Theory and Applications of Categories, Vol. 20, No. 15, 2008, pp. 504–542.

ISBELL DUALITY

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ABSTRACT. We develop in some generality the dualities that often arise when one object lies in two different categories. In our examples, one category is equational and the other consists of the topological objects in a (generally different) equational category.

1. Introduction

Many years ago, one of the authors heard Bill Lawvere say that a potential duality arises when a single object lives in two different categories. When asked, Lawvere credited the statement to John Isbell. From the literature, it seems that what is called "Isbell duality" is the one between frames with enough points and sober spaces. The common object is the boolean algebra 2 in the first category and the Sierpinski space S in the second.

The purpose of this paper is to explore the situation in some generality. Of course, there are well-known examples such as the original Stone duality and many others that follow the pattern, but we have general results that seem to be new. We study three main examples. First, the duality between $\mathbf{N}$-compact spaces, defined as spaces that are closed subspaces of a power of $\mathbf{N}$, and a class (which we have not been able to identify) of lattice ordered $\mathbf{Z}$-rings, defined as lattice-ordered rings that are algebraically and lattice-theoretically embedded in a power of $\mathbf{Z}$, (Section 6). The second is a similar duality between realcompact spaces and a class of lattice-ordered $\mathbf{R}$-algebras, (Section 7). The third is the duality between certain categories of discrete and topological abelian groups (Section 8). Other examples are briefly noted in Section 9.

Preliminary to this, we develop what appear to be new results in category theory.

There is some overlap between this article and some of the material in [Johnstone (1986)], especially his VI.4. However, Johnstone begins with different assumptions on the categories and is headed towards different conclusions. He begins with equational categories and introduces the topology by looking at the algebras for the triples (monads) associated to the hom functors. Our general theory is less restrictive and the topology is present from the beginning. He derives the duality between sober spaces and spatial frames, but in a quite different way. He and many others have done the Gelfand duality.

The first and third authors would like to thank NSERC of Canada for its support of this research. We would all like to thank McGill and Concordia Universities for partial support of the middle author's visits to Montreal.

Received by the editors 2007-11-29 and, in revised form, 2008-11-14.

Transmitted by F. William Lawvere. Published on 2008-11-17.

2000 Mathematics Subject Classification: 18A40, 18D20.

Key words and phrases: duality, fixed categories, sober objects.

As far as we are aware the duality between **N**-compact spaces and a class of **Z**-rings is new, as is the duality between a certain class of topological abelian groups and a class of discrete groups.

Another paper that overlaps this one is [Porst & Tholen]. It Section 5, the existence and some of the basic properties of both $\hom(\text{-}, Z_{\mathcal{C}})$ and $\hom(\text{-}, Z_{\mathcal{D}})$, including that the underlying sets are the respective Homs. Our paper is concerned more with constructing the homs and proving their basic properties. The referee has pointed out that the largely expository [Porst & Tholen] depends on [Dimov & Tholen (1993)] in which homs are also constructed. He has also directed us to the following additional papers on duality, [Borger, et al. (1981), Hofmann (2002), Davey (2006)].

1.1. NOTATION. *We will be considering a number of special classes of monic and epic morphisms. All our categories are concrete (with obvious underlying functors) and we will use* $\hookrightarrow$ *only for maps that, up to equivalences, are embeddings, algebraic as well as topological (when a topology is present). All epic and all other monic arrows will be denoted* $\twoheadrightarrow$ *and* $\rightarrowtail$*, respectively. If they come from a special class (e.g. extremal monics) we will say so. Another notational convention we will adopt for most of this paper is that* $|-|$ *is used to denote the underlying set of an object. The exceptions are in Sections 6 and 7 where it is used to denote the absolute value of a real number and in Section 8 where it is used to denote the discrete group underlying a topological group.*

1.2. THE SETTING. We consider categories $\mathcal{C}$ and $\mathcal{D}$ that are complete and cocomplete and have faithful limit-preserving underlying functors $|\text{-}|_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{S}et$ and $|\text{-}|_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathcal{S}et$. In addition, we specify objects $Z_{\mathcal{C}} \in \mathcal{C}$ and $Z_{\mathcal{D}} \in \mathcal{D}$ with $|Z_{\mathcal{C}}| = |Z_{\mathcal{D}}|$, denoted Z.

Let C and D be objects of $\mathcal{C}$ and $\mathcal{D}$, respectively. By a **bimorphism** of $C \times D$ to Z we mean a function $f : |C| \times |D| \longrightarrow Z$ such that:

B1. for each $c \in |C|$, $f(c, \text{-})$ underlies a morphism $D \longrightarrow Z_{\mathcal{D}}$; and

B2. for each $d \in |D|$, $f(\text{-}, d)$ underlies a morphism $C \longrightarrow Z_{\mathcal{C}}$.

These morphisms are unique because of the faithfulness of the underlying functors. We denote the set of these bimorphisms by $\mathrm{Bim}(C, D)$.

For any morphism $g : C' \longrightarrow C$ in $\mathcal{C}$, any object $D \in \mathcal{D}$, and any $f \in \mathrm{Bim}(C, D)$, we have a function $|C'| \times |D| \xrightarrow{|g| \times |D|} |C| \times |D| \xrightarrow{f} Z$. Thus if $c' \in |C'|$, then $f \circ (|g| \times |D|)(c', \text{-}) = f(|g|(c'), \text{-})$ is a morphism in $\mathcal{D}$, while for any $d \in |D|$, $f \circ (|g| \times |D|)(\text{-}, d) = f(\text{-}, d) \circ g$ and is a morphism in $\mathcal{C}$. A similar computation for $\mathcal{D}$ shows that $\mathrm{Bim} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{S}et$ is a functor. One aim in this paper is to show that, under reasonable hypotheses, this functor will be representable in the following sense:
There are functors $\hom(\text{-}, Z_{\mathcal{C}}) : \mathcal{C} \longrightarrow \mathcal{D}$ and $\hom(\text{-}, Z_{\mathcal{D}}) : \mathcal{D} \longrightarrow \mathcal{C}$ such that $\hom(\text{-}, Z_{\mathcal{C}})$ is embedded in $Z_{\mathcal{D}}^{|\text{-}|}$ and $\hom(\text{-}, Z_{\mathcal{D}})$ is embedded in $Z_{\mathcal{C}}^{|\text{-}|}$ in such a way that

$$|\hom(\text{-}, Z_{\mathcal{C}})| \cong \mathrm{Hom}_{\mathcal{C}}(\text{-}, Z_{\mathcal{C}}); \qquad |\hom(\text{-}, Z_{\mathcal{D}})| = \mathrm{Hom}_{\mathcal{D}}(\text{-}, Z_{\mathcal{D}})$$

$$\mathrm{Hom}_{\mathcal{D}}(\text{-}, \hom(\text{-}, Z_{\mathcal{C}})) \cong \mathrm{Bim}(\text{-}, \text{-}) \cong \mathrm{Hom}_{\mathcal{C}}(\text{-}, \hom(\text{-}, Z_{\mathcal{D}}))$$

and

$$\begin{tikzcd} |\hom(\text{-}, Z_{\mathcal{C}})| \arrow[r, hook] \arrow[d, "\cong"'] & |Z_{\mathcal{D}}^{|\text{-}|}| \arrow[d, "\cong"] \\ \mathrm{Hom}(\text{-}, Z_{\mathcal{C}}) \arrow[r, hook] & Z^{|\text{-}|} \end{tikzcd} \qquad \begin{tikzcd} |\hom(\text{-}, Z_{\mathcal{D}})| \arrow[r, hook] \arrow[d, "\cong"'] & |Z_{\mathcal{C}}^{|\text{-}|}| \arrow[d, "\cong"] \\ \mathrm{Hom}(\text{-}, Z_{\mathcal{D}}) \arrow[r, hook] & Z^{|\text{-}|} \end{tikzcd}$$

commute, where the bottom inclusions are the canonical ones.

2. Some categorical generalities

2.1. Fixed objects. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Let $U : \mathcal{B} \longrightarrow \mathcal{A}$ be a functor with left adjoint F. An object $A \in \mathcal{A}$ is **fixed** by UF if the inner adjunction $A \longrightarrow UFA$ is an isomorphism and an object $B \in \mathcal{B}$ is fixed by FU if the outer adjunction $FUB \longrightarrow B$ is an isomorphism. In our case of contravariant adjoints, both adjunction morphisms look inner. Thus an object $C \in \mathcal{C}$ is fixed if the canonical map $C \longrightarrow \hom(\hom(C, Z_{\mathcal{C}}), Z_{\mathcal{D}})$ is an isomorphism and $D \in \mathcal{D}$ is fixed if $D \longrightarrow \hom(\hom(D, Z_{\mathcal{D}}), Z_{\mathcal{C}})$ is. We denote by $\mathrm{Fix}(UF) \subseteq \mathcal{A}$ and $\mathrm{Fix}(FU) \subseteq \mathcal{B}$ the full subcategories of fixed objects. Nothing guarantees that these categories are non-empty, but we do have:

2.2. Theorem. *If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is left adjoint to $U : \mathcal{B} \longrightarrow \mathcal{A}$, then the restrictions of F and U define equivalences* $\mathrm{Fix}(UF) \simeq \mathrm{Fix}(FU)$.

For a proof, see [Lambek & Rattray (1979), Theorem 2.0].

2.3. The epic/extremal-monic factorization system. For the rest of this section, we will suppose that $\mathcal{A}$ is a category with finite limits and colimits. A map $m : A_1 \longrightarrow A_2$ in $\mathcal{A}$ is called an **extremal monic** if whenever $m = f \circ e$ with e epic, then e is an isomorphism. The existence of coequalizers implies that extremal monics are *monic*. We say that $\mathcal{A}$ has an epic/extremal-monic factorization system if every morphism f can be written as $f = m \circ e$ with e epic and m extremal monic. It can be shown that, for any commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, two heads, "e"] \arrow[d, "f_1"'] & A_2 \arrow[d, "f_2"] \\ A_1' \arrow[r, tail, "m"'] & A_2' \end{tikzcd}$$

with e epic and m extremal monic, there is a unique map $A_2 \longrightarrow A_1'$ making both triangles commute. That map is called the **diagonal fill-in**. This readily shows that the factorization is unique up to unique isomorphism.

Isbell showed that a sufficient condition for such a factorization system to exist is that $\mathcal{A}$ be complete and well-powered with respect to extremal monics, [Isbell (1966), Theorem 2.4].[1] We next turn to results concerning extremal well-poweredness.

We note that in any category with pullbacks and pushouts, the class of extremal monics is pullback invariant.

In the next three propositions assume that $\mathcal{A}$ has a faithful pullback-preserving underlying set functor, denoted $|\text{-}|$. We will use the well-known fact that pullback-preserving functors preserve monics. In this case this means that monics are taken to injective (that is, one-one) functions.

2.4. Proposition. *If $f : A' \rightarrowtail A$ is an extremal monomorphism in $\mathcal{A}$ such that $|f| : |A'| \longrightarrow |A|$ is surjective, then $A' \longrightarrow A$ is an isomorphism.*

Proof. If $|f|$ is surjective, then, because of the faithfulness of $|\text{-}|$ it follows that f is epic and, since f is also extremal monic, it must be an isomorphism. ∎

2.5. Proposition. *Suppose we have a diagram in $\mathcal{A}$ of the form*

$$\begin{tikzcd} & A_1 \arrow[d, "f"] \\ A_2' \arrow[r, tail, "g"'] & A_2 \end{tikzcd}$$

for which there is a function $u : |A_1| \longrightarrow |A_2'|$ such that

$$\begin{tikzcd} & {|A_1|} \arrow[dl, "u"'] \arrow[d, "|f|"] \\ {|A_2'|} \arrow[r, tail, "|g|"'] & {|A_2|} \end{tikzcd}$$

commutes. Assume either that g is extremal monic or that g is monic and $|\text{-}|$ reflects isomorphisms. Then there is a unique morphism $h : A_1 \longrightarrow A_2'$ such that $|h| = u$.

Proof. We will prove this under the hypothesis that g is extremal monic. The other case is similar. Form the pullback diagram

$$\begin{tikzcd} A_1' \arrow[r, tail, "q"] \arrow[d, "p"'] & A_1 \arrow[d, "f"] \\ A_2' \arrow[r, tail, "g"'] & A_2 \end{tikzcd}$$

[1] In reading Isbell, the statement that the category has a set of objects needs explanation. Isbell was dealing with a three-universe set theory. Although his language is different, he assumes small sets, large sets, and proper classes. His category is small-complete and small-extremally-well-powered, but may have a large set of objects.

Since extremal monics are pullback invariant, q is extremal monic. By assumption, there is a function $u : |A_1| \longrightarrow |A'_2|$ making the lower right triangle commute

$$\begin{tikzcd} |A'_1| \arrow[r, tail, "|q|"] \arrow[d, "|p|"'] & |A_1| \arrow[d, "|f|"] \arrow[dl, "u"'] \\ |A'_2| \arrow[r, tail, "|g|"'] & |A_2| \end{tikzcd}$$

As for the upper left triangle, we have that $|g| \circ u \circ |q| = |f| \circ |q| = |g| \circ |p|$. Since $|g|$ is injective, it can be cancelled on the left and the other triangle commutes. The underlying diagram is also a pullback, and one easily sees that this forces $|q| : |A'_1| \longrightarrow |A_1|$ to be an isomorphism in $\mathcal{Set}$. It follows from the preceding proposition that q is an isomorphism and then $p^{-1}q$ is the required map. ■

2.6. Proposition. *The category $\mathcal{A}$ is well-powered with respect to extremal monics. If* |-| *reflects isomorphisms, then $\mathcal{A}$ is well-powered.*

Proof. We will prove the first claim; to get the second just drop "extremal" everywhere. Suppose $A_1 \rightarrowtail A$ and $A_2 \rightarrowtail A$ are extremal subobjects with $|A_1|$ and $|A_2|$ the same subset of $|A|$. Apply Proposition 2.5 in both directions to the diagram

$$\begin{tikzcd} & A_1 \arrow[d, tail] \\ A_2 \arrow[r, tail] & A \end{tikzcd}$$

It then follows that two extremal subobjects with the same underlying subset are equal so the conclusion follows from the well-poweredness of sets. ■

From this and the previously cited theorem of Isbell's we see the following:

2.7. Theorem. *Suppose that $\mathcal{A}$ is complete with a faithful limit-preserving functor to $\mathcal{Set}$. Then $\mathcal{A}$ has an epic/extremal monic factorization system.* ■

2.8. Factorization systems and the RSAFT. In addition to the Epic/Extremal-monic factorization system, there are others. For example there is usually an Extremal-epic/Monic factorization inherited from the dual category. The general definition is a pair of subcategories $\mathcal{E}$ and $\mathcal{M}$, each containing all the isomorphisms, with the property that every arrow f has a unique, up to isomorphism, factorization $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$.

It is not required in an $\mathcal{E}/\mathcal{M}$ factorization system that every arrow in $\mathcal{E}$ be epic nor that every arrow in $\mathcal{M}$ be monic. But both conditions are satisfied in most common factorization system. Such a system will here be called **standard**. To simplify our exposition we will specialize to the standard case.

We state some of the well-known properties of factorization systems.

2.9. Proposition. *Let $\mathcal{E}/\mathcal{M}$ be a standard factorization system in the category $\mathcal{A}$. Assume that $\mathcal{A}$ has finite limits and colimits. Then the following properties—as well as their duals—hold:*

1. *The diagonal fill-in.*
2. *The converse of the diagonal fill-in: if a map m has the property that for every $e \in \mathcal{E}$ and for every commutative diagram*

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

there is a diagonal fill-in, then $m \in \mathcal{M}$.

3. *Every regular monic belongs to $\mathcal{M}$ (because every $e \in \mathcal{E}$ is epic).*
4. *$\mathcal{M}$ is invariant under pullbacks.*
5. *$\mathcal{M}$ is closed under arbitrary products.*
6. *$\mathcal{M}$ is closed under arbitrary intersections.*
7. *$\mathcal{M}$ is closed under composition (because it is a subcategory).*
8. *If $f \circ g \in \mathcal{M}$, then $g \in \mathcal{M}$.*

The argument below follows closely that of [Barr & Wells (1984), Exercise SAFT], in turn organized by G. M. Kelly.

