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KAN EXTENSIONS IN DOUBLE CATEGORIES (ON WEAK DOUBLE CATEGORIES, PART III)

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ABSTRACT.

This paper deals with Kan extensions in a weak double category. *Absolute* Kan extensions are closely related to the orthogonal adjunctions introduced in a previous paper. The *pointwise* case is treated by introducing internal comma objects, which can be defined in an arbitrary double category.

Introduction

This is a sequel to two papers on the general theory of weak (or pseudo) double categories, *'Limits in double categories'* [9] and *'Adjoint for double categories'* [10], which will be referred to as Part I and II, respectively.

In Part I, it was proved that, in a weak double category $\mathbb{A}$, all (small) double limits can be constructed from (small) products, equalisers and *tabulators*, the latter being the double limit of a vertical arrow. Part II deals with the natural notion of adjunctions $G \dashv R$ between weak double categories, where G is a *colax* double functor, while R is *lax*. It is called a *colax/lax* adjunction, and is an instance of an *orthogonal adjunction* in a particular double category (in the sense recalled here in 1.3), namely the (strict) *double category* $\mathbb{D}$bl consisting of weak double categories, lax double functors (as horizontal arrows) and colax double functors (as vertical arrows), with suitable cells (see 1.4).

Here we address Kan extensions *in* a weak double category $\mathbb{D}$, and we begin considering extensions *for* weak double categories.

Section 1 recalls some notions of Part II which are crucial here: orthogonal companions, orthogonal adjoints and $\mathbb{D}$bl. Section 2 defines right Kan extensions in a pseudo double category, and studies the absolute ones. In particular, Thm. 2.5 shows that an orthogonal adjunction amounts to an absolute Kan extension of the identity.

Section 3 introduces internal comma objects, in a *strict* double category $\mathbb{D}$; again, when this is $\mathbb{D}$bl, we find the *double comma* $F{\Downarrow}R$ defined in Part II (for a colax F and a lax R having the same codomain). The general definition of comma objects implies that the solution is, at the same time, a horizontal and a vertical double limit, as made precise

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in Thm. 3.6; the comments of 3.3 give a motivation for choosing such a strong definition (which also requires $\mathbb{D}$ to be strict).

Section 4 defines *pointwise* Kan extensions, via the previous internal comma objects. Also here, an absolute Kan extension corresponding to an orthogonal adjunction is pointwise (Thm. 4.3).

Then, Section 5 studies deeper properties of comma cells. The main result is the Pasting Theorem 5.3, analogous to the pasting properties of homotopy pullbacks in algebraic topology. It follows that pointwise extensions are 'stable under composition' (Thm. 5.4). Kan extensions *for* weak double categories are briefly considered in the last two sections, taking $\mathbb{D}$ to be $\mathbb{D}\mathrm{bl}$ or some other double category of *weak* double categories. They will be the subject of Part IV.

As to literature, let us recall that *strict double categories* where introduced by C. Ehresmann [5, 6], while *bicategories* - the weak version of 2-categories - are due to J. Bénabou [1]. *Weak* (or *pseudo*) double categories, the natural common extension of these two structures, are much more recent and have probably first appeared in published form in our Part I. Other contributions on double categories, weak or strict, are referred to in the previous Parts. Furthermore, [7, 13] deal with homotopy theory and topology, [11, 3] with computer science, [12] with theoretical physics, and [2, 4, 14] with the general theory of double categories.

Size aspects (for double categories of double categories, for instance) can be easily settled working with suitable universes. A reference I.2, or I.2.3, or I.2.3.4 relates to Part I, namely its Section 2, or Subsection 2.3, or item (4) in the latter. Similarly for Part II.

1. Preliminaries

We review some points from Part II: orthogonal companions and adjoints in a double category $\mathbb{D}$ (1.3), the double category $\mathbb{D}\mathrm{bl}$ (1.4) and some double adjunctions (1.5).

1.1. Terminology. For double categories, we use the same terminology and notation as in the previous Parts. The composite of two horizontal arrows $f\colon A \to A'$, $g\colon A' \to A''$ is written gf, while for vertical arrows $u\colon A \to\!\!\!\bullet\ B$, $v\colon B \to\!\!\!\bullet\ C$ we write $u \otimes v$ or $v{\bullet}u$, or just vu (note the reversion and the dot-marked arrows). The boundary of a double cell α, consisting of two horizontal arrows and two vertical ones, as in the left diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v"] \\
B \arrow[r, "g"'] & B'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "v"] \\
B \arrow[r, equal] & B
\end{tikzcd}
\tag{1}
$$

is displayed as $\alpha\colon (u\ (f, g)\ v)$ or also $\alpha\colon u \to v$. It is a *special cell* when the horizontal arrows are identities (as at the right). The horizontal and vertical compositions of cells are written as $(\alpha|\beta)$ and $(\frac{\alpha}{\gamma})$; or also $\alpha\,|\,\beta$ and $\alpha\otimes\gamma$. The symbols 1_A, 1_u (resp. $1^\bullet_A$, $1^\bullet_f$)

denote horizontal (resp. vertical) identities. The set of cells with boundary as in the left diagram above is written $[u\ (f,g)\ v]$.

We generally work with *pseudo double categories* (I.7.1), also called weak double categories, where the horizontal structure behaves categorically, while the composition of vertical arrows is associative up to *comparison cells* $\alpha(u,v,w)\colon (u\otimes v)\otimes w \to u\otimes(v\otimes w)$; these are *special isocells* - horizontally invertible. But we always assume that vertical identities behave *strictly*, a useful simplification, easy to obtain.

Note that, by this unitarity assumption, the vertical composite of three arrows is well-defined *whenever one of them is an identity*: in this case, the associativity isocell $\alpha(u,v,w)$ is an identity (because of its coherence with the relevant unit-comparison cells, which are assumed to be so). Consequently, the *vertical composite of three cells is well-defined whenever each of their two triples of vertical arrows falls in the previous situation*: both associativity isocells are identities (even if identities are placed at different levels). We shall refer to this situation as a *normal* ternary (vertical) composition, of arrows or cells.

A weak double category $\mathbb{A}$ contains a *bicategory* $\mathbf{V}\mathbb{A}$ of vertical arrows and special cells, as well as (*because* of unitarity) a *2-category* $\mathbf{H}\mathbb{A}$ of horizontal arrows and 'vertically special' cells (I.1.9).

Now, a *lax* double functor $R\colon \mathbb{A} \to \mathbb{X}$ between pseudo double categories (II.2.1) preserves the horizontal structure in the strict sense, and the vertical one up to *laxity comparisons*, which are special cells (the *identity* and *composition* comparison)

$$R[A]\colon 1^\bullet_{RA} \to R(1^\bullet_A)\colon RA \to\!\!\!\!\!\bullet\ RA, \qquad R[u,v]\colon Ru\otimes Rv \to R(u\otimes v)\colon RA \to\!\!\!\!\!\bullet\ RC, \tag{2}$$

for A and $u\otimes v\colon A \to\!\!\!\!\!\bullet\ B \to\!\!\!\!\!\bullet\ C$ in $\mathbb{A}$. All this has to satisfy naturality and coherence axioms. (To remember the direction of these cells, one can think of a vertical monad in $\mathbb{A}$ as a lax double functor $\underline{1} \to \mathbb{A}$, defined on the singleton double category.)

This lax R is *unitary* if, for every A in $\mathbb{A}$, the special cell $R[A]\colon 1^\bullet_{RA} \to R1^\bullet_A$ is an identity; then, by coherence, also the following cells are (for $u\colon B \to\!\!\!\!\!\bullet\ A$ and $v\colon A \to\!\!\!\!\!\bullet\ C$)

$$R[u,1^\bullet_A]\colon Ru\otimes R1^\bullet_A \to Ru, \qquad R[1^\bullet_A,v]\colon R1^\bullet_A\otimes Rv \to Rv. \tag{3}$$

Dually, a *colax* double functor $F\colon \mathbb{A} \to \mathbb{X}$ has comparison cells in the opposite direction

$$[FA]\colon F(1^\bullet_A) \to 1^\bullet_{FA}, \qquad F[u,v]\colon F(u\otimes v) \to Fu\otimes Fv. \tag{4}$$

A *pseudo* (resp. *strict*) double functor is a lax one, whose comparison cells are horizontally invertible (resp. identities); or, equivalently, a colax one satisfying the same condition. A pseudo double functor can always be made unitary.

Finally, note that, as a consequence of (3), a *unitary* lax (or colax) double functor defined on the singleton $\underline{1}$ *is necessarily strict.* More generally, this fact holds on every double category where all vertical compositions are trivial (i.e., all pairs of consecutive vertical arrows contain a vertical identity).

1.2. Some examples. Among the pseudo double categories listed in I.3, we shall mostly refer here to $\mathbb{C}$at (formed of categories, functors and profunctors, I.3.1), $\mathbb{S}$et (sets, mappings and spans, I.3.2) and $\mathbb{R}$el (sets, mappings and relations, I.3.4). In $\mathbb{C}$at, a profunctor $F\colon A \to\!\!\!\!\bullet\ B$ is defined as a functor $F\colon A^{\mathrm{op}}\times B \to \mathbf{Set}$.

The double category $\mathbb{D}$bl will be recalled below (1.4). We also consider Ehresmann's double category of quintets $\mathbb{Q}\mathbf{A}$ on a 2-category $\mathbf{A}$, where a double cell $\alpha\colon (F\ (R,S)\ G)$ is defined as a 2-cell $\alpha\colon GR \to SF$ of $\mathbf{A}$ (cf. I.1.3); in particular $\mathbb{Q}\mathbf{Cat}$ gives quintets for the 2-category of categories.

Since our attention is mostly devoted to double categories of 'categories' and 'functors', arrows in double categories will be henceforth denoted with capital letters, like $R, S, T...$ for the horizontal arrows and $F, G, H...$ for the vertical ones.

1.3. Orthogonal companions and adjoints. . Let us recall a few notions, from II.1.2-3. We are in a pseudo double category $\mathbb{D}$.

First, the horizontal morphism $R\colon A \to B$ and the vertical morphism $F\colon A \to\!\!\!\!\bullet\ B$ are presented as *companions* by assigning a pair (η, ϵ) of cells as below, called the *unit* and *counit*, satisfying the identities $\eta \,|\, \epsilon = 1^\bullet_R$, $\eta \otimes \epsilon = 1_F$

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "1\bullet"'] \arrow[dr, phantom, "\eta"] & A \arrow[d, "\bullet F"] \\
A \arrow[r, "R"'] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "R"] \arrow[d, "F\bullet"'] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, "\bullet 1"] \\
B \arrow[r, equal] & B
\end{tikzcd}
\tag{5}
$$

Given R, this is equivalent (*by unitarity*, again) to saying that the pair (F, ϵ) satisfies the following universal property:

(a) for every cell $\epsilon'\colon (F'\ (R,S)\ B)$ there is a unique cell $\lambda\colon (F'\ (A,S)\ F)$ such that $\epsilon' = \lambda \,|\, \epsilon$

$$
\begin{tikzcd}
A \arrow[r, "R"] \arrow[d, "F'\bullet"'] \arrow[dr, phantom, "\epsilon'"] & B \arrow[d, "\bullet 1"] \\
A' \arrow[r, "S"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "F'\bullet"'] \arrow[dr, phantom, "\lambda"] & A \arrow[r, "R"] \arrow[d, "\bullet F"] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, "\bullet 1"] \\
A' \arrow[r, "S"'] & B \arrow[r, equal] & B
\end{tikzcd}
\tag{6}
$$

(Dually, (F, η) is also characterised by a universal property, see II.1(b).) Therefore, if R has a vertical companion, this is determined up to a unique special isocell, *and will often be written as* R_*. Companions compose in the obvious (covariant) way. Companionship is preserved by *unitary* lax or colax double functors.

We say that $\mathbb{A}$ *has vertical companions* if every horizontal arrow has a vertical companion. For instance, in $\mathbb{C}$at, the vertical companion to a functor $R\colon A \to B$ is the associated profunctor $R_*\colon A \to\!\!\!\!\bullet\ B$, $R_*(a,b) = B(R(a), b)$; in $\mathbb{S}$et or $\mathbb{R}$el, the vertical companion of a function $R\colon A \to B$ is the associated span or relation $A \to\!\!\!\!\bullet\ B$ (often identified with R). Any double category of quintets $\mathbb{Q}\mathbf{A}$ has companions: for a map R, take the same arrow. (In fact, as proved in II.1.8, $\mathbb{Q}\mathbf{A}$ is the result of freely adding companions to the 2-category $\mathbf{A}$, viewed as a double category in the usual way.)

Below, we will denote R and R_* with the same letter and view it as a *strong arrow*, which can be placed horizontally or vertically in a diagram, as $R\colon A \to B$ or $R\colon A \to\!\!\!\bullet\; B$, respectively. In most relevant examples, such arrows can be identified without problem, or already are 'the same thing'.

Secondly, transforming companionship by vertical (or horizontal) duality, the arrows $R\colon A \to B$ and $G\colon B \to\!\!\!\bullet\; A$ are presented as *orthogonal adjoints* by a pair (ϵ, η) of cells as below

$$\begin{tikzcd} A \arrow[r, "R"] \arrow[d, "1"'] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, "G"] \\ A \arrow[r, equal] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, equal] \arrow[d, "G"'] \arrow[dr, phantom, "\eta"] & B \arrow[d, "1"] \\ A \arrow[r, "R"'] & B \end{tikzcd} \tag{7}$$

with $\epsilon \,|\, \eta = 1^\bullet_R$ and $\eta \otimes \epsilon = 1_G$. R is called the *horizontal adjoint* and G the *vertical* one. Again, given R, these relations can be described by universal properties for (G, η) or (G, ϵ) (cf. II.1.3).

The vertical adjoint of R is determined up to a special isocell and can be written R^*; vertical adjoints compose, contravariantly: $(SR)^*$ can be obtained as R^*S^*. $\mathbb{A}$ is said to *have vertical (orthogonal) adjoints* if every horizontal arrow has a vertical adjoint.

In $\mathbb{C}$at, the vertical adjoint to a functor $R\colon A \to B$ is the associated profunctor $R^*\colon B \to\!\!\!\bullet\; A$, defined by $R^*(b, a) = B(b, R(a))$. In $\mathbb{S}$et or $\mathbb{R}$el, the vertical adjoint of a function $R\colon A \to B$ is the 'reversed' span or relation $R^\sharp\colon B \to\!\!\!\bullet\; A$. On the other hand, a double category of quintets $\mathbb{Q}\mathbf{A}$ does not have, generally, all vertical adjoints, since our data amount to an adjunction in the 2-category $\mathbf{A}$, $G \dashv R$, with $\epsilon\colon GR \to 1$, $\eta\colon 1 \to RG$.

