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PROTOLOCALISATIONS OF EXACT MAL'CEV CATEGORIES

To Walter Tholen, on his sixtieth birthday

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Abstract. A protolocalisation of a regular category is a full reflective regular subcategory, whose reflection preserves pullbacks of regular epimorphisms along arbitrary morphisms. We devote special attention to the epireflective protolocalisations of an exact Mal'cev category; we characterise them in terms of a corresponding closure operator on equivalence relations. We give some examples in algebra and in topos theory.

Introduction

Consider a full reflective subcategory

$$\iota, \lambda \colon \mathcal{L} \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathcal{C}, \quad \lambda \dashv \iota$$

of a category $\mathcal{C}$. When $\mathcal{C}$ has finite limits and λ preserves them, this situation is called a *localisation*. That notion is very important since being abelian, a topos, regular, exact, homological, semi-abelian, and so on, are notions preserved under localisation. In the abelian case, being a localisation is also equivalent to λ preserving monomorphisms, or kernels, or short exact sequences.

In [4], a *protolocalisation* of a *homological* (see [3]) or a *semi-abelian* category (see [19]) is defined as a full reflective subcategory which is still regular, with a reflection which preserves short exact sequences. The aim of this paper is to extend this notion to the non-pointed case: this is done by requesting that the reflection preserves pullbacks of regular epimorphisms along arbitrary morphisms; this is equivalent to being Barr-exact and preserving pullbacks of split epimorphisms along arbitrary morphisms. We observe in particular that a protolocalisation of a regular category is entirely determined by those monomorphisms which are inverted by the reflection; this is reminiscent of a well-known property of localisations.

We devote some attention to the case of *fibered* protolocalisations: the case where the reflection is a fibration. The fibration requirement is equivalent to the *semi-left exactness* in the sense of [10]. Regularity is inherited by fibered reflections and then the protolocalisation property becomes equivalent to the class of inverted morphisms being stable under

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pullbacks along regular epimorphisms. Fibered protolocalisations are also stable under slicing.

Of course the most straightforward non-pointed version of homological (resp. semi-abelian) categories is the case of regular (resp. exact) *protomodular* categories (see [8]). Such categories are in particular *Mal'cev* categories (see [11]), that is, every reflexive relation is at once an equivalence relation. We have observed that the Mal'cev axiom is in fact sufficient for getting all our results concerning protolocalisations: the stronger axiom of protomodularity can always be avoided.

The case of epireflective protolocalisations of exact Mal'cev categories is worth a special interest. Such a protolocalisation is always *Birkhoff*, that is, the reflection is closed under regular quotients. But more importantly, the protolocalisation is then entirely determined by a corresponding closure operator on the lattices of equivalence relations (see [5]). The essential goal of the present paper is to characterise exactly those closure operators on equivalence relations which correspond to an epireflective protolocalisation.

We provide examples of protolocalisations of exact Mal'cev categories, both in algebra and in topos theory.

When this makes proofs easier, we freely use Barr's metatheorem (see [1]) allowing to develop some specific arguments "elementwise" in a regular category.

1. A review of regular, exact, Mal'cev and Goursat categories

To avoid any ambiguity, let us make clear that as many authors, we define a *regular* category to be a category *with finite limits* and pullback stable images (image = factorisation as a regular epimorphism followed by a monomorphism). A regular category is *exact* when moreover, every equivalence relation is effective, that is, is a kernel pair (see [1]).

Let us first recall a well-known *Barr–Kock theorem* (see [1]).

1.1. THEOREM. *In a regular category, consider a commutative diagram*

$$\begin{tikzcd}
S \arrow[r, shift left, "d_1^S"] \arrow[r, shift right, "d_2^S"'] \arrow[d, "h"'] \arrow[dr, phantom, "(2)"] & A \arrow[r, two heads, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & P \arrow[d, "g"] & {} \arrow[d, phantom, "\text{Diagram (BK)}"] \\
R \arrow[r, shift left, "d_1^R"] \arrow[r, shift right, "d_2^R"'] & B \arrow[r, two heads, "q"] & Q & {}
\end{tikzcd}$$

where both lines are exact forks. Then the square (1) *is a pullback if and only if one of the squares* (2) *is a pullback. In that case, both squares* (2) *are pullbacks.* ■

Let us also recall that an *exact fork* in a regular category is a diagram

$$\begin{tikzcd}
R \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & A \arrow[r, two heads, "q"] & Q
\end{tikzcd}$$

where $q = \mathsf{Coker}\,(u, v)$ is the coequaliser of u and v while (u, v) is the kernel pair of q. A *Barr-exact* functor is one which preserves exact forks.

It is easily observed that, in a regular category, the pullback of two regular epimorphisms is always a pushout:

$$\begin{tikzcd}
A \arrow[r, two heads, "v"] \arrow[d, two heads, "q"'] & B \arrow[d, two heads, "p"] \\
C \arrow[r, two heads, "u"'] & D
\end{tikzcd}$$

Indeed by regularity, the factorisation $R[q] \longrightarrow R[p]$ between kernel pairs is a regular epimorphism, from which it follows that given $f \circ v = g \circ q$, f coequalises the kernel pair of p.

A *Mal'cev* category is a category with finite limits in which every reflexive relation is an equivalence relation. It follows that *exact Mal'cev* categories are characterised among regular categories as those in which every reflexive relation is an effective equivalence relation (see [12]). A further characterisation is given by (see [11]):

1.2. Theorem. *A regular category $\mathcal{C}$ is an exact Mal'cev category if and only if, given two regular epimorphisms $f\colon A \twoheadrightarrow B$ and $g\colon A \twoheadrightarrow C$, the pushout $(Q, \overline{f}, \overline{g})$ of f and g exists, and the comparison arrow $\alpha\colon A \longrightarrow B \times_Q C$ to the pullback of $\overline{g}$ and $\overline{f}$ is a regular epimorphism:*

$$\begin{tikzcd}
A \arrow[rd, dotted, "\alpha"] \arrow[rrd, bend left, two heads, "g"] \arrow[rdd, bend right, two heads, "f"'] & & \\
& B \times_Q C \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads, "\overline{f}"] \\
& B \arrow[r, two heads, "\overline{g}"'] & Q
\end{tikzcd}$$

In general, a pushout of regular epimorphisms having the property that the factorisation to the corresponding pullback is a regular epimorphism is called a *regular pushout.* Let us observe, in particular, that in every category, a commutative diagram of split epimorphisms

$$\begin{tikzcd}
A \arrow[r, two heads, "v"] \arrow[d, shift right, "t"'] & B \arrow[d, shift right, "s"'] \\
C \arrow[r, two heads, "u"'] \arrow[u, shift right, "q"'] & D \arrow[u, shift right, "p"']
\end{tikzcd}$$

is trivially a pushout as soon as v is an epimorphism. In an exact Mal'cev category, when u and v are regular epimorphisms, we obtain thus a regular pushout.

A regular category is a Mal'cev category exactly when it is 2-permutable: given two equivalence relations R and S on the same object A, their relational composites satisfy the equality $R \circ S = S \circ R$. This property is equivalent to the fact that the join $R \vee S$ of any two equivalence relations R and S is given by $R \vee S = R \circ S$.

The property of 3-permutability of the composition of equivalence relations $R \circ S \circ R = S \circ R \circ S$ is known to be strictly weaker than the one of 2-permutability, and that it is this time equivalent to $R \vee S = R \circ S \circ R$. A regular 3-permutable category is called a *Goursat* category (see [11]). Among regular categories, Goursat categories are nicely characterised by the fact that the regular image $f(R)$ of any equivalence relation R on A along any regular epimorphism $f\colon A \twoheadrightarrow B$ is an equivalence relation on B (see [11] again). It is also true that the lattice of equivalence relations on any object is necessarily a modular lattice, a property which no longer holds for the n-permutable categories, when $n \geq 4$.

