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ANALYTIC FUNCTORS AND WEAK PULLBACKS

For the sixtieth birthday of Walter Tholen

J. ADÁMEK AND J. VELEBIL

ABSTRACT. For accessible set-valued functors it is well known that weak preservation of limits is equivalent to representability, and weak preservation of connected limits to familial representability. In contrast, preservation of weak wide pullbacks is equivalent to being a coproduct of quotients of hom-functors modulo groups of automorphisms. For finitary functors this was proved by André Joyal who called these functors analytic. We introduce a generalization of Joyal's concept from endofunctors of **Set** to endofunctors of a symmetric monoidal category.

1. Introduction

From among accessible set-valued functors those preserving limits are known to be just the representable ones. And those preserving connected limits (or, equivalently, preserving wide pullbacks) were characterized by Aurelio Carboni and Peter Johnstone [CJ] as precisely the coproducts of representables; they call these functors familially representable. Moreover, Peter Freyd and André Scedrov proved in [FS], 1.829 that weak preservation of (connected) limits implies strong preservation of them. Surprisingly, many more accessible functors weakly preserve wide pullbacks; here the characterization is: all coproducts of *symmetrized representables* by which we mean the quotient of a hom-functor $\mathscr{A}(A, -)$ modulo a group of automorphisms of A.

The latter is connected to the concept of *analytic functor* of André Joyal. In his categorical study of enumerative combinatorics, see [J_1] and [J_2], he introduced *species of structures* as functors from the category $\mathscr{B}$ of natural numbers and permutations into **Set**. Example: the species $p\colon \mathscr{B} \longrightarrow \mathbf{Set}$ of permutations assigning to n the set $p(n)$ of all permutations on n and to every permutation $\sigma\colon n \longrightarrow n$ the action of σ on $p(n)$. A set functor F is called analytic if it is the left Kan extension of a species $f\colon \mathscr{B} \longrightarrow \mathbf{Set}$. In other words, F is given on objects by the coend formula

$$FX = \int^{n:\mathscr{B}} X^n \times f(n).$$

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Example: the species p of permutations yields the analytic functor $PX = \sum_{n\in\mathbb{N}} X^n$, the free-monoid functor. Joyal characterized analytic functors as precisely the finitary set functors weakly preserving wide pullbacks.

Another category in which Joyal studied analytic functors is the category $\mathbf{Vec}_{\mathbb{K}}$ of vector spaces over a given field. Here a species is a functor from $\mathscr{B}$ to $\mathbf{Vec}_{\mathbb{K}}$. The analytic functor $F\colon \mathbf{Vec}_{\mathbb{K}} \longrightarrow \mathbf{Vec}_{\mathbb{K}}$ corresponding to f is then given by the coend formula

$$FX = \int^{n:\mathscr{B}} X^{\otimes n} \otimes f(n).$$

Example: the species of permutations yields the functor $PX = \sum_{n\in\mathbb{N}} X^{\otimes n}$, the tensor-algebra functor. We can obviously play the same game in every symmetric monoidal category $\mathscr{E}$ and define analytic endofunctors of $\mathscr{E}$ as those obtained from a species via the above coend formula. Example: The analytic endofunctors of S-sorted sets are the S-tuples of analytic endofunctors of **Set**; they weakly preserve wide pullbacks but the converse does not hold.

Weak wide pullbacks appear in several applications of category theory, e.g. in coalgebra [G] and in stable domain theory [Ta]. However, the connection between them and analytic functors is, beyond the category of sets, quite loose: in many-sorted sets Example 4.4(iii) below demonstrates that a wide pullback preserving endofunctor need not be analytic, and in vector spaces Corollary 3.9 below implies that almost no analytic functor weakly preserves wide pullbacks.

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Related Work

Ryu Hasegawa [H] presented an alternative proof of Joyal's characterization of analytic endofunctors of **Set**. In the first section we provide a new proof which is simpler than both of the previously published ones. We also prove that countable wide pullbacks are sufficient.

Analytic functors of André Joyal have recently been generalized by Brian Day [D] who defined E-analytic functors for a kernel $E(n, X)$; our definition is the special case of $E(n, X) = X^{\otimes n}$. A different approach has been taken by Marcello Fiore *et al*, see [F] and [FGHW]: for a small category $\mathscr{A}$ they form the groupoid $\mathscr{B}\mathscr{A}$ of all isomorphisms in the free completion of $\mathscr{A}$ under finite coproducts. Then a functor $F\colon [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is called analytic if it is the left Kan extension of a functor $f\colon \mathscr{B}\mathscr{A} \longrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ along canonical functor $\mathscr{B}\mathscr{A} \longrightarrow [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$. In [F] these functors are, in case $\mathscr{A}$ and $\mathscr{C}$ are groupoids, characterized as precisely the finitary functors weakly preserving quasipullbacks. Finally, M. Abbot *et al* [AAGB] study quotient containers which generalize species of structures by considering groups of automorphisms on infinite sets. The corresponding generalization of analytic functor is called an extension of the quotient container. These are precisely the accessible set functors weakly preserving wide pullbacks as we prove in Section 3.

Weak preservation of various types of limits is also studied, for endofunctors of **Set**, in [AGT]. The characterization of weak preservation of wide pullbacks in Theorem 3.3 generalizes a result of that paper; the present proof is simpler.

2. Analytic Set Functors

2.1 DEFINITION (A. Joyal). *A functor $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ is called **analytic** provided that it is the left Kan extension of a functor f from $\mathscr{B}$, the category of natural numbers and bijections, into* **Set**. *In other words, F is defined on objects X by the coend*

$$FX = \int^{n:\mathscr{B}} X^n \times f(n). \tag{2.1}$$

2.2 REMARK. For every natural number n the (symmetric) group of all permutations of n is denoted by $\mathscr{S}_n$.

(i) Suppose that a species f is given on objects by

$$f(n) = V \quad \text{and} \quad f(k) = \emptyset \qquad \text{for all } k \neq n.$$

The coend formula then expresses FX as the joint coequalizer for all the automorphisms $\bar{\sigma} = X^\sigma \times f(\sigma^{-1})$ of $X^n \times V$ for $\sigma \in \mathscr{S}_n$. In other words

$$FX = X^n \times V/{\sim} \tag{2.2}$$

where $\sim$ is the equivalence relation consisting of all pairs

$$(x, i) \sim \big(x{\cdot}\sigma, f(\sigma^{-1})(i)\big) \qquad \text{for all } x \in X^n,\ i \in V,\ \sigma \in \mathscr{S}_n. \tag{2.3}$$

(ii) Every species is an (objectwise) coproduct of species of the type (i) above. Since left Kan extension preserves coproducts, every analytic functor is a coproduct of functors given by (2.2).