2.10. Theorem. *Suppose $\mathcal{A}$ is a complete category with a standard factorization system $\mathcal{E}/\mathcal{M}$. Suppose $\mathcal{Q}$ is a* set *of objects of $\mathcal{A}$ with the property that every object $A \in \mathcal{A}$ has an $\mathcal{M}$-embedding into a product of objects from $\mathcal{Q}$. Suppose objects of $\mathcal{A}$ have only a set of $\mathcal{M}$-subobjects. Then $\mathcal{A}$ has an initial object.*

Proof. Since each object has a set of $\mathcal{M}$-subobjects and $\mathcal{M}$ is invariant under intersection, each object has a least $\mathcal{M}$-subobject. So let A be an object and A_0 its least $\mathcal{M}$-subobject. Write $A \rightarrowtail \prod_{i\in I} Q_i$ with each $Q_i \in \mathcal{Q}$. Let $J \subseteq I$ be an enumeration of the set of *distinct* objects among the Q_i. There is a canonical arrow $s : \prod_{j\in J} Q_j \longrightarrow \prod_{i\in I} Q_i$ determined by $\mathrm{proj}_i \circ s = \mathrm{proj}_j$ where j is the unique element of J for which $Q_j = Q_i$. Clearly s is a split monic and hence belongs to $\mathcal{M}$. Form the pullback

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & \prod_{j\in J} Q_j \arrow[d, tail] \\ A \arrow[r, tail] & \prod_{i\in I} Q_i \end{tikzcd}$$

Let P_0 be the least $\mathcal{M}$-subobject of P. Then P is also an $\mathcal{M}$-subobject of A so that $A_0 \subseteq P_0$ since A_0 is the least $\mathcal{M}$-subobject of A. But since P_0 has no proper $\mathcal{M}$-subobject, $A_0 = P_0$. Thus the least $\mathcal{M}$-subobject of A is in fact a subobject of a product of a distinct set of objects from $\mathcal{Q}$ which means that there is only a set of these least $\mathcal{M}$-subobjects. The product of these least subobjects is an object with at least one morphism to every object of $\mathcal{A}$. Its least $\mathcal{M}$-subobject cannot have two maps to any object, else the equalizer of those two maps would be smaller. ∎

2.11. Corollary. [Relative Special Adjoint Functor Theorem] *Suppose $\mathcal{A}$ is as above. Then any limit-preserving functor $U : \mathcal{A} \longrightarrow \mathcal{B}$ has a left adjoint.*

Proof. For any object $B \in \mathcal{B}$ the category B/U has as objects (A, f) where $A \in \mathcal{A}$ and $(f : B \longrightarrow UA) \in \mathcal{B}$. A map $g : (A, f) \longrightarrow (A', f')$ is a map $g : A \longrightarrow A'$ such that

$$\begin{tikzcd} & B \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ UA \arrow[rr, "Ug"'] & & UA' \end{tikzcd}$$

commutes. The set of objects (Q, f) with $Q \in \mathcal{Q}$ gives the set $B/\mathcal{Q}$ and the set of $g : (A, f) \longrightarrow (A', f')$ with $g \in \mathcal{E}$ (respectively, $g \in \mathcal{M}$), gives the classes $B/\mathcal{E}$ and $B/\mathcal{M}$. It is easy to see that B/U is complete and that the set $B/\mathcal{Q}$ of objects and the classes $(B/\mathcal{E})/(B/\mathcal{M})$ satisfy the conditions of the theorem so that B/U has an initial object $(FB, \epsilon B)$ which gives the adjunction. ∎

3. More categorical generalities: triples

3.1. The triple associated to an adjoint pair. Let $F : \mathcal{A} \longrightarrow \mathcal{X}$ be left adjoint to $U : \mathcal{X} \longrightarrow \mathcal{A}$ with adjunction natural transformations $\eta : \mathrm{Id} \longrightarrow UF$ and $\epsilon : FU \longrightarrow \mathrm{Id}$. Then for $T = UF$ and $\mu = U\epsilon F$, the triple $\mathbf{T} = (T, \eta, \mu)$ will be called the *triple associated to the adjunction.*

In this paper, all the triples we use arise in the following way. Let $\mathcal{A}$ be a complete category, E be an object of $\mathcal{A}$, and $F = \mathrm{Hom}(-, E) : \mathcal{A} \longrightarrow \mathcal{S}et^{\mathrm{op}}$ which is readily seen to be left adjoint to $U = E^{(-)} : \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathcal{A}$. Then $\mathbf{T}$ is the triple associated to the adjoint pair.

3.2. Definition. *Suppose $F : \mathcal{A} \longrightarrow \mathcal{X}$ is left adjoint to $U : \mathcal{X} \longrightarrow \mathcal{A}$ with adjunction morphisms $\eta : \mathrm{Id} \longrightarrow UF$ and $\epsilon : FU \longrightarrow \mathrm{Id}$. Let $T = UF$ be the associated triple. Following Paul Taylor, we say that an object A of $\mathcal{A}$ is* **T-sober** *if there is an equalizer diagram*

$$A \longrightarrow UX \rightrightarrows UY$$

In fact, we could take X and Y to have the form FB and FC so that $UX = TB$ and $UY = TC$, so the name is not a misnomer. However, this form is convenient for use. We say that A is **canonically T-sober** if the diagram

$$\begin{tikzcd} A \arrow[r, "\eta A"] & TA \arrow[r, shift left, "T\eta A"] \arrow[r, shift right, "\eta TA"'] & T^2A \end{tikzcd}$$

is an equalizer. Later, we will give an example that shows that **T**-sobriety is strictly weaker than canonical **T**-sobriety (Example 3.8). If **T** is the triple derived from the powers of E as described in 3.1 above, we will say that an object is **E-sober** (respectively, **canonically E-sober**), when it is **T**-sober (respectively, canonically **T**-sober).

3.3. PROPOSITION. *Let $\mathcal{A}$ and $\mathcal{X}$ be categories. Let $F : \mathcal{A} \longrightarrow \mathcal{X}$ be left adjoint to $U : \mathcal{X} \longrightarrow \mathcal{A}$ with adjunction morphisms $\eta : \mathrm{Id} \longrightarrow UF$ and $\epsilon : FU \longrightarrow \mathrm{Id}$. Suppose that $\mathcal{A}$ has an epic/extremal-monic factorization system. Then for an object $A \in \mathcal{A}$, ηA is an extremal monic if and only if there is an extremal monic $A \rightarrowtail UX$ for some $X \in \mathcal{X}$.*

PROOF. By the universal mapping property of ηA every map $A \longrightarrow UX$ factors through ηA. ∎

The meaning of the next result is that in a category with cokernel pairs a sufficient (and obviously necessary) condition that a **T**-sober object A be canonically **T**-sober is that the cokernel pair of ηA be embedded in an instance of T. In an abelian category, cokernel pairs may be replaced by cokernels. The result has been known in one form or another since the early days of the study of triples. It is probably due to Jon Beck, but we have been unable to find a reference.

3.4. PROPOSITION. *Let $\mathcal{A}$, $\mathcal{X}$, U, F, and **T** be as above. Suppose that there is an equalizer diagram of the form $\begin{tikzcd} A \arrow[r, "\eta A"] & TA \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & UY \end{tikzcd}$. Then A is canonically **T**-sober.*

PROOF. Define $v = U\epsilon Y \circ Tf$ and $w = U\epsilon Y \circ Tg$ both from $T^2A \longrightarrow UY$. From the diagram

$$\begin{tikzcd} TA \arrow[r, "f"] \arrow[d, "\eta TA"'] & UY \arrow[d, "\eta UY"'] \arrow[dr, "\mathrm{id}"] & \\ T^2A \arrow[r, "Tf"'] & TUY \arrow[r, "U\epsilon Y"'] & UY \end{tikzcd}$$

we see that $v \circ \eta TA = f$ and similarly $w \circ \eta TA = g$. From $f \circ \eta A = g \circ \eta A$ we get $Tf \circ T\eta A = Tg \circ T\eta A$ so that

$$v \circ T\eta A = U\epsilon Y \circ Tf \circ T\eta A = U\epsilon Y \circ Tg \circ T\eta A = w \circ T\eta A$$

Thus if $h : A' \longrightarrow TA$ satisfies $\eta TA \circ h = T\eta A \circ h$, we have

$$f \circ h = v \circ \eta TA \circ h = v \circ T\eta A \circ h = w \circ T\eta A \circ h = w \circ \eta TA \circ h = g \circ h$$

and thus we have a unique $k : A' \longrightarrow A$ such that $\eta A \circ k = h$. ∎

3.5. THEOREM. *Suppose that every* $\mathbf{T}$*-sober object is canonically* $\mathbf{T}$*-sober. Then the equalizer of two maps between* $\mathbf{T}$*-sober objects is* $\mathbf{T}$*-sober.*

PROOF. Suppose $A'' \xrightarrow{f} A \overset{g}{\underset{h}{\rightrightarrows}} A'$ is an equalizer and A and A' are $\mathbf{T}$-sober. It is immediate that $A'' \xrightarrow{f} A \overset{(g,\mathrm{id})}{\underset{(h,\mathrm{id})}{\rightrightarrows}} A' \times A$ is also an equalizer. Moreover, the two arrows $A \longrightarrow A'$ now have a common left inverse. Thus we may suppose that g and h have a common left inverse l. In the diagram

$$\begin{tikzcd}
A'' \arrow[r, "e"] & A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] \arrow[d, "\eta A"'] & A' \arrow[l, "l" description] \arrow[d, "\eta A''"] \\
& TA \arrow[r, shift left=2, "Tf"] \arrow[r, shift right=2, "Tg"'] \arrow[d, shift right=3, "\eta TA"'] \arrow[d, shift left=3, "T\eta A"] & TA' \arrow[l, "Tl" description] \arrow[d, shift right=3, "\eta TA'"'] \arrow[d, shift left=3, "T\eta A'"] \\
& T^2A \arrow[r, shift left=2, "T^2f"] \arrow[r, shift right=2, "T^2g"'] \arrow[u, "\mu A" description] & T^2A' \arrow[l, "T^2l" description] \arrow[u, "\mu A'" description]
\end{tikzcd}$$

the row and both columns are equalizers. There are many more commutativities, which we leave to the reader to calculate. There are enough of them to apply the following lemma, which will complete the proof. ■

The following folkloric result goes back, in dual form in degree zero, to the fact that the diagonal chain complex associated to a double simplicial object is homotopic to its total complex.

3.6. LEMMA. *Suppose in any category*

$$\begin{tikzcd}
A_0'' \arrow[r, "e_0"] & A_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] \arrow[d, "h"'] & A_0' \arrow[d, "h'"] \\
& A_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "g_1"'] \arrow[d, shift right=2, "k"'] \arrow[d, shift left=2, "m"] & A_1' \arrow[d, shift right=3, "k'"'] \arrow[d, shift left=3, "m'"] \\
& A_2 \arrow[r, shift left=2, "f_2"] \arrow[r, shift right=2, "g_2"'] & A_2' \arrow[l, "l_2" description] \arrow[u, "l'" description]
\end{tikzcd}$$

is a diagram in which the row and two columns are equalizers. Assume the following commutativities:

1. $f_1 h = h' f_0$;
2. $g_1 h = h' g_0$;
3. $f_2 k = k' f_1$;
4. $g_2 k = k' g_1$;
5. $f_2 m = m' f_1$;
6. $g_2 m = m' g_1$;
7. $l'k' = l'm' = \mathrm{id}$;
8. $l_2 f_2 = l_2 g_2 = \mathrm{id}$.

Then $A_0'' \xrightarrow{he_0} A_1 \underset{m'g_1=g_2m}{\overset{k'f_1=f_2k}{\rightrightarrows}} A_2'$ *is an equalizer.*

Proof. First we calculate

$$k'f_1he_0 = k'h'f_0e_0 = k'h'g_0e_0 = k'g_1he_0 = g_2khe_0 = g_2mhe_0 = m'g_1he_0$$

so that he_0 is equalized by the other two maps. It is sufficient to show this lemma holds in the category of sets. So suppose that $a_1 \in A_1$ is an element such that $f_2ka_1 = g_2ma_1$. Then apply l_2 to conclude that $ka_1 = ma_1$. But then there is a unique element $a_0 \in A_0$ such that $ha_0 = a_1$. Using l' we similarly conclude that $f_1a_1 = g_1a_1$ and then $h'f_0a_0 = f_1ha_0 = f_1a_1 = g_1a_1 = g_1ha_0 = h'g_0a_0$ and h' is monic so that $f_0a_0 = g_0a_0$ and therefore there is a unique element $a_0'' \in A_0''$ such that $e_0a_0'' = a_0$. ∎

Since the full subcategory of $\mathcal{A}$ consisting of the $\mathbf{T}$-sober objects is closed under products, we conclude immediately:

3.7. Corollary. *If $\mathcal{A}$ is complete and if every $\mathbf{T}$-sober object is canonically $\mathbf{T}$-sober, then the full subcategory of $\mathbf{T}$-sober objects is a limit-closed subcategory of $\mathcal{A}$* ∎

3.8. Example. We are indebted to George Janelidze for this example, posted on the categories net 2008-05-12. It describes a triple $\mathbf{T}$ on a complete category $\mathcal{A}$ for which the category of $\mathbf{T}$-sober objects is not complete. Hence there are $\mathbf{T}$-sober objects not canonically $\mathbf{T}$-sober.

Let $\mathcal{A}$ be the category of commutative rings, let $\mathbf{Q}$ be the field of rational numbers, $E = \mathbf{Q}[2^{1/4}]$ and $K = \mathbf{Q}[2^{1/2}]$. Although it is not necessary, it might help to imagine these are subfields of the real numbers and that the roots are the unique positive roots. Then E has a unique element with the properties of being a square root of 2, while being a square. Every power of E has such a unique element. The unicity implies that under any ring homomorphism from one power to another, this element is preserved and hence is in the equalizer of any two homomorphisms between powers of E. Thus every

E-sober ring contains a square root of 2 (but generally not a fourth root) and hence $\mathbf{Q}$ is not E-sober. On the other hand, K is the equalizer of the two automorphisms of E and $\mathbf{Q}$is the equalizer of the two automorphisms of K, which of course do not extend to automorphisms of E.

3.9. Definition. *Let $U : \mathcal{X} \longrightarrow \mathcal{A}$ be a functor. We say that U is a* **cogenerating** *functor if for any pair of distinct arrows $f, g : A \longrightarrow B$ in $\mathcal{A}$, there is an object X of $\mathcal{X}$and a map of the form $h : B \longrightarrow UX$ such that $hf \neq hg$. We will say that U is a* **coseparating** *functor if it is cogenerating and whenever $f : A \longrightarrow B$ is not an epimorphism and $g : B \longrightarrow UX$ is a regular monomorphism, there is a pair of maps $h, k : UX \longrightarrow UY$ for some object Y of $\mathcal{X}$such that $hg \neq kg$, but $hgf = kgf$.*

Suppose U has a left adjoint F and $\mathbf{T}$ is the associated triple. Then one can readily show that U-cogenerating and U-coseparating are the same as T-cogenerating and T-coseparating, respectively. This follows immediately from the fact that for any object X, the map $U\eta X : X \longrightarrow UFUX$ is a monomorphism split by the other adjunction transformation ϵUX.

In the special case that $\mathbf{T}$ comes from the powers of an object E as described in 3.1 we will say that E is **cogenerating** respectively **coseparating**, when T is. We will also say that E is a **cogenerator**, respectively, a **coseparator**.