A variety of universal algebras is a Mal'cev category if and only if its theory has a ternary (possibly derived) term $p(x, y, z)$ satisfying the axioms

$$p(x, x, y) = y, \quad p(x, y, y) = x$$

(see [24]). Any variety whose theory is equipped with a group operation is thus a Mal'cev variety, since in that case it suffices to set $p(x, y, z) = xy^{-1}z$. The variety of Heyting algebras is also a Mal'cev variety (see [21]), so that the dual category of an elementary topos is an exact Mal'cev category (see [11]). Another class of examples of exact Mal'cev categories arises from the compact Hausdorff models $\mathbb{T}(\mathsf{HComp})$ of a Mal'cev theory $\mathbb{T}$.

On the other hand, for a given variety, the Goursat property is equivalent to the existence of two ternary operations $p(x, y, z)$ and $q(x, y, z)$ such that

$$p(x, x, y) = y, \quad q(x, y, y) = x, \quad p(x, y, y) = q(x, x, y),$$

as shown in [18]. Implication algebras form a 3-permutable variety, which is not 2-permutable: recall that these algebras are equipped with a binary operation $\triangleright$ satisfying the identities

$$(x \triangleright y) \triangleright y = (y \triangleright x) \triangleright x, \quad (x \triangleright y) \triangleright x = x, \quad x \triangleright (y \triangleright z) = y \triangleright (x \triangleright z).$$

2. The protolocalisations

Let us now introduce the main notion of this paper.

2.1. Definition. *A* protolocalisation *of a regular category $\mathcal{C}$ is a full reflective subcategory*

$$\iota\colon \mathcal{L} \rightarrowtail \mathcal{C}; \; \lambda\colon \mathcal{C} \longrightarrow \mathcal{L}; \; \lambda \dashv \iota,$$

such that:

1. *$\mathcal{L}$ is still regular;*
2. *λ preserves the pullback of a regular epimorphism along an arbitrary morphism.*

In the abelian case, given a full reflective subcategory $\iota, \lambda\colon \mathcal{L}\leftrightarrows\mathcal{C}$, λ being Barr-exact is equivalent by additivity to λ being exact, that is, preserving short exact sequences; this is further equivalent to λ preserving all finite limits. Such a situation is called a localisation. But already in the semi-abelian case these equivalences no longer hold: preserving short exact sequences is stronger than being Barr-exact and weaker than preserving finite limits (see [4]). Our Proposititon 2.2 measures the gap between these two definitions of exactness. This statement is of course highly reminiscent of the definition of a protomodular category (see [8]).

2.2. Proposition. *Let $\iota, \lambda\colon \mathcal{L}\leftrightarrows\mathcal{C}$ be a full reflective and regular subcategory of a regular category $\mathcal{C}$. The following conditions are equivalent:*

1. *λ is a protolocalisation;*
2. *the following two properties hold:*
   (a) *λ is Barr-exact;*
   (b) *λ preserves the pullback of a split epimorphism along an arbitrary morphism.*

Proof. (1 ⇒ 2) is trivial. Conversely, consider Diagram (BK) where the square (1) is a pullback. All four morphisms d_i^S and d_i^R are epimorphisms split by the diagonal. By assumption, exact forks and both pullbacks (2) are preserved by λ. Theorem 1.1 implies that $\lambda(1)$ is a pullback as well. ■

2.3. Corollary. *Let $\mathcal{C}$ be a homological category. The two notions of protolocalisation in Definition 2.1 of this paper and in Definition 17 of [4] are equivalent.*

Proof. Proposition 19 in [4] shows that the definition in that paper implies our Definition 2.1. Conversely in the pointed case, our Proposition 2.2 forces the preservation of the kernel of a regular epimorphism, since this kernel is given by the pullback over the zero object. ■

Let us also mention that

2.4. Proposition. *A protolocalisation of an exact category is still exact.*

Proof. This follows from Theorem 7 in [4]. ■

Let us next recall that an object $L \in \mathcal{C}$ is *orthogonal* to a morphism $h \in \mathcal{C}$ when for every morphism f as in the following diagram

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[dl, dotted, "g"] \\
L &
\end{tikzcd}$$

there exists a unique morphism g making the triangle commutative.

The following result is well-known in the case of a localisation (see [10]). Our more general version applies clearly to the case of a protolocalisation of a regular category.

2.5. Proposition. *Let $\iota, \lambda \colon \mathcal{L} \overset{\longleftarrow}{\longrightarrow} \mathcal{C}$ be a full reflective subcategory of a regular category $\mathcal{C}$. When the reflection λ is Barr-exact, the full subcategory $\mathcal{L}$ is that of those objects of $\mathcal{C}$ orthogonal to the monomorphisms inverted by λ.*

Proof. It is well-known (see [15]) that being in $\mathcal{L}$ is equivalent to being orthogonal to all morphisms h inverted by λ, or simply orthogonal to each $\eta_A \colon A \longrightarrow \iota\lambda(A)$, the unit of the adjunction, for all $A \in \mathcal{C}$. This proves already that each $L \in \mathcal{L}$ is orthogonal to every monomorphism inverted by λ.

Conversely, assume that L is orthogonal to every monomorphism inverted by λ; we shall prove that L is orthogonal to each η_A. For that we consider the following diagram

$$\begin{tikzcd}
A \arrow[r, tail, "\delta_A"] & R_A \arrow[r, shift left, "u_A"] \arrow[r, shift right, "v_A"'] & A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, two heads, "p_A"] & \iota\lambda(A) \arrow[dl, dotted, "h"'] \\
 & & L & S_A \arrow[l, dotted, "g"] \arrow[u, tail, "s_A"']
\end{tikzcd}$$

where $\eta_A = s_A \circ p_A$ is the image factorisation of η_A, (u_A, v_A) is the kernel pair of p_A and δ_A is the diagonal of this kernel pair.

Since $\lambda(\eta_A)$ is an isomorphism, the regular epimorphism $\lambda(p_A)$ is also a monomorphism, thus an isomorphism. Thus $\lambda(s_A)$ is an isomorphism as well. On the other hand the reflection λ preserves the exact fork $(u_A, v_A; p_A)$. Since $\lambda(p_A)$ is an isomorphism, so are thus $\lambda(u_A)$ and $\lambda(v_A)$, but then also $\lambda(\delta_A)$, which is their right inverse.

Now let $L \in \mathcal{C}$ be orthogonal to every monomorphism inverted by λ. Consider a morphism $f \colon A \longrightarrow L$. Since $f \circ u_A \circ \delta_A = f = f \circ v_A \circ \delta_A$, we obtain $f \circ u_A = f \circ v_A$ by the uniqueness part of the orthogonality condition $\delta_A \perp L$. But $p_A = \mathsf{Coker}\,(u_A, v_A)$, from which there is a unique factorisation $g \colon S_A \longrightarrow L$ such that $g \circ p_A = f$. The orthogonality condition $s_A \perp L$ forces finally the existence of a unique morphism $h \colon \iota\lambda(A) \longrightarrow L$ such that $h \circ s_A = g$. ■

3. The fibered case

Given a full reflective subcategory $\iota, \lambda\colon \mathcal{L}\overset{\hookleftarrow}{\longrightarrow}\mathcal{C}$ of a category $\mathcal{C}$ with finite limits, let us write $\mathcal{E}$ for the class of those morphisms of $\mathcal{C}$ inverted by λ and $\mathcal{M}$ for the class of those morphisms of $\mathcal{C}$ *orthogonal* to all morphisms in $\mathcal{E}$. This means that $m \in \mathcal{M}$ when in every commutative square with $e \in \mathcal{E}$

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d] & \bullet \arrow[d] \arrow[ld, dotted] \\
\bullet \arrow[r, "m"'] & \bullet
\end{tikzcd}$$

there exists a unique diagonal making both triangles commutative. The pair $(\mathcal{E}, \mathcal{M})$ is a *prefactorisation system*; it is a factorisation system when moreover each arrow $f \in \mathcal{C}$ factors (necessarily uniquely) as $f = m \circ e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ (see [10]).