2.3 EXAMPLES (see [J_1], [J_2]). (i) The representable functor $(-)^n$ is, for every $n \in \mathbb{N}$, analytic. The corresponding species is given on objects by $f(n) = \mathscr{S}_n$ and $f(k) = \emptyset$ for $k \neq n$; on morphisms put $f(\sigma)\colon \tau \longmapsto \sigma{\cdot}\tau$ for $\sigma, \tau \in \mathscr{S}_n$.

(ii) The *species of permutations*

$$p(n) = \mathscr{S}_n \qquad \text{for all } n \in \mathbb{N}$$

which is the coproduct of those in (i) yields the analytic functor

$$PX = \sum_{n \in \mathbb{N}} X^n = \text{free monoid on } X.$$

(iii) Let $\mathscr{G}$ be a group of permutations, a subgroup of $\mathscr{S}_n$. We denote by

$$FX = X^n/\mathscr{G}$$

the quotient functor of X^n modulo the following equivalence $\sim_{\mathscr{G}}$:

$$x \sim_{\mathscr{G}} y \qquad \text{iff } x = y{\cdot}\sigma \text{ for some } \sigma \in \mathscr{G}.$$

These functors are called *symmetrized representables.* They are analytic: the corresponding species is the quotient of that in (i) given by

$$f(n) = \mathscr{S}_n/{\sim_{\mathscr{G}}}.$$

In fact, in the equivalence (2.3) all pairs $(x, [\mathrm{id}_n])$ fulfil

$$(x, [\mathrm{id}_n]) \sim (y, [\mathrm{id}_n]) \qquad \text{iff } x \sim_{\mathscr{G}} y,$$

and every pair $(x, [\sigma])$ is equivalent to $(x{\cdot}\sigma, [\mathrm{id}_n])$. Consequently, the equivalence relation (2.3) yields the quotient $X^n \times (\mathscr{S}_n/{\sim_{\mathscr{G}}}) \longrightarrow FX$ assigning to $(x, [\tau])$ the value $[x{\cdot}\tau]$.

(iv) The trivial species

$$e(n) = 1 \quad \text{for all } n \in \mathbb{N}$$

yields the analytic functor

$$EX = \text{finite multisets in } X.$$

In fact, (2.2) implies that

$$EX = \sum_{n \in \mathbb{N}} X^n/\mathscr{S}_n$$

and this means that in an n-tuple in X^n the order of the coordinates does not matter, but the number of repetitions does.

(v) The *species of trees* assigns to every n the set $t(n)$ of all (directed, rooted) trees on $n = \{0, 1, \dots, n-1\}$. The corresponding analytic functor is

$$TX = \text{isomorphism classes of trees labelled in } X.$$

2.4 Remark. (i) As already observed by Věra Trnková [T], every set-valued functor F is a coproduct of functors preserving the terminal object: the coproduct is indexed by $F1$ and for every $i \in F1$ we consider the subfunctor F_i of F given on objects X by $F_iX = (Ft)^{-1}(i)$ for the unique morphism $t\colon X \longrightarrow 1$. This fact immediately generalizes to functors $F\colon \mathscr{A} \longrightarrow \mathbf{Set}$ for an arbitrary category $\mathscr{A}$ having a terminal object.

(ii) We now characterize functors preserving weak pullbacks from among a special class of set functors; the same result was already published in [AT] but with a complicated proof.

2.5 DEFINITION (see [AT]). *A set functor F is called **super-finitary** if there exists a natural number n such that the elements of Fn generate all of F. That is, for every set X we have*

$$FX = \bigcup_{f\colon n \to X} Ff[Fn]. \tag{2.4}$$

2.6 THEOREM. *For a super-finitary set functor the following conditions are equivalent:*

(i) *F weakly preserves pullbacks,*

(ii) *F is analytic,*

and

(iii) *F is a coproduct of symmetrized representables.*

PROOF. (i) $\Rightarrow$ (iii) Due to 2.4(i) we can assume $F1 = 1$. Then F weakly preserves finite products. Let us choose the least number n with (2.4). Then there exists $x_0 \in Fn$ which is *minimal*, i.e. such that x_0 does not lie in $F_i[Fk]$ for any proper subobject $i\colon k \hookrightarrow n$. It follows that

$$\text{if } x_0 \in Ff[FX] \text{ for some } f\colon X \longrightarrow n, \text{ then } f \text{ is epic.} \tag{2.5}$$

In particular, every element of

$$\mathscr{G} = \big\{\sigma\colon n \longrightarrow n; F\sigma(x_0) = x_0\big\} \tag{2.6}$$

is an epimorphism, thus an isomorphism. We obtain a subgroup $\mathscr{G} \subseteq \mathscr{S}_n$, and we prove are going to that

$$F \cong (-)^n/\mathscr{G}.$$

In fact, the Yoneda transformation

$$(-)^n \longrightarrow F, \qquad u \longmapsto Fu(x_0)$$

yields a natural isomorphism $(-)^n/\mathscr{G} \cong F$. This is clear provided that we prove that

(a) every element of F has the form $Fu(x_0)$

and

(b) given $u, v \in X^n$ then $Fu(x_0) = Fv(x_0)$ iff $u \sim_{\mathscr{G}} v$.

