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A COHOMOLOGICAL DESCRIPTION OF CONNECTIONS AND CURVATURE OVER POSETS

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ABSTRACT. What remains of a geometrical notion like that of a principal bundle when the base space is not a manifold but a coarse graining of it, like the poset formed by a base for the topology ordered under inclusion? Motivated by the search for a geometrical framework for developing gauge theories in algebraic quantum field theory, we give, in the present paper, a first answer to this question. The notions of transition function, connection form and curvature form find a nice description in terms of cohomology, in general non-Abelian, of a poset with values in a group G. Interpreting a 1–cocycle as a principal bundle, a connection turns out to be a 1–cochain associated in a suitable way with this 1–cocycle; the curvature of a connection turns out to be its 2–coboundary. We show the existence of nonflat connections, and relate flat connections to homomorphisms of the fundamental group of the poset into G. We discuss holonomy and prove an analogue of the Ambrose-Singer theorem.

1. Introduction

One of the outstanding problems of quantum field theory is to characterize gauge theories in terms of their structural properties. Naturally, as gauge theories have been successful in describing elementary particle physics, there is a notion of a gauge theory in the framework of renormalized perturbation theory. Again, looking at theories on the lattice, there is a well defined notion of a lattice gauge theory.

This paper is a first step towards a formalism which adapts the basic notions of gauge theories to the exigencies of algebraic quantum field theory. If successful, this should allow one to uncover structural features of gauge theories. Some earlier ideas in this direction may be found in [17].

In mathematics, a gauge theory may be understood as a principal bundle over a manifold together with its associated vector bundles. For applications to physics, the manifold in question is spacetime but, in quantum field theory, spacetime does not enter directly as a differential manifold or even as a topological space. Instead, a suitable base for the topology of spacetime is considered as a partially ordered set (*poset*), ordered under inclusion. This feature has to be taken into account to have a variant of gauge theories within algebraic quantum field theory. To do this we adopt a cohomological approach. After all, a principal fibre bundle can be described in terms of its transition functions

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and these form a 1–cocycle in Čech cohomology with values in a group G. We develop here a 1–cohomology of a poset with values in G and regard this as describing principal bundles over spacetimes. A different 1–cohomology has already proved useful in algebraic quantum field theory: a cohomology of the poset with values in a net of observables describes the superselection sectors. The formalism developed here can be adapted to this case.

We begin by explaining the notions of simplex, path and homotopy in the context of posets showing that these notions behave in much the same way as their better known topological counterparts. We define the fundamental group of a path-connected poset which, in practice, coincides with the fundamental group of the underlying topological space. We then explain the 1–cohomology of a poset with values in G linking it to homotopy: the category of 1–cocycles is equivalent to the category of homomorphisms from the fundamental group to G.

Having defined principal bundles, we next introduce the appropriate notion of connection and curvature and investigate the set of connections on a principal bundle, these being thus associated with a particular 1–cohomology. We discuss holonomy and prove a version of the Ambrose-Singer Theorem.

We finally introduce the notion of gauge transformation and the action of the group of gauge transformations on the set of connections of a principal bundle. We also relate flat connections to homomorphisms from the fundamental group into G. We end by giving a brief outlook.

2. Homotopy of posets

In this section we analyze a simplicial set associated with a poset, having in mind two main aims: first, we want to introduce the first homotopy group of posets because we will discuss cohomology later; secondly, we want to introduce the notion of inflating simplices which is at the basis of the theory of connections. We will start by introducing symmetric simplicial sets and defining their first homotopy group. The first homotopy group of a poset K will be defined as that of a symmetric simplicial set $\Sigma_*(K)$ associated with K. The inflationary structure of $\Sigma_*(K)$ will be introduced and analyzed in the third subsection. Throughout this section, we shall consider a poset K and denote its order relation by $\leq$. References for this section are [14, 20, 22].

2.1. Symmetric simplicial sets. A *simplicial set* Σ_* is a graded set indexed by the non-negative integers equipped with maps $\partial_i : \Sigma_n \to \Sigma_{n-1}$ and $\sigma_i : \Sigma_n \to \Sigma_{n+1}$, with $0 \leq i \leq n$, satisfying the following relations

$$\begin{gathered}\partial_i\partial_j = \partial_j\partial_{i+1},\ i \geq j; \qquad \sigma_i\sigma_j = \sigma_{j+1}\sigma_i,\ i \leq j; \\ \partial_i\sigma_j = \sigma_{j-1}\partial_i,\ i < j; \qquad \partial_j\sigma_j = \partial_{j+1}\sigma_j = 1; \qquad \partial_i\sigma_j = \sigma_j\partial_{i-1},\ i > j+1.\end{gathered} \tag{1}$$

The elements of Σ_n are called *n–simplices*; the mappings ∂_i and σ_i *faces* and *degeneracies* respectively. We shall denote: the compositions $\partial_i\partial_j$, $\sigma_i\sigma_j$ by the symbols ∂_{ij} and σ_{ij}, respectively; 0–simplices by the letter a; 1–simplices by b; 2–simplices by c and generic n–simplices by x. A simplicial set Σ_* is said to be *symmetric* [8] if there are mappings $\tau_i : \Sigma_n(K) \to \Sigma_n(K)$ for $n \geq 1$ and $i \in \{0, \ldots n-1\}$, satisfying the relations

$$\begin{gathered}\tau_i \tau_i = 1, \quad \tau_i \tau_{i-1} \tau_i = \tau_{i-1} \tau_i \tau_{i-1}, \quad \tau_j \tau_i = \tau_i \tau_j \ \ i < j-1; \\ \partial_j \tau_i = \tau_{i-1} \partial_j \ \ i > j, \quad \partial_{i+1} = \partial_i \tau_i, \quad \partial_j \tau_i = \tau_i \partial_j \ \ i < j-1; \\ \sigma_j \tau_i = \tau_{i+1} \sigma_j \ \ i > j, \quad \sigma_i \tau_i = \tau_{i+1} \tau_i \sigma_{i+1}, \quad \tau_i \sigma_i = \sigma_i, \quad \sigma_j \tau_i = \tau_i \sigma_j \ \ i < j-1.\end{gathered} \tag{2}$$

Furthermore, these relations imply that $\partial_{i+1}\tau_i = \partial_i$ and $\tau_i \tau_{i+1} \sigma_i = \sigma_{i+1} \tau_i$. The mappings τ_i define an action τ of the permutation group $\mathbb{P}(n+1)$ on $\Sigma_n(K)$: it is enough to observe that relations (2) say that τ_i implements the transposition between the elements i and $i+1$, and that transpositions generate the permutation group. Two n–simplices x and y are said to have the *same orientation* if there exists an *even* permutation σ of $\mathbb{P}(n+1)$ such that $y = \tau_\sigma x$; they have a *reverse orientation* if there is an *odd* permutation σ of $\mathbb{P}(n+1)$ such that $y = \tau_\sigma x$. We denote by $[x]$ the equivalence class of n–simplices which have the same orientation as x, and by $\overline{[x]}$ the equivalence class of n–simplices whose orientation is the reverse of x. Notice that for any 0–simplex a we have $[a] = \{a\}$. For 1–simplices we have $[b] = \{b\}$, while $\overline{[b]} = \{\overline{b}\}$, where $\overline{b} \equiv \tau_0 b$. Note that $\partial_0 \overline{b} = \partial_1 b$, and $\partial_1 \overline{b} = \partial_0 b$. In the following we shall refer to $\overline{b}$ as the *reverse* of the 1–simplex b.