3.1. Definition. *By a fibered reflection of a category $\mathcal{C}$ with finite limits is meant a full reflective subcategory $\iota, \lambda\colon \mathcal{L}\overset{\hookleftarrow}{\longrightarrow}\mathcal{C}$ such that λ is a fibration (see [7, 9]).*

3.2. Proposition. *Consider a full reflective subcategory $\iota, \lambda\colon \mathcal{L}\overset{\hookleftarrow}{\longrightarrow}\mathcal{C}$ of a category $\mathcal{C}$ with finite limits. The following conditions are equivalent:*

1. *the reflection is fibered;*
2. *the reflection is semi-left-exact in the sense of [10].*

In these conditions, the corresponding prefactorisation system $(\mathcal{E}, \mathcal{M})$ is a factorisation system and λ preserves the pullbacks along morphisms of $\mathcal{M}$.

Proof. The proof of Proposition 36 in [4] applies without any change to prove the equivalence. The rest follows from [10]. ■

As a consequence (see [22]):

3.3. Corollary. *A fibered reflection of a regular category is still regular.*

Proof. By Theorem 4 in [4], it suffices to prove that λ preserves "some" pullbacks along the morphisms of $\mathcal{L}$. Since all morphisms in $\mathcal{L}$ are in particular in $\mathcal{M}$ (see [10]), the result follows at once from Proposition 3.2. ■

3.4. Proposition. *Let $\iota, \lambda\colon \mathcal{L}\overset{\hookleftarrow}{\longrightarrow}\mathcal{C}$ be a fibered reflection of a regular category $\mathcal{C}$. The following conditions are equivalent:*

1. *the reflection is a protolocalisation;*
2. *the class $\mathcal{E}$ of those morphisms inverted by λ is stable under pullbacks along regular epimorphisms.*

Proof. (1 ⇒ 2) simply because λ preserves pullbacks along regular epimorphisms. Conversely, by Corollary 3.3, we know already that $\mathcal{L}$ is regular. Next consider the following diagram

$$\begin{tikzcd}
A \arrow[r, "e'"] \arrow[d, two heads, "h'"'] \arrow[rr, bend left, "f'"] & C \arrow[r, "m'"] \arrow[d, two heads, "h''"'] & B \arrow[d, two heads, "h"] \\
X \arrow[r, "e"'] \arrow[rr, bend right, "f"'] & Z \arrow[r, "m"'] & Y
\end{tikzcd}$$

where the outer part is a pullback, with h a regular epimorphism. We split this pullback in two pieces along the $(\mathcal{E},\mathcal{M})$-factorisation $f = m \circ e$ of f. The reflection λ preserves the left hand pullback by assumption and the right hand pullback by Proposition 3.2. ■

Let us also observe that fibered protolocalisations are stable under slicing:

3.5. Proposition. *Every fibered protolocalisation* $\iota,\lambda\colon \mathcal{L}\overset{\longleftarrow}{\longrightarrow}\mathcal{C}$ *of a regular category* $\mathcal{C}$ *induces, for every object* $I \in \mathcal{C}$*, a fibered protolocalisation*

$$\iota_I,\lambda_I\colon \mathcal{L}/\lambda(I)\overset{\longleftarrow}{\longrightarrow}\mathcal{C}/I.$$

Proof. The existence of the adjunction $\lambda_I \dashv \iota_I$ is well-known (see [6], Lemma 4.3.4). Let us simply recall that

$$\lambda_I\big(C\xrightarrow{\ f\ }I\big) = \big(\lambda(C)\xrightarrow{\lambda(f)}\lambda(I)\big)$$

while $\iota_I\big(L\xrightarrow{g}\lambda(I)\big) = \big(L'\xrightarrow{g'}I\big)$ is obtained via the pullback

$$\begin{tikzcd}
L' \arrow[r] \arrow[d, "g'"'] & L \arrow[d, "g"] \\
I \arrow[r, "\eta_I"'] & \lambda(I)
\end{tikzcd}$$

where η_I is the unit of the adjunction $\lambda \dashv \iota$.

By semi-left exactness of λ (see Proposition 3.2), the upper morphism in this square is itself a unit (see [10]), thus is isomorphic to $\eta_{L'}$. But this forces

$$\lambda_I\iota_I\big(L\xrightarrow{\ g\ }\lambda(I)\big) \cong \lambda_I\big(L'\xrightarrow{\ g'\ }I\big) \cong \big(L\xrightarrow{\ g\ }\lambda(I)\big)$$

proving that the counit of the adjunction is an isomorphism. Therefore ι_I is full and faithful.

The slice categories of a regular category are still regular and both pullbacks and regular epimorphisms are computed in the slice category as in the original category. Therefore λ_I preserves pullbacks along regular epimorphisms, since so does λ. ■

4. The case of epireflections

By an *epireflection* of a *regular* category we mean a reflection having *regular* epimorphic units. Let us recall at once that an epireflection of a regular category is itself a regular category (see [4], Example 11). But moreover:

4.1. Proposition. *Every epireflective protolocalisation of a regular category is fibered.*

Proof. The pullback of a unit of the adjunction is now the pullback of a regular epimorphism, thus is preserved by the reflection. Therefore the reflection is semi-left exact (see [10]) and thus fibered (see Proposition 3.2). ■

4.2. Proposition. *Given an epireflection* $\iota, \lambda \colon \mathcal{L} \overset{\longleftarrow}{\longrightarrow} \mathcal{C}$ *of a regular category* $\mathcal{C}$*, the inclusion functor* $\iota \colon \mathcal{L} \longrightarrow \mathcal{C}$ *is Barr-exact.*

Proof. As a right adjoint, the functor ι preserves kernel pairs. The image factorisation in $\mathcal{C}$ of a regular epimorphism of $\mathcal{L}$ lies entirely in $\mathcal{L}$, since an epireflection is closed under subobjects. Thus the mono-part of the factorisation is an isomorphism and ι preserves regular epimorphisms. ■

4.3. Proposition. *Given an epireflective protolocalisation* $\iota, \lambda \colon \mathcal{L} \overset{\longleftarrow}{\longrightarrow} \mathcal{C}$ *of a regular category* $\mathcal{C}$*,*

1. λ *fixes the terminal object* $\mathbf{1}$*;*
2. λ *fixes the subobjects of* $\mathbf{1}$*;*
3. λ *preserves finite products.*

Proof. First of all, ι preserves the terminal object, proving that the terminal object of $\mathcal{C}$ lies in $\mathcal{L}$. Since we have an epireflection and $\mathbf{1} \in \mathcal{L}$, every subobject of $\mathbf{1}$ is also in $\mathcal{L}$.