To prove (a) use (2.4): it is sufficient to prove that every element $y \in Fn$ has the form $y = Fu(x_0)$. Since F weakly preserves finite products, for $x_0, y \in Fn$ there exists $z \in F(n \times n)$ with

$$F\pi_1(z) = x_0 \qquad \text{and} \qquad F\pi_2(z) = y \qquad \text{for the projections } \pi_1, \pi_2\colon n \times n \longrightarrow n.$$

By (2.4) we have $z_0 \in Fn$ with $Ff(z_0) = z$. From

$$x_0 = F\pi_1(z) = F(\pi_1 \cdot f)(z_0)$$

we conclude that $\pi_1 \cdot f \colon n \longrightarrow n$ is epimorphism, see (2.5), thus, an isomorphism. Put

$$u = \pi_2 \cdot f \cdot (\pi_1 \cdot f)^{-1},$$

then

$$y = F\pi_2(z) = F(\pi_2 \cdot f)(z_0) = F(\pi_2 \cdot f) \cdot F(\pi_1)^{-1}(x_0) = Fu(x_0)$$

as requested.

To prove (b), observe that $u \sim_{\mathscr{G}} v$ implies $u = v \cdot \sigma$ with $F\sigma(x_0) = x_0$, thus, $Fu(x_0) = Fv(x_0)$. Conversely, assuming $Fu(x_0) = Fv(x_0)$, form a pullback

$$\begin{tikzcd} & P \arrow[dl, "\bar{v}"'] \arrow[dr, "\bar{u}"] & \\ n \arrow[dr, "u"'] & & n \arrow[dl, "v"] \\ & X & \end{tikzcd}$$

Since F weakly preserves it, there exists $z \in FP$ with

$$F\bar{u}(z) = x_0 = F\bar{v}(z).$$

Express, using (a),

$$z = Ff(x_0) \qquad \text{for } f \colon n \longrightarrow P$$

then the morphisms $\bar{u} \cdot f, \bar{v} \cdot f \colon n \longrightarrow n$ lie in $\mathscr{G}$, and

$$u = u \cdot (\bar{v} \cdot f) \cdot (\bar{v} \cdot f)^{-1} = v \cdot (\bar{u} \cdot f) \cdot (\bar{v} \cdot f)^{-1}$$

which yields the desired equation $u = v.\sigma$ for

$$\sigma = (\bar{u} \cdot f).(\bar{v} \cdot f)^{-1} \in \mathscr{G}.$$

(iii) $\Rightarrow$ (ii) See Example 2.3(iii).

(ii) $\Rightarrow$ (i) Let us verify that every functor F given by (2.2) weakly preserves pullbacks

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A_1 \arrow[dr, "a_1"'] & & A_2 \arrow[dl, "a_2"] \\ & B & \end{tikzcd}$$

In fact, given elements $[x_k, i_k]$ with $Fa_1([(x_1, i_1)]) = Fa_2([(x_2, i_2)])$, we have

$$(a_1{\cdot}x_1, i_1) \sim (a_2{\cdot}x_2, i_2).$$

Thus there exists $\sigma \in \mathscr{S}_n$ with

$$a_1{\cdot}x_1{\cdot}\sigma = a_2{\cdot}x_2 \qquad \text{and} \qquad f(\sigma^{-1})(i_1) = i_2.$$

The first equality yields a unique $b\colon n \longrightarrow P$ with

$$x_1{\cdot}\sigma = p_1{\cdot}b \qquad \text{and} \qquad x_2 = p_2{\cdot}b.$$

Then the element $[(b, i_2)]$ is mapped by Fp_1 to $[(x_1{\cdot}\sigma, f(\sigma^{-1})(i_1))] = [(x_1, i_1)]$ and by Fp_2 to $[(x_2, i_2)]$, as requested. ∎

2.7 Corollary. *For every finitary functor F:* **Set** $\longrightarrow$ **Set** *the following conditions are equivalent:*

(i) *F is analytic,*

(ii) *F is a coproduct of symmetrized representables,*

(iii) *F weakly preserves countable wide pullbacks,*

(iv) *F weakly preserves wide pullbacks.*

Proof. In fact, it is sufficient to prove this for functors with $F1 = 1$. The implication (ii) $\Rightarrow$ (i) follows from 2.3(iii), and (iii) $\Rightarrow$ (ii) follows from 2.6 because F is superfinitary. In fact, assuming the contrary, for every $n \in \mathbb{N}$ we have $x_n \in FX_n$ with $x_n \notin \bigcup_{h\in X^n} Fh[Fn]$. But $F1 = 1$ implies that F weakly preserves countable products. Thus, there exists $x \in \prod_{n\in\mathbb{N}} FX_n$ with $F\pi_n(x) = x_n$ for all $n \in \mathbb{N}$. Since F is finitary, there exists a morphism $g\colon k \longrightarrow X$ and $y \in Fk$ with $x = Fg(y)$. Then $x_k = F(\pi_k{\cdot}g)(y)$, a contradiction.

Finally, (iv) $\Rightarrow$ (iii) trivially and (i) $\Rightarrow$ (iv) is proved as in 2.6. ∎

2.8 Example. The finite-powerset functor $\mathscr{P}_{\text{fin}}$ preserves weak pullbacks. This demonstrates that in 2.7(iii) pullbacks are not sufficient.

2.9 Remark. (a) The equivalence of (i), (iii) and (iv) was proved by André Joyal [J_2], and a compact proof was later presented by Ryu Hasegawa [H].

(b) Condition (iii) trivially implies that F preserves limits of cofiltered diagrams (and the formulation of Joyal used that plus weak preservation of pullbacks). In fact, let

$$p_t\colon P \longrightarrow X_t \qquad (t \in T)$$

be a cofiltered limit and assume, without loss of generality, that T has a terminal element t_0; thus we have connecting morphisms $x_t\colon X_t \longrightarrow X_{t_0}$ for $t \in T$. The limit cone is, obviously, a wide pullback of the cocone $(x_t)_{t\in T}$, thus, FP is a wide pullback of $(Fx_t)_{t\in T}$.To prove that Fp_t $(t \in T)$ is a limit cone we only need to observe that it is collectively monic. Given elements $x_1 \neq x_2$ in FP find a finite set $m\colon M \hookrightarrow X$ with $x_1, x_2 \in Fm[FM]$. Since the limit is cofiltered, there exists $t \in T$ such that $p_t{\cdot}m$ is monic, therefore $F(p_t{\cdot}m)$ is monic: recall that F preserves monomorphisms because it (weakly) preserves intersections. This implies $Fp_t(x_1) \neq Fp_t(x_2)$.