One of the aims of this section is to introduce the notion of the first homotopy group for a symmetric simplicial set associated with a poset. A purely combinatorial definition of homotopy groups for arbitrary simplicial sets has been given by Kan [10] (see also [14]). This definition is very easy to handle in the case of simplicial sets satisfying the extension condition. Recall that a simplicial set Σ_* satisfies the *extension condition* if given $0 \leq k \leq n+1$ and $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots x_{n+1} \in \Sigma_n$, with $n \geq 1$, such that

$$\partial_i x_j = \partial_{j-1} x_i, \qquad i < j \text{ and } i, j \neq k, \tag{3}$$

then there is $x \in \Sigma_{n+1}$ such that $\partial_i x = x_i$ for any $i \neq k$. Nevertheless, as we shall see in the next subsection, the simplicial set associated with a poset does not satisfy this condition in general. For an arbitrary symmetric simplicial set, however, there is an easy way to introduce the first homotopy group. The procedure introduced in [20, 22] is easier to handle than that of Kan which involves the group complex of a simplicial set. Moreover, in the case of the poset formed by a suitable base for the topology of a manifold, it leads to a first homotopy group isomorphic to that of the manifold (see next subsection).

Here we recall definitions and results from [20, 22]. Given $a_0, a_1 \in \Sigma_0$, a *path p from a_0 to a_1*, written $p : a_0 \to a_1$, is a finite ordered sequence $p = \{b_n, \ldots, b_1\}$ of 1–simplices satisfying the relations

$$\partial_1 b_1 = a_0, \quad \partial_0 b_i = \partial_1 b_{i+1} \ \text{ for } \ i \in \{1, \ldots, n-1\}, \quad \partial_0 b_n = a_1. \tag{4}$$

The *starting point* of p, written $\partial_1 p$, is the 0–simplex a_0, while the *endpoint* of p, written $\partial_0 p$, is the 0–simplex a_1. The *boundary* of p is the ordered pair $\partial p \equiv \{\partial_0 p, \partial_1 p\}$. A path

p is said to be a *loop* if $\partial_0 p = \partial_1 p$. Σ_* is said to be *pathwise connected* whenever for any pair a_0, a_1 of 0–simplices there is a path $p : a_0 \to a_1$. The set of paths is equipped with the following operations. Given two paths $p = \{b_n, \dots, b_1\}$ and $q = \{b'_k, \dots b'_1\}$, and $\partial_0 p = \partial_1 q$, their *composition* $q * p$ is defined by

$$q * p \equiv \{b'_k, \dots, b'_1, b_n, \dots, b_1\}.$$

The *reverse of* p is the path

$$\overline{p} \equiv \{\overline{b}_1, \dots, \overline{b}_n\}.$$

Note that the reverse is involutive and the composition $*$ is associative. In particular note that any path $p = \{b_n, \dots, b_1\}$ can be also seen as the composition of its 1–simplices, i.e., $p = b_n * \cdots * b_1$. An *elementary deformation* of a path p consists in replacing a 1–simplex $\partial_1 c$ of the path by a pair $\partial_0 c, \partial_2 c$, where $c \in \Sigma_2$, or, conversely in replacing a consecutive pair $\partial_0 c, \partial_2 c$ of 1–simplices of p by a single 1–simplex $\partial_1 c$. Two paths with the same endpoints are *homotopic* if they can be obtained from one another by a finite set of elementary deformations. Homotopy defines an equivalence relation $\sim$ on the set of paths with the same endpoints, which is compatible with reverse and composition, namely

$$\begin{array}{rcl} p \sim q & \iff & \overline{p} \sim \overline{q}, \\ p \sim q, \ p_1 \sim q_1 & \Rightarrow & p_1 * p \sim q_1 * q, \end{array} \tag{5}$$

(the latter equation holds whenever the r.h.s. is defined). Furthermore, for any path p, the following relations hold:

$$\begin{array}{rcl} p * \sigma_0 \partial_1 p \sim p & \text{and} & p \sim \sigma_0 \partial_0 p * p; \\ \overline{p} * p \sim \sigma_0 \partial_1 p & \text{and} & \sigma_0 \partial_0 p \sim p * \overline{p}. \end{array} \tag{6}$$

We prove the relation $\overline{p} * p \sim \sigma_0 \partial_1 p$ the others follow in a similar fashion. It is clearly enough to prove this relation for 1–simplices. So, given a 1–simplex b consider the 2–simplex $\tau_0 \sigma_1 b$. Using relations (1) and (2) we have $\partial_0 \tau_0 \sigma_1 b = \partial_1 \sigma_1 b = b$, $\partial_1 \tau_0 \sigma_1 b = \partial_0 \sigma_1 b = \sigma_0 \partial_0 b$, and $\partial_2 \tau_0 \sigma_1 b = \tau_0 \partial_2 \sigma_1 b = \tau_0 b = \overline{b}$. Hence, $b * \overline{b} \sim \sigma_0 \partial_0 b$.

Now, fix $a \in \Sigma_0$, and define

$$\pi_1(\Sigma_*, a) \equiv \{p : a \to a\}/\sim,$$

the quotient of the set of loops with endpoints a by the homotopy equivalence relation. Let $[p]$ be the equivalence class associated with the loop $p : a \to a$, and let

$$[p] \cdot [q] = [p * q], \qquad [p], [q] \in \pi_1(\Sigma_*, a).$$

As a consequence of (5), (6), $\pi_1(\Sigma_*, a)$ with this composition rule is a group: the identity is the equivalence class $[\sigma_0 a]$ associated with the degenerate 1–simplex $\sigma_0 a$; the inverse of $[p]$ is the equivalence class $[\overline{p}]$ associated with the reverse $\overline{p}$ of p. $\pi_1(\Sigma_*, a)$ is the *first homotopy group* of Σ_* based on a. If Σ_* is pathwise connected, then $\pi_1(\Sigma_*, a)$ is isomorphic to $\pi_1(\Sigma_*, a_1)$ for any $a_1 \in \Sigma_0$; this isomorphism class is the *fundamental group* of Σ_*, written $\pi_1(\Sigma_*)$. If $\pi_1(\Sigma_*)$ is trivial, then Σ_* is said to be *simply connected*.