The product of two objects is their pullback over $\mathbf{1}$. Via the image factorisations of the morphisms to $\mathbf{1}$, this pullback can be split in four pullback pieces

$$\begin{tikzcd}
A \times B \arrow[r, two heads] \arrow[d, two heads] & \bullet \arrow[r, tail] \arrow[d, two heads] & A \arrow[d, two heads] \\
\bullet \arrow[r, two heads] \arrow[d, tail] & \bullet \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "(*)"] & \bullet \arrow[d, tail] \\
B \arrow[r, two heads] & \bullet \arrow[r, tail] & \mathbf{1}
\end{tikzcd}$$

Each partial pullback is preserved by λ: $(*)$ because it is fixed by λ and the other ones by definition of a protolocalisation. ■

Let us also recall that

4.4. Definition. *An epireflection $\iota, \lambda \colon \mathcal{L}\overset{\longleftarrow}{\longrightarrow}\mathcal{C}$ of a regular category $\mathcal{C}$ is* Birkhoff *when $\mathcal{L}$ is stable in $\mathcal{C}$ under regular quotients.*

In the case of algebraic theories, the Birkhoff epireflections are those obtained by adding axioms to a theory.

4.5. Proposition. *An epireflective protolocalisation $\iota, \lambda \colon \mathcal{L}\overset{\longleftarrow}{\longrightarrow}\mathcal{C}$ of a regular category $\mathcal{C}$ is necessarily Birkhoff.*

Proof. Consider a regular epimorphism $q \colon L \twoheadrightarrow C$ in $\mathcal{C}$, with $L \in \mathcal{L}$. Its kernel pair (u, v), yields the diagram

$$\begin{tikzcd}
R \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, two heads, "\eta_R"'] & L \arrow[r, two heads, "q"] \arrow[d, "\eta_L"', "\cong"] & C \arrow[d, two heads, "\eta_C"] \\
\iota\lambda(R) \arrow[r, shift left, "\iota\lambda(u)"] \arrow[r, shift right, "\iota\lambda(v)"'] & \iota\lambda(L) \arrow[r, two heads, "\iota\lambda(q)"'] & \iota\lambda(C)
\end{tikzcd}$$

The pair (u, v) is monomorphic and η_L is an isomorphism, thus η_R is a monomorphism and therefore an isomorphism.

Since ι preserves kernel pairs, the bottom line remains a kernel pair, by the protolocalisation axiom. But $\iota\lambda(q)$ is a regular epimorphism in $\mathcal{C}$, since so are q and η_C. Thus both lines are exact forks and η_C is an isomorphism, since so are η_R and η_L. ■

5. The case of exact Mal'cev categories

We want now to study the protolocalisations in terms of closure operators on equivalence relations. For this we need to switch first to the *exact Goursat* case and finally to the *exact Mal'cev case.* The following theorem is proved in [5].

5.1. Theorem. *For an exact Goursat category $\mathcal{C}$, there is a bijection between:*

- *the Birkhoff epireflections $\iota, \lambda \colon \mathcal{L}\overset{\longleftarrow}{\longrightarrow}\mathcal{C}$;*
- *the operators associating, with every equivalence relation S on an object B, another equivalence relation $\overline{S}$ on B, and satisfying the following properties for S, T equivalence relations on B, $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow C$ morphisms of $\mathcal{C}$:*

[CL1] $S \subseteq \overline{S}$;

[CL2] $\overline{\overline{S}} = \overline{S}$;

[CL3] $S \subseteq T$ *implies* $\overline{S} \subseteq \overline{T}$;

[CL4] $\overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S})$;

[CL5] $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$ *when f is a regular epimorphism;*

[CL6] $g(\overline{S}) = \overline{g(S)}$ *when g is a regular epimorphism.*

Via this correspondence, the reflection of an object $B \in \mathcal{C}$ is given by

$$\eta_B \colon B \twoheadrightarrow B/\overline{\Delta_B}.$$

The closure of an equivalence relation S on B is given by

$$q_S^{-1}\big(R[\eta_{B/S}]\big) = R\big[\eta_{B/S} \circ q_S\big]$$

where $q_S \colon B \twoheadrightarrow B/S$ is the quotient map and $R[\bullet]$ indicates a kernel pair. This closure is also equal to

$$\overline{S} = S \vee \overline{\Delta_B}$$

where $\vee$ indicates the supremum in the lattice of equivalence relations on B and Δ_B indicates the diagonal. ■

With the terminology of [5], axioms [CL1], [CL3] and [CL4] are those for a *closure operator* on equivalence relations. In this statement, we have used freely the classical notation $f^{-1}(R)$ to indicate $(f \times f)^{-1}(R)$ and analogously, $g(R)$ for the direct image of R along the regular epimorphism $g \times g$. Notice that the inverse image of an equivalence relation is always an equivalence relation, while the direct image along a regular epimorphism is an equivalence relation precisely when the Goursat axiom holds (see [16]).

Our purpose is now to investigate the form of the additional axiom which will force a Birkhoff epireflection to become a protolocalisation. Our first characterisation theorem is:

5.2. THEOREM. *Consider a Birkhoff epireflection $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ of an exact Goursat category $\mathcal{C}$. The following conditions are equivalent:*

1. *the reflection λ is Barr-exact;*
2. *for every equivalence relation $r \colon R \rightarrowtail A \times A$ in $\mathcal{C}$, one has $\overline{\Delta_R} = r^{-1}\big(\overline{\Delta_A} \times \overline{\Delta_A}\big)$.*

Moreover, in these conditions, the reflection preserves finite powers.

PROOF. Consider the diagram

$$\begin{tikzcd}
\overline{\Delta_R} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, dotted] & R \arrow[r, two heads, "\eta_R"] \arrow[d, tail, "r = (d_1, d_2)"] & \iota\lambda(R) \arrow[d, dotted, "s = \big(\iota\lambda(d_1), \iota\lambda(d_2)\big)"] \\
\overline{\Delta_A} \times \overline{\Delta_A} \arrow[r, shift left] \arrow[r, shift right] & A \times A \arrow[r, two heads, "\eta_A \times \eta_A"'] & \iota\lambda(A) \times \iota\lambda(A)
\end{tikzcd}$$

where both lines are exact forks. An easy chase on this diagram shows that s is a monomorphism if and only if $\overline{\Delta_R} = r^{-1}\big(\overline{\Delta_A} \times \overline{\Delta_A}\big)$.

If the reflection preserves exact forks, $s = \big(\iota\lambda(d_1), \iota\lambda(d_2)\big)$ is a kernel pair, thus a monomorphism. We have seen that this forces condition 2. Conversely as already observed, condition 2 forces $s = \big(\iota\lambda(d_1), \iota\lambda(d_2)\big)$ to be a monomorphism. But since η_R is a regular epimorphism, the subobject $\iota\lambda(R) = \eta_R(R)$ is an equivalence relation by the Goursat axiom; further, by exactness, it is thus a kernel pair. So λ preserves the fact of being a kernel pair. Since λ, as a left adjoint, preserves also the coequaliser of a kernel pair, it is Barr-exact.

When these equivalent conditions are satisfied, choose $R = A \times A$ in the argument above. This forces $\overline{\Delta_{A\times A}} = \overline{\Delta_A} \times \overline{\Delta_A}$ and thus $\iota\lambda(A \times A) \cong \iota\lambda(A) \times \iota\lambda(A)$. ■

As a consequence, the closure operation can also be described by the well-known construction valid in the case of a localisation:

5.3. COROLLARY. *Consider a Barr-exact Birkhoff epireflection* $\iota, \lambda\colon \mathcal{L} \leftrightarrows \mathcal{C}$ *of an exact Goursat category* $\mathcal{C}$*. The closure of an equivalence relation* R *on* B *is also given by the following pullback:*

$$\begin{tikzcd}
\overline{R} \arrow[r, two heads] \arrow[d, tail, "r"'] & \iota\lambda(R) \arrow[d, tail, "\iota\lambda(r)"] \\
B \times B \arrow[r, two heads, "\eta_B \times \eta_B"'] & \iota\lambda(B) \times \iota\lambda(B)
\end{tikzcd}$$

PROOF. By Theorem 5.2 the statement makes sense. By Proposition 4.2, the composite $\iota\lambda$ is still Barr-exact. Applying it to the exact fork

$$\begin{tikzcd}
R \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & B \arrow[r, two heads, "q_R"] & B/R
\end{tikzcd}$$

we get that $\iota\lambda(R)$ is the kernel pair of $\iota\lambda(q_R)$. Thus the pullback in the diagram yields the kernel pair of $\iota\lambda(q_R) \circ \eta_B$, which is precisely $\overline{R}$. ■

Our next result emphasises the role of a property which is well-known for universal closure operators.