1.4. A double category of double categories. . Lax and colax double functors do not compose well. But they can be organised in a *strict* double category $\mathbb{D}$bl (introduced in II.2.2) which was, in Part II, the basis for our general notion of an *external* double adjunction, *between* (weak) double categories, and will be, here, one of the basis for Kan extensions in the same external sense.

Its objects are pseudo double categories $\mathbb{A}, \mathbb{B}, \ldots$; its horizontal arrows are the *lax* double functors $R, S \ldots$ between them; its vertical arrows are *colax* double functors $F, G \ldots$ (II.2.1). A cell α

$$\begin{tikzcd} \mathbb{A} \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & \mathbb{B} \arrow[d, "G"] \\ \mathbb{C} \arrow[r, "S"'] & \mathbb{D} \end{tikzcd} \tag{8}$$

is - loosely speaking - a 'horizontal transformation' $\alpha\colon GR \to SF$ (as stressed by the arrow we are placing in the square). But writing $GR \to SF$ is an abuse of notation: these composites are neither lax nor colax (just morphisms of double graphs, respecting the horizontal structure), and the coherence conditions of α require the individual knowledge *of the four 'functors', with their comparison cells.*

Precisely, the cell α consists of:

- the lax double functors R, S; the colax double functors F, G,

- maps $\alpha A\colon GR(A) \to SF(A)$ and cells αu in $\mathbb{D}$ (for all objects A and all vertical arrows $u\colon A \to\!\!\!\!\bullet\; A'$ in $\mathbb{A}$),

$$\begin{tikzcd} GRA \arrow[r, "\alpha A"] \arrow[d, "GRu"'] \arrow[dr, phantom, "\alpha u"] & SFA \arrow[d, "SFu"] \\ GRA' \arrow[r, "\alpha A'"'] & SFA' \end{tikzcd} \tag{9}$$

These data must satisfy the *naturality conditions* (c0), (c1) and the *coherence conditions* (c2), (c3)

$$\begin{aligned}
&\text{(c0)}\quad \alpha A'.GRf = SFf.\alpha A,\\
&\text{(c1)}\quad (GRa\,|\,\alpha v) = (\alpha u\,|\,SFa),\\
&\text{(c2)}\quad (GR[A]\,|\,\alpha 1^\bullet_A\,|\,SF[A]) = (G[RA]\,|\,1^\bullet_{\alpha A}\,|\,S[FA]),\\
&\text{(c3)}\quad (GR[u,v]\,|\,\alpha w\,|\,SF[u,v]) = (G[Ru,Rv]\,|\,(\alpha u\otimes\alpha v)\,|\,S[Fu,Fv]),
\end{aligned} \tag{10}$$

for a horizontal arrow $f\colon A \to A'$, a cell $a\colon (u\ (f,g)\ v)$, an object A and a vertical composite $w = u\otimes v$ in $\mathbb{A}$, respectively. (Actually, (c0) is a consequence of (c1); but writing it down makes things clearer.)

The horizontal and vertical compositions of cells, in $\mathbb{D}$bl, are computed in the obvious way, based in both cases on the *horizontal composition* of cells (in weak double categories), which is strictly associative. This explains, in part, the fact that $\mathbb{D}$bl is a *strict* double category, as proved in II.2.2.

1.5. External adjunctions. As we have already recalled, an orthogonal adjunction in $\mathbb{D}$bl is a *colax/lax* adjunction $G \dashv R$ between weak double categories, where G is a *colax* double functor, while R is *lax*. The cells $\epsilon\colon GR \to 1$, $\eta\colon 1 \to RG$ of (5) will now be called the counit and unit of the adjunction. Various examples can be found in II.5. We only recall two of them.

In II.5.5, we considered the pseudo double categories $\mathbb{S}$et$(= \mathbb{S}$p**Set**) and $\mathbb{C}$osp**Set** of spans and cospans of sets, linked by an obvious colax/lax adjunction $G \dashv R$

$$G\colon \mathbb{S}\mathrm{p}\mathbf{Set} \rightleftarrows \mathbb{C}\mathrm{osp}\mathbf{Set} :R, \qquad \epsilon\colon GR \to 1, \qquad \eta\colon 1 \to RG, \tag{11}$$

where, loosely speaking, G turns a span into its pushout, while R turns a cospan into its pullback. Both G and R are unitary (or can be made so). One can replace **Set** with any category **C** with pullbacks and pushouts.

A variant shows a non-unitary example (II.5.6), based on a 2-category **C** with 2-pullbacks, 2 -pushouts, comma and cocomma squares. There is then a second colax/lax adjunction $C \dashv K$, where C is constructed with cocomma and K with comma squares

$$C\colon \mathbb{S}\mathrm{p}\mathbf{C} \rightleftarrows \mathbb{C}\mathrm{osp}\mathbf{C} :K, \qquad \epsilon\colon CK \to 1, \qquad \eta\colon 1 \to KC. \tag{12}$$

2. Kan extensions in weak double categories

We introduce general and absolute Kan extensions *in* an arbitrary weak double category $\mathbb{D}$.

2.1. Kan extensions in a 2-category. Let us recall that, *in* a 2-category $\mathbf{D}$, the *right Kan extension* of an arrow S along an arrow R is an arrow $G = \mathrm{Ran}_R(S)$ equipped with a 2-cell $\epsilon\colon GR \to S$ so that the pair (G, ϵ) is universal

$$\begin{tikzcd} I \arrow[r, "R"] \arrow[d, equal] & J \arrow[r, equal] \arrow[d, "G"] \arrow[dl, phantom, "\epsilon"] & J \arrow[d, "G'"] \arrow[dl, phantom, "\tau"] \\ I \arrow[r, "S"'] & A \arrow[r, equal] & A \end{tikzcd} \quad = \quad \begin{tikzcd} I \arrow[r, "R"] \arrow[d, equal] & J \arrow[d, "G'"] \arrow[dl, phantom, "\alpha"] \\ I \arrow[r, "S"'] & A \end{tikzcd} \tag{13}$$

i.e., every similar pair (G', α) factors through the former, by a unique 2-cell $\tau\colon G' \to G$.

This extension is said to be *absolute* if, for every R', S', G' as below, the following mapping between sets of 2-cells is bijective

$$[G'R', S'G] \to [G'R'R, S'S], \qquad\qquad \tau \mapsto (\epsilon \,|\, \tau), \tag{14}$$

$$\begin{tikzcd} I \arrow[r, "R"] \arrow[d, equal] & J \arrow[r, "R'"] \arrow[d, "G"] \arrow[dl, phantom, "\epsilon"] & J' \arrow[d, "G'"] \arrow[dl, phantom, "\tau"] \\ I \arrow[r, "S"'] & A \arrow[r, "S'"'] & A' \end{tikzcd} \quad = \quad \begin{tikzcd} I \arrow[r, "R'R"] \arrow[d, equal] & J' \arrow[d, "G'"] \arrow[dl, phantom, "\alpha"] \\ I \arrow[r, "S'S"'] & A' \end{tikzcd}$$

(A more faithful rendering of the definition would have R' the identity, but we stated it in this equivalent form with an eye on our version for double categories.)

On the other hand, assuming that our 2-category $\mathbf{D}$ *has comma objects*, G is the *pointwise* right Kan extension of S along R [15] if it is stable under comma squares: for every arrow H

$$\begin{tikzcd} H{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] & J' \arrow[d, "H"] \arrow[dl, phantom, "\omega"] \\ I \arrow[r, "R"'] \arrow[d, equal] & J \arrow[d, "G"] \arrow[dl, phantom, "\epsilon"] \\ I \arrow[r, "S"'] & A \end{tikzcd} \tag{15}$$

the comma-object $H{\downarrow}R$ gives $GH = \mathrm{Ran}_P(SQ)$, via the pasted cell $\vartheta = \epsilon Q.G\omega$.

2.2. Kan extensions in a double category. We introduce now Kan extensions *in* a pseudo double category $\mathbb{D}$. Given a vertical arrow F and two horizontal arrows R, S

$$\begin{tikzcd} I \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\epsilon"] & J \arrow[d, dotted, "G"] \\ X \arrow[r, "S"'] & A \end{tikzcd} \tag{16}$$

we say that the vertical arrow $G \colon J \to A$ is the *right Kan extension* of F *along* the horizontal arrows R, S (or, *from* R *to* S), via the cell ϵ, if:

(Ran) for every $G' \colon J \to A$, the mapping $[G\ (J, A)\ G'] \to [F\ (R, S)\ G']$, $\tau \mapsto (\epsilon \,|\, \tau)$ is bijective.

The vertical arrow G is determined up to a special isocell and written $\mathrm{Ran}_{R,S}(F)$. The cell ϵ is called the *counit* of the extension, and a *right Kan square*.

Further, we say that G is an *absolute extension* if, for every R', S' and G' as below, there is a bijection:

$$[G\ (R', S')\ G'] \to [F\ (R'R, S'S)\ G'], \qquad \tau \mapsto (\epsilon \,|\, \tau), \tag{17}$$

$$\begin{tikzcd} I \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\epsilon"] & J \arrow[r, "R'"] \arrow[d, "G"] \arrow[dr, phantom, "\tau"] & J' \arrow[d, "G'"] \\ X \arrow[r, "S"'] & A \arrow[r, "S'"'] & A' \end{tikzcd} = \begin{tikzcd} I \arrow[r, "R'R"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & J' \arrow[d, "G'"] \\ X \arrow[r, "S'S"'] & A' \end{tikzcd}$$

The *pointwise* case needs comma objects in $\mathbb{D}$ and will be treated in Section 4.

2.3. Dual notions. Applying transpose duality in $\mathbb{D}$ (i.e., symmetry with respect to the 'main diagonal'), we get a left Kan extension $S = \mathrm{Lan}_{F,G}(R)$: the horizontal arrow S is the *left Kan extension* of R along the vertical arrows F, G (or, from F to G), via the cell η, if:

$$\begin{tikzcd} I \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\eta"] & J \arrow[d, "G"] \\ X \arrow[r, dotted, "S"'] & A \end{tikzcd} \tag{18}$$

(Lan) for every $S' \colon X \to A$, the mapping $[X\ (S, S')\ A] \to [F\ (R, S')\ G]$, $\tau \mapsto \eta \otimes \tau$ is bijective.

On the other hand, horizontal duality in $\mathbb{D}$ yields the notion of Kan *liftings*, which will not be studied here. In the left diagram below, the lifting (F, ϵ) gives a bijection

$\tau \mapsto (\tau \,|\, \epsilon)$

$$
\begin{tikzcd}
I \arrow[r, equal] \arrow[d, "F'"'] \arrow[dr, phantom, "\tau"] & I \arrow[r, "R"] \arrow[d, dotted, "F"'] \arrow[dr, phantom, "\epsilon"] & A \arrow[d, "G"] \\
X \arrow[r, equal] & X \arrow[r, "S"'] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
I \arrow[r, "R'"] \arrow[d, equal] \arrow[dr, phantom, "\tau"] & J \arrow[d, equal] \\
I \arrow[r, dotted, "R"'] \arrow[d, "F"'] \arrow[dr, phantom, "\epsilon"] & A \arrow[d, "G"] \\
X \arrow[r, "S"'] & A
\end{tikzcd}
\tag{19}
$$

while, at the right hand, the lifting (R, ϵ) gives a bijection $\tau \mapsto \tau \otimes \epsilon$.

2.4. Theorem. [Pasting properties] *Let $\epsilon'' = (\epsilon \,|\, \epsilon')$ be a horizontal composition of cells, in a pseudo double category $\mathbb{D}$, and ϵ be an absolute right Kan square. Then:*

$$
\begin{tikzcd}
I \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\epsilon"] & J \arrow[r, "R'"] \arrow[d, "G"] \arrow[dr, phantom, "\epsilon'"] & J' \arrow[r, dotted, "R''"] \arrow[d, "G'"] \arrow[dr, phantom, "\tau"] & J'' \arrow[d, dotted, "G''"] \\
X \arrow[r, "S"'] & A \arrow[r, "S'"'] & A' \arrow[r, dotted, "S''"'] & A''
\end{tikzcd}
\tag{20}
$$

(a) ϵ' is a right Kan square if and only if ϵ'' is;

(b) ϵ' is an absolute right Kan square if and only if ϵ'' is.

Proof. We prove (b); the proof of (a) is the same, taking R'' and S'' to be horizontal identities.

Suppose that ϵ' is an absolute right Kan square; given a cell α on the 'outer perimeter' of (20), there is precisely one $\beta\colon (G\ (R''R', S''S')\ G'')$ such that $(\epsilon \,|\, \beta) = \alpha$, and then precisely one $\tau\colon (G'\ (R'', S'')\ G'')$ such that $(\epsilon' \,|\, \tau) = \beta$.

Conversely, if $(\epsilon|\epsilon')$ is an absolute right Kan square, given a cell $\beta\colon (G\,(R''R', S''S')\,G'')$, the cell $\alpha = (\epsilon \,|\, \beta)$ yields precisely one τ such that $(\epsilon \,|\, \epsilon' \,|\, \tau) = (\epsilon \,|\, \beta)$, which means - by the universal property of ϵ - precisely one τ such that $(\epsilon' \,|\, \tau) = \beta$. ■

2.5. Theorem. [The formal vertical adjoint theorem] *In a pseudo double category $\mathbb{D}$, the horizontal arrow R has a vertical adjoint (1.3) if and only if the right Kan extension* $\mathrm{Ran}_{R,1}(1_A)$ *exists and is absolute. Then, $G = \mathrm{Ran}_{R,1}(1_A)$ is the vertical adjoint and the counit ϵ of the extension is also the ϵ-cell of the orthogonal adjunction*

$$
\begin{tikzcd}
A \arrow[r, "R"] \arrow[d, "1"'] \arrow[dr, phantom, "\epsilon"] & X \arrow[d, dotted, "G"] \\
A \arrow[r, "1"'] & A
\end{tikzcd}
\tag{21}
$$

PROOF. First, let R and G be orthogonal adjoints in $\mathbb{D}$, with cells ϵ and η (as in the diagrams below) satisfying the equations $(\epsilon\,|\,\eta) = 1^\bullet_R$, $\eta \otimes \epsilon = 1_G$.