2.10 Corollary. *A finitary set functor preserving countable limits is representable.*

In fact, from 2.7 and $F1 = 1$ we know that F is a symmetrized representable. Moreover, if $F = (-)^n/\mathscr{G}$ preserves equalizers, then $\mathscr{G}$ is the trivial group: for every $\sigma \in \mathscr{G}$ consider the equalizer e of σ and id_n, since Fe is the equalizer of $F\sigma$ and id_{Fn} it is easy to derive $\sigma = \mathrm{id}_n$.

3. Weak Wide Pullbacks

3.1 Assumption. Throughout this section $\mathscr{A}$ denotes a locally λ-presentable category. Recall that this means that (i) $\mathscr{A}$ is complete and (ii) (for some infinite cardinal λ) it has a set $\mathscr{A}_\lambda$ of objects that are λ-presentable (that is, the hom-functors preserve λ-filtered colimits) and whose closure under λ-filtered colimits is all of $\mathscr{A}$. In fact, we could require less: all we need is that $\mathscr{A}$ be cowellpowered, λ-accessible (that is, have λ-filtered colimits and satisfy (ii) above), have weak pullbacks an initial object, and (epi, strong mono) factorizations. All this holds in locally λ-presentable categories, see [AR].

Recall that a functor is called λ-accessible if it preserves λ-filtered colimits.

3.2 Definition. *By a* ***symmetrized representable functor*** *is meant a quotient of a* hom-*functor* $\mathscr{A}(A,-)$ *modulo a group* $\mathscr{G}$ *of automorphisms of* A*; notation:*

$$\mathscr{A}(A,-)/\mathscr{G}\colon \mathscr{A} \longrightarrow \mathbf{Set}\,.$$

Explicitly, to an object X *this functor assigns the set*

$$\mathscr{A}(A,X)/{\sim_{\mathscr{G}}}$$

where for $f, f'\colon A \longrightarrow X$ *we put*

$$f \sim_{\mathscr{G}} f' \qquad \text{iff} \qquad f = f'{\cdot}\sigma \text{ for some } \sigma \in \mathscr{G}.$$

3.3 Theorem. *A*n accessible functor $F\colon \mathscr{A} \longrightarrow \mathbf{Set}$ weakly preserves wide pullbacks iff it is a coproduct of symmetrized representables.

Remark. More detailed, for a λ-accessible functor $F\colon \mathscr{A} \longrightarrow \mathbf{Set}$ we prove equivalence of the following conditions:

(i) F weakly preserves wide pullbacks

and

(ii) F is a coproduct of functors $\mathscr{A}(A,-)/\mathscr{G}$ where A is λ-accessible.

Proof. (ii) $\Rightarrow$ (i) is trivial and analogous to 2.6, let us prove (i) $\Rightarrow$ (ii). Due to 2.4(i) we can assume that $F1 = 1$, thus, F weakly preserves products. Let λ be a cardinal such that $\mathscr{A}$ is a locally λ-presentable category and F is λ-accessible. Then all elements

$$x \in FA \qquad (A\ \lambda\text{-presentable})$$

generate F in the sense that every element $y \in FY$ has, for some such $x \in FA$, the form $y = Ff(x)$ where $f\colon A \longrightarrow Y$ is a morphism. Since F weakly preserves products, there is a single element $x_0 \in FA$ generating F, and since F (weakly) preserves intersections we can assume that x_0 lies in no image of Fm where $m\colon M \longrightarrow A$ is a proper subobject. It follows that x_0 is minimal, that is, (2.5) holds. (In fact given $x_0 \in Ff[FX]$, factorize $f = m{\cdot}e$ where e is epic and m is strongly monic, then x_0 lies in the image of Fm which implies that m is an isomorphism, thus, f is an epimorphism.) This element x_0 yields a surjective Yoneda transformation $\mathscr{A}(A,-) \longrightarrow F$. Since F is λ-accessible, it is easy to see that the minimality implies that A is λ-presentable. We prove

$$F \cong \mathscr{A}(A,-)/\mathscr{G}$$

for the group $\mathscr{G}$ of all automorphisms $\sigma\colon A \longrightarrow A$ with $F\sigma(x_0) = x_0$. The proof is completely analogous to that in 2.6: (a) is clear. In (b) we need to verify that the morphism

$$e = \bar{u}{\cdot}f\colon A_1 \longrightarrow A_1$$

is an isomorphism (by symmetry, so is $\bar{v}{\cdot}g$). From the minimality of x_0 we know that e is an epimorphism, since $Fe(x_0) = x_0$. We will now construct a chain

$$C\colon\ \mathbf{Ord}^{\mathrm{op}} \longrightarrow \mathscr{A}$$

with connecting morphisms $C_{ij}\colon C(i) \longrightarrow C(j)$ for $i \geq j$ such that

$$C(i) = A \qquad \text{and} \qquad C_{i+1,i} = e \qquad \text{for all } i \in \mathbf{Ord}.$$

Since $\mathscr{A}$ is cowellpowered, the object A has less than α quotients for some cardinal α, therefore the quotients

$$C_{\alpha,i}\colon A \longrightarrow A \qquad (i < \alpha)$$

are not pairwise distinct. Consequently, there exists $i < \alpha$ such that $C_{\alpha,i+1}$ and $C_{\alpha,i}$ represent the same quotient, and then from $C_{i+1,i} = e$ it follows that e is an isomorphism.

