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COMPONENTS, COMPLEMENTS AND THE REFLECTION FORMULA

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Abstract. We illustrate the formula $(\downarrow p)x = \Gamma_!(x/p)$, which gives the reflection $\downarrow p$ of a category $p : P \to X$ over X in discrete fibrations. One of its proofs is based on a "complement operator" which takes a discrete fibration A to the functor $\neg A$, right adjoint to $\Gamma_!(A \times -) : \mathbf{Cat}/X \to \mathbf{Set}$ and valued in discrete opfibrations.

Some consequences and applications are presented.

1. Introduction

The "classical" formulas (see [Lawvere, 1973] and [Paré, 1973])

$$(\downarrow p)x = \Gamma_!(x/p) \qquad ; \qquad (\uparrow p)x = \Gamma_!(p/x) \tag{1}$$

or better

$$\downarrow p = \Gamma_!(-/p) \qquad ; \qquad \uparrow p = \Gamma_!(p/-) \tag{2}$$

which display the presheaves corresponding to the reflections of a category $p : P \to X$ over a base category X in discrete fibrations and in discrete opfibrations as components of "comma" categories, do not appear at a first sight particularly pregnant or self-explicatory. However, noting that x/p is the product $x/X \times p$ in $\mathbf{Cat}/X$ (and similarly $p/x = X/x \times p$) and that x/X and X/x (corresponding to the representable presheaves) are the reflections $\uparrow x$ and $\downarrow x$ of the object $x : 1 \to X$, we get

$$(\downarrow p)x = \Gamma_!(\uparrow x \times p) \qquad ; \qquad (\uparrow p)x = \Gamma_!(\downarrow x \times p) \tag{3}$$

These are exactly the set-valued version of the formulas

$$x \in \downarrow p \iff \Gamma_!(\uparrow x \cap p) \qquad ; \qquad x \in \uparrow p \iff \Gamma_!(\downarrow x \cap p) \tag{4}$$

which clearly give the reflections of a subset p of a poset in lower and upper subsets. (Here $\Gamma_!$ is the two-valued correspective of the components functor, and gives `false` on the empty set and `true` elsewhere). Remarkably, also the most natural proof of (4),

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based on the classical complement operator which takes lower sets into upper sets (and conversely) and on the consequent law

$$\Gamma_!(\downarrow p \cap q) \quad \Longleftrightarrow \quad \Gamma_!(p \cap \uparrow q) \tag{5}$$

generalizes almost straightforwardly to the set-valued context. (See [Pisani, 2007] for a detailed discussion of the poset case, which provides useful insight and motivations.)

Recall that the fundamental adjunctions

$$\Gamma_! \dashv \Gamma^* \dashv \Gamma_* : \mathbf{Cat} \to \mathbf{Set} \tag{6}$$

between the components, discrete and points functors, can be generalized to

$$\Gamma_! \dashv \Gamma^* \dashv \Gamma_* : \mathbf{Cat}/X \to \mathbf{Set} \tag{7}$$

where for $p : P \to X$, $\Gamma_! p$ is given by the components of the total (or domain) category P, while for $S \in \mathbf{Set}$, $\Gamma^* S$ is the first projection of the product of X and the discrete category on S. In the present context, the (partially defined) "complement operator" is parametrized by sets:

$$\neg : \mathbf{Set} \times (\mathbf{Cat}/X)^{\mathrm{op}} \to \mathbf{Cat}/X$$

It takes a discrete fibration A into the "exponential" functor $\neg A := (\Gamma^* -)^A : \mathbf{Set} \to \mathbf{Cat}/X$ which is valued in discrete opfibrations (and vice versa) and allows to prove the law

$$\Gamma_!(\downarrow p \times q) \quad \Longleftrightarrow \quad \Gamma_!(p \times \uparrow q) \tag{8}$$

from which the reflection formula is easily drawn. (A related definition of "negation" is proposed in [Lawvere, 1996]).

After presenting some basic facts about categories over a base in Section 2, we develop this and other proofs of the reflection formula in Section 3. In Section 4 we treat colimits in X as given by the (partially defined) reflection of $\mathbf{Cat}/X$ in principal (or representable) discrete fibrations. Thus $\downarrow(-)$ is an intermediate step of this reflection, and we can easily derive important properties of (co)limits following [Paré, 1973]. In Section 5 we consider those categories $t : T \to X$ over X such that

$$\mathrm{hom}(t, A) \cong \mathrm{ten}(t, A) := \Gamma_!(t \times A) \tag{9}$$

for any discrete fibration or discrete opfibration A. Again motivated by the two-valued case, we call them "atoms". Any idempotent arrow in X is an atom, and the reflections of atoms are, as presheaves on X, the retracts of representable functors; so they generate the Cauchy completion of X, displaying it as the Karoubi envelope of X. Finally, in Section 6 we show how the reflection formula can be used to obtain in a very direct way the reflection of graphs in (idempotent, bijective or n-periodic) evolutive sets.

Some of these topics have been presented at the International Conference on Category Theory (CT06) held at White Point in June 2006, and under a slightly different perspective, also in the preprint [Pisani, 2005]. The preprint [Pisani, 2007], to which we will often refer, contains more details and has a more didactical style.

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2. Categories over a base

We review some facts about categories over a base that will be used in the sequel. Though most of them are standard, others seem not to be widely known.

2.1. Discrete fibrations and strong dinaturality. Let X be a fixed category and $\mathbf{y} : X \to \mathbf{Set}^{X^{\mathrm{op}}}$ be the Yoneda embedding. Let us denote by i the functor

$$i := \mathbf{y}/- : \mathbf{Set}^{X^{\mathrm{op}}} \to \mathbf{Cat}/X \tag{10}$$

which takes a presheaf $A : X^{\mathrm{op}} \to \mathbf{Set}$ into the category of elements $\mathbf{y}/A$ with its projection over X. The categories over X isomorphic to some iA are the discrete fibrations. The functor i is full and faithful:

$$\mathbf{Set}^{X^{\mathrm{op}}}(A, B) \cong \mathbf{Cat}/X(iA, iB) \tag{11}$$

Note that the right hand side is discrete even when the 2-structure of $\mathbf{Cat}/X$ is acknowledged. We denote by

$$\downarrow(-) : \mathbf{Cat}/X \to \mathbf{Set}^{X^{\mathrm{op}}} \tag{12}$$

a left adjoint of i; thus $i\downarrow(-)$ gives the reflection of $\mathbf{Cat}/X$ in the full subcategory of discrete fibrations. Similarly, we define $i : \mathbf{Set}^{X} \to \mathbf{Cat}/X$ and $\uparrow(-) \dashv i$.

2.2. Remark. For any object $x : 1 \to X$, there are bijections

$$\mathbf{Cat}/X(x, iA) \cong Ax \cong \mathbf{Set}^{X^{\mathrm{op}}}(X(-, x), A)$$

natural in $A \in \mathbf{Set}^{X^{\mathrm{op}}}$. So we get $\downarrow x \cong X(-, x)$. Since furthermore $iX(-, x) \cong X/x$, the reflection $i\downarrow x$ of an object in discrete fibrations is the slice category X/x. Similarly, given an arrow $\lambda : 2 \to X$ of the base category, with domain x and codomain y, $\downarrow\lambda \cong X(-, y)$; and the inclusions of the domain and the codomain become $X(-, \lambda)$ and the identity respectively.