Given $\alpha\colon (1^\bullet\ (R'R, S')\ G')$, the equation $(\epsilon\,|\,\tau) = \alpha$ has precisely one solution as a cell $\tau\colon (G\ (R', S')\ G')$, namely $\tau = (\eta\,|\,1^\bullet_{R'}) \otimes \alpha$ (see the left diagram below)

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "G"'] \arrow[dr, phantom, "\eta"] & X \arrow[r, "R'"] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "1^\bullet"] & X' \arrow[d, "1^\bullet"] \\
A \arrow[r, "R"'] \arrow[d, "1^\bullet"'] \arrow[drr, phantom, "\alpha"] & X \arrow[r, "R'"'] & X' \arrow[d, "G'"] \\
A \arrow[r, equal] & A \arrow[r, "S'"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "R"] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\epsilon"] & X \arrow[r, equal] \arrow[d, "G"] \arrow[dr, phantom, "\eta"] & X \arrow[d, "1^\bullet"] \\
A \arrow[r, equal] & A \arrow[r, "R"'] & X
\end{tikzcd}
\tag{22}
$$

Conversely, let $G = \mathrm{Ran}_{R,1}(1_A)$ be an absolute extension, via ϵ. Then, ϵ satisfies the universal property II.1.3(b), which says that it is a part of an orthogonal adjunction.

(Directly, one can obtain the other cell η by the equation $(\epsilon\,|\,\eta) = 1^\bullet_R$ in the right diagram, and then prove the second triangular identity $\eta \otimes \epsilon = 1_G$ by cancelling ϵ in the equation $(\epsilon\,|\,\eta\otimes\epsilon) = \epsilon$. The latter follows from the previous equation by middle-four interchange.) ■

2.6. EXAMPLES OF INTERNAL KAN EXTENSIONS. . The previous statement gives examples of absolute Kan extensions in various weak double categories.

In fact, orthogonal adjoints in $\mathbb{C}$at, $\mathbb{S}$et and $\mathbb{R}$el, have been recalled in 1.3. We have also recalled that an orthogonal adjunction in $\mathbb{D}$bl is an *external* colax/lax adjunction $G \dashv R$ *between* weak double categories, as studied in Part II (1.5); this leads to *external* Kan extensions for weak double categories, whose examination will begin in Section 6.

2.7. LEMMA. [Replacement] *In a strict double category* $\mathbb{D}$, *two cells* ϵ, λ *are given, as in the diagram below*

$$
\begin{tikzcd}
I' \arrow[r, "T"] \arrow[d, "L"'] \arrow[dr, phantom, "\lambda"] & J' \arrow[d, "L'"] \\
I \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\epsilon"] & J \arrow[d, "G"] \\
X \arrow[r, "S"'] & A
\end{tikzcd}
\tag{23}
$$

(a) Suppose there exist two cells φ, μ *as below, with* $(\varphi\,|\,\mu\otimes\lambda) = 1^\bullet_R$. *If* $GL' = \mathrm{Ran}_{T,S}(FL)$ *via* $\lambda\otimes\epsilon$, *then* $G = \mathrm{Ran}_{R,S}(F)$ *via* ϵ.

$$\begin{tikzcd}
I \arrow[r, equal] \arrow[dd, "1^\bullet"'] & I \arrow[r, "R"] \arrow[d, "M"'] \arrow[dr, phantom, "\mu"] & J \arrow[d, "M'"] \\
{} \arrow[r, phantom, "\varphi"] & I' \arrow[r, "T"] \arrow[d, "L"'] \arrow[dr, phantom, "\lambda"] & J' \arrow[d, "L'"] \\
I \arrow[r, equal] & I \arrow[r, "R"'] & J
\end{tikzcd}$$

(b) Suppose there exist three cells φ, μ, ψ as in the diagrams below, with $1_L \otimes \varphi = 1_L$ and $(\lambda \otimes \mu \,|\, \psi) = 1^\bullet_T$. Then the converse holds: if $G = \mathrm{Ran}_{R,S}(F)$ via ϵ, then $GL' = \mathrm{Ran}_{T,S}(FL)$ via $\lambda \otimes \epsilon$.

$$\begin{tikzcd}
I \arrow[r, equal] \arrow[dd, "1^\bullet"'] & I \arrow[d, "M"] \\
{} \arrow[r, phantom, "\varphi"] & I' \arrow[d, "L"] \\
I \arrow[r, equal] & I
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
I' \arrow[r, "T"] \arrow[d, "L"'] \arrow[dr, phantom, "\lambda"] & J' \arrow[r, equal] \arrow[d, "L'"] & J' \arrow[dd, "1^\bullet"] \\
I \arrow[r, "R"] \arrow[d, "M"'] \arrow[dr, phantom, "\mu"] & J \arrow[d, "M'"] \arrow[r, phantom, "\psi"] & {} \\
I' \arrow[r, "T"'] & J' \arrow[r, equal] & J'
\end{tikzcd}$$

Note. This statement will become clearer under stronger hypotheses (5.2), assuming that T is a 'vertical deformation retract' of R via λ, μ, φ, in the sense of 5.1.

Proof. (a) Take a 'test' vertical arrow $G' \colon J \,\multimap\, A$ and a cell α as in the left diagram below. Then, the cell $\lambda \otimes \alpha$ gives a (unique) special cell $\rho \colon (GL' \ (1,1) \ G'L')$ such that $(\lambda \otimes \epsilon \,|\, \rho) = \lambda \otimes \alpha$; but then

$$(\varphi \otimes 1_F \,|\, \mu \otimes \lambda \otimes \epsilon \,|\, 1_{M'} \otimes \rho) = (\varphi \otimes 1_F \,|\, \mu \otimes \lambda \otimes \alpha). \tag{24}$$

Applying on both sides of this equality the hypothesis $(\varphi \,|\, \mu \otimes \lambda) = 1^\bullet_R$, we obtain $(\epsilon \,|\, 1_{M'} \otimes \rho) = \alpha$. On the other hand, if $(\epsilon \,|\, \tau) = (\epsilon \,|\, \tau')$, then $(\lambda \otimes \epsilon \,|\, 1_{L'} \otimes \tau) = (\lambda \otimes \epsilon \,|\, 1_{L'} \otimes \tau')$

and $1_{L'}\otimes\tau = 1_{L'}\otimes\tau'$, whence $\tau = \tau'$ (precomposing with $1_{M'}$ and recalling that $L'M' = 1$).

$$
\begin{tikzcd}
I \arrow[r, equal] \arrow[dd, "1^\bullet"'] & I \arrow[r, "R"] \arrow[d, "M"'] \arrow[dr, phantom, "\mu"] & J \arrow[d, "M'"] \\
{} \arrow[r, phantom, "\varphi"] & I' \arrow[r, "T"] \arrow[d, "L"'] \arrow[dr, phantom, "\lambda"] & J' \arrow[d, "L'"] \\
I \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, phantom, "1_F"] & I \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & J \arrow[d, "G'"] \\
X \arrow[r, equal] & X \arrow[r, "S"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \arrow[r, equal] \arrow[d, "L"'] \arrow[dr, phantom, "1_L"] & I' \arrow[r, "T"] \arrow[d] \arrow[dr, phantom, "\lambda"] & J' \arrow[r, equal] \arrow[d, "L'"] & J' \arrow[dd, "1^\bullet"] \\
I \arrow[r, equal] \arrow[dd, "1^\bullet"'] & I \arrow[r, "R"] \arrow[d, "M"'] \arrow[dr, phantom, "\mu"] & J \arrow[d, "M'"] \arrow[r, phantom, "\psi"] & {} \\
{} \arrow[r, phantom, "\varphi"] & I' \arrow[r, "T"] \arrow[d, "L"'] \arrow[dr, phantom, "\beta"] & J' \arrow[r, equal] \arrow[dd, "K"] \arrow[dr, phantom, "1_K"] & J' \arrow[dd, "K"] \\
I \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, phantom, "1_F"] & I \arrow[d, "F"] & {} & {} \\
X \arrow[r, equal] & X \arrow[r, "S"'] & \mathbf{A} \arrow[r, equal] & A
\end{tikzcd}
\tag{25}
$$

(b) Now $G = \mathrm{Ran}_{R,S}(F)$ via ϵ, and we must prove that $GL' = \mathrm{Ran}_{T,S}(FL)$, via $\lambda\otimes\epsilon$. Take a cell β as in the right diagram above. Then, for the cell $(\varphi\otimes 1_F\,|\,\mu\otimes\beta)$ there exists a (unique) cell τ such that $(\epsilon\,|\,\tau) = (\varphi\otimes 1_F\,|\,\mu\otimes\beta)$; therefore, the cell $(1_{L'}\otimes\tau\,|\,\psi\otimes 1_K)$ satisfies

$$(\lambda\otimes\epsilon\,|\,1_{L'}\otimes\tau\,|\,\psi\otimes 1_K) = (\lambda\otimes(\varphi\otimes 1_F\,|\,\mu\otimes\beta)\,|\,\psi\otimes 1_K) = (1_L\otimes\varphi\otimes 1_F\,|\,((\lambda\otimes\mu\,|\,\psi)\otimes\beta)) = \beta, \tag{26}$$

because $(\lambda\otimes\mu\,|\,\psi) = 1$ and $1_L\otimes\varphi = 1$. Finally, if $(\lambda\otimes\epsilon\,|\,\rho) = (\lambda\otimes\epsilon\,|\,\rho')$, we have $(\mu\otimes\lambda\otimes\epsilon\,|\,\rho) = (\mu\otimes\lambda\otimes\epsilon\,|\,\rho')$ and, precomposing with $\varphi\otimes 1_F$, $(\epsilon\,|\,\rho) = (\epsilon\,|\,\rho')$, whence $\rho = \rho'$. ■

3. Internal comma objects

Comma objects in $\mathbb{D}$bl have been constructed in Part II, and shown to satisfy both a *horizontal* and a *vertical* universal property. Here, we define comma objects in an abstract (strict) double category $\mathbb{D}$, by a system of three universal properties based on the orthogonal companions of 1.3 and the *commutative cells* they produce. (The last notion makes sense also in weak double categories.)

3.1. Commutative cells. In the weak double category $\mathbb{D}$ four arrows are given, as below

$$
\begin{tikzcd}
\bullet \arrow[r, "R"] \arrow[d, "F"'] & \bullet \arrow[d, "G"] \\
\bullet \arrow[r, "S"'] & \bullet
\end{tikzcd}
\tag{27}
$$

In the following two cases one can define the *commutative cell on this boundary*, using a ‘flipping procedure’ already considered in II.1.6 in a more general form. It will be written as $\lambda\colon (GR = SF)$; in a diagram, such commutative cells will often be denoted by a boldface Greek letter.

(a) If the *vertical* arrows F, G are strong (with horizontal companions denoted by the same letters) *and* $GR = SF$, then λ is defined by the pasting of the left diagram below (containing the unit of G, the counit of F and two vertical identities)

$$
\begin{tikzcd}
\bullet \arrow[r, "R"] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "1^\bullet"] & \bullet \arrow[r, equal] \arrow[d, "1^\bullet"] \arrow[dr, phantom, "\eta_G"] & \bullet \arrow[d, "G_*"] \\
\bullet \arrow[r, "R", "F"'] \arrow[d, "F_*"'] \arrow[dr, phantom, "\epsilon_F"] & \bullet \arrow[r, "G", "S"'] \arrow[d, "1^\bullet"] \arrow[dr, phantom, "1^\bullet"] & \bullet \arrow[d, "1^\bullet"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "S"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, phantom, "1_F"] & \bullet \arrow[r, "R"] \arrow[d, "F"', "R"] \arrow[dr, phantom, "\epsilon_R"] & \bullet \arrow[d, "1^\bullet"] \\
\bullet \arrow[r, equal] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\eta_S"] & \bullet \arrow[r, equal] \arrow[d, "S"', "G"] \arrow[dr, phantom, "1_G"] & \bullet \arrow[d, "G"] \\
\bullet \arrow[r, "S"'] & \bullet \arrow[r, equal] & \bullet
\end{tikzcd}
\tag{28}
$$

(b) If the *horizontal* arrows R, S are strong (with vertical companions denoted by the same letters) *and* $SF = GR$ (i.e., $S_*F = GR_*$), then λ is defined by the pasting of the right diagram above.

If all the four arrows of the boundary are strong, we can proceed either way, and we get *strictly* the same result, *provided we proceed in a coherent way*, as made explicit in Lemma 3.5. Here, one should carefully distinguish between horizontal and vertical arrows, starting with a commutative square $GR = SF$ of *horizontal* arrows (for instance), then replacing F and G in diagram (27) with their vertical companions F_* and G_*, and constructing, with the given units and counits, a special isocell *realising the identification* $G_*R_* = S_*F_*$.

Let us assume we have chosen in $\mathbb{D}$ a 'coherent choice of companions', consistent with composition (also for units and counits); this choice need not be global, but might be restricted to a suitable subcategory of horizontal arrows. Then, with reference to this system, *a commutative cell is determined by its boundary* and commutative cells are closed under horizontal and vertical composition (forming a *flat* double subcategory of $\mathbb{D}$).