The chain C is defined by transfinite induction in such a way that for all $i \geq j$ we have that x_0 is a fixed point of $FC_{i,j}$. Put $C(0) = A$. Given $C(i)$ put $C(i+1) = A$ and let $C_{i+1,j} = C_{i,j}{\cdot}e$ for all $j \leq i$. If i is a limit ordinal, form a limit $l_j\colon L \longrightarrow A$ $(j < i)$ of the i-chain already defined. This is a wide pullback of all $C_{j,0}$ with $j < i$, and since F weakly preserves this wide pullback, we know from $FC_{j,0}(x_0) = x_0$ that there exists $b \in FL$ with $Fl_j(b) = x_0$ for all $j < i$. There exists $f\colon A \longrightarrow L$ with $b = Ff(x_0)$. Put

$$C_{i,j} = l_j{\cdot}f \qquad \text{for all } j < i$$

then x_0 is a fixed point of $C_{i,j}$ as requested. ∎

3.4 Corollary. *Let $\mathscr{A}$ and $\mathscr{B}$ be locally presentable categories. An accessible functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ weakly preserves wide pullbacks iff every functor $\mathscr{B}(B, F-)$ for $B \in \operatorname{obj} \mathscr{B}$ is a coproduct of symmetrized representables.*

In fact, the functors $\mathscr{B}(B, -)$ are accessible and collectively create wide weak pullbacks.

3.5 Example. An accessible functor $F\colon \mathbf{Set}^I \longrightarrow \mathbf{Set}^I$ weakly preserves wide pullbacks iff it is an I-tuple of coproducts symmetrized representables.

3.6 Remark. (i) Mark Weber introduced in [W] the concept of generic factorizations for a given functor. From his results it easily follows that only functors weakly preserving wide pullbacks can admit generic factorizations. Our Corollary 3.4 shows the converse implication for accessible functors between locally presentable categories.

(ii) The definition of analytic functor in 2.1 has an obvious infinitary variant: given a cardinal λ let $\mathscr{B}_\lambda$ be the category of all cardinals smaller than λ and all isomorphisms. We can investigate all set functors that are left Kan extensions of functors from $\mathscr{B}_\lambda$ to $\mathbf{Set}$. Each of them is clearly λ-accessible, i.e., preserves λ-filtered colimits. Theorem 3.3 gives a full characterization of these functors: they are the λ-accessible functors weakly preserving wide pullbacks. Or, equivalently, coproducts of symmetrized representables $\mathbf{Set}(A, -)/\mathscr{G}$ with $\operatorname{card} A < \lambda$.

(iii) More generally, if $\mathscr{A}$ is a locally λ-presentable category then the following conditions are equivalent for accessible functors $F\colon \mathscr{A} \longrightarrow \mathbf{Set}$:

(a) F weakly preserves wide pullbacks

and

(b) F is the left Kan extension of a functor from the category of λ-presentable objects and isomorphisms into $\mathbf{Set}$.

3.7 Remark. André Joyal considered in [J_1] also analytic endofunctors of the category

$$\mathbf{Vec}_{\mathbb{K}}$$

of vector spaces over a given field. Here we prove that, in contrast to $\mathbf{Set}$, almost none of them weakly preserves wide pullbacks:

3.8 Proposition. *Let $\mathscr{A}$ be a complete abelian category. An endofunctor F weakly preserves wide pullbacks iff it has the form $F(-) = F_0(-) + B$ where F_0 preserves limits.*

Proof. The condition is clearly sufficient. To prove that it is necessary, consider an endofunctor F preserving weak wide pullbacks and with $F0 = 0$. Then F preserves limits. In fact, F preserves weak products and intersections. We will prove that F preserves every product $\pi_i \colon X \longrightarrow X_i$ $(i \in I)$ strongly, that is, $(F\pi_i)$ is collectively monic. In fact, F preserves kernels because a kernel m of a morphism f is characterized by the weak pullback

$$\begin{tikzcd}
& {} \arrow[dl, "m"'] \arrow[dr] & \\
{} \arrow[dr, "f"'] & & 0 \arrow[dl] \\
& {} &
\end{tikzcd}$$

which F preserves. Since (π_i) is collectively monic, we have $\bigcap_{i\in I} \ker \pi_i = 0$, therefore, $\bigcap_{i\in I} \ker F\pi_i = F(\bigcap_{i\in I} \ker \pi_i) = 0$, thus, F preserves products. Consequently, it preserves limits.

To conclude the proof, let $F0$ be arbitrary, put

$$B = F0.$$

The zero morphisms $e_X \colon X \longrightarrow 0$ define short exact sequences

$$F_0 X \xrightarrow{m_X} FX \xrightarrow{Fe_X} B$$

and since Fe_X is a split epimorphism, we conclude

$$FX = F_0 X + B.$$

Thus $m \colon F_0 \longrightarrow F$ is a subfunctor such that F_0 weakly preserve wide pullbacks. In fact, this follows from F having this property and $F(-) = F_0(-) \times B$: given a wide pullback $p_i \colon P \longrightarrow A_i$ of $a_i \colon A_i \longrightarrow A$ $(i \in I)$ and given elements $x_i \in F_0 A_i$ with $F_0 a_i(x_i) = x$ for all $i \in I$, then for $(x_i, 0) \in FA_i$ we have $Fa_i(x_i, 0) = (x, 0)$. Consequently, there exists $(y, b) \in FP = F_0 P \times B$ with $F_0 a_i(y) = x_i$ for all $i \in I$. Since $F_0 0 = 0$, we conclude from the above that F_0 preserves limits. ■

3.9 Corollary. *The only endofunctors of* $\mathbf{Vec}_{\mathbb{K}}$ *weakly preserving wide pullbacks are the coproducts*

$$\mathbf{Vec}_{\mathbb{K}}(A, -) + B$$

of enriched representables and constant functors.

In fact, a limit preserving endofunctor F_0 has a left adjoint G (use Special Adjoint Functor Theorem). For $A = GK$ we have $G \cong A \otimes -$ (since G preserves copowers of K), thus, $F_0 \cong \mathbf{Vec}_{\mathbb{K}}(A, -)$.

4. Generalized Analytic Functors

Throughout this section $(\mathscr{E}, \otimes, I)$ denotes a symmetric monoidal category with colimits. We assume that $\otimes$ commutes with coproducts.