It is worth noting that for a single object, both the definition of cogenerator and of coseparator can be phrased in terms of maps to E, rather than involving its powers. For cogenerators, this is well known, but we spell it out for coseparators. Namely that E is a coseparator if and only if whenever we have $A \xrightarrow{f} B \xrightarrow{g} E^X$ with f not an epimorphism and g a regular monomorphism, then there are maps $h, k : E^X \longrightarrow E$ such that $hg \neq kg$, but $hgf = kgf$.

One way an object E can be coseparating is that it may happen that whenever $f : A \longrightarrow B$ is not an epimorphism and $g : B \longrightarrow C$ is a regular monomorphism to *any* object of $\mathcal{A}$, then there are maps $h, k : C \longrightarrow E$ such that $hg \neq kg$, but $hgf = kgf$. This stronger form is significant since it is obvious that if $E \rightarrowtail E'$ is a monomorphism, if E and E' cogenerate the same subcategory, and if E is coseparating in this stronger sense, then so is E'. This holds, in particular, if E is injective.

For example, both the unit interval and the real line are cogenerators in the category of completely regular spaces. Since the interval is injective, it is coseparating. Therefore the line is also coseparating even though it is not injective. In fact, every path connected space with more than one point is coseparating since every such space contains a copy of the unit interval (see [Whyburn, (1942), Remark 3 on page 39] for a proof of Kelley's celebrated theorem that any path in a space contains an interval). Another example is that in the category of zero-dimensional spaces, every space with more than one point is cogenerating and, once we show that the two point discrete space is coseparating, so are all the others. Coseparating objects do not seem to be rare, unlike, say, injectives. In the case of completely regular spaces, it is a challenge to find a cogenerating space that is not coseparating.

We will see in Proposition 8.11 that $\mathbf{Z}$ is coseparating in the category of $\mathbf{Z}$-cogenerated topological abelian groups. On the other hand, we do not know if $\mathbf{Z}$ is coseparating in the category of (discrete) $\mathbf{Z}$-cogenerated groups.

Before getting to these issues, we prove the main theorem which motivated the concept.

3.10. Theorem. *Let $F : \mathcal{A} \longrightarrow \mathcal{X}$ be left adjoint to $U : \mathcal{X} \longrightarrow \mathcal{A}$ and $\mathbf{T}$ be the associated triple. Assume that U is coseparating and that $\mathcal{A}$ is complete and well-powered with respect to extremal monics. Then the following are equivalent for an object A of $\mathcal{A}$:*

(1) *A is canonically $\mathbf{T}$-sober;*

(2) *A is $\mathbf{T}$-sober;*

(3) *there is an extremal monomorphism $A \longrightarrow UX$ for some object X of $\mathcal{X}$;*

(4) *$\eta A : A \longrightarrow TA$ is an extremal monomorphism.*

Proof. It is clear that (1) implies (2) implies (3) and Proposition 3.3 says that (3) implies (4). Now assume (4). Let $B \xrightarrow{g} TA$ be an equalizer of $T\eta A$ and ηTA. There is a map $f : A \longrightarrow B$ such that $\eta A = gf$. Since f is a first factor of an extremal monic it also an extremal monic and will be an isomorphism if it is epic. If not, there are maps $h, k : TA \longrightarrow UY$ such that $hg \neq kg$, but $h \circ \eta A = k \circ \eta A$. In the diagram

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[rr, bend left, "\eta A"] & B \arrow[r, "g"'] & TA \arrow[r, shift left, "\eta TA"] \arrow[r, shift right, "T\eta A"'] \arrow[d, shift right, "h"'] \arrow[d, shift left, "k"] & T^2A \arrow[d, shift right, "Th"'] \arrow[d, shift left, "Tk"] \\ & & UX \arrow[r, "\eta UX"'] & TUX \end{tikzcd}$$

we have,

$$\eta UY \circ h \circ g = Th \circ \eta TA \circ g = Th \circ T\eta A \circ g = T(h \circ \eta A) \circ g$$
$$= T(k \circ \eta A) \circ g = Tk \circ T\eta A \circ g = Tk \circ \eta TA \circ g = \eta UY \circ k \circ g$$

and ηUY is monic, split by $U\epsilon Y$, so we find that $hg = kg$, a contradiction. Thus (1) follows by Proposition 3.4. ∎

4. The leading examples

We now have to qualify what we mean by an "object living in two categories". By way of motivation, we consider a class of examples that includes all the ones that interest us.

Recall that a **varietal category** is one whose objects are sets with operations and morphisms are functions that commute with the operations.

Throughout this section, we will make two basic assumptions. First we suppose that $\mathcal{D}$ is varietal and second, that for any n-ary operation ω in the theory of $\mathcal{D}$, the corresponding function $\omega : Z^n \longrightarrow Z$ that defines the structure of $Z_{\mathcal{D}}$ underlies a morphism $Z_{\mathcal{C}}^n \longrightarrow Z_{\mathcal{C}}$. The second assumption can be interpreted to mean that not only is $Z_{\mathcal{D}}$ a model of the theory in sets, but also that its counterpart $Z_{\mathcal{C}}$ is a model in $\mathcal{C}$. Note that the n above is permitted to be an infinite cardinal.

The asymmetry is more apparent than real. If, for example, both $\mathcal{C}$ and $\mathcal{D}$ are varietal, then the assumptions are symmetric since the assertions that $Z_{\mathcal{C}}$ is a model of the theory defining $\mathcal{D}$ and that $Z_{\mathcal{D}}$ is a model of the theory defining $\mathcal{C}$, turn out to be equivalent. Each one means that for any n-ary operation ω in the theory of $\mathcal{D}$ and any m-ary operation ξ in the theory of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
(Z^n)^m \arrow[r, "\cong"] \arrow[d, "\omega^m"'] & (Z^m)^n \arrow[r, "\xi^n"] & Z^n \arrow[d, "\omega"] \\
Z^m \arrow[rr, "\xi"'] & & Z
\end{tikzcd}$$

commutes. If $\mathcal{C}$ is the category of topological spaces, then it is the category of relational models of the theory of ultrafilters and a similar diagram can be drawn, only now it will only weakly commute in the relational sense. The one serious asymmetry, in the general case we deal with in the next section, is that we will have to assume that the underlying functor on $\mathcal{D}$, but not the one on $\mathcal{C}$, reflects isomorphisms.

4.1. THE FUNCTOR $\hom(\text{-}, Z_{\mathcal{C}}) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$. For any object $C \in \mathcal{C}$, we describe an object $\hom(C, Z_{\mathcal{C}}) \in \mathcal{D}$. The underlying set of $\hom(C, Z_{\mathcal{C}})$ is $\mathrm{Hom}(C, Z_{\mathcal{C}})$. For each n-ary operation ω in the theory of $\mathcal{D}$, let $\widehat{\omega} : Z_{\mathcal{C}}^n \longrightarrow Z_{\mathcal{C}}$ be the lifting of ω on $Z_{\mathcal{C}}$. Let $\omega : \mathrm{Hom}(C, Z_{\mathcal{C}})^n \longrightarrow \mathrm{Hom}(C, Z_{\mathcal{C}})$ be the composite

$$\mathrm{Hom}(C, Z_{\mathcal{C}})^n \xrightarrow{\cong} \mathrm{Hom}(C, Z_{\mathcal{C}}^n) \xrightarrow{\mathrm{Hom}(C,\widehat{\omega})} \mathrm{Hom}(C, Z_{\mathcal{C}})$$

It is clear that $\hom(C, Z_{\mathcal{C}})$ satisfies the equations to be an object of $\mathcal{D}$. If $f : C' \longrightarrow C$ is a morphism in $\mathcal{C}$, then we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(C, Z_{\mathcal{C}})^n \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}(f,Z_{\mathcal{C}})^n"'] & \mathrm{Hom}(C, Z_{\mathcal{C}}^n) \arrow[r, "\mathrm{Hom}(C,\widehat{\omega})"] \arrow[d, "\mathrm{Hom}(f,Z_{\mathcal{C}}^n)"] & \mathrm{Hom}(C, Z_{\mathcal{C}}) \arrow[d, "\mathrm{Hom}(f,Z_{\mathcal{C}})"] \\
\mathrm{Hom}(C', Z_{\mathcal{C}})^n \arrow[r, "\cong"'] & \mathrm{Hom}(C, Z_{\mathcal{C}}^n) \arrow[r, "\mathrm{Hom}(C',\widehat{\omega})"'] & \mathrm{Hom}(C, Z_{\mathcal{C}})
\end{tikzcd}$$

which implies that $\mathrm{Hom}(f, Z_{\mathcal{C}})$ underlies a map, denoted $\hom(f, Z_{\mathcal{C}})$ in $\mathcal{D}$. The proof of functoriality is clear.

4.2. Proposition. *For any object $C \in \mathcal{C}$, there is a monomorphism* $\mathrm{hom}(C, Z_{\mathcal{C}}) \overset{\theta}{\rightarrowtail} Z_{\mathcal{D}}^{|C|}$ *such that $|\theta|$ is the canonical embedding of* $\mathrm{Hom}(C, Z_{\mathcal{C}})$ *into* $Z^{|C|}$.

Proof. Since $\mathrm{hom}(C, Z_{\mathcal{C}})$ is the set $\mathrm{Hom}(C, Z_{\mathcal{C}})$ with the structure map described above, it is sufficient to show that the inclusion $\mathrm{Hom}(C, Z_{\mathcal{C}}) \subseteq Z^{|C|}$ underlies a morphism in $\mathcal{D}$. We have, for each n-ary operation ω, a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(C, Z_{\mathcal{C}})^n \arrow[r, tail] \arrow[d, "\cong"'] & (Z^{|C|})^n \arrow[d, "\cong"] \\
\mathrm{Hom}(C, Z_{\mathcal{C}}^n) \arrow[r, tail] \arrow[d, "\mathrm{Hom}(C,\omega)"'] & (Z^n)^{|C|} \arrow[d, "\omega^{|C|}"] \\
\mathrm{Hom}(C, Z_{\mathcal{C}}) \arrow[r, tail] & Z^{|C|}
\end{tikzcd}$$

which allows the desired conclusion. ∎

4.3. Theorem. *For any objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, we have* $\mathrm{Hom}(D, \mathrm{hom}(C, Z_{\mathcal{C}})) \cong \mathrm{Bim}(C, D)$.

Proof. Suppose $f : D \longrightarrow \mathrm{hom}(C, Z_{\mathcal{C}})$ is given. Then $|f| : |D| \longrightarrow \mathrm{Hom}(C, Z_{\mathcal{C}}) \subseteq \mathrm{Hom}(|C|, Z) = Z^{|C|}$ whose exponential transpose is $\varphi : |D| \times |C| \longrightarrow Z$. Evidently, for all $d \in |D|$, $\varphi(\text{-}, d) = |f|(d) \in \mathrm{Hom}(C, Z_{\mathcal{C}})$. Using the preceding proposition, we have $D \xrightarrow{f} \mathrm{hom}(C, Z_{\mathcal{C}}) \overset{\theta}{\hookrightarrow} Z_{\mathcal{D}}^{|C|}$, so that for each $c \in |C|$, we get a $\mathcal{D}$-morphism $D \xrightarrow{f} Z_{\mathcal{D}}$ so that $\varphi(c, \text{-})$ underlies the composite. This shows that $\mathrm{Hom}(D, \mathrm{hom}(C, Z_{\mathcal{C}})) \subseteq \mathrm{Bim}(C, D)$.

Let $\varphi : |C| \times |D| \longrightarrow Z$ be a bimorphism. Apply Proposition 2.5 to the diagram

$$\begin{tikzcd}
 & D \arrow[d] \\
\mathrm{hom}(C, Z_{\mathcal{C}}) \arrow[r, hook] & Z_{\mathcal{D}}^{|C|}
\end{tikzcd}$$

whose underlying set diagram is

$$\begin{tikzcd}
 & {|D|} \arrow[d] \arrow[dl] \\
\mathrm{Hom}(C, Z_{\mathcal{C}}) \arrow[r, hook] & Z^{|C|}
\end{tikzcd}$$

which gives the opposite inclusion. ∎

5. The general case

5.1. Hypotheses. *We now return to the general case as described in 1.2. Motivated by the special case above, we will suppose that for each object $C \in \mathcal{C}$, there is subobject we will denote* $\hom(C, Z_{\mathcal{C}}) \subseteq Z_{\mathcal{D}}^{|C|}$*, for which we have the following commutative diagram*

$$\begin{tikzcd} |\hom(C, Z_{\mathcal{C}})| \arrow[r, hook] \arrow[d, "\cong"'] & |Z_{\mathcal{D}}^{|C|}| \arrow[d, "\cong"] \\ \mathrm{Hom}(C, Z_{\mathcal{C}}) \arrow[r, hook] & Z^{|C|} \end{tikzcd}$$

We will also suppose that $|\text{-}| : \mathcal{D} \longrightarrow \mathcal{S}et$ *reflects isomorphisms.*

5.2. Theorem. *The object function* $C \mapsto \hom(C, Z_{\mathcal{C}})$ *extends to a functor* $\hom(\text{-}, Z_{\mathcal{C}}) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$. *Moreover,* $\mathrm{Hom}(D, \hom(C, Z_{\mathcal{C}})) \cong \mathrm{Bim}(C, D)$.

Proof. If $f : C \longrightarrow C'$ is a morphism in $\mathcal{C}$ apply Proposition 2.5 to the diagram

$$\begin{tikzcd} \hom(C', Z_{\mathcal{C}}) \arrow[r, hook] & Z_{\mathcal{D}}^{|C'|} \arrow[d, "Z_{\mathcal{D}}^{|f|}"] \\ \hom(C, Z_{\mathcal{C}}) \arrow[r, hook] & Z_{\mathcal{D}}^{|C|} \end{tikzcd}$$

in which the vertical map in 2.5 is the composite $\hom(C', Z_{\mathcal{C}}) \hookrightarrow Z_{\mathcal{D}}^{|C'|} \longrightarrow Z_{\mathcal{D}}^{|C|}$ and whose underlying square is

$$\begin{tikzcd} \mathrm{Hom}(C', Z_{\mathcal{C}}) \arrow[r, hook] \arrow[d, "\mathrm{Hom}(f, Z_{\mathcal{C}})"'] & Z^{|C'|} \arrow[d, "Z^{|f|}"] \\ \mathrm{Hom}(C, Z_{\mathcal{C}}) \arrow[r, hook] & Z^{|C|} \end{tikzcd}$$

The second claim is proved in exactly the same way as Theorem 4.3. ∎

5.3. Proposition. *The functor* $\hom(\text{-}, Z_{\mathcal{C}}) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$ *preserves limits.*

Proof. If $C = \operatorname{colim} C_\alpha$, we get the usual map $\hom(C, Z_{\mathcal{C}}) \longrightarrow \lim \hom(C_\alpha, Z_{\mathcal{C}})$ whose underlying function is the isomorphism $\mathrm{Hom}(C, Z_{\mathcal{C}}) \xrightarrow{\cong} \lim \mathrm{Hom}(C_\alpha, Z_{\mathcal{C}})$. Since the underlying functor on $\mathcal{D}$ reflects isomorphisms, the conclusion follows. ∎

For any object $D \in \mathcal{D}$, there is a functor $\Phi_D = \mathrm{Hom}(D, \mathrm{hom}(\text{-}, Z_{\mathcal{C}})) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}et$ and it is immediate from the preceding proposition that this functor preserves limits.