3.2. Comma objects. Suppose we have, in the strict double category $\mathbb{D}$, a vertical arrow F and a horizontal arrow R with the same codomain. The (internal) *comma object* $F{\downarrow}R$ comes with two projections P, Q which are *strong arrows* (1.3) and a cell ω, called a *comma cell*

$$
\begin{tikzcd}
F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\
X \arrow[r, "R"'] & C
\end{tikzcd}
\tag{29}
$$

satisfying the following three universal properties:

(a) (*horizontal property*) For every pair of *horizontal* arrows $S\colon U \to A$, $T\colon U \to X$ and every cell $\alpha\colon (1\ (S, RT)\ F)$ there is a unique horizontal arrow L such that $PL = S$,

$QL = T$ and moreover the commutative cell $\lambda\colon (QL = T)$ gives $(\lambda\,|\,\omega) = \alpha$

$$
\begin{tikzcd}
U \arrow[rr, "S"] \arrow[d, "1^\bullet"'] & & A \arrow[d, "F"] \\
U \arrow[r, "T"'] & X \arrow[r, "R"'] \arrow[u, phantom, "\alpha"] & C
\end{tikzcd}
\;=\;
\begin{tikzcd}
U \arrow[r, "L"] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\boldsymbol{\lambda}"] & F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\
U \arrow[r, "T"'] & X \arrow[r, "R"'] & C
\end{tikzcd}
\tag{30}
$$

(b) (*vertical property*) For every pair of *vertical* arrows $G\colon V \to\!\!\!\!\bullet\; X$, $H\colon V \to\!\!\!\!\bullet\; A$ and every cell $\beta\colon (G\ (1, R)\ FH)$ there is a unique vertical arrow M such that $PM = H$, $QM = G$ and the commutative cell $\mu\colon (H = PM)$ gives $\mu\otimes\omega = \beta$

$$
\begin{tikzcd}
V \arrow[r, equal] \arrow[dd, "G"'] & V \arrow[d, "H"] \\
\arrow[r, phantom, "\beta"] & A \arrow[d, "F"] \\
X \arrow[r, "R"'] & C
\end{tikzcd}
\;=\;
\begin{tikzcd}
V \arrow[r, equal] \arrow[d, "M"'] \arrow[dr, phantom, "\boldsymbol{\mu}"] & V \arrow[d, "H"] \\
F{\downarrow}R \arrow[r, "P"'] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\
X \arrow[r, "R"'] & C
\end{tikzcd}
\tag{31}
$$

(c) (*global property*) Suppose we have four arrows L, L', M, M', as in the left diagram below. Form the commutative cells λ, μ, and let $\alpha = (\lambda\,|\,\omega)$, $\beta = \mu\otimes\omega$ (as in the previous properties)

$$
\begin{tikzcd}
W \arrow[r, "L'"] \arrow[d, "M'"'] \arrow[dr, phantom, "\sigma"] & V \arrow[r, equal] \arrow[d, "M"'] \arrow[dr, phantom, "\boldsymbol{\mu}"] & V \arrow[d, "H"] \\
U \arrow[r, "L"] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\boldsymbol{\lambda}"] & F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\
U \arrow[r, "T"'] & X \arrow[r, "R"'] & C
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[drr, phantom, "\alpha'"] & \bullet \arrow[r, equal] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d, "1^\bullet"'] \arrow[drr, phantom, "\alpha"] & \bullet \arrow[r] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r, equal] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r, phantom, "\beta'"] \arrow[d, "1^\bullet"'] & \bullet \arrow[r, phantom, "\beta"] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}
\tag{32}
$$

then, for every pair of cells (α', β') as above, linked by the coherence equation $\alpha'\otimes\alpha = (\beta'\,|\,\beta)$ (displayed at the right), there is precisely one cell $\sigma\colon (M'\ (L', L)\ M)$ such that:

$$
\begin{gathered}
\sigma\,|\,\mu = \alpha', \qquad \sigma\otimes\lambda = \beta' \\
(\alpha'\otimes\alpha = (\beta'\,|\,\beta),\ \ \alpha' \in [M'\ (L', PL)\ H],\ \ \beta' \in [M'\ (L', T)\ QM]).
\end{gathered}
\tag{33}
$$

We say that $\mathbb{D}$ *has comma objects* when all comma objects $F{\downarrow}R$ exist, for every vertical F and horizontal R having the same codomain.

To study the relationship of comma objects with double limits, it will be useful to consider two particular instances of the global property:

(d) we shall speak of the *horizontal 2-dimensional property* when we restrict M to be the *vertical* identity of $F{\downarrow}R$; and of the *vertical 2-dimensional property* when we restrict L to be the *horizontal* identity of $F{\downarrow}R$.

It is easy to see, in the examples below, that the uniqueness of λ (or μ) fails if we do not require that it be a commutative cell: one can start with a non-commutative cell λ' with the same boundary and replace it with the commutative λ such that $(\lambda\,|\,\omega) = (\lambda'\,|\,\omega)$.

3.3. Comments. . A comma object is a sort of 'symmetric double limit', related with the *horizontal* double limits dealt with in Part I *and* with the symmetrical notion of a *vertical* double limit, by transpose duality (the symmetry with respect to the main diagonal of double cells).

In fact, by flipping the strong arrow $Q\colon F{\downarrow}R \to C$ to its horizontal companion, the cell ω (29) 'becomes' a cell $\zeta = (\eta_Q\,|\,\omega)$

$$
\begin{tikzcd}
F{\downarrow}R \arrow[r, "P"] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\zeta"] & A \arrow[d, "F"] \\
F{\downarrow}R \arrow[r, "RQ"'] & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F{\downarrow}R \arrow[r, equal] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\eta_Q"] & F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\
F{\downarrow}R \arrow[r, "Q"'] & X \arrow[r, "R"'] & C
\end{tikzcd}
\tag{34}
$$

which is (or trivially amounts to) a horizontal cone for the diagram (F, R) in $\mathbb{D}$, as defined in Part I.

Symmetrically, by flipping the strong arrow $P\colon F{\downarrow}R \to A$ to its vertical companion, the cell ω becomes a cell ϑ

$$
\begin{tikzcd}
F{\downarrow}R \arrow[r, equal] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\vartheta"] & F{\downarrow}R \arrow[d, "FP"] \\
X \arrow[r, "R"'] & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F{\downarrow}R \arrow[r, equal] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\eta_P"] & F{\downarrow}R \arrow[d, "P"] \\
F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\
X \arrow[r, "R"'] & C
\end{tikzcd}
\tag{35}
$$

which is a *vertical* cone of the diagram (F, R) *in* $\mathbb{D}$ (as defined by transpose duality).

We shall prove, in Thm. 3.6, that the horizontal (resp. vertical) universal property of the comma is equivalent to the 1-dimensional universal property of the horizontal (resp. vertical) double limit of the diagram (F, R). Furthermore, the global universal property 3.2(c) is a symmetric way of 'upgrading' *both* properties to dimension 2. Thus, a comma object as defined above is *at the same time* a horizontal and a vertical double limit (in the full, 2-dimensional sense), linked by companionship of projections. Of course, requiring that both such limits exist and coincide is a very strong condition, and we can only expect our comma objects to exist in very particular double categories, having some sort of symmetry in themselves, like $\mathbb{D}$bl and the double categories of quintets (cf. 3.4). We also note that, without assuming $\mathbb{D}$ to be strict, the vertical properties should be stated in a much more complex way.

We end with remarking that it *would be possible* to deal with *right* Kan extensions using just *vertical comma objects*, defined as *vertical* double limits or, equivalently, by the vertical property 3.2(b) together with the vertical 2-dimensional property in 3.2(d) (cf. Thm. 3.6). Symmetrically, *left* Kan extensions would be dealt with *horizontal* comma objects, defined as *horizontal double limits*. However, since we need both aspects and since both kind of limits coincide wherever we need them, it seems simpler to assume this coincidence from the beginning.

3.4. Examples. (a) In the double category of quintets $\mathbb{Q}\mathbf{Cat}$, we get the usual comma category $F{\downarrow}R$, with objects $(a, x; c\colon Fa \to Rx)$, where a is an object of A, x an object of X and c a map of C.

The first two universal properties work in the usual way, with functors L, M defined as follows, on the objects:

$$\begin{aligned} &L(u) = (Su, Tu;\ \alpha u\colon FSu \to RTu) &&\text{(for } u \text{ in } U),\\ &M(v) = (Hv, Gv;\ \beta v\colon FHv \to RGv) &&\text{(for } v \text{ in } V). \end{aligned} \tag{36}$$

But the *global* universal property, in this context, is interesting, and perhaps new. The natural transformation $\sigma\colon ML' \to LM'$ must have components $\sigma(w) = (\alpha'(w), \beta'(w))$

$$\begin{tikzcd} HL'w \arrow[d, "\alpha' w"'] \\ SM'w \end{tikzcd} \qquad \begin{tikzcd} GL'w \arrow[d, "\beta' w"'] \\ TM'w \end{tikzcd} \qquad \begin{tikzcd} FHL'w \arrow[r, "\beta L'w"] \arrow[d, "F\alpha' w"'] & RGL'w \arrow[d, "R\beta' w"] \\ FSM'w \arrow[r, "\alpha M'w"'] & RTM'w \end{tikzcd} \tag{37}$$

Conversely, defining σ this way is legitimate: the square above commutes precisely because of the coherence condition $\alpha' \otimes \alpha = (\beta' \mid \beta)$. Finally:

$$(\sigma \mid \mu)(w) = P\sigma(w) = \alpha' w, \qquad (\sigma \otimes \lambda)(w) = Q\sigma(w) = \beta' w. \tag{38}$$

(b) Comma objects in $\mathbb{D}\mathrm{bl}$ have been constructed in II.2.5 and will be a crucial tool of Part IV. Here, we only recall that a colax double functor F and a lax double functor R with the same codomain have a comma pseudo double category $F{\downdownarrows}R$, forming a cell ω in $\mathbb{D}\mathrm{bl}$

$$\begin{tikzcd} F{\downdownarrows}R \arrow[r, "P"] \arrow[d, "Q"', "\bullet" marking] \arrow[dr, phantom, "\omega"] & \mathbb{A} \arrow[d, "F", "\bullet" marking] \\ \mathbb{X} \arrow[r, "R"'] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "c"] \arrow[d, "Fa"', "\bullet" marking] \arrow[dr, phantom, "="] & RX \arrow[d, "Rx", "\bullet" marking] \\ FA' \arrow[r, "c'"'] & RX' \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "c"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & RX \arrow[d, "Rv", "\bullet" marking] \\ FB \arrow[r, "d"'] & RY \end{tikzcd} \tag{39}$$

An object of the pseudo double category $F{\downdownarrows}R$ is a triple $(A, X; c\colon FA \to RX)$; a horizontal map $(a, x)\colon (A, X; c) \to (A', X'; c')$ comes from a commutative square of $\mathbb{C}$, as in the middle diagram above; a vertical arrow $(u, v; \gamma)\colon (A, X; c) \to (B, Y; d)$ comes from a cell $\gamma\colon (Fu\ (c, d)\ Rv)$ in $\mathbb{C}$, as in the right diagram above. The projections P, Q are strict double functors (and strong arrows in $\mathbb{D}\mathrm{bl}$).

The first two universal properties of 3.2 have been proved in Thm. II.2.6. The global property can be easily verified, as for $\mathbb{Q}\mathbf{Cat}$ above, noting that the coherence of the double cell σ can be easily deduced from the coherence of its 'projections' on $\mathbb{A}$ and $\mathbb{X}$, which are the given cells $(\sigma \mid \mu) = \alpha'$, $\sigma \otimes \lambda = \beta'$.

In all these examples, one can note that - coherently with the relationship already mentioned - the comma object is constructed as a (horizontal) double limit, with elements already known from the previous Parts: first the tabulator of the vertical arrow, then a pullback with the given horizontal arrow; finally, one has to take the vertical companion of the 'second' projection of this pullback.

3.5. Lemma. [Commutative cells are well determined] *Suppose we have, in the weak double category $\mathbb{D}$, a commutative diagram $GR = SF$ of four horizontal arrows. If all these arrows are strong, the commutative cell λ on the following boundary is well determined: the two constructions considered in 3.1, by flipping the horizontal or the vertical arrows, give the same result*

$$\begin{tikzcd} \bullet \arrow[r, "R"] \arrow[d, "F_*"'] \arrow[dr, phantom, "\boldsymbol{\lambda}"] & \bullet \arrow[d, "G_*"] \\ \bullet \arrow[r, "S"'] & \bullet \end{tikzcd} \qquad (40)$$

provided we use, in identifying $G_*R_* = S_*F_*$, the special isocell given by the equality $GR = SF$, via the units and counits of companionship (see the diagram below).

Proof. The proof is trivial, once the statement is correctly understood. The special isocell referred to above is constructed as follows:

$$\begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\eta_R"] & \bullet \arrow[r, equal] \arrow[d, "R_*"] \arrow[dr, phantom, "1"] & \bullet \arrow[d, "R_*"] \\ \bullet \arrow[r, "R"] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "1^\bullet"] & \bullet \arrow[r, equal] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\eta_G"] & \bullet \arrow[d, "G_*"] \\ \bullet \arrow[r, "R"] & \bullet \arrow[r, "G"] & \bullet \\ \bullet \arrow[r, "F"'] \arrow[d, "F_*"'] \arrow[dr, phantom, "\epsilon_F"] & \bullet \arrow[r, "S"'] \arrow[d, "1^\bullet"] \arrow[dr, phantom, "1^\bullet"] & \bullet \arrow[d, "1^\bullet"] \\ \bullet \arrow[r, equal] \arrow[d, "S_*"'] \arrow[dr, phantom, "1"] & \bullet \arrow[r, "S"'] \arrow[d, "S_*"'] \arrow[dr, phantom, "\epsilon_S"] & \bullet \arrow[d, "1^\bullet"'] \\ \bullet \arrow[r, equal] & \bullet \arrow[r, equal] & \bullet \end{tikzcd} \qquad (41)$$

Note that this quaternary vertical composition is uniquely determined, because of a remark in 1.1 on 'normal composition': in each of its 'ternary parts' there is at least one vertical arrow which is an identity. ■

3.6. Theorem. [Comma objects and double limits] *A comma object $F{\downarrow}R$ is at the same time the horizontal and the vertical double limit of the diagram (F, R), where the projections are companions.*

More precisely, let us consider a cell ω in the double category $\mathbb{D}$, where P, Q are strong arrows

$$\begin{tikzcd} Z \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\ X \arrow[r, "R"'] & C \end{tikzcd} \qquad \zeta \qquad \begin{tikzcd} Z \arrow[r, "P"] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\vartheta"] & A \arrow[d, "F"] \\ Z \arrow[r, "RQ"'] & C \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, equal] \arrow[d, "Q"'] & Z \arrow[d, "FP"] \\ X \arrow[r, "R"'] & C \end{tikzcd} \qquad (42)$$

and let us write $\zeta = (\eta_Q \,|\, \omega)$ and $\vartheta = \eta_P \otimes \omega$ the 'equivalent' cells obtained by flipping Q or P, as in (34)-(35).

(a) The object Z, equipped with the cell ω, satisfies the horizontal property of comma objects if and only if it satisfies the 1-dimensional universal property of the horizontal double limit of (R, F), when equipped with the horizontal arrows P, Q and the cell ζ.

(a′) The object Z, equipped with the cell ω, satisfies the vertical property of comma objects if and only if it satisfies the 1-dimensional universal property of the vertical double limit of (R, F), when equipped with the vertical arrows P, Q and the cell ϑ.

(b) Assume that ζ satisfies the 1-dimensional universal property, or equivalently that ω satisfies the horizontal property. Then, adding the 2-dimensional universal property of ζ is equivalent to adding the horizontal 2-dimensional property of ω (cf. 3.2(d)).

(b′) Assume that ϑ satisfies the 1-dimensional universal property, or equivalently that ω satisfies the vertical property. Then, adding the 2-dimensional universal property of ϑ is equivalent to adding the vertical 2-dimensional property of ω (cf. 3.2(d)).

Proof. It is sufficient to prove (a) and (b).