The symmetry isomorphisms $s_{X,Y}\colon X \otimes Y \longrightarrow Y \otimes X$ allow us to define, for every permutation σ in $\mathscr{S}_n$, the corresponding automorphism $X^{\otimes\sigma}$ of $X^{\otimes n} = X \otimes (X \otimes \cdots \otimes X))\dots)$. For example, if σ just swaps i and $i+1$ then $X^{\otimes\sigma} = X^{\otimes(i-1)} \otimes s_{X,X} \otimes X^{\otimes(n-i-1)}$. This extends to all permutations by the rule

$$X^{\otimes \mathrm{id}} = \mathrm{id}_{X^{\otimes n}} \qquad \text{and} \qquad X^{\otimes \sigma\cdot\tau} = X^{\otimes\tau}\cdot X^{\otimes\sigma}$$

since every permutation is a composite of transpositions $i \longleftrightarrow i+1$.

In fact, we obtain a functor

$$K\colon \mathscr{B}^{\mathrm{op}} \times \mathscr{E} \longrightarrow \mathscr{E}$$

given on objects by

$$(n, X) \longmapsto X^{\otimes n}$$

and on morphisms by

$$(\sigma, f) \longmapsto X^{\otimes n} \xrightarrow{X^{\otimes\sigma}} X^{\otimes n} \xrightarrow{f^{\otimes n}} Y^{\otimes n}.$$

The following definition is an obvious extension of the cases $\mathscr{E} = \mathbf{Set}$ and $\mathscr{E} = \mathbf{Vec}_{\mathbb{K}}$ considered by André Joyal in [J$_2$]:

4.1 Definition. *By a $\mathscr{E}$-**species** is meant a functor*

$$f\colon \mathscr{B} \longrightarrow \mathscr{E}.$$

*The corresponding **analytic functor** $F\colon \mathscr{E} \longrightarrow \mathscr{E}$ is the functor defined by the ordinary coend*

$$FX = \int^{n:\mathscr{B}} X^{\otimes n} \otimes f(n). \tag{4.1}$$

4.2 Remark. The coend formula (4.1) indeed defines a functor $F\colon \mathscr{E} \longrightarrow \mathscr{E}$ since $K(n,X) = X^{\otimes n}$ is functorial in X. Thus, for $h\colon X \longrightarrow Y$, $Fh\colon FX \longrightarrow FY$ is defined by

$$Fh = \int^{n:\mathscr{B}} h^{\otimes n} \otimes f(n).$$

4.3 Remarks. (i) The (objectwise) coproduct $\sum_{i\in I} f_i$ of $\mathscr{E}$-species yields a coproduct of the corresponding analytic functors. This follows from $\otimes$ commuting with coproducts.

(ii) Suppose f is a species such that for some $n \in \mathscr{B}$ we have

$$f(n) = V \qquad \text{and} \qquad f(k) = 0 \qquad \text{for all } n \neq k$$

(where 0 is the initial object). Then FX is defined by the joint coequalizer

$$\begin{tikzcd} X^{\otimes n} \otimes V \arrow[loop above, "\bar{\sigma}"] \arrow[d, "k"] & (\sigma \in \mathscr{S}_n) \\ FX \end{tikzcd} \tag{4.2}$$

of the morphisms

$$\bar{\sigma} = X^{\otimes\sigma} \otimes f(\sigma^{-1})\colon X^{\otimes n} \otimes V \longrightarrow X^{\otimes n} \otimes V. \tag{4.3}$$

This follows from the usual formula for coends together with the obvious "shortening" of zig-zags

$$\begin{tikzcd} & X^{\otimes n} \otimes V \arrow[dl, "\mathrm{id} \otimes f(\sigma)"'] \arrow[dr, "X^{\otimes\sigma} \otimes \mathrm{id}"] & \\ X^{\otimes n} \otimes V \arrow[rr, "X^{\otimes\sigma} \otimes f(\sigma^{-1})"'] & & X^{\otimes n} \otimes V \end{tikzcd}$$

(iii) Every species is a coproduct of species of type (ii) above. Thus, every analytic functor is a coproduct of functors given on objects X by the coequalizer (4.2).

4.4 Example. (i) For every $n \in \mathbb{N}$ the n-th tensor power functor

$$FX = X^{\otimes n}$$

is analytic. Its species is defined by

$$f(n) = \mathscr{S}_n \bullet I \qquad \text{and} \qquad f(k) = 0 \text{ for } k \neq n$$

on objects and by $f(\sigma)\cdot i_\tau = i_{\sigma\cdot\tau}$ on morphisms $\sigma \in \mathscr{S}_n$, where $i_\tau\colon I \longrightarrow \mathscr{S}_n \bullet I$ are the injections. In fact, the coequalizer (4.2) is easily seen to be the morphism $k\colon \mathscr{S}_n \bullet X^{\otimes n} \longrightarrow X^{\otimes}$ defined by $k\cdot i_\tau = X^{\otimes\tau}$ for all $\tau \in \mathscr{S}_n$.

(ii) The functor

$$FX = \sum_{n\in\mathbb{N}} X^{\otimes n}$$

is analytic. This is, in case $\mathscr{E} = \mathbf{Set}$, the free-monoid functor and in case $\mathscr{E} = \mathbf{Vec}_{\mathbb{K}}$ (with the tensor product) the functor of tensor algebra on X.

(iii) The analytic endofunctors of the category

$$\mathscr{A} = \mathbf{Set}^S$$

of S-sorted sets are precisely the functors

$$F\big(X_s\big)_{s\in S} = \big(F_s X_s\big)_{s\in S}$$

given by an S-tuple of analytic functors $F_s\colon \mathbf{Set} \longrightarrow \mathbf{Set}$. This follows easily from the fact that a species $f\colon \mathscr{B} \longrightarrow \mathbf{Set}^S$ is just an S-tuple of species in the sense of Section 2.

Every analytic functor is finitary and weakly preserves wide pullbacks—but not conversely. For example the functor $F\colon\ \mathbf{Set}\times\mathbf{Set} \longrightarrow \mathbf{Set}\times\mathbf{Set}$ defined by

$$F(X,Y) = (X\times Y, 1)$$

preserves limits but is not analytic.