The construction of the category of elements admits the following generalization. Let Din^*_X the category that has the functors $X^{\mathrm{op}} \times X \to \mathbf{Set}$ as objects and the strong dinatural transformations (also known as "Barr dinatural", see [Paré & Roman, 1998] and [Ghani, Uustalu & Vene, 2004]) as arrows. Given a functor $H \in \mathrm{Din}^*_X$, we define a category iH over X as follows:

- the objects over $x \in X$ are the elements of $H(x, x)$;
- given $\lambda : x \to y$ in X, there is at most one arrow from $a \in H(x, x)$ to $b \in H(y, y)$ over λ, and this is the case iff $H(x, \lambda)a = H(\lambda, y)b \in H(x, y)$.

Then one easily verifies (see [Pisani, 2007]) that this constructions is the object map of a full and faithful functor

$$i : \mathrm{Din}^*_X \to \mathbf{Cat}/X \tag{13}$$

that extends both $i : \mathbf{Set}^{X^{\mathrm{op}}} \to \mathbf{Cat}/X$ and $i : \mathbf{Set}^X \to \mathbf{Cat}/X$ (which come up when H is dummy in one variable). We are particularly interested in the following case: given $A : X^{\mathrm{op}} \to \mathbf{Set}$ and $D : X \to \mathbf{Set}$, let $H(x, y) = Ax \times Dy$, that is H is the composite

$$X^{\mathrm{op}} \times X \xrightarrow{A \times D} \mathbf{Set} \times \mathbf{Set} \xrightarrow{\times} \mathbf{Set}$$

Then iH over X is the product $iA \times iD$ in $\mathbf{Cat}/X$. Furthermore, in this case

$$\mathrm{Din}_X(H, K) \;\cong\; \mathrm{Din}^*_X(H, K) \;\cong\; \mathbf{Cat}/X(iH, iK) \tag{14}$$

that is, the dinatural transformations with domain H are also strongly dinatural, for any K.

2.3. Change of base and components. Let $f : X \to Y$ be a functor. As in any category with pullbacks, it gives rise to a pair of adjoint functors

$$f_! \dashv f^* : \mathbf{Cat}/Y \to \mathbf{Cat}/X \tag{15}$$

where $f_!$ is given by composition with f, while f^* is obtained by pulling back in $\mathbf{Cat}$. The pair $f_! \dashv f^*$ satisfies the Frobenius law, that is the morphism

$$f_!\pi_1 \wedge (\varepsilon \circ f_!\pi_2) : f_!(p \times f^*q) \to f_!p \times q \tag{16}$$

is an isomorphism for any $p \in \mathbf{Cat}/X$ and $q \in \mathbf{Cat}/Y$, ε being the counit of the adjunction.

In particular, we get the fibers over objects or arrows of X of a category over X:

$$\begin{tikzcd} Px \arrow[r] \arrow[d, "x^*p"'] & P \arrow[d, "p"] \\ 1 \arrow[r, "x"] & X \end{tikzcd} \qquad \begin{tikzcd} P\lambda \arrow[r] \arrow[d, "\lambda^*p"'] & P \arrow[d, "p"] \\ 2 \arrow[r, "\lambda"] & X \end{tikzcd} \tag{17}$$

The pullback of a discrete fibration is a discrete fibration; more precisely, the left hand diagram below commutes (up to isomorphisms), and so also the right hand square of left adjoints commutes:

$$\begin{tikzcd} \mathbf{Set}^{X^{\mathrm{op}}} \arrow[d, "i"'] & \mathbf{Set}^{Y^{\mathrm{op}}} \arrow[l, "\mathbf{Set}^f"'] \arrow[d, "i"] \\ \mathbf{Cat}/X & \mathbf{Cat}/Y \arrow[l, "f^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Set}^{X^{\mathrm{op}}} \arrow[r, "\exists_f"] & \mathbf{Set}^{Y^{\mathrm{op}}} \\ \mathbf{Cat}/X \arrow[u, "\downarrow\!(-)"] \arrow[r, "f_!"] & \mathbf{Cat}/Y \arrow[u, "\downarrow\!(-)"'] \end{tikzcd} \tag{18}$$

(In particular, the fiber $(iA)x$ of the discrete fibration iA is the discrete category on the set Ax.) We then obtain that in the diagram below

$$\begin{tikzcd} \mathbf{Set}^{X^{\mathrm{op}}} \arrow[d, "i"'] & \\ \mathbf{Cat}/X \arrow[r, "\downarrow(-)\circ f_!"] & \mathbf{Set}^{Y^{\mathrm{op}}} \arrow[ul, "\mathbf{Set}^f"'] \end{tikzcd} \tag{19}$$

$\downarrow(-) \circ f_! \dashv i \circ \mathbf{Set}^f$, and since i is full and faithful also $\downarrow(-) \circ f_! \circ i \dashv \mathbf{Set}^f$, that is the left Kan extension of $A : X^{\mathrm{op}} \to \mathbf{Set}$ along f^{op} can be computed by:

$$\exists_f A \cong \downarrow(f_!(iA)) \tag{20}$$

In the special case $Y = 1$, the diagrams (18) above become

$$\begin{tikzcd}
\mathbf{Set}^{X^{\mathrm{op}}} \arrow[d, "i"'] & \mathbf{Set} \arrow[l, "\Delta"'] \arrow[d, "i"] \arrow[dl, "\Gamma^*"'] \\
\mathbf{Cat}/X & \mathbf{Cat} \arrow[l, "X\times -"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Set}^{X^{\mathrm{op}}} \arrow[r, "\mathrm{Colim}"] & \mathbf{Set} \\
\mathbf{Cat}/X \arrow[u, "\downarrow(-)"] \arrow[ur, "\Gamma_!"] \arrow[r, "\mathrm{tot}"] & \mathbf{Cat} \arrow[u, "\pi_0"']
\end{tikzcd} \tag{21}$$

In particular, we have the functors

$$\Gamma_! \dashv \Gamma^* : \mathbf{Set} \to \mathbf{Cat}/X \tag{22}$$

Given $p : P \to X$ in $\mathbf{Cat}/X$, $\Gamma_! p$ is the set of components of the total category P while for a set S, $\Gamma^* S$ is the projection $X \times S \to X$ of the product of X and the discrete category on S. Furthermore, the discrete functor Γ^* has a right adjoint Γ_*, giving the set of points of p, that is its sections.

The formula (20) now becomes

$$\mathrm{Colim} A \;\cong\; \pi_0(\mathrm{tot}(iA)) \;\cong\; \Gamma_!(iA) \tag{23}$$

which expresses the colimit of a presheaf by the components of its category of elements. (Similarly, $\mathrm{Lim} A = \Gamma_*(iA)$.) This can be generalized as follows. Let us denote by

$$\mathrm{Coend}^* : \mathrm{Din}^*_X \to \mathbf{Set}$$

a left adjoint to the functor $\Delta : \mathbf{Set} \to \mathrm{Din}^*_X$, defined in the usual way. Then

$$\begin{tikzcd}
\mathrm{Din}^*_X \arrow[d, "i"'] & \\
\mathbf{Cat}/X & \mathbf{Set} \arrow[l, "\Gamma^*"'] \arrow[ul, "\Delta"']
\end{tikzcd} \tag{24}$$

commutes, and as above we get

$$\mathrm{Coend}^* H \;\cong\; \Gamma_!(iH) \tag{25}$$

While in general $\mathrm{Coend}^* H$ and the usual $\mathrm{Coend} H$ are different (see [Pisani, 2007]), by the remark at the end of section 2.1, they coincide for $H(x, y) = Ax \times Dy$:

$$A \otimes D \;\cong\; \Gamma_!(iH) \;\cong\; \Gamma_!(iA \times iD) \tag{26}$$

Thus we are led to consider $\Gamma_!(iA \times iD)$ as a generalized tensor product, and we pose

$$\mathrm{ten} := \Gamma_!(- \times -) : \mathbf{Cat}/X \times \mathbf{Cat}/X \to \mathbf{Set} \tag{27}$$

calling it the "tensor functor".