5.4. THEOREM. *Consider a Birkhoff epireflection* $\iota, \lambda\colon \mathcal{L} \leftrightarrows \mathcal{C}$ *of an exact Mal'cev category* $\mathcal{C}$*. The following conditions are equivalent:*

1. λ *preserves the pullback of two regular epimorphisms;*
2. *given two equivalence relations* R, S *on an object* $B \in \mathcal{C}$, $\overline{R \wedge S} = \overline{R} \wedge \overline{S}$.

Of course in these conditions, λ *is Barr-exact.*

Proof. ($1 \Rightarrow 2$). By exactness, the lattice of equivalence relations on B is isomorphic to the lattice of regular quotients of B. Moreover, writing q_T for the quotient of B by an equivalence relation T, one has always the following diagram of regular epimorphisms

$$\begin{tikzcd}
B \arrow[dr, two heads, "q_{R\wedge S}"] \arrow[drr, two heads, "q_S"] \arrow[ddr, two heads, "q_R"'] & & \\
& B/(R\wedge S) \arrow[r, two heads, "p_S"'] \arrow[d, two heads, "p_R"] \arrow[dr, phantom, "(*)"] & B/S \arrow[d, two heads, "t_S"] \\
& B/R \arrow[r, two heads, "t_R"'] & B/(R\vee S)
\end{tikzcd}$$

where the outside part is trivially a pushout. By Theorem 1.2, the factorisation of this pushout through the pullback of t_R and t_S is still a regular epimorphism: then trivially, this pullback must be $B/(R\wedge S)$. In other words, in the diagram above, the square $(*)$ is a pullback.

The pullback $(*)$ is preserved by the reflection λ, since it is constituted of regular epimorphisms. We obtain so a new commutative diagram, where η indicates the unit of the adjunction and the square is still a pullback.

$$\begin{tikzcd}
B \arrow[r, two heads, "\eta_B"] & \iota\lambda(B) \arrow[dr, two heads, "\iota\lambda(q_{R\wedge S})"] \arrow[drr, two heads, "\iota\lambda(q_S)"] \arrow[ddr, two heads, "\iota\lambda(q_R)"'] & & \\
& & \iota\lambda\big(B/(R\wedge S)\big) \arrow[r, two heads, "\iota\lambda(p_S)"'] \arrow[d, two heads, "\iota\lambda(p_R)"] & \iota\lambda\big(B/S\big) \arrow[d, two heads, "\iota\lambda(t_S)"] \\
& & \iota\lambda\big(B/R\big) \arrow[r, two heads, "\iota\lambda(t_R)"'] & \iota\lambda\big(B/(R\vee S)\big)
\end{tikzcd}$$

Since the diagram is commutative and the pair $\big(\iota\lambda(p_R), \iota\lambda(p_S)\big)$ is monomorphic, the kernel pair of $\iota\lambda(q_{R\wedge S}) \circ \eta_B$ is the intersection of the kernel pairs of $\iota\lambda(q_R) \circ \eta_B$ and $\iota\lambda(q_S) \circ \eta_B$. This means precisely $\overline{R\wedge S} = \overline{R} \wedge \overline{S}$.

($2 \Rightarrow 1$). Consider the pullback $(\star)$ of two regular epimorphisms f, g in $\mathcal{C}$ and its image by $\iota\lambda$.

$$\begin{tikzcd}
B \arrow[r, two heads, "a"] \arrow[d, two heads, "b"'] \arrow[dr, phantom, "(\star)"] & A \arrow[d, two heads, "g"] & & \iota\lambda(B) \arrow[r, two heads, "\iota\lambda a"] \arrow[d, two heads, "\iota\lambda(b)"'] \arrow[dr, phantom, "\iota\lambda(\star)"] & \iota\lambda(A) \arrow[d, two heads, "\iota\lambda(g)"] \\
D \arrow[r, two heads, "f"'] & C & & \iota\lambda(D) \arrow[r, two heads, "\iota\lambda(f)"'] & \iota\lambda(C)
\end{tikzcd}$$

In a regular category, the pullback of two epimorphisms is also a pushout, proving that $(\star)$ and thus $\lambda(\star)$ are pushouts. The pushout in $\mathcal{C}$ of $\iota\lambda(a)$ and $\iota\lambda(b)$ is a quotient of $\iota\lambda(A)$; by the Birkhoff axiom, it lies in $\mathcal{L}$. Thus the pushout in $\mathcal{C}$ is also the pushout in $\mathcal{L}$. In other words, $\iota\lambda(\star)$ is a pushout in $\mathcal{C}$.

But in the exact Mal'cev category $\mathcal{C}$, Theorem 1.2 implies that the factorisation

$$\alpha\colon \iota\lambda(B) \twoheadrightarrow \iota\lambda(A) \times_{\iota\lambda(C)} \iota\lambda(D)$$

through the pullback is a regular epimorphism. If we prove that α is an isomorphism, then $\iota\lambda(\star)$ will be a pullback, thus $\lambda(\star)$ as well. Of course, it suffices to prove that α is a monomorphism.

Writing $R[\bullet]$ to indicate a kernel pair, we must prove that $R[\alpha]$ is the diagonal. Since η_B is a regular epimorphism, this is equivalent to prove that $R[\alpha\circ\eta_B] = R[\eta_B]$. By definition of a pullback, $R[a]\wedge R[b] = \Delta_B$, thus

$$R[\eta_B] = \overline{\Delta_B} = \overline{R[a]\wedge R[b]}.$$

On the other hand, considering analogously the monomorphic pair constituted of the two projections of the pullback $\iota\lambda(A)\times_{\iota\lambda(C)}\iota\lambda(D)$, we get

$$R[\alpha\circ\eta_B] = R[\iota\lambda(a)\circ\eta_B]\wedge R[\iota\lambda(b)\circ\eta_B] = R[\eta_A\circ a]\wedge R[\eta_D\circ b] = \overline{R[a]}\wedge\overline{R[b]}.$$

By assumption, this yields the result. ■

Let us recall that a category is *arithmetical* when it is exact, Mal'cev and its lattices of equivalence relations are distributive (see [23]).

5.5. Corollary. *Consider a Birkhoff epireflection* $\iota,\lambda\colon \mathcal{L}\overset{\longleftarrow}{\longrightarrow}\mathcal{C}$ *of an arithmetical category* $\mathcal{C}$*. The functor* λ *always preserves the pullback of two regular epimorphisms.*

Proof. It is proved in [5], Proposition 3.13, that under our assumptions $f\big(\overline{R\wedge S}\big) = \overline{f(R)}\wedge\overline{f(S)}$ for every regular epimorphism f. One concludes by choosing f to be the identity and applying Theorem 5.4. ■

Let us now introduce a notion which will prove to be an adequate substitute for the notion of normal subobject, in the absence of a zero object.