2.2. Homotopy of a poset. Underlying cohomology is what is called the simplicial category Δ^+ that can be realized in various ways. The simplest way is to take the objects of Δ^+ to be the finite positive ordinals, $\Delta_n = \{0, 1, \dots, n\}$, with $n \geq 0$, and to take the arrows to be the monotone mappings. All these monotone mappings are compositions of two particular simple types of mapping; the injective monotone mappings from one ordinal to the succeeding ordinal denoted $d_i : (n-1) \to n$, with $i \in \{0, 1, \dots, n\}$, and defined as

$$d_i(k) \equiv \begin{cases} k & k < i \ , \\ k+1 & \text{otherwise} \ ; \end{cases}$$

and the surjective monotone mappings from one ordinal to the preceding one denoted $s_i : (n+1) \to n$, with $i \in \{0, 1, \dots, n\}$, and defined as

$$s_i(k) \equiv \begin{cases} k & k \leq i \ , \\ k-1 & \text{otherwise} \ . \end{cases}$$

The mappings d_i and s_i satisfy relations which are dual to the relations (1) satisfied by ∂_i and σ_i respectively.

We may also regard Δ_n as a partially ordered set, namely as the set of its non-void subsets ordered under inclusion. We denote this poset by $\widetilde{\Delta}_n$. Any map, in particular a monotone one, $m : \Delta_n \to \Delta_p$ induces, in an obvious way, an order-preserving map of the partially ordered sets $\widetilde{\Delta}_n$ and $\widetilde{\Delta}_p$, denoted by $\widetilde{m}$. We can then define a *singular n–simplex* of a poset K to be an order preserving map $f : \widetilde{\Delta}_n \to K$. We denote the set of singular $n-$simplices by $\Sigma_n(K)$, and call the *simplicial set* of K the set $\Sigma_*(K)$ of all singular simplices. Note that a map $m : \Delta_n \to \Delta_p$ induces a map $m^* : \Sigma_p(K) \to \Sigma_n(K)$, where $m^*(f) \equiv f \circ \widetilde{m}$ with $f \in \Sigma_p(K)$. In particular, the faces and degeneracies mapping are defined by

$$\partial_i : \Sigma_n(K) \to \Sigma_{n-1}(K), \text{ where } \partial_i \equiv d_i^*,$$
$$\sigma_i : \Sigma_n(K) \to \Sigma_{n+1}(K), \text{ where } \sigma_i \equiv s_i^*.$$

One can easily check that ∂_i and σ_i satisfy the relations (1). A 0–simplex a is just an element of the poset. Inductively, for $n \geq 1$, an $n-$simplex x is formed by $n+1$ $(n-1)-$simplices $\partial_0 x, \dots, \partial_n x$, whose faces are constrained by the relations (1), and by a 0–simplex $|x|$ called the *support* of x such that $|\partial_0 x|, \dots, |\partial_n x| \leq |x|$. The ordered set $\{\partial_0 x, \dots, \partial_n x\}$, denoted ∂x, is called the *boundary* of x. We say that an n–simplex is *degenerate* if it is of the form $\sigma_i y$ for some $(n-1)$–simplex y and for some $i \in \{0, 1, \dots, n-1\}$. In particular we have $|\sigma_i y| = |y|$.

The next step is to show that $\Sigma_*(K)$ is a symmetric simplicial set. Given a 1–simplex b, let $\tau_0 b$ be the 1–simplex defined by $|\tau_0 b| \equiv |b|$ and such that

$$\partial_0 \tau_0 b \equiv \partial_1 b, \quad \partial_1 \tau_0 b \equiv \partial_0 b. \tag{7}$$

By induction for $n \geq 2$: given an n–simplex x, for $i \in \{0, \dots n-1\}$, let $\tau_i x$ be the n–simplex defined by $|\tau_i x| \equiv |x|$ and

$$\partial_j \tau_i x \equiv \tau_{i-1} \partial_j x \ \ i > j, \quad \partial_i \tau_i x \equiv \partial_{i+1} x, \quad \partial_{i+1} \tau_i x \equiv \partial_i x, \quad \partial_j \tau_i x \equiv \tau_i \partial_j x \ \ i < j-1. \tag{8}$$

One can easily see that the mappings $\tau_i : \Sigma_n(K) \to \Sigma_n(K)$, for $n \geq 1$ and $i \in \{0, \ldots, n-1\}$, satisfy the relations (2).

We now are concerned with the homotopy of a poset K, that is, the homotopy of the symmetric simplicial set $\Sigma_*(K)$ as defined in the previous subsection. To begin with we show that $\Sigma_*(K)$ does not fulfil the extension condition in general. To this end we recall that a poset K is said to be *directed* whenever for any pair $\mathcal{O}_1, \mathcal{O}_2 \in K$, there is $\mathcal{O}_3 \in K$ such that $\mathcal{O}_1, \mathcal{O}_2 \leq \mathcal{O}_3$.

We have the following

2.3. LEMMA. *$\Sigma_*(K)$ is pathwise connected and satisfies the extension condition if, and only if, K is directed.*

PROOF. ($\Leftarrow$) Clearly $\Sigma_*(K)$ is pathwise connected. Let $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in \Sigma_n(K)$ satisfy the compatibility relations. As K is directed, we can find $\mathcal{O} \in K$ such that $\mathcal{O} \geq |x_0|, \ldots, |x_{k-1}|, |x_{k+1}|, \ldots, |x_{n+1}|$. Let x_k be the n–simplex defined by $|x_k| \equiv \mathcal{O}$ and

$$\partial_i x_k \equiv \partial_{k-1} x_i \quad i < k, \qquad \partial_k x_k \equiv \partial_k x_{k+1}, \qquad \partial_i x_k \equiv \partial_k x_{i+1}, \quad i > k.$$

Let w be the $n+1$–simplex defined by $|w| \equiv \mathcal{O}$ and $\partial_i w \equiv x_i$. ($\Rightarrow$) Consider two 0–simplices a_0, a_1. Since $\Sigma_*(K)$ is pathwise connected there is a path $p : a_0 \to a_1$ of the form $p = b_n * \cdots * b_2 * b_1$. Using the extension condition with respect to b_2, b_1, we can find a 2–simplex c such that $\partial_2 c = b_1$ and $\partial_0 c = b_2$. Hence, $|c| \geq a_0 = \partial_1 b_1, \partial_0 b_2$. Apply again the extension condition with respect to $b_3, \partial_1 c$. Then, there is a 2–simplex c_1 such that $\partial_0 c_1 = b_3$ and $\partial_2 c = \partial_1 c$. Hence, $|c_1| \geq \partial_{11} c = a_0, \partial_0 b_3$. The proof follows by iterating this procedure until b_n. ∎

The condition on a poset to be directed is too restrictive. In fact, *if K is directed, then $\Sigma_*(K)$ has a contracting homotopy. Hence, K is simply connected* [20, 22] (it could be easily seen that $\Sigma_*(K)$ is simply connected also according to Kan's definition of homotopy).