2.4. Exponentials and complements. The category **Cat** is not locally cartesian closed, that is exponentials in $\mathbf{Cat}/X$ may not exist. When they do, the objects of p^q over x are the functors $Px \to Qx$, while the arrows of p^q over λ are the functors $P\lambda \to Q\lambda$ over 2 (see diagram 17).

Discrete fibrations and opfibrations are exponentiable. The following particular case will be needed in the sequel (see [Pisani, 2007]):

2.5. Proposition. *If iA is a discrete fibration and iD a discrete opfibration over X, then the exponential $(iD)^{iA}$ in $\mathbf{Cat}/X$ is a discrete opfibration (and dually $(iA)^{iD}$ is a discrete fibration).*

■

Explicitly, as a covariant presheaf $(iD)^{iA}$ is the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{A \times D} \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \xrightarrow{\hom} \mathbf{Set} \tag{28}$$

that is, $(iD)^{iA}x = Dx^{Ax}$ and

$$\begin{aligned} (iD)^{iA}f = Df^{Af} \quad : \quad & Dx^{Ax} \to Dy^{Ay} \\ & h \mapsto Df \circ h \circ Af \end{aligned}$$

2.6. Definition. *Given $p : P \to X$ in $\mathbf{Cat}/X$, a right adjoint to* $\mathrm{ten}(p, -) = \Gamma_!(p \times -)$: $\mathbf{Cat}/X \to \mathbf{Set}$ *is called "the complement of p" and is denoted by $\neg p$:*

$$\mathrm{ten}(p, -) \dashv \neg p : \mathbf{Set} \to \mathbf{Cat}/X \tag{29}$$

It is easy to see that p has a complement iff the exponential $(\Gamma^* S)^p$ exists for any set S. In this case we have

$$\neg p \cong (\Gamma^* -)^p \tag{30}$$

In particular, since $\Gamma^* S$ is both a discrete fibration and a discrete opfibration for any set S, Proposition 2.5 gives the following correspective of the fact that classical complementation on the subsets of a poset takes lower sets into upper sets (and conversely):

2.7. Corollary. *Any discrete fibration iA has a complement valued in discrete opfibrations (and conversely). Explicitly, as a covariant presheaf $(\neg\, iA)S = (\Gamma^* S)^{iA} =$ $\mathbf{Set}(A-, S)$, that is:*

$$\begin{aligned} (\neg\, iA)Sx &= S^{Ax} \\ (\neg\, iA)Sf &= S^{Af} \quad : \quad S^{Ax} \to S^{Ay} \\ & \qquad\qquad\quad h \mapsto h \circ Af \end{aligned}$$

■

2.8. Remark. The adjunction $\mathrm{ten}(iA,-) \dashv \neg\, iA : \mathbf{Set} \to \mathbf{Cat}/X$ restricts to

$$A \otimes - \dashv \mathbf{Set}(A-, \_\_) : \mathbf{Set} \to \mathbf{Set}^X$$

a particular case of the closed structure of the bicategory of distributors.

We now prove an "adjunction-like" property of the tensor functor on $\mathbf{Cat}/X$ which will allow us to derive directly the reflection formula.

2.9. Proposition. *Let X be a category, iA a discrete fibration and iD a discrete opfibration on X. Then for any $p : P \to X$ in $\mathbf{Cat}/X$ there are natural bijections*

$$\mathrm{ten}(p, iA) \cong \mathrm{ten}(i{\uparrow}p, iA) \qquad ; \qquad \mathrm{ten}(p, iD) \cong \mathrm{ten}(i{\downarrow}p, iD)$$

Proof. We prove the first one, the other being symmetrical. For any set S

$$\begin{array}{c} \mathrm{ten}(p, iA) \to S \\ \hline p \to (\neg\, iA)S \\ \hline i{\uparrow}p \to (\neg\, iA)S \\ \hline \mathrm{ten}(i{\uparrow}p, iA) \to S \end{array}$$

where we have used Proposition 2.7. The result follows by Yoneda. ∎

The following proposition can be interpreted as the fact that complementation in $\mathbf{Cat}/X$ is classical, when restricted to discrete fibrations and discrete opfibrations; in particular, it is possible to "recover" a discrete fibration (or presheaf) from its complement (see [Pisani, 2007]):

2.10. Proposition. *If iA and iB are both discrete fibrations (or both discrete opfibrations), then there is a natural bijection*

$$\hom(iA, iB) \cong \hom(\neg\, iB, \neg\, iA)$$

∎

3. The reflection formula

The values of the "slice functors"

$$X/- : X \to \mathbf{Cat}/X \qquad ; \qquad -/X : X \to \mathbf{Cat}/X \tag{31}$$

used in the following propositions are understood with their canonical projection $X/x \to X$ and $x/X \to X$.

3.1. Proposition. *The left adjoint* $\downarrow(-) \dashv i$ *is given on a* $p : P \to X$ *by*

$$\downarrow p \cong \mathrm{ten}(-/X, p)$$

or equivalently $\downarrow p \cong \Gamma_!(-/p)$.

3.2. Proposition. *The right adjoint* $i \dashv (-)^\circ$ *is given on a* $p : P \to X$ *by*

$$p^\circ \cong \mathrm{hom}(X/-, p)$$

We begin with a proof of Proposition 3.1, by means of the complement operator, which has a striking similarity with that (straightforward) of Proposition 3.2.

In [Pisani, 2007] it is checked directly that the above formulas actually give the desired adjoints. For the left adjoint, this check is here superseded by reviewing other proofs of Proposition 3.2, that rest on standard completeness properties; namely on Kan extensions of set functors and on colimits of presheaves respectively.

3.3. The reflection formula via complementation. We begin by noticing that for any discrete fibration iA

$$Ax \cong \mathrm{ten}(x, iA) \cong \mathrm{hom}(x, iA) \tag{32}$$

Indeed, in general for any $p : P \to X$, $\mathrm{ten}(x, p) = \Gamma_!(Px)$ and $\mathrm{hom}(x, p) = \Gamma_*(Px)$ are the components and the objects of the fiber category over x; since iA has discrete fibers the result follows.

3.4. Proposition. *Given a category* $p : P \to X$ *over* X*, the presheaf* $\downarrow p : X^{\mathrm{op}} \to \mathbf{Set}$ *acts on objects as follows*

$$(\downarrow p)x = \mathrm{ten}(x/X, p)$$

Proof.

$$\frac{\dfrac{\dfrac{(\downarrow p)x}{\mathrm{ten}(x, i\downarrow p)}}{\mathrm{ten}(i\uparrow x, i\downarrow p)}}{\mathrm{ten}(i\uparrow x, p)}$$

where we used twice the adjunction-like properties of Proposition 2.9. The result now follows from the fact that $i\uparrow x \cong iX(x, -) \cong x/X$ (see Remark 2.2). ∎

It is worth stressing the strong similarity between the above derivation of the reflection formula and the following derivation of the coreflection formula:

3.5. PROPOSITION. *Given a category $p : P \to X$ over X, the presheaf $p^\circ : X^{\mathrm{op}} \to \mathbf{Set}$ acts on objects as follows*

$$(p^\circ)x = \hom(X/x, p)$$

PROOF.

$$\frac{\dfrac{\dfrac{(p^\circ)x}{\hom(x, ip^\circ)}}{\hom(i\!\downarrow\! x, ip^\circ)}}{\hom(i\!\downarrow\! x, p)}$$

where only the adjunction laws have been used. ■

To complete the proofs of propositions 3.1 and 3.2, note that the action of the arrows λ in X on the fibers follows from the form of the reflection $\downarrow\!\lambda$ (see Remark 2.2).