5.6. Definition. *In a category* $\mathcal{C}$ *with finite limits, a monomorphism* $f\colon A\rightarrowtail B$ *is saturated for an equivalence relation* R *on* B *when the following diagram is a pullback*

$$\begin{tikzcd} f^{-1}(R) \arrow[r, tail] \arrow[d, two heads, "d_1"'] & R \arrow[d, two heads, "d_1"] \\ A \arrow[r, tail, "f"'] & B \end{tikzcd}$$

Written elementwise, this condition means

$$a \in A,\ (a, b) \in R \quad \Rightarrow \quad b \in A.$$

Of course the "elementwise" description is exactly the usual notion of a subset saturated for an equivalence relation. Notice also that in the pointed case, the equivalence class of $0 \in A$ is always saturated: thus kernel subobjects are special instances of saturated subobjects.

We have then:

5.7. Theorem. *Let $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ be a Birkhoff epireflection of an exact Mal'cev category. The following conditions are equivalent:*

1. *λ is a protolocalisation;*
2. *the corresponding closure operator satisfies the additional axiom:*

[CL7] $\overline{f^{-1}(R)} \wedge f^{-1}(\overline{S}) = \overline{f^{-1}(R \wedge S)}$
for two equivalence relations R, S on B and a subobject $f \colon A \rightarrowtail B$ saturated for R.

Proof. ($1 \Rightarrow 2$). Notice that the inclusion

$$\overline{f^{-1}(R \wedge S)} \subseteq \overline{f^{-1}(R)} \wedge f^{-1}(\overline{S})$$

always holds, simply because

$$\overline{f^{-1}(R \wedge S)} \subseteq \overline{f^{-1}(R)}, \quad \overline{f^{-1}(R \wedge S)} \subseteq \overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S}).$$

For the other inclusion, consider $(a, a') \in \overline{f^{-1}(R)} \wedge f^{-1}(\overline{S})$. We have thus

$$a'' \in A,\ (a, a'') \in R,\ (a'', a') \in \overline{\Delta_A}, \quad b \in B,\ (a, b) \in S,\ (b, a') \in \overline{\Delta_B}.$$

Since $\overline{f^{-1}(R)} \subseteq f^{-1}(\overline{R})$, we have by Theorem 5.4

$$(a, a') \in \overline{R} \wedge \overline{S} = \overline{R \wedge S} = (R \wedge S) \vee \overline{\Delta_B} = (R \wedge S) \circ \overline{\Delta_B} = \overline{\Delta_B} \circ (R \wedge S).$$

This means the existence of

$$c \in B,\ (a, c) \in R \wedge S,\ (c, a') \in \overline{\Delta_B}.$$

By saturation of A for R, we get $c \in A$. But then since both (a'', a) and (a, c) are in R, we obtain

$$(a'', c) \in f^{-1}(R) \wedge f^{-1}(\overline{\Delta_B}) \subseteq \overline{\Delta_A}.$$

And

$$(a, c) \in R \wedge S,\ (c, a'') \in \overline{\Delta_A},\ (a'', a') \in \overline{\Delta_A} \quad \Rightarrow \quad (a, a') \in \overline{f^{-1}(R \wedge S)}.$$

($2 \Rightarrow 1$). Notice first that putting $f = \mathsf{id}_B$ in condition 2, we obtain $\overline{R \wedge S} = \overline{R} \wedge \overline{S}$ for every two equivalence relations on B. Thus Theorem 5.4 applies and it remains to consider the case of pulling back a regular epimorphism along a monomorphism. In Diagram (BK), assume thus that the square (1) is a pullback, with g a monomorphism; we must prove that $\lambda(1)$ is a pullback. Since λ is Barr-exact by Theorem 5.4, this is equivalent to proving that $\iota\lambda(2)$ is a pullback.

Since g is a monomorphism, a chase on Diagram (BK) proves that $S = f^{-1}(R)$. Since the squares (2) are pullbacks, f is saturated for R and the assumption 2 of the statement applies. Let us prove that the square $\iota\lambda(2)$ — let us say, for the index 1 — is a pullback. The Barr-exactness of λ allows us to apply Theorem 5.2. We write f as a canonical inclusion.

An element of $\iota\lambda(S)$ has the form $\big(\eta_A(a), \eta_A(a')\big)$ for some $(a, a') \in S$. It is mapped by $\iota\lambda(d_1)$ on $\eta_A(a)$ and by $\iota\lambda(h)$ on $\big(\eta_A(a), \eta_A(a')\big)$. But two elements x, y of an object X are identified in $\iota\lambda(X)$ when $(x, y) \in \overline{\Delta_X}$. And by Theorem 5.2, given an equivalence relation T on X and two pairs $(x, y) \in T$, $(x', y') \in T$,

$$\big((x, y), (x', y')\big) \in \overline{\Delta_T} \quad \Leftrightarrow \quad (x, x') \in \overline{\Delta_X},\ (y, y') \in \overline{\Delta_X}.$$

To prove that $\iota\lambda(2)$ is a pullback, we must first prove that two elements in $\iota\lambda(A)$ and $\iota\lambda(R)$ which agree in $\iota\lambda(B)$ come from an element of $\iota\lambda(S)$. This means

> If $a \in A$, $(b, b') \in R$, $\big(b, a\big) \in \overline{\Delta_B}$,
> there exists $(a', a'') \in S$ such that $(a', a) \in \overline{\Delta_A}$, $\big((a', a''), (b, b')\big) \in \overline{\Delta_R}$.

Observe at once that, by the Mal'cev axiom

$$(a, b) \in \overline{\Delta_B},\ (b, b') \in R \quad \Rightarrow \quad (a, b') \in R \circ \overline{\Delta_B} = \overline{\Delta_B} \circ R.$$

This proves the existence of $b'' \in B$ such that

$$(a, b'') \in R, \quad (b'', b') \in \overline{\Delta_B}.$$

By saturation of A for R, we have then

$$b'' \in A, \quad (a, b'') \in S.$$

To conclude, it suffices to put $a' = a$ and $a'' = b''$.

Second, we must prove that the pair $\big(\iota\lambda(d_1), \iota\lambda(h)\big)$ is monomorphic.

If $(a,a')\in S$, $(a'',a''')\in S$, $(a,a'')\in\overline{\Delta_A}$, $\big((a,a'),(a'',a''')\big)\in\overline{\Delta_R}$,
then $\big((a,a'),(a'',a''')\big)\in\overline{\Delta_S}$.

Observe that

$$(a',a)\in S,\ (a,a'')\in\overline{\Delta_A},\ (a'',a''')\in S \quad\Rightarrow\quad (a',a''')\in S\vee\overline{\Delta_A}=\overline{S}.$$

Together with $(a',a''')\in\overline{\Delta_B}$ and assumption 2 in the statement, this proves that

$$(a',a''')\in\overline{S}\wedge f^{-1}(\overline{\Delta_B})=\overline{f^{-1}(R\wedge\Delta_B)}=\overline{f^{-1}(\Delta_B)}=\overline{\Delta_A}.$$

■

6. Examples

First of all let us observe that:

6.1. Example. Let $\iota,\lambda\colon\mathcal{L}\overset{\longleftarrow}{\longrightarrow}\mathcal{C}$ be a fibered protolocalisation of a homological category. Then for every object $I\in\mathcal{C}$, we get a corresponding fibered protolocalisation

$$\iota_I,\lambda_I\colon\mathcal{L}/\lambda(I)\overset{\longleftarrow}{\longrightarrow}\mathcal{C}/I.$$

Proof. By Proposition 3.5 and Corollary 2.3. A wide supply of fibered protolocalisations of homological categories can be found in [4]. The slice categories are still regular protomodular, thus Mal'cev, but generally no longer pointed. ■

In fact most examples given in [4] in the pointed case have also a non-pointed counterpart, but in most cases the proofs are substantially different. Let us focus on the most striking cases.

6.2. Example. The category BoRng_1 of Boolean unital rings — which is equivalent to the category of Boolean algebras — is an epireflective protolocalisation of the category VNReg_1 of commutative unital von Neumann regular rings. Both categories are exact Mal'cev.