We will say that a poset K is *pathwise connected*, *simply connected* whenever $\Sigma_*(K)$ is pathwise connected and simply connected, respectively. Furthermore, we define $\pi_1(K, a) \equiv \pi_1(\Sigma_*(K), a)$ and $\pi_1(K) \equiv \pi_1(\Sigma_*(K))$ and call them, respectively, the *first homotopy group of K*, with base the 0–simplex a, and the *fundamental group of K*.

The link between the first homotopy group of a poset and the corresponding topological notion can be achieved as follows. Let M be an arcwise connected manifold and let K be a base for the topology of M whose elements are arcwise and simply connected, open subsets of M. Consider the poset formed by ordering K under *inclusion*. Then K is pathwise connected and $\pi_1(M) = \pi_1(K)$, where $\pi_1(M)$ is the fundamental group of M [22, Theorem 2.18].

2.4. INFLATIONARY STRUCTURE. We deal with the inflationary structure of the simplicial set associated with a poset. We will describe the basic properties of this structure and use them to give a definition of an inflationary structure for an abstract symmetric simplicial set. As we shall see later, theory of connections on a poset is based on this structure.

A 1−simplex b is said to be *inflating* whenever

$$\partial_1 b \leq \partial_0 b \quad \text{and} \quad |b| = \partial_0 b \tag{9}$$

By induction for $n \geq 1$: an n−simplex x is said to be *inflating* whenever all its $(n-1)$–faces $\partial_0 x, \ldots, \partial_n x$ are inflating $(n-1)$–simplices and $|x| = \partial_{123\cdots(n-1)}x$. Any 0–simplex will be regarded as inflating. We will denote the set of inflating n–simplices by $\Sigma^{\text{inf}}_n(K)$. Given a monotone mapping $m : \Delta_p \to \Delta_n$ then $m^*(x) = x \circ \widetilde{m}$ is inflating if x is. Thus $\Sigma^{\text{inf}}_*(K)$ is a simplicial subset of $\Sigma_*(K)$.

Note that the 0–subsimplices of an inflating n–simplex form a totally ordered subset of K. For instance, c is an inflating 2–simplex if, and only if,

$$\partial_{11}c = \partial_{12}c \leq \partial_{02}c = \partial_{10}c \leq \partial_{00}c = \partial_{01}c = |c|.$$

Other properties of inflating simplices are shown in the following lemma.

2.5. Lemma. *The following assertions hold.*

(i) $\Sigma^{\text{inf}}_0(K) = \Sigma_0(K)$;

(ii) Given $b, b' \in \Sigma^{\text{inf}}_1(K)$. *If* $\partial_1 b = \partial_0 b'$ *and* $\partial_0 b = \partial_1 b'$, *then* $b = b' = \sigma_0 a$ *for some* 0*–simplex* a.

(iii) Given $b, b' \in \Sigma^{\text{inf}}_1(K)$. *If* $\partial_0 b = \partial_1 b'$, *then there is* $c \in \Sigma^{\text{inf}}_2$ *with* $\partial_0 c = b'$ *and* $\partial_2 c = b$.

(iv) Given $x, x' \in \Sigma^{\text{inf}}_n(K)$. *If* $\partial_i x = \partial_i x'$ *for* $i = 0, \ldots, n$, *then* $x = x'$.

(v) For any $x \in \Sigma_n(K)$, *there are* $b_0, b_1, \ldots, b_n \in \Sigma^{\text{inf}}_1(K)$ *such that* $\partial_1 b_k = \partial_{01\cdots\widehat{k}\cdots n}x$ *for* $0 \leq k \leq n$, *and* $\partial_0 b_0 = \partial_0 b_1 = \cdots = \partial_0 b_n$.

Proof. (i) and (ii) are an easy consequence of the definition of inflating simplices. (iii) Define c by taking $\partial_2 c = b$, $\partial_0 c = b'$ and $\partial_1 c \equiv (\partial_0 b', \partial_1 b)$. (iv) is obvious. (v) Define $|b_k|$ and $\partial_0 b_k$ to be $|x|$ and $\partial_1 b_k \equiv \partial_{012\cdots\widehat{k}\cdots n}x$, for $0 \leq k \leq n$. ∎

The simplicial set $\Sigma^{\text{inf}}_*(K)$ does not, in general, satisfy the extension condition (3). In fact, suppose there is a pair of 1–simplices $b_0, b_1 \in \Sigma^{\text{inf}}_1(K)$ such that $\partial_0 b_0 = \partial_0 b_1$ but neither $\partial_1 b_0 \leq \partial_1 b_1$ nor $\partial_1 b_0 \geq \partial_1 b_1$. Then, according to the definition of inflating simplex, an inflating 2–simplex c such that $\partial_0 c = b_0$ and $\partial_1 c = b_1$ does not exist. It is easy to see that this is the case when K is a pathwise connected but not totally ordered poset.

The properties listed in Lemma 2.5 does not involve the notion of support of a simplex. This makes it possible to introduce an inflationary structure for abstract symmetric simplicial sets, too.

2.6. Definition. *A symmetric simplicial set Σ_* has an* ***inflationary structure****, if there is a simplicial subset Σ'_* of Σ_* enjoying the properties $(i)-(v)$ of the above lemma. The elements of Σ'_* will be called* ***inflating simplices****.*

Some observations on this definition are in order. Let Σ_* be a symmetric simplicial set with an inflationary structure Σ'_*. *First*, Property (ii) says that Σ'_* is in general not symmetric. *Secondly*, Property (iv) is a *uniqueness* condition: an inflating n–simplex is completely defined by its boundary. So sometimes, in the following, we will use the symbol (a, a') to denote the inflating 1–simplex b such that $\partial_0 b = a$ and $\partial_1 b = a'$. *Thirdly*, consider two inflating 1–simplices of the form (a_2, a_1) and (a_1, a_0). Property (iii) implies that there is $c \in \Sigma'_*$ such that $\partial_2 c = (a_1, a_0)$ and $\partial_0 c = (a_2, a_1)$. Since $\partial_1 c \in \Sigma'_*$, with $\partial_{11} c = a_0$, $\partial_{01} c = a_2$, then $\partial_1 c = (a_2, a_0)$. *Fourthly*, Property (v) says that the set of 0–subsimplices of an n–simplex admits an upper bound.

An important property of symmetric simplicial sets with an inflationary structure is shown in the following lemma.

2.7. Lemma. *Let Σ_* be an symmetric simplicial set with an inflationary structure Σ'_*. Then writing $a \leq a'$ whenever there is $b \in \Sigma'_1$ with $\partial_1 b = a$ and $a' \leq \partial_0 b$ defines a partial order relation on Σ_0.*

Proof. By (i) and (ii), the relation $\leq$ is reflexive and antisymmetric. Given $a_0, a_1, a_2 \in \Sigma_0$ such that $a_0 \leq a_1$ and $a_1 \leq a_2$. By (iii) there is an inflating 2–simplex c with $\partial_2 c = (a_1, a_0)$, $\partial_0 c = (a_2, a_1)$. Clearly, $\partial_1 c = (a_2, a_0)$, hence $a_0 \leq a_2$. ■

We denote the poset associated with Σ_0 by Lemma 2.7 by the symbol $K(\Sigma_0)$.

