$$(G-\mathbf{Sets}_{df})^{\mathrm{op}} \xhookrightarrow{i} (R_G^+)^{\mathrm{op}} \xrightarrow{P} \mathrm{Spt},$$

where i is the inclusion. Similarly, if f is a morphism in $\mathrm{PreSpt}(R_G^+, K)$, let f_{df} be the induced morphism in $\mathrm{PreSpt}(G-\mathbf{Sets}_{df})$. Thus, there is the restriction functor

$$i_*\colon \mathrm{PreSpt}(R_G^+, K) \to \mathrm{PreSpt}(G-\mathbf{Sets}_{df}), \quad P \mapsto i_*(P) = P_{df}.$$

Now we make a few observations about the above model category structures and i_*. As before, let f be a morphism in $\mathrm{PreSpt}(R_G^+, K)$. It is easy to see that if f is a cofibration in M_{in} or $\mathrm{M}_{L(in)}$, then $i_*(f)$ is a cofibration in $\mathrm{PreSpt}(G-\mathbf{Sets}_{df})$ (equipped with the local injective model category structure described above). Similarly, since any projective cofibration is an objectwise cofibration, if f is a cofibration in M_{pr} or $\mathrm{M}_{L(pr)}$, then $i_*(f)$ is a cofibration in $\mathrm{PreSpt}(G-\mathbf{Sets}_{df})$.

If f is an objectwise weak equivalence, then $i_*(f)$ is also an objectwise weak equivalence. Thus, for each integer n and for any object C in $G-\mathbf{Sets}_{df}$, $\pi_n(f(C))$ is an isomorphism, so that the map $\pi_n(i_*(f))$ of presheaves is an isomorphism. Hence, the associated map $\tilde{\pi}_n(i_*(f))$ of sheaves on $G-\mathbf{Sets}_{df}$ is an isomorphism, implying that $i_*(f)$ is a local stable equivalence. Thus, if f is a weak equivalence in M_{in} or M_{pr}, then $i_*(f)$ is a weak equivalence in $\mathrm{PreSpt}(G-\mathbf{Sets}_{df})$.

Now suppose that f is a weak equivalence between fibrant presheaves in the model category $\mathrm{PreSpt}^{L(in)}(R_G^+, K)$. Since a global fibration is a fibration in $\mathrm{M}_{L(pr)}$, f is also a weak equivalence between fibrant presheaves in $\mathrm{PreSpt}^{L(pr)}(R_G^+, K)$. It follows (see [CHSW08, pg. 561]) that f is an objectwise weak equivalence. Thus, we can conclude that if f is a weak equivalence between globally fibrant presheaves in $\mathrm{PreSpt}^{L(in)}(R_G^+, K)$, then $i_*(f)$ is a weak equivalence in $\mathrm{PreSpt}(G-\mathbf{Sets}_{df})$.

Given the last statement, it is natural to wonder if i_* preserves globally fibrant objects: if P is a fibrant presheaf in $\mathrm{PreSpt}^{L(in)}(R_G^+, K)$, one can ask if $i_*(P)$ is fibrant. Similarly, one can ask if i_* preserves all global fibrations. Since global fibrations in

$\mathrm{PreSpt}^{L(in)}(R_G^+, K)$ are defined to be the maps with the right lifting property with respect to all trivial cofibrations, the definition of local stable equivalence implies that to answer these questions, one needs a good understanding of sheaves on the site (R_G^+, K). In particular, these matters could be considered when K is $\mathcal{K}$ or $\mathcal{M}$. However, we have not succeeded in carrying out such an analysis.

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