Proof. It is proved in [4], Example 49, that the category BoRng of boolean rings (not necessarily with unit) is an epireflective protolocalisation of the category VNReg of von Neumann regular rings (not necessarily with unit). In all four categories pullbacks are computed as in Set and the regular epimorphisms are the surjective homomorphisms. Therefore it suffices, given a unital von Neumann regular ring R, to prove that its boolean reflection $\eta_R\colon R\twoheadrightarrow\iota\lambda(R)$ in the non unital case is also its reflection in the unital case. But since η_R is surjective, $\eta_R(1)$ is a unit in $\iota\lambda(R)$. Moreover if $f\colon R\longrightarrow A$ is a morphism of unital rings with A boolean, the unique factorisation $\lambda(R)\longrightarrow A$ in the non-unital case preserves the units, since so do η_R and f. ■

6.3. Example. Let $\mathcal{C}$ be an arithmetical category (see [23]); write $\mathsf{Eq}(\mathcal{C})$ for the category of equivalence relations in $\mathcal{C}$. Consider

$$\Delta\colon \mathcal{C} \longrightarrow \mathsf{Eq}(\mathcal{C}), \quad \chi\colon \mathsf{Eq}(\mathcal{C}) \longrightarrow \mathcal{C}$$

where $\Delta(A)$ is the discrete equivalence relation on A, while $\chi(A,R)$ is the quotient of A by the equivalence relation R. This is an epireflective protolocalisation between exact Mal'cev categories, but not a localisation.

Proof. An exact Mal'cev category $\mathcal{C}$ is arithmetical if and only if every groupoid is an equivalence relation (see [23]). The category of internal groupoids in an exact Mal'cev category is itself exact Mal'cev (see [16]). Thus in the conditions of the statement, the category $\mathsf{Eq}(\mathcal{C})$ is exact Mal'cev as well.

A regular epimorphism $f\colon (A,R) \longrightarrow (B,S)$ in $\mathsf{Eq}(\mathcal{C})$ is such that both $f\colon A \longrightarrow B$ and its factorisation $f'\colon R \longrightarrow S$ are regular epimorphisms in $\mathcal{C}$ (see [17]). In particular, $S = f(R)$.

The functor χ is trivially left adjoint to Δ and the unit of the adjunction is the quotient map

$$\eta_{(A,R)}\colon (A,R) \twoheadrightarrow (A/R, \Delta_{A/R}).$$

which is a regular epimorphism in $\mathsf{Eq}(\mathcal{C})$.

To prove that χ is a protolocalisation, let us consider a pullback $(*)$ in $\mathsf{Eq}(\mathcal{C})$ and its image by χ, with f a regular epimorphism in $\mathsf{Eq}(\mathcal{C})$.

$$\begin{tikzcd}
(P,U) \arrow[r, two heads, "p_C"] \arrow[d, "p_A"'] \arrow[dr, phantom, "(*)"] & (C,T) \arrow[d, "g"] & P/U \arrow[r, two heads, "\chi(p_C)"] \arrow[d, "\chi(p_A)"'] \arrow[dr, phantom, "\chi(*)"] & C/T \arrow[d, "\chi(g)"] \\
(A,R) \arrow[r, two heads, "f"'] & (B,S) & A/R \arrow[r, two heads, "\chi(f)"'] & B/S
\end{tikzcd}$$

Since $S = f(R)$, the square

$$\begin{tikzcd}
A \arrow[r, two heads, "f"] \arrow[d, two heads, "\eta_{(A,R)}"'] & B \arrow[d, two heads, "\eta_{(B,S)}"] \\
A/R \arrow[r, two heads, "\chi(f)"'] & B/S
\end{tikzcd}$$

is a pushout of regular epimorphisms in $\mathcal{C}$. Since the category $\mathcal{C}$ is exact Mal'cev, the factorisation

$$\alpha\colon A \longrightarrow A/R \times_{B/S} B$$

through the pullback is a regular epimorphism by Theorem 1.2. Let us prove that the surjectivity of α forces the diagram $\chi(*)$ to be a pullback.

First we must show that

Given $a \in A$, $c \in C$ such that $\big(f(a), g(c)\big) \in S$,
there exists $(a', c') \in P$ such that $(a, a') \in R$ and $(c, c') \in T$.

But $\big(f(a), g(c)\big) \in A/R \times_{B/S} B$ and since α is surjective,

$$\exists a' \in A \quad (a, a') \in R, \quad f(a') = g(c).$$

It suffices to put $c' = c$.

Next we must prove that the pair $\big(\chi(p_A), \chi(p_C)\big)$ is monomorphic. That is

If $f(a) = g(c)$, $f(a') = g(c')$, $(a, a') \in R$, $(c, c') \in T$,
then $\big((a, c), (a', c')\big) \in U$.

This is immediate because pullbacks in $\mathsf{Eq}(\mathcal{C})$ are computed componentwise.

To observe that we do not have a localisation, it suffices to prove that χ does not preserve monomorphisms. Indeed, $(A, \Delta_A) \rightarrowtail (A, A \times A)$ is a monomorphism mapped by χ on $A \longrightarrow \mathbf{1}$. ■

Let us recall that the dual of a topos is an exact, Mal'cev category (see [11]).

6.4. Example. Let $\mathcal{A}$ be a small filtered category. This yields a protolocalisation between exact Mal'cev categories

$$\Big(\Delta, \mathsf{lim} \colon \mathsf{Set} \leftrightarrows \widehat{\mathcal{A}}\Big)^{\mathsf{op}}$$

where $\widehat{\mathcal{A}}$ is the topos of presheaves on $\mathcal{A}$ and $\Delta(X)$ is the constant presheaf on X. This protolocalisation is generally not a localisation.

Proof. For facility we work at once with Set and $\widehat{\mathcal{A}}$, not with their duals. Since $\mathcal{A}$ is connected, Δ is full and faithful. We must prove that its right adjoint, the functor lim, preserves the pushout of a monomorphism along an arbitrary morphism. Let us recall that in a topos, the pushout of a monomorphism is also a pullback (see [20]). Moreover in the topos of sets — and more generally, in every boolean topos — the pushout of a monomorphism always has the form

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail] & Y \arrow[d, tail] \\
X \amalg Z \arrow[r, "f \amalg \mathsf{id}_Z"'] & Y \amalg Z
\end{tikzcd}$$

Thus in $\widehat{\mathcal{A}}$, such a formula holds pointwise.

Let us consider the pushout (1) in $\widehat{\mathcal{A}}$ and the corresponding diagram (2) of limits in Set:

$$\begin{tikzcd}
K \arrow[r, "\varphi"] \arrow[d, tail, "\alpha"'] \arrow[dr, phantom, "(1)"] & H \arrow[d, tail, "\beta"] & \mathsf{lim}\, K \arrow[r, "\mathsf{lim}(\varphi)"] \arrow[d, tail, "\mathsf{lim}\,\alpha"'] \arrow[dr, phantom, "(2)"] & \mathsf{lim}\, H \arrow[d, tail, "\mathsf{lim}\,\beta"] \\
F \arrow[r, "\psi"'] & G & \mathsf{lim}\, F \arrow[r, "\mathsf{lim}(\psi)"'] & \mathsf{lim}\, G
\end{tikzcd}$$

The right adjoint lim preserves limits, thus the square (2) is a pullback with $\mathsf{lim}\,\alpha$, $\mathsf{lim}\,\beta$ monomorphisms. Writing $\mathsf{lim}\,\alpha$ and $\mathsf{lim}\,\beta$ as canonical inclusions, this proves already that

$$\mathsf{lim}\, H, \quad \mathsf{lim}\,\psi\big(\mathsf{lim}\, F \setminus \mathsf{lim}\, K\big)$$

are disjoint subsets of $\mathsf{lim}\, G$. It remains to prove that they cover $\mathsf{lim}\, G$, while $\mathsf{lim}\,\psi$ is injective on $\mathsf{lim}\, F \setminus \mathsf{lim}\, K$.