An observation is in order. Assume that a poset K is given. Then it is clear that $K(\Sigma_0(K)) = K$. Conversely, assume that a symmetric simplicial set Σ_* with an inflationary structure Σ'_* is given. Let $\Sigma_*(K(\Sigma_0))$ be the simplicial set associated with the poset $K(\Sigma_0)$. Then, whilst $\Sigma^{\mathrm{inf}}_*(K(\Sigma_0))$ and Σ'_* are simplicial equivalent, the same does not hold for $\Sigma_*(K(\Sigma_0))$ and Σ_*, because of the supports of the n–simplices of $\Sigma_*(K(\Sigma_0))$.

Examples. Lemma 2.7 is the key to giving examples of an inflationary structure on a symmetric simplicial set. The basic strategy is to define a simplicial set Σ^{inf}_* using the simplicial category Δ^+ taking the partial ordering into account and another such Σ_* using the symmetric simplicial category and forgetting the partial ordering. Here are some examples to illustrate this procedure, the first being just a variant of our basic example. Let L be a poset and define $\Sigma^{\mathrm{inf}}_n(L)$ to be the set of order-preserving mappings from $\{0, 1, 2, \ldots, n\}$ to L and $\Sigma_n(L)$ to be an arbitrary mapping from $\{0, 1, 2, \ldots, n\}$ to L. $\Sigma^{\mathrm{inf}}_*(L)$ is a simplicial subset of $\Sigma_*(L)$ and it is easily checked that it defines an inflationary structure on $\Sigma_*(L)$. Note that $\Sigma^{\mathrm{inf}}_*(L)$ is as defined previously but is now included in a larger symmetric simplicial set.

Now let G be a partially ordered group and use a formulation in terms of homogeneous simplices so that $\Sigma^{\mathrm{inf}}_*(G)$ and $\Sigma_*(G)$ can be defined as if G were a partially ordered set but we now have an action of G on the left compatible with the inclusion of $\Sigma^{\mathrm{inf}}_*(G)$ in $\Sigma_*(G)$.

Let A be a partially ordered affine space and let $\Sigma_n(A)$ be the set of affine mappings from the standard simplex Δ_n to A. Given such an affine map x write $x \in \Sigma_n^{\text{inf}}(A)$ if the vertices of x are totally ordered $a_0 \le a_1 \le \cdots \le a_n$. It is easily verified that we get an inflationary structure if A is directed.

Let $\mathcal{U} := \{U_a\}$ be a covering of a set X ordered under inclusion and define, as usual, $\Sigma_*(\mathcal{U})$ taking a n–simplex x to be an ordered set of indices, $\{a_0, a_1, \ldots, a_n\}$ such that $\cap_{i=0}^n U_{a_i}$ is non-void. We say that x is inflating if $a_0 \ge a_1 \ge \cdots \ge a_n$ and get in this way a simplicial subset $\Sigma_*^{\text{inf}}(\mathcal{U})$. All the conditions for getting an inflationary structure on $\Sigma_*(\mathcal{U})$ are fulfilled except for (v). But this, too, is satisfied if we take $\mathcal{U}$ to be a base of open sets for the topology of a topological space X.

The last example could be generalized by taking a poset P and defining $\Sigma_n(P)$ to be the ordered sets $\{a_0, a_1, \ldots, a_n\}$ of elements of P such that there is an $a \in P$ with $a \le a_0, a_1, \ldots, a_n$.

3. Cohomology of posets

The present section deals with the, in general non-Abelian, cohomology of a pathwise connected poset K with values in a group G. The first part is devoted to explaining the motivation for studying the non-Abelian cohomology of a poset and to defining an n–category. The general theory is developed in the second part: we introduce the set of n–cochains, for $n = 0, 1, 2, 3$, the coboundary operator, and the cocycle identities up to the 2^{nd}-degree. In the last part we study the 1–cohomology, in some detail, relating it to the first homotopy group of a poset. It is worth stressing that definitions and results, presented in this section, on the cohomology of K rely only on the fact that $\Sigma_*(K)$ is a symmetric simplicial set. Hence they admit an obvious generalization to abstract symmetric simplicial sets.

3.1. Preliminaries. The cohomology of the poset K with values in an Abelian group A, written additively, is the cohomology of the simplicial set $\Sigma_*(K)$ with values in A. To be precise, one can define the set $\mathrm{C}^n(K, A)$ of n–cochains of K with values in A as the set of functions $v : \Sigma_n(K) \to A$. The coboundary operator d defined by

$$\mathrm{d}v(x) = \sum_{k=0}^{n} (-1)^k \, v(\partial_k x), \qquad x \in \Sigma_n(K),$$

is a mapping $\mathrm{d} : \mathrm{C}^n(K, A) \to \mathrm{C}^{n+1}(K, A)$ satisfying the equation $\mathrm{dd}v = \iota$, where ι is the trivial cochain. This allows one to define the n–cohomology groups. For a non-Abelian group G no choice of ordering gives the identity $\mathrm{dd}v = \iota$.

One motivation for studying the cohomology of a poset K with values in a non-Abelian group comes from the algebraic approach to quantum field theory. The leading idea of this approach [9] is that all the physical content of a quantum system is encoded in the observable net, an inclusion preserving correspondence which associates to any open and bounded region of Minkowski space the algebra generated by the observables measurable within that region. The collection of these regions forms a poset when ordered under inclusion. A 1–cocycle equation arises in studying charged sectors of the observable net: the charge transporters of sharply localized charges are 1–cocycles of the poset taking values in the group of unitary operators of the observable net [15]. The attempt to include more general charges in the framework of algebraic quantum field theory, charges of electromagnetic type in particular, has led one to derive higher cocycles equations, up to the third degree [16, 17]. The difference, with respect to the Abelian case, is that a n–cocycle equation needs n composition laws. Thus in non-Abelian cohomology instead, for example, of trying to take coefficients in a non-Abelian group the n–cocycles take values in an n–category associated with the group. The cocycles equations can be understood as pasting together simplices, and, in fact, a n–cocycle can be seen as a representation in an n–category of the algebra of an oriented n–simplex [23].

Before trying to learn the notion of an n–category, it helps to recall that a category can be defined in two equivalent manners. One definition is based on the set of objects and the corresponding set of arrows. However, it is possible to define a category referring only to the set of arrows. Namely, a category is a set $\mathcal{C}$, whose elements are called arrows, having a partial and associative composition law $\diamond$, and such that any element of $\mathcal{C}$ has left and right $\diamond$-units. This amounts to saying that (i) $(f \diamond g) \diamond h$ is defined if, and only if, $f \diamond (g \diamond h)$ is defined and they are equal; (ii) $f \diamond g \diamond h$ is defined if, and only if, $f \diamond g$ and $g \diamond h$ are defined; (iii) any arrow g has a left and a right unit u and v, that is $u \diamond g = g$ and $g \diamond v = g$. In this formulation the set of objects are the set of units.