(a) Indeed, the horizontal universal property (30) can be rewritten without commutative cells, saying that

$$\begin{tikzcd} U \arrow[rr, "S"] \arrow[d, "1^\bullet"'] \arrow[drr, phantom, "\alpha"] & & A \arrow[d, "F"] \\ U \arrow[r, "T"'] & X \arrow[r, "R"'] & C \end{tikzcd} \quad = \quad \begin{tikzcd} U \arrow[r, "L"] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "1^\bullet"] & F{\downarrow}R \arrow[r, "P"] \arrow[d, "1^\bullet"] \arrow[dr, phantom, "\zeta"] & A \arrow[d, "F"] \\ U \arrow[r, "L"'] & F{\downarrow}R \arrow[r, "RQ"'] & C \end{tikzcd} \qquad (43)$$

- for every pair of horizontal arrows $S\colon U \to A$, $T\colon U \to X$ and every cell $\alpha\colon (1\ (S, RT)\ F)$ there is a unique horizontal arrow L such that $PL = S$, $QL = T$ and $(1^\bullet_L \,|\, \zeta) = \alpha$. Which is equivalent to saying that $F{\downarrow}R$ is the *one-dimensional horizontal double limit* of the diagram (F, R) *in* $\mathbb{D}$, as defined in Part I.

(b) Let us begin by writing down the 2-dimensional property of ζ as a horizontal limit, in a simplified form which takes advantage of the 1-dimensional property, already considered.

A cell σ as below

$$\begin{tikzcd} U' \arrow[r, "L'"] \arrow[d, "H\bullet"'] \arrow[dr, phantom, "\sigma"] & F{\downarrow}R \arrow[r, "P"] \arrow[d, "\bullet 1^\bullet"] \arrow[dr, phantom, "\zeta"] & A \arrow[d, "\bullet F"] \\ U'' \arrow[r, "L''"'] & F{\downarrow}R \arrow[r, "RQ"'] & C \end{tikzcd} \tag{44}$$

produces two cells

$$\pi = (\sigma \,|\, 1^\bullet_P)\colon (H\ (PL', PL'')\ 1^\bullet_A), \qquad \rho = (\sigma \,|\, 1^\bullet_Q)\colon (H\ (QL', QL'')\ 1^\bullet_X), \tag{45}$$

satisfying the following relations with the cells $\zeta' = (1^\bullet_{L'} \,|\, \zeta)$, $\zeta'' = (1^\bullet_{L''} \,|\, \zeta)$ which determine the horizontal arrows L', L'':

$$\pi \otimes \zeta'' = \zeta' \otimes (\rho \,|\, 1^\bullet_R), \tag{46}$$

since they both coincide with $(\sigma \,|\, \zeta)$.

The 2-dimensional universal property says that this mapping is bijective. Precisely, given three arrows and two cells satisfying the following bonds

$$\begin{array}{lll} L'\colon U' \to F{\downarrow}R, & L''\colon U'' \to F{\downarrow}R, & H\colon U' \to\!\!\!\!\!\bullet\;\, U'', \\ \pi\colon (H\ (PL', PL'')\ 1^\bullet_A), & \rho\colon (H\ (QL', QL'')\ 1^\bullet_X), & \pi \otimes \zeta'' = \zeta' \otimes (\rho \,|\, 1^\bullet_R), \end{array} \tag{47}$$

$$\begin{tikzcd} U' \arrow[rr, "PL'"] \arrow[d, "H\bullet"'] \arrow[drr, phantom, "\pi"] & & A \arrow[d, "\bullet 1^\bullet"] \\ U'' \arrow[r, "L''"] \arrow[d, "1^\bullet\bullet"'] \arrow[dr, phantom, "1^\bullet"] & F{\downarrow}R \arrow[r, "P"] \arrow[d, "\bullet 1^\bullet"] \arrow[dr, phantom, "\zeta"] & A \arrow[d, "F\bullet"] \\ U'' \arrow[r, "L''"'] & F{\downarrow}R \arrow[r, "RQ"'] & C \end{tikzcd} \quad = \quad \begin{tikzcd} U' \arrow[r, "L'"] \arrow[d, "1^\bullet\bullet"'] \arrow[dr, phantom, "1^\bullet"] & F{\downarrow}R \arrow[rr, "P"] \arrow[d, "1^\bullet\bullet"] \arrow[drr, phantom, "\zeta"] & & A \arrow[d, "\bullet F"] \\ U' \arrow[r, "L'"] \arrow[d, "H\bullet"'] \arrow[drr, phantom, "\rho"] & F{\downarrow}R \arrow[r, "Q"] & X \arrow[r, "R"] \arrow[d, "1^\bullet\bullet"'] \arrow[dr, phantom, "1^\bullet"] & C \arrow[d, "\bullet 1^\bullet"] \\ U'' \arrow[rr, "QL''"'] & & X \arrow[r, "R"'] & C \end{tikzcd}$$

there is precisely one cell $\sigma\colon (H\ (L', L'')\ 1^\bullet)$ satisfying (45), i.e.:

$$(\sigma \,|\, 1^\bullet_P) = \pi, \qquad (\sigma \,|\, 1^\bullet_Q) = \rho. \tag{48}$$

It will be useful to remark that the coherence condition expressed in the diagram above can be equivalently written as follows, by flipping the (strong) projection Q in the right-hand part (ϵ_Q is its counit)

$$\pi \otimes \zeta'' = (\rho' \,|\, \omega), \qquad (\rho = (\rho' \,|\, \epsilon_Q)), \tag{49}$$

$$\begin{tikzcd} U' \arrow[r, "PL'"] \arrow[d, "H\bullet"'] \arrow[dr, phantom, "\pi"] & A \arrow[d, "\bullet 1^\bullet"] \\ U'' \arrow[r, "PL''"] \arrow[d, "1^\bullet\bullet"'] \arrow[dr, phantom, "\zeta''"] & A \arrow[d, "\bullet F"] \\ U'' \arrow[r, "RQL'"'] & C \end{tikzcd} \quad = \quad \begin{tikzcd} U' \arrow[r, "L'"] \arrow[d, "H\bullet"'] \arrow[dr, phantom, "\rho'"] & F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q\bullet"] \arrow[dr, phantom, "\omega"] & A \arrow[d, "\bullet F"] \\ U'' \arrow[r, "QL''"'] & X \arrow[r, "R"'] & C \end{tikzcd}$$

Now, always by flipping the projections P, Q, we can translate the data of (47) into the following ones, arranged as in the global property of ω, and actually as in the *horizontal* 2-dimensional property, because of the presence of the *vertical* identity of $F{\downarrow}R$:

$$
\begin{tikzcd}
U' \arrow[r, "L'"] \arrow[d, "H"'] \arrow[dr, phantom, "?"] & F{\downarrow}R \arrow[r, equal] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\boldsymbol{\mu}"] & F{\downarrow}R \arrow[d, "P"] \\
U'' \arrow[r, "L''"] \arrow[d, "1^\bullet"'] \arrow[dr, phantom, "\boldsymbol{\lambda}"] & F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\
U'' \arrow[r, "QL''"'] & X \arrow[r, "R"'] & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "L'"] \arrow[d, "H"'] \arrow[drr, phantom, "\pi'"] & \bullet \arrow[r, equal] & \bullet \arrow[d, "P"] \\
\bullet \arrow[r] \arrow[d, "1^\bullet"'] \arrow[drr, phantom, "\zeta''"] & \bullet \arrow[r] & \bullet \arrow[d, "F"] \\
\bullet \arrow[r, "QL''"'] & \bullet \arrow[r, "R"'] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "L'"] \arrow[d, "H"'] & \bullet \arrow[r, equal] \arrow[d, "1^\bullet"'] & \bullet \arrow[d, "P"] \\
\bullet \arrow[r, phantom, "\rho'"] \arrow[d, "1^\bullet"'] & \bullet \arrow[r, phantom, "\vartheta"] \arrow[d, "Q"'] & \bullet \arrow[d, "F"] \\
\bullet \arrow[r, "QL''"'] & \bullet \arrow[r, "R"'] & \bullet
\end{tikzcd}
\tag{50}
$$

where

- $\lambda\colon (QL'' = QL'')$ and $\mu\colon (P = P)$ are commutative cells,
- $\zeta'' = (\lambda \,|\, \omega) = (1^\bullet_{L''} \,|\, \zeta)$, as above, and $\vartheta = \mu \otimes \omega$,
- $\pi'\colon (H\ (L', PL'')\ P)$ corresponds to $\pi\colon (H\ (PL', PL'')\ 1^\bullet_A)$, by flipping P,
- $\rho'\colon (H\ (L', QL'')\ Q)$ corresponds to $\rho\colon (H\ (QL', QL'')\ 1^\bullet_X)$, by flipping Q (as in (48)).

The equivalence of the two universal properties follows now from the following two remarks:

- by flipping P, the coherence condition $\pi \otimes \zeta'' = (\rho' \,|\, \omega)$ of (49) is equivalent to $\pi' \otimes \zeta'' = (\rho' \,|\, \vartheta)$, i.e. the coherence condition of the global property (50),
- similarly, the conditions on σ in (48), i.e. $(\sigma \,|\, 1^\bullet_P) = \pi$ and $(\sigma \,|\, 1^\bullet_Q) = \rho$, are equivalent to the conditions for the global property (50), i.e. $(\sigma \,|\, \mu) = \pi'$ and $\sigma \otimes \lambda = \rho'$. ∎

4. Pointwise Kan extensions

Pointwise extensions in a strict double category $\mathbb{D}$ are defined, using the comma objects of the previous section.

4.1. The pointwise case. (a) Assume that $\mathbb{D}$ has all comma objects (3.2). We say that G is the *pointwise right Kan extension* of F, *from R to S*, via ϵ, if

$$
\begin{tikzcd}
H{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & J' \arrow[d, "H"] \\
I \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\epsilon"] & J \arrow[d, "G"] \\
X \arrow[r, "S"'] & A
\end{tikzcd}
\tag{51}
$$

- for every vertical arrow $H\colon J' \to J$, $GH = \mathrm{Ran}_{P,S}(FQ)$, via $\omega\otimes\epsilon$,

(where ω is the comma cell of $H{\downarrow}R$). Then, G *is* indeed a Kan extension, as we prove below.

(b) We also want a slightly more general definition, where $\mathbb{D}$ need not have all comma objects. Let $\mathbb{D}_0$ be a double subcategory of $\mathbb{D}$ containing all its objects, all its horizontal arrows, some *distinguished vertical arrows* (closed under identities and composition, of course) and all the cells whose vertical arrows are distinguished.

Then, if $\mathbb{D}_0$ has comma objects, we say that $G = \mathrm{Ran}_{R,S}(F)$ is *pointwise on the distinguished vertical arrows* of $\mathbb{D}$ if the previous condition holds for every *distinguished* vertical arrow $H\colon J' \to J$. Note that F and G need not be distinguished: the present generalisation does not amount to pointwise Kan extensions in $\mathbb{D}_0$. The fact that G is indeed a Kan extension still holds.

4.2. Theorem. *In a pseudo double category $\mathbb{D}$ with comma objects, every pointwise Kan extension is a Kan extension. More generally, this fact also holds for a Kan extension which is pointwise on a choice of distinguished vertical arrows of $\mathbb{D}$, in the sense of 4.1(b).*

Proof. The proof is based on the Replacement Lemma (2.7). Take, in definition 4.1, $J' = J$ and $H = 1^\bullet_J$ (which is necessarily distinguished, according to the previous definition). By hypothesis, $G = \mathrm{Ran}_{P,S}(FQ)$. Now, the vertical universal property of $J{\downarrow}R$ gives a vertical arrow M and a (commuative) cell μ such that

$$\begin{tikzcd} I \arrow[r, "R"] \arrow[d, "M"'] \arrow[dr, phantom, "\boldsymbol{\mu}"] & J \arrow[d, "1^\bullet"] \\ J{\downarrow}R \arrow[r, "P"] \arrow[d, "L"'] \arrow[dr, phantom, "\omega"] & J \arrow[d, "1^\bullet"] \\ I \arrow[r, "R"'] & J \end{tikzcd} \qquad \begin{aligned} &PM = R, \\ &LM = 1^\bullet_I, \\ &\mu\otimes\omega = 1^\bullet_R. \end{aligned} \tag{52}$$

Applying 2.7(a), the fact that $G = \mathrm{Ran}_{P,S}(FQ)$ via $\omega\otimes\epsilon$ implies that $G = \mathrm{Ran}_{R,S}(F)$ via ϵ. ∎

4.3. Theorem. [Adjunctions are pointwise extensions] *Let $\mathbb{D}$ be a pseudo double category with comma objects. Then an absolute right Kan extension $G = \mathrm{Ran}_{R,1}(1_A)$, as in Theorem 2.5, is always a pointwise extension.*

Proof. With the notation of 2.5, consider the solid left diagram below (omitting G' and τ), where G is vertically adjoint to R, with cells ϵ, η

$$
\begin{tikzcd}
H{\downarrow}R \arrow[r, "P"] \arrow[d, "Q\bullet"'] \arrow[dr, phantom, "\omega"] & J \arrow[r, equal] \arrow[d, "H\bullet"] \arrow[ddr, phantom, "\tau"] & J \arrow[dd, dotted, "\bullet G'"] \\
A \arrow[r, "R"] \arrow[d, "1^\bullet\bullet"'] \arrow[dr, phantom, "\epsilon"] & X \arrow[d, "\bullet G"] & \\
A \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
J \arrow[r, equal] \arrow[dd, "GH\bullet"'] \arrow[ddr, phantom, "="] & J \arrow[r, equal] \arrow[d, "1^\bullet\bullet"'] \arrow[dr, phantom, "\boldsymbol{\mu}"] & J \arrow[d, "\bullet 1^\bullet"] \\
& H{\downarrow}R \arrow[r, "P"'] \arrow[d, "Q\bullet"'] \arrow[dr, phantom, "\omega"] & J \arrow[d, "\bullet H"] \\
A \arrow[r, equal] & A \arrow[r, "R"'] & X
\end{tikzcd}
\;=\; 1_H \otimes \eta. \tag{53}
$$

In this situation, the cell $\eta\colon (G\ (1, R)\ 1)$ yields, with the vertical universal property of the comma, a unique commutative cell μ such that $\mu\otimes\omega = 1_H\otimes\eta$ (see the right diagram above).