3.6. THE REFLECTION FORMULA VIA KAN EXTENSION. We begin by noting that the usual coend formula for the left Kan extension of a presheaf $A : X^{\mathrm{op}} \to \mathbf{Set}$ along a functor $f : X \to Y$

$$(\exists_f A)y \;\cong\; A \otimes Y(y, f-) \tag{33}$$

may be rewritten as

$$\exists_f A \;\cong\; \mathrm{ten}(iA, -/f) \tag{34}$$

Indeed, by (26), $A \otimes Y(y, f-) \;\cong\; \mathrm{ten}(iA, i(Y(y, f-)))$ and $i(Y(y, f-)) \;\cong\; y/f$.

By the second of diagrams (18)

$$\exists_f \downarrow\! p \;\cong\; \downarrow\!(f_! p) \tag{35}$$

for any $p : P \to X$. In particular

$$\downarrow\! p \;\cong\; \downarrow\!(p_! 1_P) \;\cong\; \exists_p \downarrow\! 1_P \;\cong\; \exists_p \Delta 1 \tag{36}$$

where 1_P is the terminal discrete fibration $\mathrm{id}_P : P \to P$ over P. So by the (34) above we conclude

$$\downarrow\! p \;\cong\; \exists_p \Delta 1 \;\cong\; \mathrm{ten}(i\Delta 1, -/p) \;\cong\; \Gamma_!(1_P \times -/p) \;\cong\; \Gamma_!(-/p) \tag{37}$$

as desired. (Conversely, the reflection formula can be used to derive the coend formula (33) or (34), as shown in [Pisani, 2007].)

3.7. The reflection formula via colimits of presheaves. We begin by observing that for any $p : P \to X$ there is a bijection

$$\mathrm{Cone}(p, x) \;\cong\; \mathrm{Cone}(1, X(p-, x)) = \mathrm{Cone}(1, \mathbf{Set}^p \downarrow x) \tag{38}$$

natural in x, and that by Yoneda

$$A \;\cong\; \mathbf{Set}^{\mathbf{y}} \downarrow A \tag{39}$$

for any presheaf $A : X^{\mathrm{op}} \to \mathbf{Set}$ (on the right hand side, A is considered as an object of $\mathbf{Set}^{X^{\mathrm{op}}}$). Then we have the following chain of bijections, natural in A:

$$\begin{array}{c}
\mathbf{Cat}/X(p, iA) \\ \hline
\mathbf{Cat}/X(p_! 1_P, iA) \\ \hline
\mathbf{Cat}/P(1_P, p^*(iA)) \\ \hline
\mathbf{Cat}/P(i\Delta 1, i(\mathbf{Set}^p A)) \\ \hline
\mathbf{Set}^{P^{\mathrm{op}}}(\Delta 1, \mathbf{Set}^p A) \\ \hline
\mathbf{Set}^{P^{\mathrm{op}}}(\Delta 1, \mathbf{Set}^p(\mathbf{Set}^{\mathbf{y}} \downarrow A)) \\ \hline
\mathbf{Set}^{P^{\mathrm{op}}}(\Delta 1, \mathbf{Set}^{\mathbf{y} \circ p} \downarrow A) \\ \hline
\mathrm{Cone}(1, \mathbf{Set}^{\mathbf{y} \circ p} \downarrow A) \\ \hline
\mathrm{Cone}(\mathbf{y} \circ p, A) \\ \hline
\mathbf{Set}^{X^{\mathrm{op}}}(\mathrm{Colim}(\mathbf{y} \circ p), A)
\end{array}$$

Thus,

$$\downarrow p \;\cong\; \mathrm{Colim}(\mathbf{y} \circ p) \tag{40}$$

and since colimits in $\mathbf{Set}^{X^{\mathrm{op}}}$ are computed pointwise, we get

$$(\downarrow p)x \;\cong\; (\mathrm{Colim}(\mathbf{y} \circ p))x \;\cong\; \mathrm{Colim}((\mathbf{y} \circ p)x) \;\cong\; \mathrm{Colim} X(x, p-) \;\cong\; \Gamma_!(x/p) \tag{41}$$

as desired.

4. Colimits as reflections

For any fixed $p : P \to X$, we have the following bijections, natural in x:

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{\mathbf{Cat}/X(p, X/x)}{\mathbf{Cat}/X(p_!1_P, X/x)}}{\mathbf{Cat}/P(1_P, p^*(X/x))}}{\mathbf{Cat}/P(1_P, p/x)}}{\mathbf{Cat}/P(i\Delta 1, iX(p-, x))}}{\mathbf{Set}^{P^{\mathrm{op}}}(\Delta 1, X(p-, x))}}{\mathrm{Cone}(1, X(p-, x))}}{\mathrm{Cone}(p, x)}$$

(where (38) is used again). That is,

$$\mathbf{Cat}/X(p, X/-) \cong \mathrm{Cone}(p, -) \tag{42}$$

4.1. Remark. Of course, this can be seen directly: a morphism $p \to X/x$ over X takes an object $a \in P$ (over pa) to an object $\lambda_a : pa \to x$ (over pa) of X/x; and an arrow $u : a \to b$ in P (over pu) to a commutative triangle

$$\begin{tikzcd} pa \arrow[d, "pu"'] \arrow[dr, "\lambda_a"] & \\ pb \arrow[r, "\lambda_b"'] & x \end{tikzcd}$$

(Composition is preserved automatically.)

So $p : P \to X$ has a colimit iff there is a universal arrow from p to $X/-$. If this is the case, the universal arrow is a limiting cone $p \to X/\mathrm{Colim}\, p$. Letting p vary, we in fact have two different modules "cone"

$$\mathrm{Cone} : X^P \to X \qquad ; \qquad \mathrm{Cone} : \mathbf{Cat}/X \to X$$

both representable on the right, and so also two different operative definitions of the colimit *functor*: either as the partially defined left adjoint

$$\mathrm{Colim} : X^P \to X$$

to the functor $\Delta : X \to X^P$, or as the partially defined left adjoint

$$\mathrm{Colim} : \mathbf{Cat}/X \to X$$

to the functor $X/- : X \to \mathbf{Cat}/X$. Dually, we have two limit functors

$$\mathrm{Lim} : X^P \to X \qquad ; \qquad \mathrm{Lim} : (\mathbf{Cat}/X)^{\mathrm{op}} \to X$$

respectively right adjoint to $\Delta : X \to X^P$ and to $-/X : X \to (\mathbf{Cat}/X)^{\mathrm{op}}$.

4.2. Remark. The two colimit functors (which agree on each single $p : P \to X$ as an object of two different categories) can be merged in a single

$$\mathrm{Colim} : \mathbf{Cat}/X^* \to X$$

(see [Mac Lane, 1971], [Paré, 1973] and [Kan, 1958]) where $\mathbf{Cat}/X^*$ is the Grothendieck category associated to the 2-representable functor $\mathbf{Cat}(-,X) : \mathbf{Cat} \to \mathbf{Cat}$ (which includes both $\mathbf{Cat}/X$ and all the X^P as subcategories). Indeed, we have the further module "cone" (which includes the other ones)

$$\mathrm{Cone}(p,x) := \mathbf{Cat}/X^*(p, \delta x)$$

where $\delta : X \to \mathbf{Cat}/X^*$ is the full and faithful functor which sends an object $x \in X$ to the corresponding category over X.