(iv) Let $\mathscr{E}$ be a symmetric monoidal closed category. The *endomorphism operad* of an object Z, see [MSS], is the analytic functor whose species is

$$f(n) = [Z^{\otimes n}, Z] \qquad \text{for all } n\in\mathscr{B}.$$

(v) For every group $\mathscr{G} \subseteq \mathscr{S}_n$ we define the functor

$$FX = X^{\otimes n}/\mathscr{G}$$

called *symmetrized tensor power* on objects by the joint coequalizer of all $X^{\otimes\sigma}$, $\sigma\in\mathscr{G}$:

$$\begin{tikzcd} X^{\otimes n} \arrow[loop above, "X^{\otimes\sigma}"] \arrow[d] \\ X^{\otimes n}/\mathscr{G} \end{tikzcd} \qquad (\sigma\in\mathscr{G})$$

This functor is analytic as proved below.

Consequently, coproducts of symmetrized tensor powers are analytic. For example in $\mathbf{Vec}_{\mathbb{K}}$ the functor of symmetric algebra on X

$$FX = \sum_{n\in\mathbb{N}} X^{\otimes n}/\mathscr{S}_n$$

is analytic.

4.5 Remark. (1) The reason why we do not work with $\mathscr{E}$-functors is that important examples of analytic functors are not $\mathscr{E}$-functors. For $\mathscr{E} = \mathbf{Vec}_{\mathbb{K}}$ this means linearity on hom-sets, and it is easy to see that neither the tensor-algebra functor nor the symmetric-algebra functor are linear on hom-sets.

(2) In case of $\mathbf{Set}$ the elegant definition of analytic functors was: they are the Kan extensions of species. This, however, does not work in general. For $\mathbf{Vec}_{\mathbb{K}}$ the Kan extension would be given by $\int^{n:\mathscr{B}} X^n \times f(n)$.

4.6 Observation. Every species in $\mathbf{Set}$, $f\colon \mathscr{B} \longrightarrow \mathbf{Set}$ yields a $\mathscr{E}$-species

$$f_{\mathscr{E}}\colon \mathscr{B} \longrightarrow \mathscr{E}$$

by putting

$$f_{\mathscr{E}}(n) = f(n) \bullet I$$

and analogously on morphisms. For the corresponding analytic functor $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ we denote by $F_{\mathscr{E}}\colon \mathscr{E} \longrightarrow \mathscr{E}$ the analytic endofunctor of $\mathscr{E}$.

In the following examples we consider the monoidal category $\mathbf{Vec}_{\mathbb{K}}$ of vector spaces on a field $\mathbb{K}$.

(i) The free monoid functor P, see 2.3(ii), yields the above tensor algebra functor

$$P_{\mathbf{Vec}_{\mathbb{K}}} X = \sum_{n\in\mathbb{N}} X^{\otimes n}.$$

(ii) The finite multiset functor E of 2.3(iv) yields

$$E_{\mathbf{Vec}_{\mathbb{K}}} X = \sum X^{\otimes n}/\mathscr{S}_n \qquad \text{(symmetric algebra on } X\text{)}.$$

(iii) The tree species, 2.3(v), yields the functor

$$T_{\mathbf{Vec}_{\mathbb{K}}} X = \text{free pre-Lie algebra on } X,$$

see [BL]. Also the free Lie algebras define an analytic functor, see [J$_2$].

4.7 Proposition. *The analytic functors $F_{\mathscr{E}}$ obtained from analytic set functors F are precisely the coproducts of symmetrized tensor powers.*

Proof. (1) Consider the species f defining the analytic set functor $FX = X^n/\mathscr{G}$ in 2.3(iii). The corresponding analytic endofunctor $F_{\mathscr{E}}$ of $\mathscr{E}$ is given by the coequalizer (4.2):

$$\begin{tikzcd}
(\mathscr{S}_n/{\sim_{\mathscr{G}}}) \bullet X^{\otimes n} \arrow[loop above, "\bar\sigma"] \arrow[d, "c"] \\
F_{\mathscr{E}} X
\end{tikzcd}$$

We will verify that $F_{\mathscr{E}}$ is the symmetrized power $X^{\otimes n}/\mathscr{G}$. For every $[\tau] \in \mathscr{S}_n/{\sim_{\mathscr{G}}}$ let $i_{[\tau]}$ be the coproduct injection of $(\mathscr{S}_n/{\sim_{\mathscr{G}}}) \bullet X^{\otimes n}$, then $\bar\sigma$ is defined by

$$\bar\sigma{\cdot}i_{[\tau]} = i_{[\sigma^{-1}\cdot\tau]}{\cdot}X^{\otimes\sigma}.$$

The property $c{\cdot}\bar\sigma = c$ thus yields

$$c{\cdot}i_{[\tau]} = c{\cdot}i_{[\sigma^{-1}\cdot\tau]}{\cdot}X^{\otimes\sigma}$$

for all $\sigma, \tau \in \mathscr{S}_n$. By putting $\sigma = \tau$ we get

$$c{\cdot}i_{[\sigma]} = c{\cdot}i_{[\mathrm{id}]}{\cdot}X^{\otimes\sigma}.$$

For all $\sigma \in \mathscr{G}$ we thus see that, since $[\sigma] = [\mathrm{id}]$, the morphism $c_0 = c \cdot i_{[\mathrm{id}]} \colon X^{\otimes n} \longrightarrow F_{\mathscr{E}} X$ coequalizes $X^{\otimes \sigma}$, $\sigma \in \mathscr{G}$. To prove that c_0 is the joint coequalizer (and therefore $F_{\mathscr{E}} X = X^{\otimes n}/\mathscr{G}$), let $f \colon X^{\otimes n} \longrightarrow Y$ also coequalize $X^{\otimes \sigma}$, $\sigma \in \mathscr{G}$. Define $g \colon (\mathscr{S}_n/{\sim_{\mathscr{G}}}) \bullet X^{\otimes n} \longrightarrow Y$ by

$$g \cdot i_{[\tau]} = f \cdot X^{\otimes \tau}.$$

Then for all $\sigma \in \mathscr{S}_n$ we have

$$\begin{aligned} g \cdot \bar{\sigma} \cdot i_{[\tau]} &= g \cdot i_{[\sigma^{-1} \cdot \tau]} \cdot X^{\otimes \sigma} \\ &= f \cdot X^{\otimes \sigma^{-1} \cdot \tau} \cdot X^{\otimes \sigma} \\ &= f \cdot X^{\otimes \tau} \\ &= g \cdot i_{[\tau]}. \end{aligned}$$

Thus g coequalizes all $\bar{\sigma}$, $\sigma \in \mathscr{S}_n$, and then it uniquely factorizes through c. Consequently, $f = g \cdot i_{[\mathrm{id}]}$ uniquely factorizes through $c_0 = c \cdot i_{[\mathrm{id}]}$.