A second cell $\sigma\colon (1^\bullet\ (P, 1)\ L)$ comes from the global universal property of ω (3.2(c)), with $\omega' = \omega\otimes\epsilon$ and $\eta' = 1_H\otimes\eta$ (since the right-hand equality below plainly holds)

$$
\begin{tikzcd}
H{\downarrow}R \arrow[r, "P"] \arrow[d, "1^\bullet\bullet"'] \arrow[dr, phantom, "\sigma"] & J \arrow[r, equal] \arrow[d, "L\bullet"] \arrow[dr, phantom, "\boldsymbol{\mu}"] & J \arrow[d, "\bullet 1^\bullet"] \\
H{\downarrow}R \arrow[r, equal] \arrow[d, "Q\bullet"'] \arrow[dr, phantom, "1_Q"] & H{\downarrow}R \arrow[r, "P"] \arrow[d, "Q\bullet"] \arrow[dr, phantom, "\omega"] & J \arrow[d, "\bullet H"] \\
A \arrow[r, equal] & A \arrow[r, "R"'] & X
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "P"] \arrow[d, "1^\bullet\bullet"'] \arrow[drr, phantom, "1^\bullet"] & \bullet \arrow[r, equal] & \bullet \arrow[d, "\bullet 1^\bullet"] \\
\bullet \arrow[r, equal] \arrow[d, "Q\bullet"'] \arrow[drr, phantom, "\omega"] & \bullet \arrow[r] & \bullet \arrow[d, "\bullet H"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "R"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "P"] \arrow[dd, "1^\bullet\bullet"'] & \bullet \arrow[r, equal] \arrow[dd] & \bullet \arrow[dd, "\bullet 1^\bullet"] \\
\bullet \arrow[r, phantom, "\omega'"] & \bullet \arrow[r, phantom, "\eta'"] & \bullet \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "R"'] & \bullet
\end{tikzcd}
\tag{54}
$$

(One can note that we are only using the *vertical 2-dimensional property*, cf. 3.2(d)). We have thus:

$$(\sigma \,|\, \mu) = 1^\bullet_P, \qquad\qquad \sigma\otimes 1_Q = \omega\otimes\epsilon. \tag{55}$$

Now, given $G'\colon J \to\!\!\!\bullet\, A$ and $\alpha\colon (Q\ (P, 1)\ G')$ on the outer perimeter of the left diagram (53), the existence of a cell $\tau\colon (GH\ (1, 1)\ G')$ such that $(\omega\otimes\epsilon \,|\, \tau) = \alpha$ determines it, as $\tau = \mu\otimes\alpha$

$$\mu\otimes\alpha = \mu\otimes(\omega\otimes\epsilon \,|\, \tau) = (\mu\otimes\omega\otimes\epsilon \,|\, \tau) = (1_H\otimes\eta\otimes\epsilon \,|\, \tau) = \tau, \tag{56}$$

where both ternary vertical compositions, $\mu\otimes\omega\otimes\epsilon$ and $1_H\otimes\eta\otimes\epsilon$, are normal (1.1).

Conversely, letting $\tau = \mu\otimes\alpha$, the problem is solved using (55):

$$(\omega\otimes\epsilon \,|\, \tau) = (\omega\otimes\epsilon \,|\, \mu\otimes\alpha) = (\sigma\otimes 1_Q \,|\, \mu\otimes\alpha) = (\sigma \,|\, \mu)\otimes\alpha = \alpha. \tag{57}$$

■

5. Pasting properties of comma objects

Pasting properties are proved in Thm. 5.3; as a consequence, pointwise extensions are 'stable under composition' (Thm. 5.4).

5.1. VERTICAL RETRACTS. Recalling some classical facts will give a better understanding of comma objects and their properties. In homotopy theory, a comma square corresponds to a *standard homotopy pullback* (determined by a similar universal property, where cells are homotopies); diagrammatic lemmas for the latter generally amount to saying that some 'comparison map' is a *homotopy equivalence.* For instance, this is the case for a pasting of homotopy pullbacks; or between the ordinary pullback of a fibration and its homotopy pullback. In the 2-category **Cat**, one obtains *future retracts* (i.e., full reflective subcategories) or *past retracts* (i.e., full coreflective subcategories), according to the direction of cells (cf. [8], 1.6).

Here, we shall obtain somewhat similar results. We say that the horizontal arrow T is a *vertical deformation retract of* the horizontal arrow R, via the cells λ, μ, φ (as in the diagram below) if we have

$$\begin{aligned} &\lambda \otimes \mu = 1^\bullet_T && (ML = 1,\ M'L' = 1), \\ &(\varphi \,|\, \mu \otimes \lambda) = 1^\bullet_R, \quad 1_L \otimes \varphi = 1_L && (L'M' = 1). \end{aligned} \tag{58}$$

$$\begin{tikzcd} \bullet \arrow[r, "T"] \arrow[d, "L"'] \arrow[dr, phantom, "\lambda"] & \bullet \arrow[d, "L'"] \\ \bullet \arrow[r, "R"] \arrow[d, "M"'] \arrow[dr, phantom, "\mu"] & \bullet \arrow[d, "M'"] \\ \bullet \arrow[r, "T"'] & \bullet \end{tikzcd} \quad = \quad 1^\bullet_T, \qquad\qquad \begin{tikzcd} \bullet \arrow[r, equal] \arrow[dd, "1^\bullet"'] \arrow[ddr, phantom, "\varphi"] & \bullet \arrow[r, "R"] \arrow[d, "M"'] \arrow[dr, phantom, "\mu"] & \bullet \arrow[d, "M'"] \\ & \bullet \arrow[r, "T"'] \arrow[d, "L"'] \arrow[dr, phantom, "\lambda"] & \bullet \arrow[d, "L'"] \\ \bullet \arrow[r, equal] & \bullet \arrow[r, "R"'] & \bullet \end{tikzcd} \quad = \quad 1^\bullet_R,$$

In particular, we shall often encounter the case where R, T are *strong arrows*, $L' = M' = 1$ and λ, μ are *commutative cells.* For this case, it is sufficient to have:

$$R = TM, \quad ML = 1, \qquad\qquad 1_L \otimes \varphi = 1_L, \quad \varphi \otimes 1_M = 1_M. \tag{59}$$

Then $T = TML = RL$, and the commutative cells $\lambda\colon (T = RL)$, $\mu\colon (R = TM)$ give $\lambda \otimes \mu = 1^\bullet_T$ and $(\varphi \,|\, \mu \otimes \lambda) = \varphi \otimes 1_R = \varphi \otimes 1_M \otimes 1_T = 1_R$.

5.2. LEMMA. [Replacement Lemma, II] *In a strict double category $\mathbb{D}$, two cells ϵ, λ are* *given, as in the left diagram below (and as in the Replacement Lemma 2.7)*

$$\begin{tikzcd}
I' \arrow[r, "T"] \arrow[d, "L"'] \arrow[dr, phantom, "\lambda"] & J' \arrow[d, "L'"] \\
I \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\epsilon"] & J \arrow[d, "G"] \\
X \arrow[r, "S"'] & A
\end{tikzcd} \tag{60}$$

If the horizontal arrow T is a vertical deformation retract of the horizontal arrow R, via the cells λ, μ, φ (as in (58)), then the following two conditions are equivalent:

$$G = \mathrm{Ran}_{R,S}(F) \ \ \textit{via } \epsilon, \qquad\qquad GL' = \mathrm{Ran}_{T,S}(FL) \ \ \textit{via } \lambda\otimes\epsilon. \tag{61}$$

PROOF. It is a straightforward consequence of the previous Replacement Lemma, 2.7. ■

5.3. THEOREM. [Pasting Theorem] *Let $\mathbb{D}$ be a strict double category. Consider the pasting of two comma cells, in the left diagram, and the comma ω'' of the vertical composite FF', in the middle diagram*

$$\begin{tikzcd}
F'{\downarrow}P \arrow[r, "P'"] \arrow[d, "Q'"'] \arrow[dr, phantom, "\omega'"] & B \arrow[d, "F'"] \\
F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\
X \arrow[r, "R"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
(FF'){\downarrow}R \arrow[r, "P''"] \arrow[dd, "Q''"'] \arrow[ddr, phantom, "\omega''"] & B \arrow[dd, "FF'"] \\
& \\
X \arrow[r, "R"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[dd, "1^\bullet"'] \arrow[ddr, phantom, "\varphi"] & F'{\downarrow}P \arrow[r, "P'"] \arrow[d, "M"'] \arrow[dr, phantom, "\boldsymbol{\mu}"] & B \arrow[d, "1^\bullet"] \\
& (FF'){\downarrow}R \arrow[r, "P''"] \arrow[d, "L"'] \arrow[dr, phantom, "\boldsymbol{\lambda}"] & B \arrow[d, "1^\bullet"] \\
\bullet \arrow[r, equal] & F'{\downarrow}P \arrow[r, "P'"'] & B
\end{tikzcd} \tag{62}$$

Then the strong arrow P'' (a projection of the comma of the composite) is a vertical deformation retract (5.1) of P' (a projection of the iterated comma), in the strong sense of (59):

- there exist commutative cells λ, μ and a comparison cell φ, as displayed in the right diagram above, satisfying:

$$ML = 1, \qquad 1_L\otimes\varphi = 1_L, \qquad \varphi\otimes 1_M = 1_M. \tag{63}$$

Note. Recall that $ML = 1$ gives $\lambda\otimes\mu = 1$ (and actually amounts to that, since our cells are commutative).

Proof. First, by the vertical universal property of ω'', there is a unique vertical arrow $M\colon F'{\downarrow}P \to\!\!\!\!\!\bullet\;(FF'){\downarrow}R$ such that

$$P''M = P', \qquad Q''M = QQ', \qquad \mu\otimes\omega'' = \omega'\otimes\omega, \tag{64}$$

$$\begin{tikzcd} F'{\downarrow}P \arrow[r, "P'"] \arrow[d, "M\bullet"'] \arrow[dr, phantom, "\boldsymbol{\mu}"] & B \arrow[d, "1^\bullet"] \\ (FF'){\downarrow}R \arrow[r, "P''"] \arrow[d, "Q''\bullet"'] \arrow[dr, phantom, "\omega''"] & B \arrow[d, "FF'"] \\ X \arrow[r, "R"'] & C \end{tikzcd} \quad = \quad \begin{tikzcd} F'{\downarrow}P \arrow[r, "P'"] \arrow[d, "Q'\bullet"'] \arrow[dr, phantom, "\omega'"] & B \arrow[d, "F'"] \\ F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q\bullet"'] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\ X \arrow[r, "R"'] & C \end{tikzcd}$$

with the commutative cell $\mu\colon (P' = P''M)$.

Now, to define the second comparison L, we begin by constructing a vertical arrow $N\colon (FF'){\downarrow}R \to\!\!\!\!\!\bullet\; F{\downarrow}R$, by the vertical universal property of $F{\downarrow}R$ (3.2(b))

$$PN = F'P'', \qquad QN = Q'', \qquad \nu\otimes\omega = \omega'' \qquad (\nu\colon (F'P'' = PN)), \tag{65}$$

$$\begin{tikzcd} (FF'){\downarrow}R \arrow[r, equal] \arrow[d, "N\bullet"'] \arrow[dr, phantom, "\boldsymbol{\nu}"] & (FF'){\downarrow}R \arrow[d, "F'P''"] \\ F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q\bullet"'] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\ X \arrow[r, "R"'] & C \end{tikzcd} \quad = \quad \omega'', \qquad \begin{tikzcd} (FF'){\downarrow}R \arrow[r, "P''"] \arrow[d, "N\bullet"'] \arrow[dr, phantom, "\boldsymbol{\nu}"] & B \arrow[d, "F'"] \\ F{\downarrow}R \arrow[r, "P"'] & A \end{tikzcd}$$

Flipping horizontally the strong arrow P'', the cell ν will also be written as at the right hand, above.

Then, we have a vertical $L\colon (FF'){\downarrow}R \to\!\!\!\!\!\bullet\; F'{\downarrow}P$, by the vertical universal property of $F'{\downarrow}P$

$$P'L = P'', \qquad Q'L = N, \qquad \lambda\otimes\omega' = \nu\colon (F'P'' = PN). \tag{66}$$

$$\begin{tikzcd} (FF'){\downarrow}R \arrow[r] \arrow[d, "L\bullet"'] \arrow[dr, phantom, "\boldsymbol{\lambda}"] & B \arrow[d, "1^\bullet"] \\ F'{\downarrow}P \arrow[r] \arrow[d, "Q'\bullet"'] \arrow[dr, phantom, "\omega'"] & B \arrow[d, "F'"] \\ F{\downarrow}R \arrow[r, "P"'] & A \end{tikzcd} \quad = \nu$$

The equality $\lambda\otimes\mu = 1^\bullet_{P''}$ (i.e., $ML = 1$) is then detected by the vertical universal property of $(FF'){\downarrow}R$

$$(\lambda\otimes\mu)\otimes\omega'' = \lambda\otimes\omega'\otimes\omega = \nu\otimes\omega = \omega''. \tag{67}$$

As a first step for constructing φ, the global universal property of ω gives a cell ρ satisfying the following equations, where $\overline{\omega} = \omega' \otimes \omega$ (notice that P' and Q should be 'flipped', to follow the pattern of 3.2(c))

$$
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[dd, "Q'"'] & F'{\downarrow}P \arrow[r, "P'"] \arrow[d, "M"'] \arrow[dr, phantom, "\boldsymbol{\mu}"] & B \arrow[d, "1^\bullet"] \\
\arrow[r, phantom, "\rho"] & (FF'){\downarrow}R \arrow[r, "P''"] \arrow[d, "N"'] \arrow[dr, phantom, "\boldsymbol{\nu}"] & B \arrow[d, "F'"] \\
\bullet \arrow[r, equal] \arrow[d, "Q"'] \arrow[dr, phantom, "1"] & F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & A \arrow[d, "F"] \\
\bullet \arrow[r, equal] & X \arrow[r, "R"'] & C
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "Q'"'] \arrow[drr, phantom, "\omega'"] & \bullet \arrow[r, "P'"] & \bullet \arrow[d, "F'"] \\
\bullet \arrow[r, equal] \arrow[d, "Q"'] \arrow[drr, phantom, "\omega"] & \bullet \arrow[r] & \bullet \arrow[d, "F"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "R"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "Q'"'] & \bullet \arrow[r, "P'"] \arrow[d] & \bullet \arrow[d, "F'"] \\
\bullet \arrow[d, "Q"'] \arrow[r, phantom, "1"] & \bullet \arrow[d] \arrow[r, phantom, "\overline{\omega}"] & \bullet \arrow[d, "F"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "R"'] & \bullet
\end{tikzcd}
\tag{68}
$$

$$(\rho \,|\, (\mu \otimes \nu)) = \omega', \qquad \rho \otimes 1_Q = 1_{QQ'}. \tag{69}$$

It will be useful to note that $1_L \otimes \rho = 1_N$, as it results applying $\lambda \otimes - = (1_L \,|\, \lambda) \otimes -$ to all terms of diagram (68). In fact, $\lambda \otimes \mu = 1$, $\lambda \otimes \omega' = \nu$ and $\lambda \otimes (\omega' \otimes \omega) = \nu \otimes \omega = \omega''$.

$$
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "N"'] \arrow[dr, phantom, "1_L \otimes \rho"] & (FF'){\downarrow}R \arrow[r, "P''"] \arrow[d, "N"'] \arrow[dr, phantom, "\boldsymbol{\nu}"] & B \arrow[d, "F'"] \\
\bullet \arrow[r, equal] \arrow[d, "Q"'] \arrow[dr, phantom, "1"] & F'{\downarrow}P \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & B \arrow[d, "F"] \\
\bullet \arrow[r, equal] & X \arrow[r, "R"'] & C
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "N"'] \arrow[drr, phantom, "\boldsymbol{\nu}"] & \bullet \arrow[r, "P''"] & \bullet \arrow[d, "F'"] \\
\bullet \arrow[r, equal] \arrow[d, "Q"'] \arrow[drr, phantom, "\omega"] & \bullet \arrow[r] & \bullet \arrow[d, "F"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "R"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "Q'"'] & \bullet \arrow[r, "P''"] \arrow[d] & \bullet \arrow[d, "F'"] \\
\bullet \arrow[d, "Q"'] \arrow[r, phantom, "1"] & \bullet \arrow[d] \arrow[r, phantom, "\omega''"] & \bullet \arrow[d, "F"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "R"'] & \bullet
\end{tikzcd}
\tag{70}
$$

Since putting 1_N in the upper-left cell above also solves these equations, these solutions coincide.