An n–category is a set $\mathcal{C}$ with an ordered set of n partial composition laws. This means that $\mathcal{C}$ is a category with respect to any such composition law $\diamond$. Moreover, if $\times$ and $\diamond$ are two such composition laws with $\diamond$ greater than $\times$, written $\times \prec \diamond$, then:

1. every $\times$-unit is a $\diamond$-unit;
2. $\times$-composition of $\diamond$-units, when defined, leads to $\diamond$-units;
3. the following relation, called the *interchange law*, holds:

   $$(f \times h) \diamond (f_1 \times h_1) = (f \diamond f_1) \times (h \diamond h_1),$$

   whenever the right hand side is defined.

An arrow f is said to be a *k–arrow*, for $k \leq n$, if it is a unit for the $k+1$ composition law. To economize on brackets, from now on we adopt the convention that if $\times \prec \diamond$, then a $\times$–composition law is to be evaluated before a $\diamond$–composition. For example, the interchange law reads

$$f \times h \diamond f_1 \times h_1 = (f \diamond f_1) \times (h \diamond h_1).$$

It is surprising that with this convention all the brackets disappear from the coboundary equations (see below).

That an n–category is the right set of coefficients for a non-Abelian cohomology can be understood by borrowing from [3] the following observation. Assume that $\times$ is Abelian, that is, $f \times g$ equals $g \times f$ whenever the compositions are defined. Assume that $\diamond$–units are $\times$–units. Let $1, 1'$ be, respectively, a left and a right $\diamond$–unit for f and g. By using the interchange law we have

$$f \diamond g = 1 \times f \diamond g \times 1' = (1 \diamond g) \times (f \diamond 1') = g \times f.$$

Hence $\diamond$ equals $\times$ and both composition laws are Abelian. Furthermore, if $\star$ is a another composition law such that $\times \prec \star \prec \diamond$, then $\times = \star = \diamond$.

3.2. Non-Abelian cohomology. Our approach to the non-Abelian cohomology of a poset will be based on n–categories. But it is worth pointing out that crossed complexes could be used instead (see [5]). In this approach cocycles turn out to be morphisms from a crossed complex associated with $\Sigma_*(K)$ to a suitable target crossed complex. This approach might be convenient for studying higher homotopy groups of K, but these are outside the scope of the present paper.

Our first aim is to introduce an n–category associated with a group G to be used as set of coefficients for the cohomology of the poset K. To this end, we draw on a general procedure [19] associating to an n–category $\mathcal{C}$, satisfying suitable conditions, an $(n+1)$–category $\mathcal{I}(\mathcal{C})$. This construction allows one to define the $(n+1)$–coboundary of a n–cochain in $\mathcal{C}$ as an $(n+1)$–cochain in $\mathcal{I}(\mathcal{C})$, at least for $n = 0, 1, 2$. For the convenience of the reader we review this construction in Appendix A.

Before starting to describe non-Abelian cohomology, we introduce some notation. The elements of a group G will be indicated by Latin letters. The composition of two elements g, h of G will be denoted by gh, and by e the identity of G. Let $Inn(G)$ be the group of inner automorphisms of G. We will use Greek letters to indicate the elements of $Inn(G)$. By $\alpha\tau$ we will denote the inner automorphism of G obtained by composing α with τ, that is $\alpha\tau(h) \equiv \alpha(\tau(h))$ for any $h \in G$. The identity of this group, the identity automorphism, will be indicated by ι. Finally given $g \in G$, the equation

$$g\,\alpha = \tau\, g$$

means $g\alpha(h) = \tau(h)g$ for any $h \in G$.

The categories nG . In degree 0, this is simply the group G considered as a set. In degree 1 it is the category $1G$ having a single object, the group G, and as arrows the elements of the group. Composition of arrows is the composition in G. So we identify this category with G. Observe that the arrows of $1G$ are invertible. By applying the procedure provided in [19] (see Appendix A) we see that $\mathcal{I}(1G)$ is a 2–category, denoted by $2G$, whose set of arrows is

$$2G \equiv \{(g, \tau) \mid g \in G, \quad \tau \in Inn(G)\}, \tag{10}$$

and whose composition laws are defined by

$$\begin{array}{rcll} (g,\tau)\times(h,\gamma) & \equiv & (g\tau(h),\tau\gamma), & \\ (g,\tau)\diamond(h,\gamma) & \equiv & (gh,\gamma), & \text{if } \sigma_h\gamma=\tau, \end{array} \tag{11}$$

where σ_h is the inner automorphism associated with h. Some observations on $2G$ are in order. Note that the composition $\times$ is always defined. Furthermore, the set of 1–arrows is the set of those elements of $2G$ of the form (e,τ). Finally, all the 2–arrows are invertible. Now, in Appendix A we show that $2G$ satisfies all the properties required to define the 3–category $\mathcal{I}(2G)$. However, it is not convenient to use this category as coefficients for the cohomology, but a category, denoted by $3G$, which turns out to be isomorphic to $\mathcal{I}(2G)$ (Lemma A.2). The 3–category $3G$ is the set

$$3G \equiv \{(g,\tau,\gamma) \mid g\in\mathcal{Z}(G),\quad \tau,\gamma\in Inn(G)\}, \tag{12}$$

where $\mathcal{Z}(G)$ is the centre of G, with the following three composition laws

$$\begin{array}{rcll} (g,\tau,\gamma)\times(g',\tau',\gamma') & \equiv & (gg',\tau\tau',\gamma\tau\gamma'\tau^{-1}), & \\ (g,\tau,\gamma)\diamond(g',\tau',\gamma') & \equiv & (gg',\tau',\gamma\gamma',), & \text{if } \tau=\gamma'\tau' \\ (g,\tau,\gamma)\cdot(g',\tau',\gamma') & \equiv & (gg',\tau,\gamma), & \text{if } \tau=\tau',\ \gamma=\gamma'. \end{array} \tag{13}$$

Note that $\cdot$ is Abelian. The set of 1-arrows $(3G)_1$ is the subset of elements of $3G$ of the form (e,γ,ι), where ι denotes the identity automorphism; 2–arrows $(3G)_2$ are the elements of $3G$ of the form (e,τ,γ). Finally, if G is Abelian, then $\times=\diamond=\cdot$ and the categories $2G$ and $3G$ are nothing but the group G.