5.4. Proposition. *For any $D \in \mathcal{D}$, the functor Φ_D is representable.*

Proof. It suffices to find a solution set. Let $f : D \longrightarrow \mathrm{hom}(C, Z_{\mathcal{C}})$ be an element of $\Phi_D(C)$. For all $d \in |D|$ we have $|f|(d) \in \mathrm{hom}(C, Z_C)$. Let $\tilde{f}$ be the map from $C \longrightarrow Z_{\mathcal{C}}^{|D|}$ for which $\pi_d \tilde{f} = |f|(d)$. Factor $\tilde{f}$ as $C \xrightarrow{e} C' \xrightarrow{m} Z_{\mathcal{C}}^{|D|}$ with e epic and m extremal monic. By transposing, we have a commutative triangle

$$\begin{tikzcd} & {|D|} \arrow[dl] \arrow[d, "|f|"] \\ \mathrm{Hom}(C', Z_{\mathcal{C}}) \arrow[r, "\mathrm{Hom}(e, Z_{\mathcal{C}})"'] & \mathrm{Hom}(C, Z_{\mathcal{C}}) \end{tikzcd}$$

which underlies the open triangle

$$\begin{tikzcd} & D \arrow[d, "f"] \\ \mathrm{hom}(C', Z_{\mathcal{C}}) \arrow[r, "\mathrm{hom}(e, Z_{\mathcal{C}})"'] & \mathrm{hom}(C, Z_{\mathcal{C}}) \end{tikzcd}$$

Since e is epic, $\mathrm{Hom}(e, Z_{\mathcal{C}})$ is injective, which implies, since the underlying functor on $\mathcal{D}$ is faithful, that $\mathrm{hom}(e, Z_{\mathcal{C}})$ is monic and we can apply Proposition 2.5 to the above triangles to get the map $f' : D \longrightarrow \mathrm{hom}(C', Z_{\mathcal{C}})$ that is the element of $\Phi_D(C')$ for which $\Phi_D(e)(f') = f$. Thus the set of all elements of the form $f' \in \mathrm{Hom}(C', Z_{\mathcal{C}})$, taken over all extremal subobjects of $Z_{\mathcal{C}}^{|D|}$, is a solution set, which shows that Φ_D is representable. ∎

We denote by $\mathrm{hom}(D, Z_{\mathcal{D}})$ the object that represents Φ_D. It satisfies

$$\mathrm{Hom}(C, \mathrm{hom}(D, Z_{\mathcal{D}})) \cong \mathrm{Hom}(D, \mathrm{hom}(C, Z_{\mathcal{C}}))$$

which means that $\mathrm{hom}(\text{-}, Z_{\mathcal{D}})$ is the object function of a functor $\mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{C}$ that is adjoint on the right to the functor $\mathrm{hom}(\text{-}, Z_{\mathcal{C}})$. It follows from the preceding proof that $\mathrm{hom}(D, Z_{\mathcal{D}})$ is an extremal subobject of $Z_{\mathcal{C}}^{|D|}$, a fact we will need later.

Putting this together with 5.2, we get the following:

5.5. Theorem. *Under the conditions of 1.2 and 5.1,* $\mathrm{hom}(\text{-}, Z_{\mathcal{C}})$ *extends to a functor* $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$ *which is adjoint on the right to a functor* $\mathrm{hom}(\text{-}, Z_{\mathcal{D}}) : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{C}$. *It satisfies*

$$\mathrm{Hom}(C, \mathrm{hom}(D, Z_{\mathcal{D}})) \cong \mathrm{Bim}(C, D) \cong \mathrm{Hom}(D, \mathrm{hom}(C, Z_{\mathcal{C}}))$$ ∎

5.6. Notation. When there is no chance of confusion, we will usually denote by C^*, the object $\hom(C, Z_{\mathcal{C}})$ of $\mathcal{D}$. Similarly, D^* will denote the object $\hom(D, Z_{\mathcal{D}})$ of $\mathcal{C}$. Obviously, this notation must be used with care. We will usually use the notation $C, C', C_0, \ldots$ for objects of $\mathcal{C}$and similarly for objects of $\mathcal{D}$.

5.7. The underlying functor of hom. We wish to show that $|\hom| = \mathrm{Hom}$. When $C \in \mathcal{C}$, we have assumed that $|C^*| \cong \mathrm{Hom}(C, Z_{\mathcal{C}})$, so it will be necessary to do this only for D^*. We will have to make an additional assumption (see 5.10 below), which actually holds in every case in which the internal homs exist. But first, there is some housekeeping to carry out.

Suppose $U : \mathcal{A} \longrightarrow \mathcal{S}et$ is representable by an object R. If U has a left adjoint F, then $R = F1$. Then the set of unary operations is the set of natural transformations $U \longrightarrow U$, which is the set of natural transformations $\mathrm{Hom}(R, \text{-}) \longrightarrow \mathrm{Hom}(R, \text{-}) \cong \mathrm{Hom}(R, R) \cong UR$. Thus the unary operations are in one-one correspondence with the the elements $\omega \in UR$. Let us examine how this works. Given $\omega \in UR$ we must interpret ω as a unary operator on each object $A \in \mathcal{A}$. So if $a \in UA$ then we must define $\omega a \in UA$. But $a \in UA$ corresponds to $\widehat{a} \in \mathrm{Hom}(R, A)$ (as R represents U) in which case we define $\omega a = U(\widehat{a})(\omega)$.

Now we return to the situation we are dealing with. Let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ consisting of all objects that are extremal subobjects of a power of $Z_{\mathcal{C}}$.

5.8. Proposition. *The category $\mathcal{C}_0$ is complete.*

Proof. Suppose $\{C_i\}$ is a family of objects of $\mathcal{C}_0$. Let $C_i \hookrightarrow Z_{\mathcal{C}}^{X_i}$. Then we claim that $\prod C_i \hookrightarrow Z_{\mathcal{C}}^{\sum X_i}$. Suppose $C \twoheadrightarrow C'$ is epic and we have a commutative square

$$\begin{tikzcd} C \arrow[r, two heads] \arrow[d] & C' \arrow[d] \\ \prod C_i \arrow[r, tail] & Z_{\mathcal{C}}^{\sum X_i} \end{tikzcd}$$

For each i, we have

$$\begin{tikzcd} C \arrow[r, two heads] \arrow[d] & C' \arrow[d] \arrow[dl] \\ C_i \arrow[r, tail] & Z_{\mathcal{C}}^{X_i} \end{tikzcd}$$

and these diagonal maps $C' \longrightarrow C_i$ combine to give a map $C' \longrightarrow \prod C_i$ with the required commutativity. Now Proposition 2.9(2) implies that $\prod C_i \hookrightarrow Z_{\mathcal{C}}^{\sum X_i}$.

If $C' \longrightarrow C \rightrightarrows C''$ is an equalizer, then C' is an extremal subobject of whatever power of $Z_{\mathcal{C}}$ that C is. ■

5.9. Proposition. *The restriction of* $|\text{-}|$ *to* $\mathcal{C}_0$ *has a left adjoint.*

Proof. Let $\mathcal{E}_0$ and $\mathcal{M}_0$ consist of those maps in $\mathcal{C}_0$ that are epics (respectively, extremal monics), in $\mathcal{C}$. This is obviously a factorization system in $\mathcal{C}_0$. Each object of $\mathcal{C}_0$ has only a set of $\mathcal{M}_0$-subobjects and each object has an $\mathcal{M}_0$-embedding into a power of $Z_{\mathcal{C}}$. We now apply the RSAFT, Corollary 2.11, to show the existence of the left adjoint. ∎

5.10. That additional assumption. There is an equational theory associated to each concrete category. This theory has for its n-ary operations the natural transformations $U^n \longrightarrow U$. Any equation satisfied by these natural transformations becomes an equation of the theory. An algebra for this set (or even class) of operations is a set X, which is equipped with a map $X^n \longrightarrow X$ for each n-ary operation in such a way that all the equations of the theory are satisfied. Evidently, every object of the form UA is a model of the theory and we get a full embedding of $\mathcal{A}$ into the category of all models.

Note that, in general, the operations of the theory are functions on the underlying set, not morphisms in the category. In the very special case that they are morphisms in the category, it is easy to see that the category of algebras is closed. This happens in the category of modules over a commutative ring and, in particular, in abelian groups. For ω an n-ary operation in the theory of $|\text{-}|$ on $\mathcal{C}$, a **lifting** of ω to $Z_{\mathcal{D}}$ is a morphism $\widehat{\omega} : Z_{\mathcal{D}}^n \longrightarrow Z_{\mathcal{D}}$ such that $|\widehat{\omega}| = \omega$. As we see in the following theorem, the additional hypothesis is that every unary operation in the theory of $\mathcal{C}$ lifts to $Z_{\mathcal{D}}$.

5.11. Theorem. *The following are equivalent:*

(1) *Every operation on* $|\text{-}| : \mathcal{C} \longrightarrow \mathcal{S}et$ *lifts to* $Z_{\mathcal{D}}$*;*

(2) *Every unary operation on* $|\text{-}| : \mathcal{C} \longrightarrow \mathcal{S}et$ *lifts to* $Z_{\mathcal{D}}$*;*

(3) $|D^*| = \mathrm{Hom}(D, Z_{\mathcal{D}})$ *as subobjects of* $Z^{|D|}$.

Proof.

(1) ⇒ (2) Obvious.

(2) ⇒ (3) Let $F : \mathcal{S}et \longrightarrow \mathcal{C}_0$ be the left adjoint to the underlying functor. Since $F1$ represents the underlying functor, we have $\mathrm{Bim}(F1, D) \cong \mathrm{Hom}(F1, \mathrm{hom}(D, Z_{\mathcal{D}})) \cong |\mathrm{hom}(D, Z_{\mathcal{D}})|$. Therefore, it suffices to show that $\mathrm{Bim}(F1, D) \cong \mathrm{Hom}(D, Z_{\mathcal{D}})$. Let $\varphi : |F1| \times |D| \longrightarrow Z$ be a bimorphism and let $\eta \in F1$ be the generator. Then $f = \varphi(\eta, \text{-}) \in \mathrm{Hom}(D, Z_{\mathcal{D}})$. We claim that there is a one-one correspondence under which φ corresponds to f. First f determines φ. Let $d \in |D|$ be given. Then $\varphi(\text{-}, d)$ must underlie a morphism of $\mathcal{C}_0$ and so it is determined by $\varphi(\eta, d)$ which is $f(d)$. Conversely, let $f \in \mathrm{Hom}(D, Z_{\mathcal{D}})$ be given. Define $\varphi : |F1| \times |D| \longrightarrow Z$ by letting $\varphi(\omega, d) = \omega(f(d))$. Then $\varphi(\omega, \text{-}) = \omega f$ which, by hypothesis, underlies a map of D. For fixed $d \in |D|$, we see that $\varphi(-, d)$ maps ω to $\omega(f(d))$. But to evaluate ω at $f(d)$, we consider the morphism (in $\mathcal{C}$) $F1 \longrightarrow Z_{\mathcal{C}}$ that sends η to $f(d)$ and evaluate this map at ω. This readily shows that $\varphi(\text{-}, d)$ underlies a morphism of $\mathcal{C}$. Thus φ is a bimorphism.

(3) ⇒ (1) Since D^* is an object of $\mathcal{C}$ it admits all the operations of the theory. This means that every operation in the theory of $\mathcal{C}$ acts on the underlying set $|D^*| = \mathrm{Hom}(D, Z_{\mathcal{D}})$. Thus if ω is an n-ary operation, we have

$$\mathrm{Hom}(D, Z^n_{\mathcal{D}}) \cong \mathrm{Hom}(D, Z_{\mathcal{D}})^n \xrightarrow{\omega(\mathrm{hom}(D, Z_{\mathcal{D}}))} \mathrm{Hom}(D, Z_{\mathcal{D}})$$

which is readily seen to be natural in D. The Yoneda Lemma implies that natural transformation $\mathrm{Hom}(\text{-}, Z^n_{\mathcal{D}}) \longrightarrow \mathrm{Hom}(\text{-}, Z_{\mathcal{D}})$ is induced by a map $\widehat{\omega} : Z^n_{\mathcal{D}} \longrightarrow Z_{\mathcal{D}}$. ■

In the situation of Section 4, $Z_{\mathcal{C}}$ is a model of the operations of $\mathcal{D}$ and, as we saw at the beginning of that section, this means that $Z_{\mathcal{D}}$ is also a model of any operations on $\mathcal{C}$. Thus in that case the conclusion follows with no further hypotheses. These observations lead to,

5.12. THEOREM. *Suppose $\mathcal{D}$ is a full subcategory of an equational category. Under the hypotheses of 5.10, we have that $|\mathrm{hom}(\text{-}, Z_{\mathcal{D}})| \cong \mathrm{Hom}(\text{-}, Z_{\mathcal{D}})$ in such a way that*

$$\begin{tikzcd}
{|\mathrm{hom}(\text{-}, Z_{\mathcal{D}})|} \arrow[r, hook] \arrow[d, "\cong"'] & {|Z_{\mathcal{D}}^{|\text{-}|}|} \arrow[d, "\cong"] \\
\mathrm{Hom}(\text{-}, Z_{\mathcal{D}}) \arrow[r, hook] & Z^{|\text{-}|}
\end{tikzcd}$$

commutes.

PROOF. By the preceding theorem it is sufficient to show that every unary operation on $\mathcal{C}$ applied to Z lifts to a morphism on $Z_{\mathcal{D}}$. The object $\mathrm{hom}(C, Z_{\mathcal{C}})$ is in $\mathcal{D}$ so that any n-ary operation ω on $\mathcal{D}$, determines a function $|\mathrm{hom}(C, Z_{\mathcal{C}})|^n \longrightarrow |\mathrm{hom}(C, Z_{\mathcal{C}})|$. Since $|\mathrm{hom}(C, Z_{\mathcal{C}})| = \mathrm{Hom}(C, Z_{\mathcal{C}})$, and the underlying functor preserves products this leads to $\mathrm{Hom}(C, Z^n_{\mathcal{C}}) \longrightarrow \mathrm{Hom}(C, Z_{\mathcal{C}})$. This map is readily shown to be natural in $\mathcal{C}$ (similar to the argument in 4.1) and therefore comes from a map $Z^n_{\mathcal{C}} \longrightarrow Z_{\mathcal{C}}$, a lifting of ω to $Z_{\mathcal{C}}$. Since this is a map in $\mathcal{C}$, it must commute with any operation on $\mathcal{C}$. In particular, for any unary operation ξ in the theory of $\mathcal{C}$, the square

$$\begin{tikzcd}
Z^n \arrow[r, "\omega"] \arrow[d, "\xi^n"'] & Z \arrow[d, "\xi"] \\
Z^n \arrow[r, "\omega"'] & Z
\end{tikzcd}$$

commutes. Since $Z_{\mathcal{D}}$ is an object in an equational category, this means that ξ lifts to a morphism $Z_{\mathcal{D}} \longrightarrow Z_{\mathcal{D}}$. ■

6. 0-dimensional spaces and $\mathbf{Z}$-rings

In this example, we begin with $\mathcal{C} = \mathcal{T}op$ and $\mathcal{D}$ the category of lattice-ordered rings, henceforth known as LO-rings. A **lattice-ordered ring** is a ring with identity that is also a lattice such that $x \geq 0$ and $y \geq 0$ implies $xy \geq 0$ and $x \geq y$ if and only if $x - y \geq 0$. We will use without further mention the fact that any (weak) inequality can be expressed as an equation: $x \leq y$ if and only if $x \vee y = y$. Homomorphisms must preserve the ring-with-identity as well as the lattice structure.

An LO-ring is called a $\mathbf{Z}$-ring if it is a sub-LO-ring of a power of $\mathbf{Z}$. Clearly $\mathbf{Z}$-rings satisfy all equations and all the Horn sentences satisfied by $\mathbf{Z}$. A weak-$\mathbf{Z}$-ring is a quotient of a $\mathbf{Z}$-ring by a map that preserves all the structure and satisfies all equations, although not all Horn sentences. The category of weak Z-rings is the closure of $\mathbf{Z}$ under products, subobjects, and homomorphic images in the category of LO-rings. For example, any non-standard ultrapower of $\mathbf{Z}$ is a weak-$\mathbf{Z}$-ring. It therefore satisfies all first order sentences (including all the *finite* Horn sentences) satisfied by $\mathbf{Z}$, but is non-Archimedean. Examples of equations that are satisfied by weak-$\mathbf{Z}$-rings, but not all LO-rings are the commutativity of multiplication and the fact that for all x, $x^2 \geq 0$.