The global universal property of ω' gives a cell φ satisfying the following conditions, with $\mu \otimes \nu = \zeta$ (we know from (68) that $(\rho \,|\, \zeta) = \omega'$)

$$
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[dd, "1^\bullet"'] & F'{\downarrow}P \arrow[r, "P'"] \arrow[d, "M"'] \arrow[dr, phantom, "\boldsymbol{\mu}"] & B \arrow[d, "1^\bullet"] \\
\arrow[r, phantom, "\varphi"] & (FF'){\downarrow}R \arrow[r, "P''"] \arrow[d, "L"'] \arrow[dr, phantom, "\boldsymbol{\lambda}"] & B \arrow[d, "1^\bullet"] \\
\bullet \arrow[r, equal] \arrow[d, "Q'"'] \arrow[dr, phantom, "1"] & F'{\downarrow}P \arrow[r] \arrow[d, "Q'"'] \arrow[dr, phantom, "\omega'"] & B \arrow[d, "F'"] \\
\bullet \arrow[r, equal] & F{\downarrow}R \arrow[r, "P"'] & A
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "1^\bullet"'] \arrow[drr, phantom, "1^\bullet"] & \bullet \arrow[r, "P'"] & \bullet \arrow[d, "1^\bullet"] \\
\bullet \arrow[r, equal] \arrow[d, "Q'"'] \arrow[drr, phantom, "\omega'"] & \bullet \arrow[r] & \bullet \arrow[d, "F'"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "P"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "1^\bullet"'] & \bullet \arrow[r, "P'"] \arrow[d] & \bullet \arrow[d, "1^\bullet"] \\
\bullet \arrow[d, "Q'"'] \arrow[r, phantom, "\rho"] & \bullet \arrow[d] \arrow[r, phantom, "\boldsymbol{\zeta}"] & \bullet \arrow[d, "F'"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "P"'] & \bullet
\end{tikzcd}
\tag{71}
$$

$$\varphi \mid (\mu \otimes \lambda) = 1^\bullet_{P'}, \qquad\qquad \varphi \otimes 1_{Q'} = \rho. \tag{72}$$

Now, the equalities $1_L \otimes \varphi = 1_L$ and $\varphi \otimes 1_M = 1_M$ of the thesis are detected by the global universal properties of ω' and ω''. The first comes from the fact that, applying $\lambda \otimes - = (1_L \mid \lambda) \otimes -$ on the top of diagram (71), we get

$$\lambda \otimes \mu \otimes \lambda = \lambda, \qquad 1_L \otimes \rho = 1_N, \qquad \lambda \otimes \mu \otimes \nu = \nu, \tag{73}$$

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "L"'] \arrow[dr, phantom, "1_L \otimes \varphi"] & (FF')\downarrow R \arrow[r, "P''"] \arrow[d, "L"'] \arrow[dr, phantom, "\boldsymbol{\lambda}"] & B \arrow[d, "1^\bullet"] \\
\bullet \arrow[r, equal] \arrow[d, "Q'"'] \arrow[dr, phantom, "1"] & F'\downarrow P \arrow[r] \arrow[d, "Q'"'] \arrow[dr, phantom, "\omega'"] & B \arrow[d, "F'"] \\
\bullet \arrow[r, equal] & F\downarrow R \arrow[r, "P"'] & A
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "L"'] \arrow[drr, phantom, "\boldsymbol{\lambda}"] & \bullet \arrow[r, "P''"] & \bullet \arrow[d, "1^\bullet"] \\
\bullet \arrow[r, equal] \arrow[d, "Q'"'] \arrow[drr, phantom, "\omega'"] & \bullet \arrow[r] & \bullet \arrow[d, "F'"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "P"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "L"'] \arrow[dd, phantom, bend left, "1"] & \bullet \arrow[r, "P''"] \arrow[d] \arrow[dd, phantom, bend left, "\omega''"] & \bullet \arrow[d, "1^\bullet"] \\
\bullet \arrow[d, "Q"'] & \bullet \arrow[d] & \bullet \arrow[d, "F"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "P"'] & \bullet
\end{tikzcd} \tag{74}$$

and these equations are also solved by putting 1_L in the upper-left cell above.

The second comes from a similar uniqueness argument applied to the following equations:

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "M"'] \arrow[dr, phantom, "\varphi \otimes 1_M"] & (FF')\downarrow R \arrow[r, "P''"] \arrow[d, "M"'] \arrow[dr, phantom, "\boldsymbol{\mu}"] & B \arrow[d, "1^\bullet"] \\
\bullet \arrow[r, equal] \arrow[d, "Q''"'] \arrow[dr, phantom, "1"] & (FF')\downarrow R \arrow[r] \arrow[d, "P''"'] \arrow[dr, phantom, "\omega''"] & B \arrow[d, "FF'"] \\
\bullet \arrow[r, equal] & X \arrow[r, "R"'] & C
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "M"'] \arrow[drr, phantom, "\boldsymbol{\mu}"] & \bullet \arrow[r, "P''"] & \bullet \arrow[d, "1^\bullet"] \\
\bullet \arrow[r, equal] \arrow[d, "Q''"'] \arrow[drr, phantom, "\omega''"] & \bullet \arrow[r] & \bullet \arrow[d] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "R"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "M"'] \arrow[dd, phantom, bend left, "1"] & \bullet \arrow[r, "P''"] \arrow[d] \arrow[dd, phantom, bend left, "\overline{\omega}"] & \bullet \arrow[d, "1^\bullet"] \\
\bullet \arrow[d, "Q''"'] & \bullet \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "R"'] & \bullet
\end{tikzcd} \tag{75}$$

which are also solved by 1_M. ∎

5.4. THEOREM. [Pointwise stability] *Let $\mathbb{D}$ be a strict double category. If $G = \mathrm{Ran}_{R,S}(F)$ is a pointwise right Kan extension, then for every vertical arrow $H\colon J' \to J$ the extension $GH = \mathrm{Ran}_{P,S}(FQ)$ (see the diagram below) is still a pointwise extension, via the cell $\omega \otimes \epsilon$.*

PROOF. Follows from the first Replacement Lemma (2.7) together with the Pasting Theorem 5.3.

Take an arbitrary $H' \colon K' \to\!\!\!\!\bullet\ K$ and its comma $H'{\downarrow}P$, as in the left diagram below

$$
\begin{tikzcd}
H'{\downarrow}P \arrow[r, "P'"] \arrow[d, "Q'"'] \arrow[dr, phantom, "\omega'"] & J'' \arrow[d, "H'"] \\
H{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega"] & J' \arrow[d, "H"] \\
I \arrow[r, "R"'] \arrow[d, "F"] \arrow[dr, phantom, "\epsilon"] & J \arrow[d, "G"] \\
X \arrow[r, "S"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
H'{\downarrow}P \arrow[r, "P'"] \arrow[d, "L"'] \arrow[dr, phantom, "\boldsymbol{\lambda}"] & J'' \arrow[d, "1^\bullet"] \\
(HH'){\downarrow}R \arrow[r, "P''"] \arrow[d, "Q''"'] \arrow[dr, phantom, "\omega''"] & J'' \arrow[d, "HH'"] \\
I \arrow[r, "R"'] & J
\end{tikzcd}
\tag{76}
$$

We have to prove that $(GH)H' = \mathrm{Ran}_{P',S}(FQQ')$, via $\omega'\otimes(\omega\otimes\epsilon)$. Forming the comma $(HH'){\downarrow}R$, in the right diagram, the pointwise property of G with respect to the composite HH' says that

$$G(HH') = \mathrm{Ran}_{P'',S}(FQ''), \quad \text{via } \omega''\otimes\epsilon. \tag{77}$$

Now, the Pasting Theorem 5.3 says that the projection P'' is a vertical deformation retract of P', with a commutative comparison cell λ such that $\lambda\otimes\omega'' = \omega'\otimes\omega$. Applying the Replacement Lemma 2.7(b) to (2), we get the thesis: $(GH)H' = \mathrm{Ran}_{P',S}(FQQ')$, via $\lambda\otimes(\omega''\otimes\epsilon) = \omega'\otimes\omega\otimes\epsilon$. ■

6. Kan extensions for double categories

We briefly examine now the case when $\mathbb{D}$ is a double category of double categories. One of such settings will be studied in Part IV, together with its relations with double limits.

6.1. External Kan extensions. In a (pseudo) double category $\mathbb{D}$, we have defined a right Kan extension $G = \mathrm{Ran}_{R,S}(F)$ via ϵ, as in the left diagram below (see 2.2)

$$
\begin{tikzcd}
\mathbb{I} \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\epsilon"] & \mathbb{J} \arrow[d, dotted, "G"] \\
\mathbb{X} \arrow[r, "S"'] & \mathbb{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I} \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\eta"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{X} \arrow[r, dotted, "S"'] & \mathbb{A}
\end{tikzcd}
\tag{78}
$$

and a *left* Kan extension $S = \mathrm{Lan}_{F,G}(R)$ via η, as in the right diagram above (2.3).

These notions will be used letting $\mathbb{D}$ be *one of the three* 'settings' for weak double categories listed below. Thus, the vertices $\mathbb{I}, \mathbb{J}, \mathbb{X}, \mathbb{A}$ will be pseudo double categories, all arrows will be some kind of 'weak' double functors, and ϵ or η some kind of horizontal transformation $GR \to SF$.

6.2. Colax right Kan extensions. For $\mathbb{D} = \mathbb{D}\mathrm{bl}$ (1.4), the general definition $G = \mathrm{Ran}_{R,S}(F)$ says that G is the *colax* right Kan extension of the *colax double functor* F, along the *lax* double functors R, S (or from R to S), via the cell ϵ (the *counit* of the extension). The *pointwise* case is defined as in 4.1, since $\mathbb{D}\mathrm{bl}$ has all comma objects (3.4(b)).

Symmetrically, $S = \mathrm{Lan}_{F,G}(R)$ means that S is the *lax* left Kan extension of a *lax double functor* R, along the *colax* double functors F, G, via the cell η (the *unit* of the extension).

We have already seen that, in a colax/lax adjunction $G \dashv R$ between weak double categories, the left adjoint is an absolute right Kan extension of this type, $G = \mathrm{Ran}_{R,1}(1)$ (2.5), and is always a pointwise extension (4.3). Symmetrically, the right adjoint is an absolute left Kan extension $R = \mathrm{Ran}_{1,G}(1)$ (since transpose duality exchanges left and right adjoints).

Note that, here, transposing $\mathbb{D}$ is equivalent to applying 'internal' *horizontal duality* to the vertices, arrows and cells of diagram (78); in fact the latter reverses the direction of cells and exchanges lax with colax double functors.

6.3. Unitary colax right Kan extensions. Let $\mathbb{D}\mathrm{bl}_u$ be the cell-wise full double subcategory of $\mathbb{D}\mathrm{bl}$ where the *vertical* arrows are *unitary* colax double functors, while the horizontal ones are general. By restriction, also $\mathbb{D}\mathrm{bl}_u$ has all comma objects (since in $\mathbb{D}\mathrm{bl}$ the projections of a comma are strict double functors).

Taking $\mathbb{D} = \mathbb{D}\mathrm{bl}_u$ gives a *unitary colax* right Kan extension of a *unitary colax double functor* $F\colon \mathbb{I} \mathrel{\bullet\!\!\!\to} \mathbb{X}$, along two *lax* double functors. The interest of restricting vertical arrows to be unitary is shown by a few examples below (7.2, 7.3).

The symmetric notion we are interested in, *unitary lax left* Kan extensions along *colax* double functors, lives in $\mathbb{D} = \mathbb{D}\mathrm{bl}^u$, where the *horizontal* arrows are restricted to be *unitary* (lax). Again, *internal* horizontal duality on all data exchanges these situations.

6.4. Unitary lax right Kan extensions. . Finally, we write $\mathrm{Lx}\mathbb{D}\mathrm{bl}_u$ the double category of weak double categories, lax double functors (horizontally) and *unitary* lax double functors (vertically); a cell $\alpha\colon GR \to SF$, as in the diagram above, simply is a horizontal transformation of the composed lax double functors.

(Thus, $\mathrm{Lx}\mathbb{D}\mathrm{bl}_u$ is a substructure of the double category of quintets $\mathrm{Lx}\mathbb{D}\mathrm{bl} = \mathbb{Q}\mathbf{LxDbl}$, over the 2-category **LxDbl** of weak double categories, lax double functors and their horizontal transformations, already considered in II.2.2; the whole $\mathrm{Lx}\mathbb{D}\mathrm{bl}$ is not of interest for the present extensions, see 7.4.) Taking $\mathbb{D} = \mathrm{Lx}\mathbb{D}\mathrm{bl}_u$ gives a *unitary lax* right Kan extension of a *unitary lax double functor* $F\colon \mathbb{I} \mathrel{\bullet\!\!\!\to} \mathbb{X}$, along two *lax* double functors. But here, *we can always compose* SF obtaining an (arbitrary) lax double functor, and reduce our data to the (equivalent) case where the left arrow is an identity

$$\begin{tikzcd} \mathbb{I} \arrow[r, "R"] \arrow[d, "1^{\bullet}"'] & \mathbb{J} \arrow[d, "G^{\bullet}"] \arrow[dl, phantom, "\epsilon"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \end{tikzcd} \tag{79}$$

Now, $\mathrm{Lx}\mathbb{D}\mathrm{bl}_u$ does not have all comma objects, but its double subcategory $\mathrm{Lx}\mathbb{D}\mathrm{bl}_p = \mathbb{D}\mathrm{bl}_p$ (with *unitary pseudo* double functors as vertical arrows) does: they are again a part of comma objects in $\mathbb{D}\mathrm{bl}$. Speaking of *pointwise* Kan extensions in the present setting we will always mean *pointwise on unitary pseudo double functors*, as defined in 4.1(b).