The isomorphisms

$$\mathrm{Cone}(p,-) \cong \mathbf{Cat}/X(p, X/-) \cong \mathbf{Cat}/X(i\!\downarrow\! p, X/-) \cong \mathrm{Cone}(i\!\downarrow\! p, -)$$

show that the colimits of a functor are determined by its reflection: $\mathrm{Colim}\, p$ exists iff $\mathrm{Colim}(i\!\downarrow\! p)$ exists, and if this is the case they are isomorphic. Thus, if p and q in $\mathbf{Cat}/X$ have isomorphic reflections then

$$\mathrm{Colim}\, p \cong \mathrm{Colim}\, q \tag{43}$$

either existing if the other one does. More than that, the same remains true after composing with any functor $f : X \to Y$:

4.3. Proposition. *Two categories $p : P \to X$ and $q : Q \to X$ over X have isomorphic reflections $i\!\downarrow\! p \cong i\!\downarrow\! q$ if and only if for any $f : X \to Y$*

$$\mathrm{Colim}(f \circ p) \cong \mathrm{Colim}(f \circ q)$$

either side existing if the other one does.

Proof. One direction is a direct consequence of equation (40). The other follows from the isomorphisms

$$\begin{aligned}\mathrm{Cone}(f \circ p, -) &\cong \mathbf{Cat}/Y(f_! p, Y/-) \cong \mathbf{Cat}/X(p, f^*(Y/-)) \cong \mathbf{Cat}/X(i\!\downarrow\! p, f/-)\\ &\cong \mathbf{Cat}/Y(f_!(i\!\downarrow\! p), Y/-) \cong \mathrm{Cone}(f \circ (i\!\downarrow\! p), -)\end{aligned}$$

Alternatively, one could use (35) and (43). ■

Following [Paré, 1973], by applying Proposition 4.3 to the special cases $q = \mathrm{id}_X$ and $q = x : 1 \to X$ we obtain:

4.4. Corollary.

1. *The functor $p : P \to X$ is final iff $i\!\downarrow\! p \cong \mathrm{id}_X$ (that is, $\downarrow\! p \cong \Delta 1$).*
2. *The object $x \in X$ is the absolute colimit of $p : P \to X$ iff $i\!\downarrow\! p \cong X/x$ (that is, $\downarrow\! p \cong X(-,x)$).*

■

So a functor has an absolute colimit iff the "intermediate step" $i{\downarrow}(-)$ of the reflection of a category over X in principal discrete fibrations, is in fact already the final step.

4.5. Remark. Since $\mathrm{Colim} \dashv X/- \;\cong\; i \circ \mathbf{y}$ and i is full and faithful

$$\begin{tikzcd} \mathbf{Set}^{X^{\mathrm{op}}} \arrow[d, "i"'] & \\ \mathbf{Cat}/X \arrow[r, "\mathrm{Colim}"] & X \arrow[ul, "\mathbf{y}"'] \end{tikzcd} \tag{44}$$

also $\mathrm{Colim} \circ i \dashv \mathbf{y}$, so that the left adjoint to the Yoneda embedding has the well-known form: it takes a presheaf A to the colimit $\mathrm{Colim} iA$ of its category of elements over X.

By composing the adjunctions $f_! \dashv f^*$ and $\mathrm{Colim} \dashv Y/-$

$$\begin{tikzcd} \mathbf{Cat}/X \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=35, "\mathrm{Colim}(f\circ -)"] & \mathbf{Cat}/Y \arrow[l, shift left, "f^*"] \arrow[r, shift left, "\mathrm{Colim}"] \arrow[r, phantom, "\perp"] & Y \arrow[l, shift left, "Y/-"] \arrow[ll, bend left=35, "f/-"] \end{tikzcd} \tag{45}$$

we get

$$\mathrm{Colim}(f \circ -) \dashv f/- : Y \to \mathbf{Cat}/X \tag{46}$$

And since $f/- \;\cong\; i \circ \mathbf{Set}^f \circ \mathbf{y}$

$$\begin{tikzcd} \mathbf{Set}^{X^{\mathrm{op}}} \arrow[d, "i"'] & \\ \mathbf{Cat}/X \arrow[r, "\mathrm{Colim}(f\circ -)"] & Y \arrow[ul, "\mathbf{Set}^f\circ\mathbf{y}"'] \end{tikzcd} \tag{47}$$

as above we get

$$\mathrm{Colim}(f \circ i-) \dashv \mathbf{Set}^f \circ \mathbf{y} \tag{48}$$

where the left adjoint takes a presheaf A to the weighted colimit $A * f \;\cong\; \mathrm{Colim}(f \circ iA)$, displaying it as a colimit.

In the particular case $Y = \mathbf{Set}$, the adjunction (46) is the (dual of) $\mathrm{ten}(A, -) \dashv \neg A$ of section 2.4, while the adjunction (48) becomes the (dual of) $A \otimes - \dashv \mathbf{Set}(A-, \_\_)$.

5. Atoms and their reflections

We say that a category $t : T \to X$ over X is a "left atom" if there are bijections

$$\mathrm{ten}(t, iA) \;\cong\; \mathrm{hom}(t, iA)$$

natural in $A \in \mathbf{Set}^{X^{\mathrm{op}}}$. Dually, t is a "right atom" if

$$\mathrm{ten}(t, iD) \cong \mathrm{hom}(t, iD)$$

natural in $D \in \mathbf{Set}^X$. We say that t is an atom if it is both a left and a right atom. We shall see that the distinction between left and right atoms is illusory. Observe that any object $x : 1 \to X$ of the base category X is an atom (see (32)).

5.1. Proposition. *For any functor $f : X \to Y$ there are bijections*

$$\mathrm{ten}_X(p, f^*q) \cong \mathrm{ten}_Y(f_!p, q)$$

natural in $p \in \mathbf{Cat}/X$ and $q \in \mathbf{Cat}/Y$.

Proof. Using the Frobenius law (16), we have:

$$\mathrm{ten}_X(p, f^*q) = \Gamma_!(p \times f^*q) \cong \Gamma_!(f_!(p \times f^*q)) \cong \Gamma_!(f_!p \times q) = \mathrm{ten}_Y(f_!p, q) \tag{49}$$

■

5.2. Corollary. *Let $f : X \to Y$ any functor. Then for any atom $t : T \to X$ in $\mathbf{Cat}/X$, $f_!t : T \to Y$ is an atom in $\mathbf{Cat}/Y$.*

Proof. We have the following chain of natural bijections:

$$\mathrm{ten}_Y(f_!x, D) \cong \mathrm{ten}_X(x, f^*D) \cong \mathrm{hom}_X(x, f^*D) \cong \mathrm{hom}_Y(f_!x, D)$$

■

We now show that idempotent arrows of the base category are atoms. This is essentially the fact that in the graph corresponding to an idempotent endomapping, the fixed points (or loops) correspond to the components. Let $\mathbf{e}$ be the monoid which represents the idempotent arrows of categories, that is the one whose unique non-identity arrow is idempotent.