The set of n–cochains. The next goal is to define the set of n–cochains. Concerning 0– and 1–cochains nothing changes with respect to the Abelian case, i.e., 0– and 1–*cochains* are, respectively functions $v:\Sigma_0(K)\to G$ and $u:\Sigma_1(K)\to G$. A 2–*cochain* w is a pair of mappings (w_1,w_2), where $w_i:\Sigma_i(K)\to(2G)_i$, for $i=1,2$ enjoying the relation

$$w_2(c)\diamond w_1(\partial_1 c) = w_1(\partial_0 c)\times w_1(\partial_2 c)\diamond w_2(c), \qquad c\in\Sigma_2(K). \tag{14}$$

Thus associated with a 2–simplex c there is a 2–arrows $w_2(c)$ whose left and right $\diamond$–units are computed from the boundary of c using w_1. This equation and the definition of the composition laws in $2G$ entail that a 2–cochain w is of the form

$$\begin{array}{rcll} w_1(b) & = & (e,\tau_b), & b\in\Sigma_1(K), \\ w_2(c) & = & (v(c),\tau_{\partial_1 c}), & c\in\Sigma_2(K), \end{array} \tag{15}$$

where $v:\Sigma_2(K)\to G$, $\tau:\Sigma_1(K)\to Inn(G)$ are mappings satisfying the equation[1]

$$v(c)\,\tau_{\partial_1 c} = \tau_{\partial_0 c}\,\tau_{\partial_2 c}\,v(c), \qquad c\in\Sigma_2(K). \tag{16}$$

[1] Equation (16) means that $v(c)\,\tau_{\partial_1 c}(h)=\tau_{\partial_0 c}(\tau_{\partial_2 c}(h))\,v(c)$ for any $h\in G$.

This can be easily shown. In fact, according to the definition of $2G$ a 2–cochain w is of the form $w_1(b) = (e, \tau_b)$ for $b \in \Sigma_1(K)$, and $w_2(c) = (v(c), \beta_c)$ for $c \in \Sigma_2(K)$. Now, the l.h.s. of equation (14) is defined if, and only if, $\tau_{\partial_1 c} = \beta_c$ for any 2–simplex c. This fact and equation (14) entail (16) and (15), completing the proof.

A 3–*cochain* x is 3–tuple (x_1, x_2, x_3) where $x_i : \Sigma_1(K) \to (3G)_i$, for $i = 1, 2, 3$, satisfying the following equations

$$x_2(c) \diamond x_1(\partial_1 c) = x_1(\partial_0 c) \times x_1(\partial_2 c) \diamond x_2(c), \tag{17}$$

for any 2–simplex c, and

$$x_3(d) \cdot x_1(\partial_{01} d) \times x_2(\partial_3 d) \diamond x_2(\partial_1 d) = x_2(\partial_0 d) \times x_1(\partial_{23} d) \diamond x_2(\partial_2 d) \cdot x_3(d), \tag{18}$$

for any 3–simplex d. Proceeding as above, in Appendix A we show that these equations and the composition laws of $3G$ entail that a 3–cochain x is of the form

$$\begin{array}{ll} x_1(b) = (e,\ \tau_b,\ \iota), & b \in \Sigma_1(K), \\ x_2(c) = (e,\ \tau_{\partial_1 c},\ \gamma_c), & c \in \Sigma_2(K), \\ x_3(d) = (v(d),\ \tau_{\partial_{12} d},\ \gamma_{\partial_0 d}\, \gamma_{\partial_1 d}), & d \in \Sigma_3(K), \end{array} \tag{19}$$

where $\tau : \Sigma_1(K) \to Inn(G)$, $v : \Sigma_3(K) \to Z(G)$, while $\gamma : \Sigma_2(K) \to Inn(G)$ is the mapping defined as

$$\gamma_c \equiv \tau_{\partial_0 c}\, \tau_{\partial_2 c}\, \tau^{-1}_{\partial_1 c}, \qquad c \in \Sigma_2(K). \tag{20}$$

This concludes the definition of the set of cochains. We will denote the set of n–cochains of K, for $n = 0, 1, 2, 3$, by $\mathrm{C}^n(K, G)$.

Just a comment on the definition of 1–cochains: unlike the usual cohomological theories 1–cochains are neither required to be invariant under oriented equivalence of simplices nor to act trivially on degenerate simplices. However, as we shall see later, 1–cocycles and connections fulfil these properties.

The coboundary and the cocycle identities. The next goal is to define the coboundary operator d. Given a 0–cochain v, then

$$\mathrm{d}v(b) \equiv v(\partial_0 b)\, v(\partial_1 b)^{-1}, \qquad b \in \Sigma_1(K). \tag{21}$$

Given a 1-cochain u, then

$$\begin{array}{ll} (\mathrm{d}u)_1(b) \equiv (e,\ \mathrm{ad}(u(b))), & b \in \Sigma_1(K), \\ (\mathrm{d}u)_2(c) \equiv (w_u(c),\ \mathrm{ad}(u(\partial_1 c))), & c \in \Sigma_2(K), \end{array} \tag{22}$$

where w_u is the mapping from $\Sigma_2(K)$ to G defined as

$$w_u(c) \equiv u(\partial_0 c)\, u(\partial_2 c)\, u(\partial_1 c)^{-1}, \qquad c \in \Sigma_2(K), \tag{23}$$

while $\mathrm{ad}(\cdot)$ means the adjoint action, that is, for instance, $\mathrm{ad}(u(\partial_1 c))$ is the inner automorphism associated with $u(\partial_1 c)$. Finally, given a 2-cochain w of the form (15), then

$$\begin{aligned}
&(\mathrm{d}w)_1(b) \equiv \big(e,\ \tau_b,\ \iota\big), && b \in \Sigma_1(K),\\
&(\mathrm{d}w)_2(c) \equiv \big(e,\ \tau_{\partial_1 c},\ \gamma_c\big), && c \in \Sigma_2(K),\\
&(\mathrm{d}w)_3(d) \equiv \big(x_w(d),\ \tau_{\partial_{12} d},\ \gamma_{\partial_0 d}\,\gamma_{\partial_2 d}\big), && d \in \Sigma_3(K),
\end{aligned} \tag{24}$$

where γ is the function from $\Sigma_2(K)$ to $Inn(G)$ defined by τ as in (20), and x_w is the mapping $x_w : \Sigma_3(K) \to \mathcal{Z}(G)$ defined as

$$x_w(d) \equiv v(\partial_0 d)\, v(\partial_2 d)\, \big(\tau_{\partial_{01} d}(v(\partial_3 d))\, v(\partial_1 d)\big)^{-1} \tag{25}$$

for any 3–simplex d. Now, we call the *coboundary operator* d the mapping $\mathrm{d} : \mathrm{C}^n(K,G) \to \mathrm{C}^{n+1}(K,G)$ defined for $n = 0,1,2$ by the equations (21), (22) and (24) respectively. This definition is well posed as shown by the following

3.3. Lemma. *For $n = 0,1,2$, the coboundary operator* d *is a mapping* $\mathrm{d} : \mathrm{C}^n(K,G) \to \mathrm{C}^{n+1}(K,G)$, *such that*

$$\mathrm{dd}v \in ((k+2)G)_{k+1}, \qquad v \in \mathrm{C}^k(K,G)$$

for $k = 0,1$.