We first prove that second assertion. Consider two distinct elements $(y_A)_{A\in\mathcal{A}}$, $(z_A)_{A\in\mathcal{A}}$ in $\mathsf{lim}\, F \setminus \mathsf{lim}\, K$. There is thus an index A such that $y_A \notin K(A)$ and an index A' such that $z_{A'} \notin K(A')$. There is also an index A'' such that $y_{A''} \neq z_{A''}$. By filteredness we can find in $\mathcal{A}$

$$\begin{tikzcd}
A \arrow[dr, "a"] & \\
A' \arrow[r, "a'"] & B \\
A'' \arrow[ur, "a''"'] &
\end{tikzcd}$$

Since our presheaves are contravariant and the family $(y_A)_{A\in\mathcal{A}}$ is compatible, necessarily $y_B \notin K(B)$ because $y_A \notin K(A)$. Analogously $z_B \notin K(B)$ and $y_B \neq z_B$. Since ψ_B is injective on $F(B) \setminus K(B)$, $\psi_B(y_B) \neq \psi_B(z_B)$, proving that $\psi(y_A)_{A\in\mathcal{A}} \neq \psi(z_A)_{A\in\mathcal{A}}$.

To prove the first assertion, consider $\big(x_A \in G(A)\big)_{A\in\mathcal{A}} \in \mathsf{lim}\, G$. If this family is not in $\mathsf{lim}\, H$, at least one of the elements x_A is not in $H(A)$. Each such x_A has then the form $x_A = \psi_A(y_A)$ for a unique $y_A \in F(A) \setminus K(A)$. The family of all these y_A is then compatible, simply because the family of the corresponding x_A is compatible and $F(A) \setminus K(A)$ is mapped injectively by ψ_A in the pushout $G(A)$. We shall now extend that family y_A to all the objects $B \in \mathcal{A}$.

Fix thus $B \in \mathcal{A}$ and choose $A \in \mathcal{A}$ such that $x_A \notin \mathsf{lim}\, H$. By filteredness there exist

$$\begin{tikzcd}
A \arrow[r, "a"] & A' & B. \arrow[l, "b"']
\end{tikzcd}$$

Again, since $x_A \notin \mathsf{lim}\, H$, the same holds for $x_{A'}$. Thus $y_{A'}$ is already defined and it suffices to put $y_B = F(b)(y_{A'})$. This definition is independent of the choices of A' and b. Indeed if $c\colon B \longrightarrow A''$ is another possibility, by filteredness we can complete the span (b, c) in a commutative square:

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "c"'] & A' \arrow[d, "d"] \\
A'' \arrow[r, "e"'] & A'''
\end{tikzcd}$$

We have then

$$F(b)(y_{A'}) = F(b)F(d)(y_{A'''}) = F(c)F(e)(y_{A'''}) = F(c)(y_{A''}).$$

This shows that the definition of y_B is independent of the choices of A' and b, but implies at the same time that the extended family $(y_A)_{A\in\mathcal{A}}$ is compatible. Observe further that by naturality of ψ and compatibility of the family $(x_A)_{A\in\mathcal{A}}$ in G

$$\psi_B(y_B) = \psi_B F(b)(y_{A'}) = G(b)\psi_{A'}(y_{A'}) = G(b)(x_{A'}) = x_B.$$

Thus the family $(y_A)_{A\in\mathcal{A}}$ lies in $\mathsf{lim}\, F \setminus \mathsf{lim}\, K$ and is mapped on $(x_A)_{A\in\mathcal{A}}$.

To prove that we do not have in general a co-localisation, take for $\mathcal{A}$ the poset $(\mathbb{N}, \leq)$. It suffices to prove that the limit functor does not preserve epimorphisms. The following data define an epimorphism $\varphi\colon F \Rightarrow G$

$$\varphi_n\colon F(n) = \mathbb{N} \twoheadrightarrow G(n) = \{0, 1, \ldots, n\}, \quad \varphi_n(m) = \min\{m, n\}$$

where F is thus the constant functor on $\mathbb{N}$ and the restriction

$$G(n+1) \longrightarrow G(n)$$

maps $n+1$ on n and is the identity elsewhere. The function $\mathsf{lim}\,\varphi$ is not surjective: the compatible family $\big(n \in G(n)\big)_{n\in\mathbb{N}}$ does not belong to the image of $\mathsf{lim}\,\varphi$. ■

The observant reader will have noticed that we did not use the full strength of the filteredness of $\mathcal{A}$: only the slightly weaker requirements that $\mathcal{A}$ is connected and every span can be completed in a commutative square (these requirements imply in particular that two objects can always be mapped in a third one). An example of a non-filtered category satisfying these more general requirements is given by the free monoid on one generator viewed as a category with a single object. But even that weaker notion of filteredness, called *protofilteredness* in [4], is not necessary for having a protolocalisation, as our following example shows.

6.5. Example. Every monoid M induces an epireflective protolocalisation between exact Mal'cev categories

$$\Big(\Delta, \mathsf{Fix}\colon \mathsf{Set} \overset{\longleftarrow}{\longrightarrow} \mathrm{M}{-}\mathsf{Set}\Big)^{\mathsf{op}}$$

where $\mathrm{M}{-}\mathsf{Set}$ is the topos of M-sets, $\Delta(X)$ is the set X provided with the trivial action $mx = x$ for all $m \in M$ and

$$\mathsf{Fix}(X, \chi) = \big\{x \in X \big| \forall m \in M\ mx = x\big\}$$

is the set of fixed points of the M-set (X, χ). This protolocalisation is generally not a localisation.

Proof. Viewing the monoid M as a category with a single object, we come back to the situation described in Example 6.4 … except that M is generally not a filtered (nor even protofiltered) category.

Again we work at once in Set and $\mathrm{M}{-}\mathsf{Set}$, not in their duals. Trivially this time we have a mono-co-reflection. Given a pushout of M-sets (where $\amalg$ indicates the coproduct in Set, not in $\mathrm{M}{-}\mathsf{Set}$)

$$\begin{tikzcd}
(A,\alpha) \arrow[r, tail, "f"] \arrow[d, tail] & (B,\beta) \arrow[d, tail] \\
(A \amalg C,\gamma) \arrow[r, tail, "f \amalg \mathsf{id}"'] & (B \amalg C,\delta)
\end{tikzcd}$$

it suffices clearly to prove that every fixed point x of $(B \amalg C,\delta)$ which is not in B is the image of a fixed point y of $(A \amalg C,\gamma)$ which lies in C. But x is the image of a unique $y \in C$ and it suffices to prove that y is fixed. Since the bottom arrow is the identity on C and x is fixed, it suffices to prove that given $m \in M$, then my is still in C. But $my \in A$ would imply that its image $mx = x$ is in B, which is not the case.

Taking for M the monoid $(\mathbb{N},+)$, which is the free monoid on one generator, $\mathrm{M}{-}\mathsf{Set}$ is equivalent to the topos of sets (X,σ) provided with an endomorphism. The morphism

$$(\mathbf{1} \amalg \mathbf{1},\tau) \twoheadrightarrow (\mathbf{1},\mathsf{id})$$

with τ the twisting isomorphism is an epimorphism, mapped on $\emptyset \rightarrowtail \mathbf{1}$ by the functor Fix. This proves that we do not have a co-localisation. ■

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