5.3. Proposition. *Any idempotent $e : \mathbf{e} \to X$ in X is an atom. For any discrete fibration or discrete opfibration D on X, $\mathrm{ten}(e, iD) \cong \mathrm{hom}(e, iD)$ is the set $\mathrm{fix}De$ of the elements of Dx fixed by $De : Dx \to Dx$.*

Proof. For any discrete fibration or discrete opfibration iD,

$$\mathrm{ten}(e, iD) \cong \Gamma_!(e \times iD) \cong \Gamma_!(e^*(iD)) \cong \Gamma_!\, i(\mathbf{Set}^e D) \cong \mathrm{Colim}(\mathbf{Set}^e D)$$

$$\mathrm{hom}(e, iD) \cong \mathrm{hom}(1_{\mathbf{e}}, e^*(iD)) \cong \Gamma_*(e^*(iD)) \cong \Gamma_*\, i(\mathbf{Set}^e D) \cong \mathrm{Lim}(\mathbf{Set}^e D)$$

But for any functor $e : \mathbf{e} \to \mathcal{C}$, the limit and the colimit of e, when they exist, are canonically isomorphic (see [Borceux, 1994]). If $\mathcal{C} = \mathbf{Set}$ these are given by the fixed points of the idempotent mapping. ■

Note that in this context it is important to distinguish between idempotent arrows as defined above, and arrows $e : 2 \to X$ which happen to be idempotent endomorphisms; an arrow $\lambda : 2 \to X$ is an atom iff it is an isomorphism.

5.4. PROPOSITION. *For any $t : T \to X$, t is a right atom iff*

$$\downarrow t \otimes - \;\cong\; \mathrm{Nat}(\uparrow t, -) : \mathbf{Set}^X \to \mathbf{Set}$$

Dually, t is a left atom iff

$$- \otimes \uparrow t \;\cong\; \mathrm{Nat}(\downarrow t, -) : \mathbf{Set}^{X^{\mathrm{op}}} \to \mathbf{Set}$$

In particular, for any idempotent atom $e : \mathbf{e} \to X$,

$$\downarrow e \otimes D \;\cong\; \mathrm{Nat}(\uparrow e, D) \;\cong\; \mathrm{fix} De \qquad ; \qquad A \otimes \uparrow e \;\cong\; \mathrm{Nat}(\downarrow e, A) \;\cong\; \mathrm{fix} Ae$$

PROOF. Recalling Proposition 2.9, one has the following bijections, natural in $D \in \mathbf{Set}^X$:

$$\mathrm{Nat}(\uparrow t, D) \;\cong\; \mathrm{hom}(i \uparrow t, iD) \;\cong\; \mathrm{hom}(t, iD)$$

$$\downarrow t \otimes D \;\cong\; \mathrm{ten}(i \downarrow t, iD) \;\cong\; \mathrm{ten}(t, iD)$$

■

5.5. COROLLARY. *For any idempotent $e : x_0 \to x_0$ in X,*

$$\downarrow e \;\cong\; \mathrm{fix} X(-, e)$$

that is $\downarrow e$ the subfunctor of $\downarrow x_0 = X(-, x_0)$ given by the arrows fixed by composition with e:

$$\lambda : x \to x_0 \in (\downarrow e)x \quad \Longleftrightarrow \quad e \circ \lambda = \lambda$$

For any idempotents $e : x \to x$ and $e' : y \to y$ in X, $\mathrm{Nat}(\downarrow e, \downarrow e')$ is the set of arrows $\lambda : x \to y$ such that $\lambda \circ e = \lambda = e' \circ \lambda$.

PROOF. For the first part, we have

$$(\downarrow e)x \;\cong\; \downarrow e \otimes \uparrow x \;\cong\; \mathrm{Nat}(\uparrow e, \uparrow x) \;\cong\; \mathrm{fix}((\uparrow x)e) \;\cong\; \mathrm{fix} X(x, e)$$

The second part follows from the first, since $\mathrm{Nat}(\downarrow e, \downarrow e') \;\cong\; \mathrm{fix}(\downarrow e')e$. ■

By Yoneda, any idempotent in $\mathbf{Set}^{X^{\mathrm{op}}}$ on a representable $X(-, x_0)$ has the form

$$X(-, e) : X(-, x_0) \to X(-, x_0)$$

for a unique idempotent $e : x_0 \to x_0$ in X.

5.6. COROLLARY. *Let $A : X^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf on X. Then A is a retract in $\mathbf{Set}^{X^{\mathrm{op}}}$ of the representable functor $X(-, x_0)$, associated to the idempotent $X(-, e) : X(-, x_0) \to X(-, x_0)$, if and only if $A \cong \downarrow e$.*

PROOF. The retract A is the limit of $X(-, e) : \mathbf{e} \to \mathbf{Set}^{X^{\mathrm{op}}}$, and thus is given pointwise by $Ax \cong \mathrm{fix} X(x, e)$. The result then follows by Corollary 5.5. ■

5.7. Proposition. *Let* $A : X^{\rm op} \to \mathbf{Set}$ *and* $D : X \to \mathbf{Set}$ *be such that*

$$- \otimes D \;\cong\; \mathrm{Nat}(A, -) : \mathbf{Set}^{X^{\rm op}} \to \mathbf{Set}$$

Then A *is a retract in* $\mathbf{Set}^{X^{\rm op}}$ *of a representable functor.*

Proof. Let $u \in A \otimes D \;\cong\; \Gamma_!(iA \times iD)$ be a universal element of $- \otimes D$. Then u is the component of a pair $\langle a, d\rangle$ over, say, $x_0 \in X$:

$$u = [\langle a, d\rangle], \quad a \in Ax_0,\, d \in Dx_0$$

Let $\iota : A \to X(-, x_0)$ be the unique morphism in $\mathbf{Set}^{X^{\rm op}}$ such that $(\iota \otimes D)u = [\langle \mathrm{id}_{x_0}, d\rangle]$:

$$[\langle \mathrm{id}_{x_0}, d\rangle] = (\iota \otimes D)u \;\cong\; \Gamma_!(\iota \times iD)[\langle a, d\rangle] = [\langle \iota a, d\rangle]$$

and let $\rho : X(-, x_0) \to A$ be the unique morphism in $\mathbf{Set}^{X^{\rm op}}$ such that $\rho\,\mathrm{id}_{x_0} = a$. Then

$$(\rho \circ \iota) \otimes D \;\cong\; \Gamma_!((\rho \circ \iota) \times iD) : [\langle a, d\rangle] \;\mapsto\; [\langle a, d\rangle]$$

so that $\rho \circ \iota = \mathrm{id}_A$, as required. ∎

5.8. Proposition. *Let* $A : X^{\rm op} \to \mathbf{Set}$ *and* $D : X \to \mathbf{Set}$*. The following are equivalent:*

1. $A \otimes - \;\cong\; \mathrm{Nat}(D, -) : \mathbf{Set}^{X} \to \mathbf{Set}$.
2. $- \otimes D \;\cong\; \mathrm{Nat}(A, -) : \mathbf{Set}^{X^{\rm op}} \to \mathbf{Set}$.
3. $A \;\cong\; {\downarrow} e$ *and* $D \;\cong\; {\uparrow} e$*, for an idempotent atom* e*.*
4. $A \;\cong\; {\downarrow} t$ *and* $D \;\cong\; {\uparrow} t$*, for an atom* t*.*

Proof. Trivially 3 implies 4, and 4 implies 1 and 2 by Proposition 5.4. From Proposition 5.7 and Corollary 5.6 it follows that 2 implies that $A \;\cong\; {\downarrow} e$. Furthermore,

$$D \;\cong\; {\downarrow}{-} \otimes D \;\cong\; \mathrm{Nat}(A, {\downarrow}{-}) \;\cong\; \mathrm{Nat}({\downarrow} e, {\downarrow}{-}) \;\cong\; {\downarrow}{-} \otimes {\uparrow} e \;\cong\; {\uparrow} e$$

where Proposition 5.4 have been used again. Thus 2 implies 3. Since 1 and 2 are dual and 3 is autodual, also 1 implies 3. ∎

5.9. Remark. In [Kelly & Schmitt, 2005] it is shown, in an enriched context, the equivalence between 1 and 2 and the fact that A and D are adjoint modules.