We describe some of the operations and equations satisfied by weak-$\mathbf{Z}$-rings. In any LO-ring R, for any $r \in R$, we let $r^+ = r \vee 0$, $r^- = (-r) \vee 0$ and $|r| = r^+ + r^- = r^+ \vee r^-$. Of course, $r = r^+ - r^-$. We also denote by $r^!$ the derived operation $r \mapsto |r| \wedge 1$. In $\mathbf{Z}$, this operation takes any non-zero element to 1 and fixes 0. This is an idempotent with the property that $r^! r = r$. Note that $0 \leq r^! \leq 1$. These equations (including inequalities, since they can be reduced to equations) are then satisfied by all weak-$\mathbf{Z}$-rings. We also note that in a weak-$\mathbf{Z}$-ring all squares are non-negative, since squares are non-negative in $\mathbf{Z}$.

An example of an LO-ring that is not a weak $\mathbf{Z}$-ring is the field $\mathbf{Q}$ of rational numbers since $(1/2)^! = 1/2$, which is not idempotent.

6.1. Proposition. *The following properties hold when e and f are idempotents and r and s are arbitrary elements in a weak-$\mathbf{Z}$-ring R:*

(1) $e = e^!$;

(2) $e \wedge f$ *is idempotent;*

(3) $e \wedge f = ef$;

(4) $e \vee f = e + f - ef$.

(5) $(rs)^! = r^! s^!$;

(6) *if* $r = es$, *then* $r^! \leq e$;

Proof.

(1) We observe that $e = e^2 \geq 0$ and the same is true for $1-e$, which implies that $0 \leq e \leq 1$. But then $e^! = |e| \wedge 1 = e$.

(2) $0 \leq e \wedge f \leq 1$ so that $(e \wedge f)^! = e \wedge f \wedge 1 = e \wedge f$ and hence is idempotent.

(3) From $0 \leq e \leq 1$ and $f \leq 1$, we conclude that $ef \leq f$ and similarly that $ef \leq e$ and hence $ef \leq e \wedge f$. The other direction comes from $0 \leq e \wedge f \leq e$ and $0 \leq e \wedge f \leq f$ implies that $e \wedge f = (e \wedge f)^2 \leq ef$.

(4) $e \vee f = 1 - ((1-e) \wedge (1-f))$.

(5) This equation (in the form $(rs)^! = (rs)^! \wedge (r^! s^!) = (r^! s^!)$) is trivial in $\mathbf{Z}$ and hence is true in any weak-$\mathbf{Z}$-ring.

(6) If $r = es$, then $r^! = e^! s^! = e^! \wedge s^! \leq e$. ∎

If R is a weak-$\mathbf{Z}$-ring, an ideal $I \subseteq R$ is called a **convex ideal** if it contains, for each $r \in I$, the interval $[0, r]$.

The development below is modeled on that for completely regular spaces as found, for example, in [Gillman & Jerison (1960), Chapter 5]. One big difference is that in $\mathbf{Z}$-rings, every convex ideal is also absolutely convex.

6.2. Proposition. *In any weak-$\mathbf{Z}$-ring, we have the inequality*

$$|(x_1 \wedge y) - (x_2 \wedge y)| \leq |x_1 - x_2|$$

Proof. It is sufficient to prove this in $\mathbf{Z}$. Since both sides are invariant under the interchange of x_1 and x_2, we can suppose $x_1 \leq x_2$ and look at the three cases of the location of y with respect to the interval $[x_1, x_2]$. ∎

6.3. Proposition. *If I is a convex ideal in a weak-$\mathbf{Z}$-ring, then the following are equivalent:*

1. $r \in I$;
2. $|r| \in I$;
3. $r^! \in I$.

Proof. In fact if $r \in I$, then $r^2 \in I$ and $0 \leq |r| \leq r^2$ so $|r| \in I$. If $|r| \in I$, then from $0 \leq r^! \leq |r|$, we see that $r^! \in I$. If $r^! \in I$, then from $r = r^! r$, we see that $r \in I$. ∎

6.4. Proposition. *Every convex ideal is generated by idempotents. Conversely, every ideal generated by idempotents is convex.*

Proof. The first claim is an immediate consequence of the previous proposition. For the second, let I be an ideal generated by idempotents. Then $r \in I$ if and only if there is an idempotent e for which $r = er$. This will hold (for example, in $\mathbf{Z}^D$) if and only if $r^! \leq e$. For any $s \in [0, r]$ it will be the case that $s = r^! s$ and, *a fortiori*, that $s = es$. ∎

6.5. THEOREM. *Let R be a weak-$\mathbf{Z}$-ring and $I \subseteq R$ an ideal. Then the following are equivalent.*

(1) *R/I is an LO-ring and the canonical map $R \longrightarrow R/I$ is a homomorphism of LO-rings;*

(2) *R/I is ordered;*

(3) *I is convex.*

PROOF. That (1) implies (2) implies (3) is obvious. So let I be a convex ideal in R. For $x, y \in R$, let us write $x \simeq_I y$ when $x - y \in I$. If $x_1 \simeq_I x_2$, then we see from Proposition 6.2 that, for any $y \in R$, we have $x_1 \wedge y \simeq_I x_2 \wedge y$. If $x_1 \simeq_I x_2$ and $y_1 \simeq_I y_2$, then two applications of this congruence imply that $x_1 \wedge y_1 \simeq_I x_2 \wedge y_2$, so that R/I has $\wedge$ and the quotient mapping preserves them. As for $\vee$, we have $x \vee y = -(-x \wedge -y)$. ∎

6.6. REMARK. For the rest of this section, R denotes a weak-$\mathbf{Z}$-ring and ideals of R will be assumed proper. By a **maximal convex ideal** we mean a proper ideal that is maximal among convex ideals. Since being a convex ideal is an inductive property, there is a maximal convex ideal containing any convex ideal.

We denote by $B(R)$ the boolean algebra of idempotents of R with the usual boolean operations. In particular, if e and f are idempotents, then $e \wedge f = ef$ and $e \vee f = e + f - ef$. Although $B(R)$ is a ring, it is not a subring of R since it is not closed under addition, although it is closed under sup, inf, and multiplication, and contains 0 and 1.

Since convex ideals are generated by their idempotents, we have an easy proof of:

6.7. PROPOSITION. *There is a one-one correspondence between convex ideals of R and boolean ideals of $B(R)$ such that maximal convex ideals correspond to maximal boolean ideals.* ∎

6.8. PROPOSITION. *Let R be a weak-$\mathbf{Z}$-ring and I a convex ideal. Then the following are equivalent:*

(1) I is maximally convex;

(2) I is prime;

(3) R/I is an integral domain;

(4) R/I is totally ordered.

PROOF. (1) and (2) are equivalent by the preceding Proposition and the same fact for boolean rings. (2) and (3) are obviously equivalent. (3) implies (4) because $r^+r^- = 0$ implies that either $r^- = 0$ (in which case $r \geq 0$), or $r^- = 0$ (in which case $r \leq 0$). (4) implies (1) because if e is any idempotent, then either $e \leq 1 - e$ in which case, multiplying by e gives $e \leq 0$, whence $e = 0$ or $1 - e \leq e$, which implies that $e = 1$. Since there are no proper idempotents, there are no proper convex ideals. ∎

6.9. PROPOSITION. *Let R be a weak-$\mathbf{Z}$-ring that is an integral domain. Then no element of R lies between two integers and every element of $R - \mathbf{Z}$ is infinite (either greater than every integer or less than every integer).*

PROOF. For any integer n, the equation

$$\big((x \wedge n) - x\big)\big(x \wedge (n+1) - (n+1)\big) = 0$$

holds in $\mathbf{Z}$ and therefore in any weak-$\mathbf{Z}$-ring, since it expresses the fact that for every integer x either $x \le n$ or $n + 1 \le x$. This equation, in a domain R, says the same thing about R. ∎

6.10. SPEC AND Spec_0. Let R be a weak-$\mathbf{Z}$-ring. We let $\operatorname{Spec}(R)$ denote the set of maximal convex ideals of R, topologized by the Zariski topology. Since every ideal is generated by idempotents, we can take as a basis of open sets, those of the form $\{M \in \operatorname{Spec}(R) \mid e \notin M\}$ where e is an idempotent of R. Since every maximal ideal contains e or $1-e$ (their product is 0) and not both this is the same as $\{M \in \operatorname{Spec}(R) \mid 1-e \in M\}$. It will be convenient to denote by $N(e)$ the set of maximal ideals that contain the idempotent e. Clearly this set is clopen since its complement is $N(1-e)$. For idempotents d and e, we see that $N(d) \cup N(e) = N(d \wedge e)$ and $N(d) \cap N(e) = N(d \vee e)$ so that these sets are a basis for both the open and the closed sets.

Let $\operatorname{Spec}_0(R) \subseteq \operatorname{Spec}(R)$ consist of those maximal ideals M for which $R/M = \mathbf{Z}$. Since there are no non-identity ring homomorphisms $\mathbf{Z} \longrightarrow \mathbf{Z}$, we see that $\operatorname{Spec}_0(R) = \operatorname{Hom}(R, \mathbf{Z})$. We topologize $\operatorname{Spec}_0(R)$ as a subspace of $\operatorname{Spec}(R)$. A subspace of a compact Hausdorff totally disconnected space is completely regular Hausdorff and totally disconnected. There is a basis for the clopens that consists of all sets of the form $N(e) \cap \operatorname{Spec}_0(R)$. We will show below that the topology on $\operatorname{Spec}_0(R)$ is the same as the topology on $\hom(R, \mathbf{Z})$. The functor in the other direction takes the space X to $\hom(X, \mathbf{Z})$, the set $\operatorname{Hom}(X, \mathbf{Z})$ with LO-ring structure induced by the structure on $\mathbf{Z}$.

6.11. PROPOSITION. *The topology on $\operatorname{Spec}_0(R)$ is inherited from the product topology on $\mathbf{Z}^{|R|}$.*

PROOF. Although we have used the sets $N(e)$ to define the topology, we could use $N(r)$ for any element r since when I is any convex ideal, $r \in I$ if and only if $r^! \in I$. One subbase for the closed sets in the product topology on $\mathbf{Z}^{|R|}$ consists of sets of the form $V(r, n) = \{f : |R| \longrightarrow \mathbf{Z} \mid f(r) = n\}$ for $r \in R$ and $n \in Z$. Then $V(r, n) \cap \operatorname{Spec}_0(R) = N(r - n)$, which is closed. Conversely, any set of the form $N(r) = V(n, 0) \cap \operatorname{Spec}_0(R)$ is clopen. ∎

From 6.7 and the known properties of Stone duality, we get:

6.12. PROPOSITION. *For any weak-$\mathbf{Z}$-ring R,* $\operatorname{Spec}(R)$ *is compact, Hausdorff, and totally disconnected.* ∎

The proof of the following proposition is straightforward. It implies, among other things, that fixed weak-$\mathbf{Z}$-rings are $\mathbf{Z}$-rings.

6.13. PROPOSITION. *Let R be a weak-$\mathbf{Z}$-ring and $X = \hom(R, \mathbf{Z})$. Then R is a $\mathbf{Z}$-ring if and only if $R \longrightarrow R^{**} = \hom(X, \mathbf{Z})$ is injective.*

6.14. LEMMA. *If R is a $\mathbf{Z}$-ring, then $\operatorname{Spec}_0(R)$ is dense in $\operatorname{Spec}(R)$.*

PROOF. For some set X there is an embedding $R \rightarrowtail \mathbf{Z}^X$. For $x \in X$, denote by $p_x : R \longrightarrow \mathbf{Z}$ the composite $R \longrightarrow \mathbf{Z}^X \xrightarrow{p_x} \mathbf{Z}$. Recall that for an idempotent $e \in R$, the set $N(e) = \{M \in \operatorname{Spec}(R) \mid e \in M\}$ is a basic neighbourhood in $\operatorname{Spec}(R)$. Since e is idempotent, $p_m(e)$ can only be 0 or 1. If $e \neq 1$, then there is some $x \in X$ for which $p_x(e) = 0$ and then $\ker(p_x) \in N(e)$. ∎

6.15. EXAMPLES. We look at $\operatorname{Spec}(R)$, $\operatorname{Spec}_0(R)$, $B(R)$, and R^{**} for several subrings of $\mathbf{Z}^{\mathbf{N}}$.

(1) $R = \mathbf{Z}^{\mathbf{N}}$. In this case, $B(R) = 2^{\mathbf{N}}$, the set of all subsets of $\mathbf{N}$, $\operatorname{Spec}(R)$ is the Stone space of $2^{\mathbf{N}}$, which is $\beta\mathbf{N}$, and $\operatorname{Spec}_0(R) = \mathbf{N}$, since R mod any non-principal ultrafilter is a non-standard model. Also $R^{**} = \hom(\mathbf{N}, \mathbf{Z}) = \mathbf{Z}^{\mathbf{N}} = R$.

(2) R consists of the bounded sequences in $\mathbf{Z}^{\mathbf{N}}$. In this case, $B(R)$ is still $2^{\mathbf{N}}$ since the characteristic function of any subset is bounded. Therefore $\operatorname{Spec}_0(R) = \operatorname{Spec}(R) = \beta\mathbf{N}$ since no quotient contains an unbounded element. Then $R^{**} = \hom(\beta\mathbf{N}, \mathbf{Z}) = R$ since every bounded function on $\mathbf{N}$ has a unique extension to $\beta\mathbf{N}$.

(3) R consists of the functions in $\mathbf{Z}^{\mathbf{N}}$ that are eventually constant. The characteristic function of a subset of $\mathbf{N}$ lies in R if and only if the set is either finite or cofinite, so $B(R)$ consists of the finite/cofinite subset algebra. Then $\operatorname{Spec}_0(R) = \operatorname{Spec}(R) = \mathbf{N}^*$, the one-point compactification of $\mathbf{N}$, again since all elements of R are bounded; $R^{**} = \hom(\mathbf{N}^*, \mathbf{Z})$ clearly consists of all the functions that are eventually constant, that is $R^{**} = R$.

(4) R consists of all $f : \mathbf{N} \longrightarrow \mathbf{Z}$ of polynomial growth. This means that there is an n depending on f such that $|f(m)| \leq m^n$ for all sufficiently large m. (Actually, this could be stated for all $m > 1$ without changing the class.) This class is closed under the lattice operations in $\mathbf{Z}^{\mathbf{N}}$ and thus is a $\mathbf{Z}$-ring. The characteristic function of every subset is in R, so evidently $B(R) = 2^{\mathbf{N}}$ and $\operatorname{Spec}(R) = \beta\mathbf{N}$. We claim that for every $M \in \beta X - X$ there is a function in R that is unbounded mod M. In fact, the inclusion function $\mathbf{N} \hookrightarrow \mathbf{Z}$ is an unbounded function since it exceeds every integer with finitely many exceptions and a non-principal ultrafilter contains no finite set, so that it is greater than n modulo any such ultrafilter. Thus R/M is non-standard and $\operatorname{Spec}_0(R) = \mathbf{N}$ and $R^{**} = \mathbf{Z}^{\mathbf{N}} \neq R$.

(5) We say that $f : \mathbf{N} \longrightarrow \mathbf{Z}$ is eventually a polynomial if there is a polynomial p in one variable such that $f(m) = p(m)$ for all sufficiently large m. Let R be the subset of $\mathbf{Z}^{\mathbf{N}}$ consisting of all the eventually polynomial functions. We claim that R is closed under the lattice operations. For example, let us calculate $f \vee g$ for two such functions. Suppose f

is eventually p and g is eventually q. The claim is that $p \vee q$ is eventually either p or q and hence that is the case for $f \vee g$. But $p - q$ is a polynomial and has only finitely many real roots, so it crosses the x-axis only finitely often and hence is eventually positive or negative (leaving out the case $p = q$). In the first case, $p \vee q$ is eventually p and in the second it is eventually q.

A characteristic function is eventually polynomial if and only if it is eventually 0 or eventually 1, so $B(R)$ is again the finite/cofinite boolean algebra and $\mathrm{Spec}(R) = \mathbf{N}^*$. On the other hand, $\mathrm{Spec}_0(R) = \mathbf{N}$ since there are unbounded functions and $R^{**} = \mathbf{Z}^{\mathbf{N}} \neq R$. Notice also that $B(R^{**}) = 2^{\mathbf{N}} \neq B(R)$. This is the only one of the five examples for which these boolean algebras are unequal.