(2) Every analytic functor $F \colon \mathbf{Set} \longrightarrow \mathbf{Set}$ yields as $F_{\mathscr{E}}$ a coproduct of symmetrized tensor powers. This follows from (1) and 2.7: F is a coproduct of symmetrized representables and the corresponding species is a coproduct of the species in (1).

(3) Every coproduct of symmetrized tensor powers is of the form $F_{\mathscr{E}}$ for the corresponding coproduct F of symmetrized representables—again, this follows from (1). ■

4.8 Example. There are important examples of analytic functors on $\mathbf{Vec}_{\mathbb{K}}$ that are not obtained from set functors:

(i) The functor

$$FX = X \otimes X/(x \otimes x = 0)$$

of the tensor square modulo the subspaces spanned by all the vectors $x \otimes x$. This is given by the species

$$f(2) = \mathbb{K} \times \mathbb{K} \qquad \text{and} \qquad f(k) = 0 \quad \text{for } k \neq 2$$

which assigns to the transposition $\sigma \colon 2 \longrightarrow 2$ the linear map

$$f(\sigma) \colon (u_0, u_1) \longmapsto (u_0, -u_1).$$

In fact, the coequalizer (4.2) yields the equivalence $\sim$ on $X \otimes X \otimes (\mathbb{K} \otimes \mathbb{K})$ given by

$$x \otimes y \otimes (u_0, u_1) \sim y \otimes x \otimes (u_0, -u_1) = -y \otimes x \otimes (u_0, u_1)$$

and the representatives of the form $x \otimes y \otimes (0, 1)$ present the quotient space as precisely FX above.

(ii) For every $n \in \mathbb{N}$ let A_n be the subspace of the tensor power $X^{\otimes n}$ spanned by all vectors $x_1 \otimes \cdots \otimes x_n$ with $x_i = x_j$ for some $i \neq j$. Then the functor

$$FX = \sum_{n \in \mathbb{N}} X^{\otimes n}/A_n \qquad \text{(external algebra of } X\text{)}$$

is analytic. The corresponding species is defined on objects by

$$f(n) = \mathbb{K}^n$$

and on morphisms so that it assigns to the transposition $\sigma_i\colon 0 \longleftrightarrow i$ (for $i = 1, \dots, n-1$) the linear map

$$(u_0, u_1, \dots, u_{n-1}) \longmapsto (u_0, u_1, \dots, u_{i-1}, -u_i, u_{i+1}, \dots, u_{n-1}).$$

4.9 Corollary. *Analytic endofunctors of presheaf categories $\mathscr{E} = [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ are precisely the finitary endofunctors weakly preserving wide pullbacks.*

We also know that these are precisely the functors $F\colon \mathscr{E} \longrightarrow \mathscr{E}$ that are objectwise coproducts of symmetrized representables, that is, for each $C \in \mathscr{C}$ the composite $ev_C{\cdot}F\colon [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathbf{Set}$ has the form

$$ev_C{\cdot}F = \sum_{i \in I} \mathscr{E}(A_i, -)/\mathscr{G}_i \qquad \text{with } A_i \text{ finitely presentable.}$$

See Corollary 3.4.

4.10 Example. Analytic endofunctors of **Gra** are precisely those carried by the analytic endofunctor of $\mathbf{Set} \times \mathbf{Set}$, see 4.4(iv).

5. Conclusions and Open Problems

André Joyal [J_2] introduced analytic endofunctors of **Set** and characterized them as precisely the finitary endofunctors weakly preserving wide pullbacks. We have presented a short proof of that result. Ryu Hasegawa observed in [H] that the category of species, $[\mathscr{B}, \mathbf{Set}]$, is equivalent to the category of analytic functors and weakly cartesian transformations (that is, those for which all naturality squares are weak pullbacks).

In the present paper we study analytic endofunctors in a symmetric monoidal category $\mathscr{E}$. This is a special case of the concept introduced recently by Brian Day [D] who studies Fourier transforms depending on a choice of a kernel; we concentrated on the kernel $X^{\otimes n}$. The case of $\mathscr{E} = \mathbf{Vec}_{\mathbb{K}}$, the monoidal category of vector spaces, has also been studied by Joyal [J_2] who presented a number of examples of important analytic functors; none of them weakly preserves wide pullbacks, as we proved in Proposition 3.8. An open problem is a characterization of the morphisms between analytic functors (corresponding to the transformations between species). An abstract condition has been formulated in [D], however, in the important case of vector spaces a more concrete characterization is desirable.

We also study weak preservation of wide pullbacks by functors $F\colon \mathscr{A} \longrightarrow \mathbf{Set}$. In case $\mathscr{A}$ is locally presentable and F is accessible, we have proved that precisely the coproducts of symmetrized representables weakly preserve wide pullbacks. In contrast, weak preservation of connected limits is equivalent to familial representability: see [CJ] and [FS].

Here accessibility cannot be omitted: the power-set functor weakly preserves wide pullbacks but it is not accessible. A characterization of general set functors preserving weak wide pullbacks is an open problem.

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Jiří Adámek
University of Technology of Braunschweig
Germany

Jiří Velebil
Technical University in Prague
Czech Republic

Email: `Adamek@iti.cs-tu.bs.de`
`Velebil@math.feld.cvut.cz`

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