By Proposition 5.4 and the equivalence between 1 and 2 in Proposition 5.8, it follows

5.10. Corollary. *Any right atom is also a left atom, and vice versa.*

∎

By Proposition 5.8, for any atom $t : T \to X$ there exists an idempotent $e : x \to x$ in X such that $\uparrow t \cong \uparrow e$ and $\downarrow t \cong \downarrow e$. This idempotent is not unique, since for example if e splits as $x \xrightarrow{r} y \xrightarrow{i} x$, then $\downarrow e \cong \downarrow y$.

5.11. Corollary. *For an atom $T \xrightarrow{t} X$ the following are equivalent:*

1. *t has a limit.*
2. *t has a colimit.*
3. *t has an absolute limit.*
4. *t has an absolute colimit.*

Proof. Indeed, all the above are equivalent for any $e : \mathbf{e} \to X$ (see [Borceux, 1994]), and so by propositions 4.3 and 4.4 they are equivalent also for t. ∎

5.12. Remark. If the conditions of the above proposition are satisfied, that is if $\downarrow t \cong \downarrow x$ for an object $x \in X$, we could say that t "converges" to its (co)limit x. In particular, any split idempotent in X converges to its retracts, and a category is Cauchy complete iff any atom converges. Furthermore, any functor $f : X \to Y$ is "continuous": if t converges to x, then $f_! t$ converges to fx (see Corollary 5.2 and Proposition 4.3).

Recall that the Cauchy completion of a category can be obtained as the full subcategory of $\mathbf{Set}^{X^{\mathrm{op}}}$ generated by the retracts of representable functors (see for example [Borceux, 1994]). But we have seen that the latter have the form $\downarrow t$ or $\downarrow e$, and so we get the first part of the following proposition. The second part is in Corollary 5.5.

5.13. Proposition. *The reflections of atoms (or of idempotent atoms) in discrete fibrations generate the Cauchy completion of X. Furthermore, given two idempotent arrows $e : x \to x$ and $e' : y \to y$ in X, $\hom(\downarrow e, \downarrow e')$ is the set of arrows $f : x \to y$ such that $f \circ e = f = e' \circ f$. So the Cauchy completion is the same thing as the "Karoubi envelope" of X* (see [La Palme, Reyes & Zolfaghari, 2004], [Lambek & Scott, 1986] and [Lawvere, 1989]).

∎

6. Graphs and evolutive sets

As a particular case of the reflection formula, consider $X = \mathcal{N}$, the monoid of natural numbers, as the base category. Then we have the adjunction

$$\uparrow(-) \dashv i : \mathbf{Set}^{\mathcal{N}} \to \mathbf{Cat}/\mathcal{N} \tag{50}$$

with the usual formula: for any $p : P \to \mathcal{N}$,

$$(\uparrow p)\star \cong \mathrm{ten}(\mathcal{N}/\star, p) \tag{51}$$

where $\star$ is the only object of $\mathcal{N}$, so that $\mathcal{N}/\star$ is (opposite of) the *poset* of natural numbers, with its projection on $\mathcal{N}$. The action is given by the right shift:

$$k : [n, a] \mapsto [n + k, a] \qquad (a \in P)$$

Among the categories over $\mathcal{N}$ there are the unique factorization lifting (UFL) functors (see [Bunge & Niefield, 2000]), which include discrete fibrations and discrete opfibration, and which can be identified with (irreflexive) graphs as follows. For any $G \in \mathbf{Gph}$ we have the UFL functor $\mathcal{F}! : \mathcal{F}G \to \mathcal{F}\mathrm{L} \cong \mathcal{N}$, where L is the terminal graph (the loop), and $\mathcal{F} : \mathbf{Gph} \to \mathbf{Cat}$ is the free category functor; slightly improperly, we denote by $\mathcal{F}$ also the resulting functor $\mathcal{F} : \mathbf{Gph} \to \mathbf{Cat}/\mathcal{N}$. Conversely, for any UFL functor $p : P \to \mathcal{N}$ we get a graph G by the pullback below (in $\mathbf{Gph}$):

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\eta^*|p|"'] & {|P|} \arrow[d, "|p|"] \\ \mathrm{L} \arrow[r, "\eta"] & {|\mathcal{N}|} \end{tikzcd} \tag{52}$$

that is, the arrows of G are those of $|P|$ which are over $1 \in \mathcal{N}$. It is easy to see that we so obtain an equivalence between $\mathbf{Gph}$ and the full subcategory of $\mathbf{Cat}/\mathcal{N}$ of UFL functors. Furthermore, since UFL functors are closed under pullbacks, the functor $\mathcal{F} : \mathbf{Gph} \to \mathbf{Cat}/\mathcal{N}$ preserves products: $\mathcal{F}(G \times H) \cong \mathcal{F}G \times \mathcal{F}H$. And since a graph and the free category on it have the same components, $\mathcal{F} : \mathbf{Gph} \to \mathbf{Cat}/\mathcal{N}$ also preserves components: $\Gamma_! G \cong \Gamma_! \mathcal{F}G$. Thus in the particular case that $p : P \to \mathcal{N}$ is an UFL functor $\mathcal{F}G \to \mathcal{N}$, the reflection formula (51) takes the following form:

$$\begin{aligned} (\uparrow\mathcal{F}G)\star \cong \mathrm{ten}(\mathcal{N}/\star, \mathcal{F}G) \cong \mathrm{ten}(\mathcal{F}\mathrm{A}, \mathcal{F}G) &\cong \Gamma_!(\mathcal{F}\mathrm{A} \times \mathcal{F}G) \\ \cong \Gamma_!(\mathcal{F}(\mathrm{A} \times G)) &\cong \Gamma_!(\mathrm{A} \times G) \end{aligned} \tag{53}$$

where $\mathrm{A} \in \mathbf{Gph}$ is the infinite anti-chain:

$$\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$$

Thus the formula (53) gives a left adjoint to $\mathbf{Set}^{\mathcal{N}} \to \mathbf{Gph}$, or equivalently the reflection $(-)^-$ of graphs in those with a bijective domain mapping: given $G \in \mathbf{Gph}$, the nodes of G^- are the components of $\mathrm{A} \times G$, while the only arrow out of the node $[n, x] \in \Gamma_!(\mathrm{A} \times G)$, is

$$[n, x] \to [n + 1, x]$$

that is, the evolution of G^- is given by the shift of the anti-chain A.

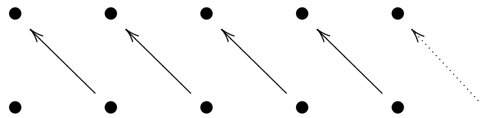
6.1. Remark. "Dually", the coreflection G° is given by the set of its chains $\mathbf{Gph}(\mathrm{A}^{\mathrm{op}}, G)$, with the evolution given by the shift of the generic chain A^{op}:

$$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots$$

6.2. Remark. Similarly, one gets the reflection formula for graphs over a base graph G in graph (op)fibrations over G (see [Boldi & Vigna, 2002]); or equivalently, the left adjoint to the inclusion $\mathbf{Set}^{\mathcal{F}G} \to \mathbf{Gph}/G$ (see [Pisani, 2007] and [Pisani, 2005]).