6.16. Fixed spaces. Recall that a space X is *fixed* if the canonical map $X \longrightarrow X^{**}$ is an isomorphism. Here we give a useful characterization of the fixed spaces. We begin by observing that a subspace inclusion in the category of completely regular (or even Hausdorff) spaces is epic if and only if the image is dense. It follows immediately that an extremal monomorphism is precisely the inclusion of a closed subspace. As the following proposition shows these are also the regular monics.

6.17. Proposition. *For any space X, the map $X \longrightarrow X^{**}$ has dense image.*

Proof. For a map $f : X \longrightarrow \mathbf{Z}$ and an integer $m \in \mathbf{Z}$, let

$$U_m(f) = \{\theta : \hom(X, \mathbf{Z}) \longrightarrow \mathbf{Z} \mid \theta \text{ is a map in } \mathcal{D} \text{ and } \theta(f) = m\}$$

These sets form a subbase for the topology on X^{**} induced by the product topology on $\mathbf{Z}^{\hom(X,\mathbf{Z})}$. Suppose that the homomorphism $\theta : \hom(X, \mathbf{Z}) \longrightarrow \mathbf{Z}$ of LO-rings is in the closure of X. Let $U_{m_1}(f_1) \cap U_{m_2}(f_2) \cap \cdots \cap U_{m_n}(f_n)$ be a basic neighbourhood of θ, which means that $\theta(f_i) = m_i$, for $i = 1 \ldots, n$. We want to find an $x \in X$ such that $f_i(x) = m_i$, for $i = 1, \ldots, n$. If no such x exists, then at least one of the $(f_i(x) - m_i)^! = 1$ for all $x \in X$ and then $\prod_{i=1}^n (1 - (f_i(x) - m_i)^!) = 0$ for all $x \in X$, which implies that $\prod(1 - (f_i - m_i)^!) = 0$. But θ is a map of LO-rings, which preserves the (-)$^!$ operation. It follows that $\theta\left(\prod(1 - (f_i - m_i)^!)\right) = 0$, which contradicts the hypothesis that $\theta(f_i) = m_i$ for all i. ∎

In the statement of the next theorem, we have used the notion of $\mathbf{N}$-compactness, which was introduced as part of a more general notion in [Engelking & Mrowka] as an obvious analog of realcompactness. Recall that a space is **realcompact**, if it is a closed subspace of a power of $\mathbf{R}$ and it is **N-compact**, [Engelking & Mrowka], if it is a closed subspace of a power of $\mathbf{N}$. A useful summary of the properties of $\mathbf{N}$-compact spaces is found in [Schlitt, (1991), Introduction]

Many of the results proved below can be found in [Eda, et al. (1989)]. Unfortunately, the book in which this appears is out of print and widely unavailable. We thank the referee for bringing it to our attention. However, given its general unavailability, we have retained our original development in order that the theorems and proofs be more widely available.

From Theorem 3.10 and the discussion preceding it, we conclude that, 6.18. Theorem. *Let X be a topological space. Then the following are equivalent:*

(1) *X is $\mathbf{N}$-compact;*

(2) *X is $\mathbf{N}$-sober;*

(3) *X is a canonically $\mathbf{N}$-sober;*

(4) *X is fixed under the adjunction.*

■

The following result will be needed for the analysis of fixed rings.

6.19. Proposition. *A 0-dimensional Lindelöf space is $\mathbf{N}$-compact.*

Proof. Let X be such a space and $\Phi = \mathrm{Hom}(X, \mathbf{Z})$. In order to sort out the various spaces and function spaces, we will adopt the following notation for this proof. Small roman letters will denote elements of X and elements of $\mathbf{N}^\Phi$; roman capitals will denote subsets of X. Small Greek letters will denote functions $X \longrightarrow \mathbf{N}$ and Greek capitals will denote sets of such functions. We will show that X is homeomorphic to a closed subspace of $\mathbf{N}^\Phi$ by evaluation. This means that $\varphi(x) = x(\varphi)$ where x denotes an element of X on the left side of that equation and the corresponding element of $\mathbf{N}^\Phi$ on the right. Continuity is clear. Since X is 0-dimensional, Φ separates points. If $U \subseteq X$ is clopen and χ_U is its characteristic function, then $\{f : \Phi \longrightarrow \mathbf{N} \mid f(\chi_U) = 1\}$ is a clopen set in $\mathbf{N}^\Phi$ whose intersection with the image of X is U. Hence X has the subspace topology. For the rest of this proof, we will treat X as a subspace of $\mathbf{N}^\Phi$.

If φ, $\psi \in \Phi$ we let $\varphi\psi$ denote their pointwise product. Now suppose that $f \in \mathbf{N}^\Phi$ is in $\mathrm{cl}(X) - X$. For U a clopen subset of X, the function χ_U takes on only the values 0 and 1 on X. It follows that when $f \in \mathrm{cl}(X)$, $f(\chi_U)$ can only be 0 or 1. We will say that f **adjoins** U when $f(\chi_U) = 1$. Note that when $V \subseteq U$ and f does not adjoin U then it does not adjoin V. The reason is that $\chi_U\chi_V = \chi_V$ and the equation $s(\varphi\psi) = x(\varphi)s(\psi)$ which holds for $x \in X$ extends by continuity to $\mathrm{cl}(X)$.

Since $f \notin X$, there is for each $x \in X$ a function $\varphi_x \in \Phi$ such that $x(\varphi_x) \neq f(\varphi_x)$. Let $u : \mathbf{N} \longrightarrow \mathbf{N}$ be the characteristic function of $\{x(\varphi_x)\}$. Then $u \circ \varphi_x$ is the characteristic function of $\varphi^{-1}\varphi(x) = U_x \subseteq N^\Phi$, which contains x and not f. Since X is Lindelöf, a countable set, say U_1, U_2, $\cdots$, of these clopen neighbourhoods covers X and f does not adjoin any of them. If we replace U_n by $V_n = U_n - \bigcup_{i<n} U_i$ we get a cover by disjoint clopen subsets of X and f does not adjoin any of them either. Now let φ be the function that takes the value n on V_n. Let $v : \mathbf{N} \longrightarrow \mathbf{N}$ be the characteristic function of $\{f(\varphi)\}$. Then $f(v \circ \varphi) = 1$ (otherwise we can find a neighbourhood of f which misses X). But $v \circ \varphi$ is the characteristic function of V_n for $n = f(\varphi)$, so this contradicts the fact that f does not adjoin V_n. ■

6.20. Fixed rings. In contrast to the situation with fixed spaces, we know rather little about fixed rings. One important property is this:

6.21. Theorem. *For any 0-dimensional completely regular space X, the ring* $\hom(X, \mathbf{Z})$ *is fixed.*

Proof. Since X^{**} is fixed, it follows from Theorem 2.2 that $\hom(X^{**}, \mathbf{Z})$ is a fixed ring. Thus it is sufficient to show that $X \rightarrowtail X^{**}$ induces an isomorphism $\hom(X^{**}, R) \longrightarrow \hom(X, R)$. Since the underlying set functor on rings reflects isomorphisms, it suffices to show that $\mathrm{Hom}(X^{**}, \mathbf{Z}) \longrightarrow \mathrm{Hom}(X, \mathbf{Z})$ is surjective since the fact that $X \rightarrowtail X^{**}$ is dense guarantees that it is injective. This amounts to showing that every continuous map $X \longrightarrow \mathbf{Z}$ has a continuous extension to X^{**}.

If $f \in R = \hom(X, \mathbf{Z})$ and $\varphi \in X^{**} = \hom(R, \mathbf{Z})$ then the map $|X^{**}| \times |R| \longrightarrow \mathbf{Z}$ given by $(\varphi, f) \mapsto \varphi(f)$ is an element of $\mathrm{Bim}(X^{**}, R)$ and thus continuous on X^{**}. But this implies that the extension of f to X^{**} given by $\varphi \mapsto \varphi(f)$ continuously extends f on X^{**}. ∎

We now introduce a necessary, but not sufficient, condition that a ring be fixed under the adjunction. The condition suffices to show that the rings of Examples 6.15, 4 and 5, are not fixed.

Let $\mathbf{Th}$ denote the theory whose n-ary operations are of the continuous functions $\mathbf{Z}^n \longrightarrow \mathbf{Z}$ and whose equations are those satisfied by these functions. Among the operations are all the LO-ring operations but there will obviously be many more, including in particular, infinitary operations. Among the unary operations there are four, $\alpha_1, \cdots, \alpha_4$, with the property that $|n| = \sum_{i=1}^{4} \alpha_i^2$. This characterizes the elements that are greater than or equal to 0 in the partial order as the sums of four squares. If X is any space and $\omega : \mathbf{Z}^n \longrightarrow \mathbf{Z}$ is continuous, then we have the composite map

$$\mathrm{Hom}(X, \mathbf{Z})^n \cong \mathrm{Hom}(X, \mathbf{Z}^n) \xrightarrow{\mathrm{Hom}(X,\omega)} \mathrm{Hom}(X, \mathbf{Z})$$

so that $\hom(X, \mathbf{Z})$ is a model of $\mathbf{Th}$. It is evident that for any $X \longrightarrow Y$, the induced $\hom(Y, \mathbf{Z}) \longrightarrow \hom(X, \mathbf{Z})$ is a morphism of models. Thus the category of fixed rings is a subcategory of the category $\mathbf{Th}$-Alg of $\mathbf{Th}$-algebras. The inclusion $\mathrm{Fix}(\mathcal{D}) \longrightarrow \mathbf{Th}$-Alg is full since the composite $\mathrm{Fix}(\mathcal{D}) \longrightarrow \mathbf{Th}\text{-Alg} \longrightarrow \mathcal{D}$ is. Since fixed rings are, by definition, of the form $\hom(X, \mathbf{Z})$ for some X, we conclude:

6.22. Theorem. *The category* $\mathrm{Fix}(\mathcal{D})$ *is a full subcategory of* $\mathbf{Th}$-Alg. ∎

To see that the converse is false, we first show by example that $\mathbf{Z}$ is not injective for inclusions of closed subspaces of 0-dimensional spaces.

Let S denote the Sorgenfrey space, which is the real line with the topology in which basic open sets have the form $[a, b)$. It is known that S is Lindelöf and it is obviously 0-dimensional so that Proposition 6.19 implies that it is $\mathbf{N}$-compact. The space $S \times S$ is not Lindelöf, but it is still $\mathbf{N}$-compact and we will see that it has a closed subspace and an $\mathbf{N}$-valued map on that subspace that does not extend to the whole space.

6.23. Proposition. *Let $Y = S \times S$. Then there is a closed subspace $X \subseteq Y$ and a function $\varphi : X \longrightarrow \mathbf{Z}$ that does not extend to Y.*

Proof. Let $X = \{(x, -x) \mid x \in S\}$. One easily sees that X is discrete since $[x, x+1) \times [-x, -x+1)$ is a neighbourhood of $(x, -x)$ that contains no other point of X. It is closed because it is closed in the ordinary topology which is coarser. Finally, as is well known, the characteristic function of the rational points in X cannot be extended to Y. ∎

6.24. Proposition. *There are $\mathbf{Th}$-algebras that are not fixed rings.*

Proof. Suppose that Y is $\mathbf{N}$-compact (and therefore fixed) and $f : X \subseteq Y$ is the inclusion of a closed subspace (so X is also fixed) for which $Y^{**} \longrightarrow X^{**}$ is not surjective. Then there is an equalizer diagram $X \xrightarrow{f} Y \underset{e}{\overset{d}{\rightrightarrows}} W$ in which $W = Y +_X Y$ is also fixed. Apply hom(-, $\mathbf{Z}$) to get

$$W^{**} \underset{e^{**}}{\overset{d^{**}}{\rightrightarrows}} Y^{**} \xrightarrow{f^{**}} X^{**}$$

Since Fix($\mathcal{C}$) is equivalent to Fix($\mathcal{D}$), this diagram is a coequalizer in Fix($\mathcal{D}$). These are maps of $\mathbf{Th}$-algebras of course and f^{**} can be factored $Y^{**} \xrightarrow{g} R \xrightarrow{h} X^{**}$ in that category, where g is surjective and h injective. Since h is monic, the fact that $f^{**}d^{**} = f^{**}e^{**}$ implies that $gd^{**} = ge^{**}$. If we apply the hom functor and use the fact that X and Y are fixed, we get $X \xrightarrow{h^{**}} R^{**} \xrightarrow{g^{**}} Y$. The arrow h^{**} is monic since $f = g^{**}h^{**}$ is and g^{**} is monic since g is epic. Thus $g^{**} : R^{**} \longrightarrow Y$ is a subobject of Y that includes the equalizer of d and e. Then $dg^{**} = eg^{**}$ implies that h^{**} is an isomorphism. But then $R^{**} = R^{****} = X^{**}$ which strictly contains R and then R is a $\mathbf{Th}$-algebra that is not in Fix($\mathcal{D}$). ∎

6.25. The weak-$\mathbf{Z}$-rings. Our example that shows that not every LO-ring is a weak-$\mathbf{Z}$-ring uses the fact that $\mathbf{Q}$ has elements between 0 and 1 that are not idempotent. This raises the question whether that condition characterizes weak-$\mathbf{Z}$-rings. The answer, with the help of a private communication from Andreas Blass is no.

If R is an LO-ring in which every element between 0 and 1 is idempotent, then for any maximal convex ideal I, the quotient R/I has no element between 0 and 1 since it contains no proper idempotent. If $R = S/J$ where S is a $\mathbf{Z}$-ring and J a convex ideal, then $R/I = S/M$ where M is some maximal convex ideal of S. If $S \subseteq \mathbf{Z}^n$ for some cardinal n, then from $B(S) \subseteq B(\mathbf{Z}^n)$ and Stone duality, we conclude that $\mathrm{Spec}(\mathbf{Z}^n) \longrightarrow \mathrm{Spec}(S)$ is surjective and so there is a maximal ideal $N \subseteq Z^n$ such that $N \cap S = M$ and then $R \cong \mathbf{Z}^n/N$. Since the latter is a non-standard model of $\mathbf{Z}$, we could then conclude that R/I could be embedded in a non-standard model of $\mathbf{Z}$. The following paragraph from Andreas Blass shows that one cannot infer that every LO-domain with every non-integer infinite is embeddable in a non-standard model of $\mathbf{Z}$.

> Not all such rings can be embedded in an ultrapower of $\mathbf{Z}$. The collection of diophantine equations that have integer solutions is recursively enumerable

but (by Matijasevich's negative solution of Hilbert's 10th problem), not recursive. So its complement isn't recursively enumerable and, in particular, is not the same as the set of diophantine equations for which Peano arithmetic (PA) proves that there's no solution (as the latter set is r.e.). So there is a diophantine equation D that has no solution but PA can't prove that fact. Thus, there is a model of PA in which D has a solution. Such a model is the non-negative part of a ring that satisfies your hypotheses but can't be embedded into an ultrapower of $\mathbf{Z}$ (because D has no solutions in any such ultrapower).

Note that the counterexample ring in the preceding paragraph looks a great deal like $\mathbf{Z}$, since PA proves a lot of number theory. And the same argument applies, mutatis mutandis, to stronger theories, like ZFC, which will make the ring look even more like $\mathbf{Z}$.

7. Tychonoff spaces and real-LO-algebras

We will now carry out a similar analysis for Tychonoff spaces and, in this section, we restrict to such spaces, unless explicitly mentioned otherwise. For a fuller and somewhat different treatment of these questions, see [Gillman & Jerison (1960), Chapter 5].