This framework is perhaps the most adequate for studying *pointwise right extensions and their relationship with double limits*, and will be dealt with in Part IV.

Here, the symmetric notion of interest, *unitary colax left* Kan extensions along *colax* double functors, lives in $\mathbb{D} = \mathrm{Cx}\mathbb{D}\mathrm{bl}^u$, where the *horizontal* arrows are *unitary* colax double functors and the vertical ones are colax. Again, it comes from *internal* horizontal duality on all data. This setting is adequate for studying *pointwise left extensions and their relationship with double colimits*,

7. Examples

We end with some examples showing how the different settings we have considered in the last section can influence the resulting Kan extension.

7.1. Remarks. Some important cases arise when $\mathbb{J}$ is the terminal double category $\underline{1}$, or the *formal vertical arrow* $\underline{2}$, i.e. the double category which has one vertical arrow $0 \nrightarrow 1$ and is otherwise trivial.

Note that a *unitary* lax (or colax) double functor defined on $\underline{1}$ or $\underline{2}$ *is necessarily strict*, as already remarked at the end of 1.1. Therefore, unitary *lax* or *colax* double functors defined on $\mathbb{J} = \underline{1}$ or $\underline{2}$ coincide, and the 'colax case' $G = \mathrm{Ran}_{R,S}(F)$ (displayed below, at the left) is more general than the 'lax' one, $G = \mathrm{Ran}_{R,S}(1)$ (displayed at the right, as in 6.4)

$$\begin{tikzcd} \mathbb{I} \arrow[r, "R"] \arrow[d, "F"'] & \mathbb{J} \arrow[d, "G"] \arrow[dl, phantom, "\epsilon"] \\ \mathbb{X} \arrow[r, "S"'] & \mathbb{A} \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{I} \arrow[r, "R"] \arrow[d, "1"'] & \mathbb{J} \arrow[d, "G"] \arrow[dl, phantom, "\epsilon"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \end{tikzcd} \tag{80}$$

7.2. Double limits as pointwise unitary (co)lax extensions. By the previous remarks, the following situation can equivalently be viewed in $\mathbb{D}\mathrm{bl}_u$ or $\mathrm{Lx}\mathbb{D}\mathrm{bl}_u$.

Let $A = \lim(S)$ be the (horizontal) double limit of a lax double functor $S\colon \mathbb{I} \to \mathbb{A}$, as defined in Part I.

It is easy to see that this amounts to saying that the *unitary* double functor $A\colon \underline{1} \to \mathbb{A}$ is the *pointwise unitary* (colax or lax) right Kan extension of S along the projection $R\colon \mathbb{I} \to \underline{1}$, via the limit cone $\epsilon\colon AR \to S$ (see the right diagram above, with $\mathbb{J} = \underline{1}$).

On the other hand, a *lax* (resp. *colax*) double functor $T\colon \underline{1} \to \mathbb{A}$ is a *monad* (resp. a *comonad*) in the bicategory $\mathbf{V}\mathbb{A}$ of objects, vertical arrows and special cells of $\mathbb{A}$, and the universal such can be quite different from the limit, as shown below (7.3, 7.4).

One can note that the *unitary colax* extension $G = \mathrm{Ran}_{R,S}(F)$, as in the left diagram (80), would give the 'generalised double limit' of a composite $SF\colon \mathbb{I} \nrightarrow \mathbb{X} \to \mathbb{A}$, where the colax F need not be the identity. These generalised double limits can still be constructed from the elementary limits of Part I; but the relevance of this fact is not clear, and we shall not develop it here.

7.3. Unitary colax versus colax. Comparing the setting $\mathbb{D}\mathrm{bl}_u$ with $\mathbb{D}\mathrm{bl}$, it is easy to construct examples where the *unitary* colax Kan extension on the singleton (the limit) is different from the ‘’general’ colax Kan extension, and actually more interesting.

Let $\mathbb{A} = \mathbb{Q}\mathbf{Cat}$, the double category of quintets on the 2-category $\mathbf{Cat}$, so that $\mathbf{V}\mathbb{A}$ is the cell-dual of $\mathbf{Cat}$, with reversed cells.

An ordinary monad $(\mathbf{A}, T, \eta, \mu)$ can be viewed as a *strict double functor* $T\colon \underline{m} \to \mathbb{A}$, where $\underline{m}$ is the *formal vertical monoid*, i.e. the strict double category generated by a vertical arrow $t\colon 0 \to\!\!\!\!\bullet\; 0$, with two special cells $e\colon 1 \to t$, $m\colon t^2 \to t$ linked by the monoid axioms.

Its *non-unitary colax* Ran on the projection $\underline{m} \to \underline{1}$ coincides with T, viewed as a colax double functor $T\colon \underline{1} \to \mathbb{A}$ (a comonad in $\mathbf{V}\mathbb{A}$ and a monad in $\mathbf{Cat}$). But its double limit, i.e. the *unitary* (colax or lax) Ran (see 7.1), is the category of algebras $\mathbf{A}^T$, with cone based on the forgetful functor

$$\begin{aligned} &\epsilon_0 = U^T\colon \mathbf{A}^T \to \mathbf{A}, && \epsilon_{1_0} = \epsilon\colon TU^T \to U^T\colon \mathbf{A}^T \to \mathbf{A}, \\ &\epsilon(A, a\colon TA \to A) = a\colon TA \to A. && \end{aligned} \tag{81}$$

Indeed, let us apply our construction theorem for double limits (I.5.5) to $T\colon \underline{m} \to \mathbb{A}$. One begins (I.6.6) by replacing the vertical arrows $U = 1^\bullet_A$, T, T^2 with their tabulators and the natural transformations η, μ (viewed as cells of $\mathbb{Q}\mathbf{Cat}$) with the corresponding arrows

$$\begin{tikzcd} \mathbb{A} \arrow[r, equal] \arrow[d, "T"'] \arrow[dr, phantom, "\eta"] & \mathbb{A} \arrow[d, "U"] \\ \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, equal] \arrow[d, "T"'] \arrow[dr, phantom, "\mu"] & \mathbb{A} \arrow[d, "T^2"] \\ \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd} \qquad \top U \xleftarrow{\top\eta} \top T \xrightarrow{\top\mu} \top T^2 \tag{82}$$

where $\top T$ has objects $(A, B, f\colon TA \to B)$; similarly for $\top U$ and $\top T^2$; and

$$\begin{aligned} &\top\eta\colon \top T \to \top U, && (\top\eta)(A, B, f\colon TA \to B) = (A, B, f.\eta A\colon A \to B), \\ &\top\mu\colon \top T \to \top T^2, && (\top\mu)(A, B, f\colon TA \to B) = (A, B, f.\mu A\colon T^2A \to B). \end{aligned} \tag{83}$$

Then, as explained in I.6.7, one adds the following arrows d, p, q, c, to take into account the fact that U is a vertical identity and T^2 a vertical composite (the objects of the ‘iterated tabulator’ $\top(T,T)$ are specified below)

$$\begin{tikzcd} \top U & \top T \arrow[l, "\top\eta"'] \arrow[r, "\top\mu"] & \top T^2 \\ \mathbb{A} \arrow[u, "d"] & & \top(T,T) \arrow[u, "c"'] \arrow[ul, shift left, "q"] \arrow[ul, shift right, "p"'] \end{tikzcd} \tag{84}$$

$d(A) = (A, A, 1_A)$,
$p(A, B, C; f\colon TA \to B; g\colon TB \to C) = (A, B, f)$,
$q(A, B, C; f\colon TA \to B; g\colon TB \to C) = (B, C, g)$,
$c(A, B, C; f\colon TA \to B; g\colon TB \to C) = (A, C, g.Tf\colon T^2A \to C)$.

Finally, the double limit of T is the ordinary limit of the diagram (84). It is a full subcategory of $\top T$ (because d is monic, and the pair p, q jointly monic); and it contains precisely those objects $(A, B, f\colon TA \to B)$ of $\top T$ which are algebras

$$B = A, \qquad f.\eta A = 1_A; \qquad f.\mu A = f.Tf. \tag{85}$$

7.4. Unitary lax versus lax. One can give a similar, simpler example to compare the setting $\mathrm{Lx}\mathbb{D}\mathrm{bl}_u$ with $\mathrm{Lx}\mathbb{D}\mathrm{bl}$.

With $\mathbb{I} = \underline{1}$, again, a lax double functor $S\colon \underline{1} \to \mathbb{Q}\mathbf{Cat}^h$ (with values in the horizontally opposite double category) amounts to a monad on the category $S(0)$. Its double limit A, corresponding to the *unitary* right Kan extension, is the category of Eilenberg-Moore algebras, while the *general* lax right Kan extension is trivial and coincides with S.

7.5. Local products. The following works both in $\mathbb{D}\mathrm{bl}_u$ and $\mathrm{Lx}\mathbb{D}\mathrm{bl}_u$. Take for $\mathbb{I}$ the strict double category $\underline{2}_I$, having two objects, 0 and 1, and a small set I of vertical arrows $i\colon 0 \mathrel{\bullet\!\!\to} 1$. Let $R\colon \underline{2}_I \to \underline{2}$ be the obvious projection, with $R(i) = (0 \mathrel{\bullet\!\!\to} 1)$

$$\begin{tikzcd}
\underline{2}_I \arrow[r, "R"] \arrow[d, "1\bullet"'] & \underline{2} \arrow[d, dotted, "\bullet G"] \arrow[dl, phantom, "\epsilon"] \\
\underline{2}_I \arrow[r, "\mathbf{u}"'] & \mathbb{A}
\end{tikzcd} \tag{86}$$

A unitary (necessarily strict) double functor $\mathbf{u}\colon \underline{2}_I \to \mathbb{A}$ is a family of parallel vertical arrows $u_i\colon A \mathrel{\bullet\!\!\to} B$. The pointwise unitary, *lax or colax*, Kan extension $G = \mathrm{Ran}_{R,u}(1)$ amounts to an arrow $u\colon A_0 \mathrel{\bullet\!\!\to} A_1$ equipped with a family of special cells $\sigma_i\colon u \to u_i$, universal in the obvious sense. We shall call it the *local product* of the family.

This does not exist in $\mathbb{Q}\mathbf{Set}$ or $\mathbb{Q}\mathbf{Cat}$ (showing that this extension is independent of the existence of double limits). But it does exist in $\mathbb{R}\mathrm{el}$ (the double category of sets, mappings and relations, see I.3.4), where it gives the intersection of our relations (even for a large I). It also exists (for a small I) in $\mathbb{S}\mathrm{et}$, where it is constructed with a limit. And in $\mathbb{C}\mathrm{at}$, where we have

$$\begin{aligned} &u\colon A^{\mathrm{op}} \times B \to \mathbf{Set}, && u(a, b) = \textstyle\prod_i u_i(a, b), \\ &p_i\colon u \to u_i, && p_i(a, b)\colon u(a, b) \to u_i(a, b). \end{aligned} \tag{87}$$

7.6. Lax versus colax. This example is an extension of the previous one. It is based on the strict double category $\underline{3}$, which is again a *vertical ordinal*, generated by two vertical arrows $0 \mathrel{\bullet\!\!\to} 1 \mathrel{\bullet\!\!\to} 2$. Of course, unitary lax or colax double functors $\underline{3} \to \mathbb{A}$ no longer coincide.

Take as $\mathbb{I}$ the (vertical) double category freely generated by three objects $0, 1, 2$ and two sets of vertical arrows

$$u'_i \colon 0 \to\!\!\!\!\!\!\bullet\;\; 1, \qquad u''_j \colon 1 \to\!\!\!\!\!\!\bullet\;\; 2 \qquad\qquad (i \in I, j \in J). \tag{88}$$

It has thus vertical arrows $u''_j u'_i \colon 0 \to\!\!\!\!\!\!\bullet\;\; 2$ indexed on $I \times J$, and is horizontally discrete. Form now the following diagram

$$\begin{tikzcd} \mathbb{I} \arrow[r, "R"] \arrow[d, "1^\bullet"'] & \underline{3} \arrow[d, dotted, "G^\bullet"] \arrow[dl, phantom, "\epsilon"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \end{tikzcd} \tag{89}$$

where R is the obvious projection (preserving objects), $\mathbb{A} = \mathbb{R}\mathrm{el}$, and S is a strict double functor.

First, let us consider the setting $\mathbb{D} = \mathrm{Lx}\mathbb{D}\mathrm{bl}_u$ and let G be the unitary *lax* right Kan extension. Plainly, G is computed by intersection of parallel relations (the 'local products' of 7.5), and is a pointwise extension

$$\begin{array}{ll} G(0 \to\!\!\!\!\!\!\bullet\;\; 1) = \bigcap_i F(u'_i), & G(1 \to\!\!\!\!\!\!\bullet\;\; 2) = \bigcap_j F(u''_i), \\ G(0 \to\!\!\!\!\!\!\bullet\;\; 2) = \bigcap_{ij} F(u''_i u'_i), & G(0 \to\!\!\!\!\!\!\bullet\;\; 1) \otimes G(1 \to\!\!\!\!\!\!\bullet\;\; 2) \subset G(0 \to\!\!\!\!\!\!\bullet\;\; 2). \end{array} \tag{90}$$

Note that G is not strict (i.e., not colax), generally: one can easily construct a finite example where both $G(0 \to\!\!\!\!\!\!\bullet\;\; 1)$ and $G(1 \to\!\!\!\!\!\!\bullet\;\; 2)$ are empty, while $G(0 \to\!\!\!\!\!\!\bullet\;\; 2)$ is not.

On the other hand, the unitary *colax* right Kan extension, in (89), also exists, and has $G'(0 \to\!\!\!\!\!\!\bullet\;\; 2) = G(0 \to\!\!\!\!\!\!\bullet\;\; 1) \otimes G(1 \to\!\!\!\!\!\!\bullet\;\; 2)$; therefore, it is not pointwise.

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