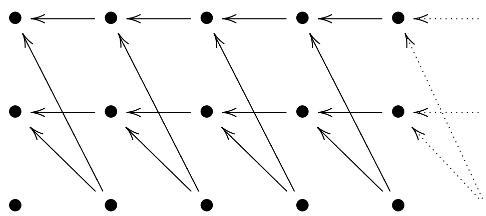
6.3. Examples.

1. Let G be the graph $\bullet \longrightarrow \bullet$ which is not an evolutive set because there are no arrows out of the second node. Applying the reflection formula, we multiply G by the anti-chain A, getting the graph



   with an infinite number of components, which are the nodes of the reflection of G. Furthermore, the action on it is given by translation, which sends any component in the one on its right. So G^- is the chain $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \dashrightarrow$, wherein the missing codomains in G have been added. On the other hand, there are no chains in G, so that the coreflection G° is void: the nodes with no codomains have been deleted.

2. Let G be the graph $\begin{tikzcd} \bullet \arrow[loop left] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[loop right] \end{tikzcd}$ which is not an evolutive set because there are two arrows out of one node. Applying the reflection formula, we multiply G by the anti-chain A, getting the two-components graph



   So the reflection of G has two nodes. Furthermore, the action on it is given again by right translation, so that the second component is a fixed point and we get

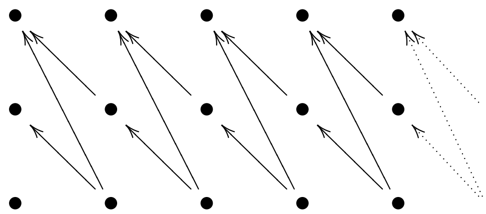
$$G^- = \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[loop right] \end{tikzcd}$$

   wherein the multiple codomains in G have been identified. On the other hand, there are four chains in G so that G° has four nodes:

$$G^\circ = \begin{tikzcd} \bullet \arrow[loop left] & \bullet \arrow[l] & \bullet \arrow[r] & \bullet \arrow[loop right] \end{tikzcd}$$

   that is, the nodes with multiple codomains have been split.

3. If $G = \begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r] & \bullet \arrow[from=1-1, bend left] \end{tikzcd}$, $\mathrm{A} \times G$ is the graph



so that $G^- = \bullet \longrightarrow \bullet\circlearrowleft$ again. One should compare this example with the technique used in [La Palme, Reyes & Zolfaghari, 2004] to compute the same reflection.

4. An example where both the phenomena of adding and identifying codomains, are present is given by the graph $\bullet \longleftarrow \bullet \longrightarrow \bullet$, whose reflection is again the chain.

5. If $G_1 = \circlearrowright\bullet \longrightarrow \bullet$, $G_2 = \circlearrowright\bullet \longrightarrow \bullet\circlearrowleft$ and $G_3 = \circlearrowright\bullet\circlearrowleft$, then G_1^-, G_2^- and G_3^- are all the loop (identification prevails in G_1^-). As for coreflections, G_1° is the loop, G_2° is the sum of the loop and the anti-chain with a loop added at its end, and G_3° is the set of sequences in a two-element set, under the action $s \mapsto s(1 + \_\_)$.

Now, let $\mathbf{e}$ the "idempotent arrow" category of Section 5, and $f : \mathcal{N} \to \mathbf{e}$ the functor which takes $1 \in \mathcal{N}$ to the non-identity arrow of $\mathbf{e}$. The composite

$$\mathbf{Set}^{\mathbf{e}} \xrightarrow{\mathbf{Set}^f} \mathbf{Set}^{\mathcal{N}} \xrightarrow{\quad i \quad} \mathbf{Cat}/\mathcal{N}$$

has a left adjoint

$$l := \exists_f \circ {\uparrow}(-) \;\cong\; {\uparrow}(-) \circ f_! : \mathbf{Cat}/\mathcal{N} \to \mathbf{Set}^{\mathbf{e}}$$

(see diagrams 18). Then (recalling Proposition 5.1)

$$(lp)\star \;\cong\; \mathrm{ten}_{\mathbf{e}}(\mathbf{e}/\star, f_! p) \;\cong\; \mathrm{ten}_{\mathcal{N}}(f^*(\mathbf{e}/\star), p) \tag{54}$$

where $\star$ is the only object of $\mathbf{e}$, so that $\mathbf{e}/\star$ is the category $\bullet \longleftarrow \bullet\circlearrowleft$ (only non-identity arrows drawn) with its projection on $\mathbf{e}$, and $f^*(\mathbf{e}/\star) \;\cong\; \mathcal{F}\mathrm{E}$ where E is the *graph* $\bullet \longleftarrow \bullet\circlearrowleft$. Thus, as above, the nodes of the reflection G^- of the graph G in idempotent evolutive sets are the components of the product $\mathrm{E} \times G$ in $\mathbf{Gph}$. The action is given again by the "shift" in E. (Of course, one has also the corresponding coreflection formula: $\mathbf{Gph}(\mathrm{E}, G)$ gives the nodes of the coreflection G°.)

Similarly, to obtain the nodes of the reflections of a graph G in bijective or n-periodic evolutive sets, we have to take the components of the products $\mathrm{Z} \times G$ and $\mathrm{Z}_n \times G$ respectively, with the graphs Z and Z_n below

$$\mathrm{Z} \;=\; \cdots\!\!\dashrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \dashrightarrow \tag{55}$$

$$\mathrm{Z}_n \;=\; \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[dl, dotted, bend left] \\ \bullet \arrow[u] & \end{tikzcd} \quad (n \text{ arrows}) \tag{56}$$

6.4. Examples.

1. It is easy to see that if G is connected (that is, $\Gamma_! G = 1$) then $\mathrm{E} \times G$ has $n+1$ components, where n is the number of nodes that are not codomains of any arrow. So the reflection in idempotent evolutive sets acts on each component by maintaining such nodes, and collapsing the rest of it to the fixed point.

2. The reflection G^- of a graph G in n-periodic evolutive sets is obtained by taking the components of $\mathrm{Z}_n \times G$. Since, as it is easily checked,

$$\mathrm{Z}_k \times \mathrm{Z}_n = \gcd(k,n) \cdot \mathrm{Z}_{\mathrm{lcm}(k,n)}$$

we deduce that Z_k^- has $\gcd(k,n)$ nodes and so, being connected,

$$\mathrm{Z}_k^- = \mathrm{Z}_{\gcd(k,n)}$$

Since any bijective evolutive set is a sum of cycles, $G = \sum_{k=1}^{\infty} S_k \cdot \mathrm{Z}_k$, its n-periodic reflection is given by

$$G^- = \Big(\sum_{k=1}^{\infty} S_k \cdot \mathrm{Z}_k \Big)^- = \sum_{k=1}^{\infty} S_k \cdot \mathrm{Z}_k^- = \sum_{k=1}^{\infty} S_k \cdot \mathrm{Z}_{\gcd(k,n)}$$

6.5. Remark. Considering **Gph** as a presheaf category (rather than a subcategory of **Cat**/$\mathcal{N}$), all the above reflections become instances of Kan extensions. But the present approach gives a more direct and intuitive way of computing them.

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