There are some serious differences. The notion of convex ideal separates into two classes, the convex ideals and the absolutely convex ideals ([Gillman & Jerison (1960), 5.1, 5.2]). The definitions of r^+, r^-, and $|r|$ remain the same, but there is no analog of $r^!$. Although there is such an operation in $\mathbf{R}$, its values are not generally idempotent and it leads nowhere. Every maximal convex ideal is absolutely convex and the quotient is a totally ordered field containing $\mathbf{R}$. We let $\mathrm{Spec}(R)$ and $\mathrm{Spec}_0(R)$ denote the set of all convex maximal ideals, respectively, those for which the quotient is $\mathbf{R}$. The latter ideals are called **real** and the others **hyper-real** in [Gillman & Jerison (1960)].

We topologize $\mathrm{Spec}(R)$ by letting a basis of open sets be $U(r) = \{M \mid r \notin M\}$ for an element $r \in R$. This is a base since $U(r) \cap U(s) = U(rs)$, since maximal ideals are prime. We topologize $\mathrm{Spec}_0(R)$ as a subspace of $\mathrm{Spec}(R)$.

7.1. PROPOSITION. *The topology on* $\mathrm{Spec}_0(R)$ *is the one induced by the inclusion of* $\mathrm{Spec}_0(R) = \mathrm{Hom}(R, \mathbf{R}) \subseteq \mathbf{R}^{|R|}$.

PROOF. The sets of the form $\{\theta : R \longrightarrow \mathbf{R} \mid \theta(r) \in (a, b)\}$, in which $r \in R$ and $a < b$ are real numbers, are a subbase for the topology of $\mathbf{R}^{|R|}$. The intersection of this set with Spec_0 consists of those LO-algebra homomorphisms $\varphi : R \longrightarrow \mathbf{R}$ for which $\varphi((r-a)^+) \neq 0$ and $\varphi((b-r)^+) \neq 0$, since for any $s \in R$, $\varphi(s^+) \neq 0$ if and only if $\varphi(s^+) > 0$. Obviously, $\{\varphi \mid \varphi(r) \neq 0\}$ is the intersection with $\mathrm{Spec}_0(R)$ of $\{\theta \mid \theta(r) \neq 0\}$ which is open in the product topology. ∎

We have, by analogy with 6.18 the following. Once more we use Theorem 3.10 and the discussion preceding it. Only the part that (2) implies (4) needs modification, so we include the argument. The spaces that satisfy these conclusions are called **realcompact**, [Gillman & Jerison (1960), 5.9], rather than $\mathbf{R}$-compact

7.2. THEOREM. *Let X be a topological space. Then the following are equivalent:*

(1) *X is realcompact;*

(2) *X is $\mathbf{R}$-sober;*

(3) *X is canonically $\mathbf{R}$-sober;*

(4) *X is a fixed space.*

PROOF. The equivalence of the first three is immediate from Theorem 3.10. To see that (2) implies (4), it will be sufficient to show that X is dense in $\mathrm{Spec}_0(\hom(X, \mathbf{R}))$ since it is closed there. It will be convenient here to use the definition of $\mathrm{Spec}_0(\hom(X, \mathbf{R}))$ as a subspace of the space of maximal ideals (since $\mathbf{R}$ is rigid). We must show that every non-empty open subset of $\mathrm{Spec}_0(\hom(X, \mathbf{R}))$ meets X. A basic open set is $U(f) = \{f \mid f \notin M\}$ for some $f : X \longrightarrow \mathbf{R}$. If $U(f) \neq \emptyset$, f is not identically 0, so there is some $x \in X$ with $f(x) \neq 0$. Let $M_x = \{g : X \longrightarrow \mathbf{R} \mid g(x) = 0\}$. This is a maximal ideal. Thus $M_x \in U(f)$ and $h \equiv h(x) \pmod{M_x}$ and so the homomorphism determined by M_x is evaluation at x. ∎

8. $\mathbf{Z}$-groups

8.1. DEFINITIONS, NOTATION, AND PRELIMINARY REMARKS. In this section, all groups are abelian and “free” means “free abelian”. We let $\mathcal{C}$ denote the category of topological abelian groups and $\mathcal{D}$ the category of abelian groups. If we let Z be the circle group, we would be generalizing Pontrjagin–Van-Kampen duality. This is interesting, but quite well known. Instead, will use the group $\mathbf{Z}$ of integers. In this section $|C|$ denotes the discrete group underlying a topological group C.

As usual, we have that the contravariant functors $\hom_{\mathcal{C}}(-, \mathbf{Z}) : \mathcal{C} \longrightarrow \mathcal{D}$ and $\hom_{\mathcal{D}}(-, \mathbf{Z}) : \mathcal{D} \longrightarrow \mathcal{C}$ are adjoint on the right. We want to work out what topological groups are fixed under the composite $\mathcal{C} \longrightarrow \mathcal{D} \longrightarrow \mathcal{C}$ and what groups are fixed under the composite $\mathcal{D} \longrightarrow \mathcal{C} \longrightarrow \mathcal{D}$.

Generally speaking an object of $\mathcal{C}$ will be denoted by a C and an object of $\mathcal{D}$ by a D. One exception to the latter is that P will generally be used for a free group in $\mathcal{D}$. If X is a free generating set for P, we write $P = X \cdot \mathbf{Z}$.

When C and C' are objects of $\mathcal{C}$, we will write $C' \hookrightarrow C$, to mean that the object C' is embedded, algebraically and topologically in C. For discrete groups, $D' \hookrightarrow D$ simply means it is embedded. For an object C of $\mathcal{C}$, we write $|C|$ for the underlying discrete group.

We note that since $(-)^*$ is a right adjoint, it preserves limits. In this contravariant context, this means it takes an exact sequence

$$D'' \longrightarrow D \longrightarrow D' \longrightarrow 0$$

in $\mathcal{D}$ to an exact sequence

$$0 \longrightarrow D'^* \longrightarrow D^* \longrightarrow D''^*$$

in which the first map is a topological embedding. Similarly, when

$$C'' \longrightarrow C \longrightarrow C' \longrightarrow 0$$

is exact in $\mathcal{C}$, so is

$$0 \longrightarrow C'^* \longrightarrow C^* \longrightarrow C''^*$$

For the latter, it is critical that the given $C \longrightarrow C'$ be a quotient mapping. We will actually apply the latter only in the case that C and C' are discrete.

There is, as usual, a canonical map $D \longrightarrow D^{**}$ and similarly for an object of $\mathcal{C}$. We will say that an object is fixed when the double dualization map is an isomorphism.

8.2. Fixed groups in $\mathcal{C}$. We will show that the fixed groups of $\mathcal{C}$ are those that are the kernel of a homomorphism between powers of $\mathbf{Z}$, that is the $\mathbf{Z}$-sober groups.

From adjointness, it follows that when $P = X \cdot \mathbf{Z}$, then its dual is $P^* = \mathbf{Z}^X$.

There is an obvious necessary condition that a topological group be fixed. If D is an abelian group, it has a resolution by free groups:

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow D \longrightarrow 0$$

where P_0 and P_1 are free. Dualizing, we get an exact sequence

$$0 \longrightarrow D^* \longrightarrow P_0^* \longrightarrow P_1^*$$

which enables us to conclude:

8.3. Proposition. *A necessary condition that a topological group be fixed is that it be $\mathbf{Z}$-sober.* ■

In order to prove that that condition is also sufficient, we begin with:

8.4. Proposition. *Free groups are fixed as objects of $\mathcal{D}$.*

Proof. Let $P = X \cdot \mathbf{Z}$ be free. Since $P^* = \mathbf{Z}^X$, we must show that $(\mathbf{Z}^X)^* \cong X \cdot \mathbf{Z}$. It is known (but far from obvious) that this is true even without continuity when X has non-measurable cardinality. But it is much easier using continuity. The kernel of a map $f : \mathbf{Z}^X \longrightarrow \mathbf{Z}$ has to be an open subgroup. Every open subgroup contains one of the form $\mathbf{Z}^{X-X_0}$ for some finite subset $X_0 \subseteq X$. This means that f factors as $\mathbf{Z}^X \longrightarrow \mathbf{Z}^{X_0} \xrightarrow{f_0} \mathbf{Z}$, which means that

$$\hom(\mathbf{Z}^X, \mathbf{Z}) \cong \operatorname{colim} \hom(\mathbf{Z}^{X_0}, \mathbf{Z}) \cong \operatorname{colim} X_0 \cdot \mathbf{Z} \cong X \cdot \mathbf{Z}$$ ■

8.5. Proposition. *Suppose C is a topological subgroup of P^* with P free. Any continuous homomorphism $f : C \longrightarrow \mathbf{Z}$ factors through the image of C in P_0^* for some finitely generated subgroup $P_0^* \subseteq P^*$.*

Proof. Let $P = X \cdot \mathbf{Z}$. Since the kernel of f is open it must contain a set of the form $A \cap \mathbf{Z}^{X - X_0}$ for some finite subset $X_0 \subseteq X$. ∎

8.6. Proposition. *Suppose D is a subgroup of a finitely generated free group P. Then for any $f : D \longrightarrow \mathbf{Z}$ there is a positive integer n such that nf extends to a homomorphism on P.*

Proof. Consider the diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d, "f"'] & P \\ \mathbf{Z} \arrow[r] & \mathbf{Q} \end{tikzcd}$$

Since $\mathbf{Q}$ is injective, f extends to a homomorphism $g : P \longrightarrow \mathbf{Q}$. If n is the lcm of the denominators of all the fractions needed to express g on the generators of P, it is clear that ng is $\mathbf{Z}$-valued. ∎

8.7. Proposition. *Suppose P is free and C is a topological subgroup of P^*. For any continuous homomorphism $f : C \longrightarrow \mathbf{Z}$, there is an integer n such that nf extends to a homomorphism $P^* \longrightarrow \mathbf{Z}$.*

Proof. From 8.5, we know that f factors through the image C_0 of C in P_0^* for some finitely generated subgroup $P_0 \subseteq P$. We now apply the previous proposition to the inclusion $C_0 \longrightarrow P_0^*$, both topologized discretely. ∎

8.8. Corollary. *If C is a topological subgroup of P^*, then the cokernel of $P \longrightarrow C^*$ is torsion.* ∎

8.9. Corollary. *If $C \rightarrowtail C'$ is an inclusion of $\mathbf{Z}$-cogenerated topological groups, then the cokernel of $C'^* \longrightarrow C^*$ is torsion.*

Proof. Let $C' \rightarrowtail P^*$ be an embedding. For any $\varphi : C \longrightarrow \mathbf{Z}$ there is an $n > 0$ and a $\psi : P^* \longrightarrow \mathbf{Z}$ such that $\psi | C = n\varphi$. Then $\psi | C'$ is an element of C'^* whose restriction to C is $n\varphi$. ∎

8.10. Theorem. *A topological group is fixed if and only if it is $\mathbf{Z}$-sober.*

Proof. We have one direction in 8.3. Suppose that $0 \longrightarrow C \longrightarrow P_0^* \longrightarrow P_1^*$ is exact. If we let $T = \operatorname{coker}(P_0 \longrightarrow C^*)$, then we have a sequence

$$P_1 \longrightarrow P_0 \longrightarrow C^* \longrightarrow T \longrightarrow 0$$

The subsequence $P_1 \longrightarrow P_0 \longrightarrow C^*$ is not exact, but the composite is 0. From Corollary 8.8, we know that T is torsion, and hence has no non-zero homomorphisms to $\mathbf{Z}$. Thus we have a sequence

$$0 \longrightarrow C^{**} \longrightarrow P_0^* \longrightarrow P_1^*$$

The initial three term subsequence is exact and the remaining composite is 0 so that C^{**} is a subgroup of the kernel of $P_0^* \longrightarrow P_1^*$, which is C, while C is canonically embedded in it. The composite $C \longrightarrow C^{**} \longrightarrow C$ is the identity since when followed by the inclusion into P_0^* it is the inclusion. When the composite of monics is an isomorphism, both factors are isomorphisms as well. ∎

8.11. PROPOSITION. *Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ consisting of those topological abelian groups that are cogenerated by $\mathbf{Z}$. Then $\mathbf{Z}$ is coseparating in $\mathcal{C}_0$.*

PROOF. Suppose that $f : A \longrightarrow B$ is not epic in $\mathcal{C}_0$ and $g : B \longrightarrow C$ is a regular monic, which implies it is a closed subgroup. Then there is a map $\varphi : B \longrightarrow \mathbf{Z}$ for which $\varphi \neq 0$ but $u\varphi f = 0$. From 8.9, there is an $n > 0$ and $\psi : C \longrightarrow \mathbf{Z}$ such that $\psi g = n\varphi$. Since $\varphi \neq 0$, also $n\varphi \neq 0$ while $\varphi f = 0$ implies the same for $n\varphi$. ∎

8.12. COROLLARY. *If there is an extremal monic $C \longrightarrow \mathbf{Z}^X$ in the category of $\mathbf{Z}$-cogenerated topological abelian groups, then C is canonically $\mathbf{T}$-sober.*

Of course to apply this, we have to know what the extremal monics are. Certainly regular monics are extremal. In any category, regular monics are extremal, so any fixed object is canonically $\mathbf{T}$-sober. On the other hand any $\mathbf{T}$-sober object is fixed, so we see that an extremal monic into a power of $\mathbf{Z}$ is regular. Thus the canonically $\mathbf{Z}$-sober objects of $\mathcal{D}$ are just the fixed ones.

8.13. COROLLARY. Fix$(\mathcal{C})$ *is the limit closure of $\mathbf{Z}$ in $\mathcal{C}$.*

PROOF. We have seen in Theorem 8.10 that every fixed group is in the limit closure of $\mathbf{Z}$. The other direction follows immediately from Corollary 3.7. ∎

8.14. THEOREM. *For any object D of $\mathcal{D}$, D^* is fixed.*

PROOF. Let $P_1 \longrightarrow P_0 \longrightarrow D \longrightarrow 0$ be a free resolution of D. This gives an exact sequence $0 \longrightarrow D^* \longrightarrow P_0^* \longrightarrow P_1^*$ and the conclusion follows from 8.10. ∎

8.15. THEOREM. *A group in $\mathcal{C}$ is fixed if and only if it is canonically $\mathbf{Z}$-sober.*

PROOF. According to 3.4 a $\mathbf{Z}$-cogenerated group A is canonically $\mathbf{Z}$-sober if and only the cokernel B of $A \longrightarrow \mathbf{Z}^{|A^*|}$ is $\mathbf{Z}$-cogenerated. And that happens if and only if $B \longrightarrow B^{**}$ is monic. The significance of the map $A \longrightarrow \mathbf{Z}^{|A^*|}$ is that it dualizes to $|A^*| \cdot \mathbf{Z} \longrightarrow A^*$ which is obviously surjective. Thus from $0 \longrightarrow A \longrightarrow \mathbf{Z}^{|A^*|} \longrightarrow B \longrightarrow 0$, we get the exact sequence $0 \longrightarrow B^* \longrightarrow |A^*| \cdot \mathbf{Z} \longrightarrow A^* \longrightarrow 0$ whose second dual is the second row of the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & \mathbf{Z}^{|A^*|} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^{**} \arrow[r] & \mathbf{Z}^{|A^*|} \arrow[r] & B^{**} &
\end{tikzcd}$$

The snake lemma implies that $\ker(B \longrightarrow B^{**}) \cong \operatorname{coker}(A \longrightarrow A^{**})$, from which the equivalence is immediate. ∎

8.16. Fixed objects of $\mathcal{D}$. Here we will characterize fixed objects of $\mathcal{D}$ as those that have a pure embedding into a power of $\mathbf{Z}$. Recall that if D is an abelian group, a subgroup $D' \subseteq D$ is said to be **pure** if every element of D' is as divisible in D' as it is in D. It is easily seen that when D is torsion free, then $D' \subseteq D$ is pure if and only if D/D' is torsion free. This is equivalent (in the torsion-free case) to the assumption that whenever $x \in D$ and $nx \in D'$ for some positive integer n, then $x \in D'$.

8.17. Proposition. *Let D be $\mathbf{Z}$-group. Then D^{**}/D is torsion.*

Proof. Let $P \twoheadrightarrow D \longrightarrow 0$ be exact with P free. Then $0 \longrightarrow D^* \longrightarrow P^*$ is exact, whence we have, from 8.8 that there is an exact sequence $P \longrightarrow D^{**} \longrightarrow T \longrightarrow 0$ with T torsion. The map $P \longrightarrow D{*}{*}$ factors as $P \twoheadrightarrow D \rightarrowtail D^{**}$ so that $D^{**}/D \cong T$. ∎

8.18. Proposition. *For any discrete group D, the bijection $|D^*| \longrightarrow D^*$ induces a pure inclusion $D^{**} \hookrightarrow |D^*|^*$ so that the quotient $|D^*|^*/D^{**}$ is torsion free.*

Proof. If $\varphi : |D^*| \longrightarrow \mathbf{Z}$ is a homomorphism and $n > 0$ an integer such that $n\varphi$ is continuous on D^*, then the kernel of $n\varphi$ is an open subgroup of D^*. But $\ker n\varphi = \ker \varphi$ so the latter is also open in D^* and hence $\varphi \in D^{**}$. ∎

8.19. Theorem. *Let D be a $\mathbf{Z}$-group and X be a set of generators for the abelian group* $\operatorname{Hom}(D, \mathbf{Z})$*. Then D is fixed if and only if* $\operatorname{coker}(D \longrightarrow \mathbf{Z}^X)$ *is torsion free.*

Proof. Since X generates $\operatorname{Hom}(D, \mathbf{Z})$ the induced map $X \cdot \mathbf{Z} \longrightarrow |D^*|$ is surjective so we have an exact sequence $0 \longrightarrow C \longrightarrow X \cdot \mathbf{Z} \longrightarrow D^{**} \longrightarrow 0$ of discrete groups in $\mathcal{C}$. Applying the duality functor gives us a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] \arrow[d, "f"'] & \mathbf{Z}^X \arrow[r] \arrow[d] & D' \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & |D^*|^* \arrow[r] & \mathbf{Z}^X \arrow[r] & C^* &
\end{tikzcd}$$

The snake lemma gives us that $\operatorname{coker} f \cong \ker g$. If D is fixed, then $D \cong D^{**}$ so that the preceding proposition implies that $\operatorname{coker} f$ is torsion free and hence so is $\ker g$. Since the image of g is a subgroup of the torsion free group C^*, we see that D' is also torsion free.

For the converse, suppose that D' is torsion free. Then we see that $\ker f$ is also torsion free. Then D^{**}/D is isomorphic to a subgroup of $\operatorname{coker} f = |D^*|^*/D$ and is thereby torsion free, while we have just seen that it is torsion. ∎

8.20. Theorem. *Suppose there is a pure embedding $D \longrightarrow \mathbf{Z}^X$ for some set X. Then the canonical embedding $D \longrightarrow \mathbf{Z}^{\mathrm{Hom}(D,\mathbf{Z})}$ is pure.*

Proof. There is a simple argument using the snake lemma, but we will give a direct proof. An mapping $D \longrightarrow \mathbf{Z}^X$ is determined by it product projections which gives a function $u : X \longrightarrow \mathrm{Hom}(D, \mathbf{Z})$. To say that D is pure in $\mathbf{Z}^X$ means that for all integers $n > 0$, if $d \in D$ is an element such that $u(x)(d)$ is divisible by n for all $x \in X$, then d is divisible by n in D. The contrapositive is that if d is not divisible in D by any integer, then for all $n > 0$, there is an $x \in X$ such that n does not divide $u(x)(d)$. Since for each $x \in X$ there is a such a function, then there is one in the entire set $\mathrm{Hom}(D, \mathbf{Z})$. ∎

8.21. Theorem. *For any object C of $\mathcal{C}$, C^* is fixed.*

Proof. Let $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow |C| \longrightarrow 0$ be a free resolution of $|C|$. We have an exact sequence

$$0 \longrightarrow |C|^* \longrightarrow P_0^* \longrightarrow P_1^*$$

which shows that $|C|^*$ has a pure embedding into P_0^*. But $C^* \longrightarrow |C|^*$ is also pure just as in the proof of Proposition 8.18 and so C^* is a pure subgroup of P_0^*. ∎

If $D \hookrightarrow D'$ we write $\mathrm{pc}(D)$ (for **pure closure**) for the set of elements of D' that have a non-zero multiple in D. Evidently, $\mathrm{pc}(D)$ is the smallest pure subgroup of D' that contains D.

8.22. Theorem. *Suppose D is $\mathbf{Z}$-cogenerated. Then $D^{**} = \mathrm{pc}(D)$ in $\mathbf{Z}^{\mathrm{Hom}(D,\mathbf{Z})}$.*

Proof. From the preceding theorem, we know that $\mathrm{pc}(D)$ is fixed. The inclusion $D \longrightarrow \mathrm{pc}(D)$ induces $(\mathrm{pc}(D))^*$ and then $\mathrm{pc}(D) \longrightarrow D^{**}$. Since $D^{**} \subseteq \mathbf{Z}^{\mathrm{Hom}(D,\mathbf{Z})}$, we have $D \subseteq \mathrm{pc}(D) \subseteq D^{**}$ as subobjects of $\mathbf{Z}^{\mathrm{Hom}(D,\mathbf{Z})}$. Since D^* is fixed, we know that $X \cdot \mathbf{Z} \longrightarrow D^{***} = D^*$ is fixed and it follows from Proposition 8.18 that $D^{**} \longrightarrow \mathbf{Z}^{\mathrm{Hom}(D,\mathbf{Z})}$ is pure. But $\mathrm{pc}(D)$ is the smallest pure subgroup of $\mathbf{Z}^{\mathrm{Hom}}$ that contains D and so we must have that $D^{**} = \mathrm{pc}(D)$. ∎

8.23. Remark. We do not know if every pure subgroup of a power of $\mathbf{Z}$ is a regular subobject of a (possibly different) power of $\mathbf{Z}$. If there are such examples, then $\mathrm{Fix}(\mathcal{D})$ is larger than the limit closure of $\mathbf{Z}$. Since a pure subobject of a pure subobject is pure, it follows that $\mathrm{Fix}(\mathcal{D})$ is complete.

8.24. Example. As an application, we see that any power of $\mathbf{Z}$ is, as an object of $\mathcal{D}$, fixed. For $F = \mathrm{Hom}(-, \mathbf{Z}) : \mathcal{D} \longrightarrow \mathcal{S}et^{\mathrm{op}}$ is left adjoint to $U = \mathbf{Z}^{(-)} : \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathcal{D}$. Since, for any set X,

$$\begin{tikzcd} UX \arrow[r, "\eta UX"] & TUX \arrow[r, shift left, "T\eta UX"] \arrow[r, shift right, "\eta TUX"'] & T^2UX \end{tikzcd}$$

is a split equalizer, it follows that

$$0 \longrightarrow UX \xrightarrow{\eta UX} TUX \xrightarrow{T\eta UX - \eta TUX} T^2UX$$

is exact and hence TUX/UX is torsion free, being embedded in a power of $\mathbf{Z}$. From [Eda, et al. (1989), Corollary 6.9], we know that when X is non-measurable, $\left(\mathbf{Z}^X\right)^* \cong X \cdot \mathbf{Z}$ and thus the second dual is $\mathbf{Z}^X$, but this shows it for all cardinalities, even though the first dual is more complicated.

8.25. Example. We describe an abelian $\mathbf{Z}$-cogenerated group G with an element t such that every homomorphism $G \longrightarrow \mathbf{Z}$ takes t to an even integer but t is not twice any other element of G. Thus the canonical map $G \longrightarrow \mathbf{Z}^{\mathrm{Hom}(G,\mathbf{Z})}$ is a non-pure embedding.

We begin with the group $A = \mathbf{Z}^{\mathbf{N}} \times \mathbf{N} \cdot \mathbf{Z}$. Let $e_n \in \mathbf{Z}^{\mathbf{N}}$ be the element with 1 in the nth coordinate and 0 in all others. Similarly, let g_n be the similar element in $\mathbf{N} \cdot \mathbf{Z}$. Let H be the subgroup of A generated by all elements of the form $(e_n - e_1, 2g_1 - 2g_n)$ and let $G = (\mathbf{Z}^{\mathbf{N}} \times \mathbf{N} \cdot \mathbf{Z})/H$. In G elements of the form $(e_n, -2g_n)$ are equal for all n.

We treat elements of A as elements of G with equality being congruence mod H.

8.26. Lemma. *Let $\varphi : \mathbf{Z}^{\mathbf{N}} \times 0 \longrightarrow \mathbf{Z}$ be a homomorphism. For any integer k there is a homomorphism $\psi : G \longrightarrow \mathbf{Z}$ such that $\psi(0, g_1) = k$ and $\psi(u, 0) = 2\varphi(u)$ for all $u \in \mathbf{Z}^{\mathbf{N}}$.*

Proof. Let $\psi_0 : A \longrightarrow \mathbf{Z}$ by

$$\psi_0(u, \sum k_n g_n) = 2\varphi(u) + \sum k_n \varphi(e_n - e_1) + k \sum k_n$$

It is a simple computation to show that ψ_0 vanishes on H and therefore induces the required map ψ.

8.27. Lemma. *G is Z-cogenerated.*

Proof. Let $(u, \sum k_n g_n)$ be a non-zero element of G. If every k_n is an even integer, this element is equal mod H to an element of the form $(v, \sum k_n g_1)$. If $\sum k_n = 0$ there is a homomorphism $\mathbf{Z}^{\mathbf{N}} \longrightarrow \mathbf{Z}$ that does not vanish on v and the preceding lemma provides $\psi : G \longrightarrow \mathbf{Z}$ with the required property. If $\sum k_n \neq 0$, then the preceding lemma applied to the 0 map on $\mathbf{Z}^{\mathbf{N}}$ provides $\psi : G \longrightarrow \mathbf{Z}^{\mathbf{N}}$ for which $\psi(0, g_1) = 1$. Then $\psi(v, \sum k_n g_1) = \sum k_n$. Finally, we consider the case that k_m is odd for at least one m. In that case, let $\psi_0 : A \longrightarrow \mathbf{Z}$ be a homomorphism such that $\psi_0(e_m, 0) = 2$, $\psi_0(0, g_m) = 1$, $\psi_0(e_n, 0) = \psi_0(0, g_n) = 0$ for all $n \neq m$. The first equation is possible using the projection of $\mathbf{Z}^X$ on the mth coordinate. It is clear that $\psi_0(u, 0)$ is even for all $u \in \mathbf{Z}^X$. Then $\psi_0(e_m - e_1, 2g_1 - 2g_m) = 2 - 2 = 0$ and so ψ_0 induces $\psi : G \longrightarrow \mathbf{Z}$ for which $\psi(u, \sum k_n g_m) = \psi(u, 0) + k_m \neq 0$ since $\psi(u, 0)$ is even and k_m is odd. ■

8.28. Lemma. *Let $t \in G$ be the common value of $(e_n, -2g_n)$. Then $\varphi(t)$ is even for every $\varphi : G \longrightarrow \mathbf{Z}$.*

Proof. Since $\mathrm{Hom}(\mathbf{Z}^{\mathbf{N}}, \mathbf{Z}) \cong \mathbf{N} \cdot \mathbf{Z}$ there is an integer n such that $\varphi(e_n, 0) = 0$. But then $\varphi(t) = -2\varphi(g_n)$. ■

8.29. LEMMA. *There is no element $s \in G$ with $2s = t$.*

PROOF. We will show that there is a map $\varphi : G \longrightarrow \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$ for which $\varphi(t) = 1$. There is a canonical map $f : \mathbf{Z}^{\mathbf{N}} \times \mathbf{N} \cdot \mathbf{Z} \longrightarrow (\mathbf{Z}_2)^{\mathbf{N}} \times \mathbf{N} \cdot \mathbf{Z}_2$ and $f(H)$ is the subgroup generated by all elements of the form $(e_1 - e_n, 0)$ and so f induces a map $G \longrightarrow (\mathbf{Z}_2)^{\mathbf{N}}/f(H) \times \mathbf{N} \cdot \mathbf{Z}_2$ and under this map $f(t) = f(e_1, 0) = f(e_2, 0) = \cdots$. There is a map to $\mathbf{Z}_2$ from the subgroup of $(\mathbf{Z}_2)^{\mathbf{N}}$ generated by t that takes t to 1. The injectivity of $\mathbf{Z}_2$ in the category of $\mathbf{Z}_2$-modules allows the extension of this map to the whole group and provides the required mapping. ∎

9. Concluding remarks

9.1. GENERAL AIMS. The main goals of this paper are to define what is meant by saying that an object Z lives in two different categories, $\mathcal{C}$ and $\mathcal{D}$, and to explore the duality that usually results. We show that when the categories have well-behaved underlying set functors, the Hom-functors, $\mathrm{Hom}_{\mathcal{C}}(-, Z)$ and $\mathrm{Hom}_{\mathcal{D}}(-, Z)$ can usually be lifted to contravariant functors between $\mathcal{C}$ and $\mathcal{D}$ which are adjoint on the right. The duality between a subcategory of $\mathcal{C}$ and a subcategory of $\mathcal{D}$ then follows from the contravariant equivalence between the full subcategories of fixed objects, as in Theorem 2.2. We note that many familiar dualities arise in this way and we construct some new examples.

We also develop some useful categorical tools to help us identify the fixed objects. In particular, the fixed objects are often the Z-sober objects. When dealing with Hausdorff spaces, the Z-sober objects may be the "$\mathbf{Z}$-compact" objects, but for a non-Hausdorff space, this won't work. For example, Let S denote the **Sierpinski space**, the two point space with one point open and the other not. The notion of "S-compact" is usually left undefined (and if we extend the standard definition as a closed subspace of a power, we do not seem to get a useful concept–an S-compact space would have at most one closed point.) The notion of an object coseparating the subcategory it cogenerates often proves to be quite useful. Every injective object is coseparating, but being coseparating is much weaker than being injective. In fact it is not easy to find objects in reasonable categories that are not coseparating. Further categorical tools are discussed in Sections 2 and 3.

9.2. MORE EXAMPLES. Here we list some well-known additional examples of this kind of duality. Most of these are found in [Johnstone (1974)]. The example of linearly compact vector spaces (see after the table for the definition) is found in [Lefschetz (1942)], where it was introduced to make homology and cohomology more nearly dual.

$\mathcal{C}$	$\mathcal{D}$	Z	Fix($\mathcal{C}$)	Fix($\mathcal{D}$)
Topological abelian groups	Abelian groups	$\mathbf{R}/\mathbf{Z}$	compact abelian groups	abelian groups
Topological spaces	Spatial frames	S	sober spaces	spatial frames
Topological spaces	Boolean algebras	$\mathbf{2}$	Stone spaces	Boolean algebras
Sets	Complete atomic boolean algebras	$\mathbf{2}$	Sets	Complete atomic boolean algebras
Linearly compact vector spaces	Vector spaces	K	Linearly compact vector spaces	Vector spaces
Complete sup semi-lattices	Complete sup semi-lattices	$\mathbf{2}$	Complete sup semi-lattices	Complete sup semi-lattices

The vector spaces are over some field K. A linearly topologized vector space is one that has a neighbourhood basis at 0 consisting of linear subspaces. Such a space is linearly compact if every cover by affine translates of such open neighbourhoods has a finite subcover. It turns out (and that is the essence of the duality) that such a space is necessarily a cartesian power of the field.

A complete sup semi-lattice is of course a complete lattice but the category has maps that preserve infinite sups but not necessarily any infs. This category is the only one we are currently aware of in which $\mathcal{C} = \mathcal{D} = \mathrm{Fix}(\mathcal{C}) = \mathrm{Fix}(\mathcal{D})$. Usually, this duality appears as a duality between complete sup semi-lattices and complete inf semi-lattices, with right adjoint as the dual of a morphism. But in the framework of this paper, it appears naturally as an auto-duality on the category of complete sup semi-lattices. Needless to say, there is a similar duality on the category of complete inf semi-lattices.

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