Proof. The proof of the first part of the statement follows easily from the definition of d, except that the function x_w, as defined in (25), takes values in $\mathcal{Z}(G)$. Writing, for brevity, v_i for $v(\partial_i d)$ and τ_{ij} for $\tau_{\partial_{ij}}$, and using relations (1) and equation (16) we have

$$v_0\, v_2\, \tau_{12} = v_0\, \tau_{02}\, \tau_{22}\, v_2 = v_0\, \tau_{10}\, \tau_{22}\, v_2 = \tau_{00}\, \tau_{20}\, \tau_{23}\, v_0\, v_2,$$

moreover

$$\begin{aligned}
\tau_{01}(v_3)\, v_1\, \tau_{12} &= \tau_{01}(v_3)\, v_1\, \tau_{11} = \tau_{01}(v_3)\, \tau_{01}\, \tau_{21}\, v_1\\
&= \tau_{01}(v_3\, \tau_{13})\, v_1 = \tau_{01}(\tau_{03}\, \tau_{23}\, v_3)\, v_1 = \tau_{01}\, \tau_{03}\, \tau_{23}\, \tau_{01}(v_3)\, v_1,
\end{aligned}$$

Hence both $v(\partial_0 d)\, v(\partial_2 d)$ and $\tau_{\partial_{01} d}(v(\partial_3 d))\, v(\partial_1 d)$ intertwine from $\tau_{\partial_{12} d}$ to $\tau_{\partial_{01} d}\, \tau_{\partial_{03} d}\, \tau_{\partial_{23} d}$. This entails that they differ only by an element of $\mathcal{Z}(G)$, proving that x_w takes values in $\mathcal{Z}(G)$. Now, it is very easy to see that $\mathrm{dd}v \in (2G)_1$, for any 0–cochain v. So, let us prove that $\mathrm{dd}u \in (3G)_2$ for any 1–cochain u. Note that

$$\begin{aligned}
&(\mathrm{d}u)_1(b) = (e,\ \mathrm{ad}(u(b))), && b \in \Sigma_1(K)\\
&(\mathrm{d}u)_2(c) = (w_u(c),\ \mathrm{ad}(u(\partial_1 c))), && c \in \Sigma_2(K),
\end{aligned}$$

where w_u is defined by (23). Then the proof follows once we have shown that

$$w_u(\partial_0 d)\, w_u(\partial_2 d) = \mathrm{ad}(u(\partial_{01} d))\big(w_u(\partial_3 d)\big)\, w_u(\partial_1 d), \qquad (*)$$

for any 3–simplex d. In fact, by (24) this identity entails that

$$\begin{aligned}
&(\mathrm{dd}u)_1(b) = (e,\ \mathrm{ad}(u(b)),\ \iota), && b \in \Sigma_1(K)\\
&(\mathrm{dd}u)_2(c) = (e,\ \mathrm{ad}(u(\partial_1 c)),\ \mathrm{ad}(w_u(c))) && c \in \Sigma_2(K),\\
&(\mathrm{dd}u)_3(d) = (e,\ \mathrm{ad}(u(\partial_{12} d)),\ \mathrm{ad}\big(w_u(\partial_0 d) w_u(\partial_2 d)\big)), && d \in \Sigma_3(K).
\end{aligned}$$

So let us prove (∗). Given $d \in \Sigma_3(K)$ and using relations (1), we have

$$\begin{aligned}
\mathrm{ad}(u(\partial_{01}d))\big(w_u(\partial_3 d)\big)\, w_u(\partial_1 d) &= u(\partial_{01}d)\, w_u(\partial_3 d)\, u(\partial_{01}d)^{-1}\, w_u(\partial_1 d)\\
&= u_{01}\,(u_{03}\,u_{23}\,u_{13}^{-1})\,u_{01}^{-1}\,(u_{01}\,u_{21}\,u_{11}^{-1}) = u_{01}\,u_{03}\,u_{23}\,u_{11}^{-1}\\
&= u_{01}\,u_{03}\,u_{02}^{-1}\,u_{02}\,u_{23}\,u_{11}^{-1} = u_{00}\,u_{20}\,u_{10}^{-1}\,u_{02}\,u_{22}\,u_{12}^{-1}\\
&= w_u(\partial_0 d)\, w_u(\partial_2 d),
\end{aligned}$$

where we have used the notation introduced above. This completes the proof. ■

In words this lemma says that if v is a 0–cochain, then dd$v$ is a 2–unit of $2G$; if u is a 1–cochain, then dd$v$ is a 3–unit of $3G$.

We now are in a position to define an n–cocycle and an n–coboundary.

3.4. Definition. *For $n = 0, 1, 2$, an n–cochain v is said to be an **n–cocycle** whenever*

$$\mathrm{d}v \in \big((n+1)G\big)_n.$$

*It is said to be an **n–coboundary** whenever*

$$v \in \mathrm{d}\big((n-1)G\big)$$

(for $n = 0$ this means that $v(a) = e$ for any 0–simplex a). We will denote the set of n–cocycles by $\mathrm{Z}^n(K,G)$, and the set of n–coboundaries by $\mathrm{B}^n(K,G)$.

Lemma 3.3 entails that $\mathrm{B}^n(K,G) \subseteq \mathrm{Z}^n(K,G)$ for $n = 0, 1, 2$. Although it is outside the scope of this paper, we note that this relation also holds for $n = 3$. One can check this assertion by using the 3–cocycle given in [16].

It is very easy now to derive the cocycle equations. A 0–cochain v is a 0–*cocycle* if

$$v(\partial_0 b) = v(\partial_1 b), \qquad b \in \Sigma_1(K). \tag{26}$$

A 1–cochain z is a 1–*cocycle* if

$$z(\partial_0 c)\, z(\partial_2 c) = z(\partial_1 c), \qquad c \in \Sigma_2(K). \tag{27}$$

Let $w = (w_1, w_2)$ be 2–cochain of the form $w_1(b) = (e, \tau_b)$ for $b \in \Sigma_1(K)$, $w_2(c) = (v(c), \tau_{\partial_1 c})$ for $c \in \Sigma_2(K)$, where v and τ are mappings satisfying (16). Then w is a 2–*cocycle* if

$$v(\partial_0 d)\, v(\partial_2 d) = \tau_{\partial_{01} d}\big(v(\partial_3 d)\big)\, v(\partial_1 d), \qquad d \in \Sigma_3(K). \tag{28}$$

In the following we shall mainly deal with 1–cohomology. Our purpose will be to show that the notion of 1–cocycle admits an interpretation as a principal bundle over a poset and that this kind of bundle admits connections. The 2–coboundaries enter the game as the curvature of connections. Since K is pathwise connected, it turns out that any 0–cocycle v is a constant function. Thus the 0–cohomology of K yields no useful information.

3.5. 1–Cohomology. This section is concerned with 1–cohomology. In the first part we introduce the category of 1–cochains and the basic notions that will be used throughout this paper. The second part deals with 1–cocycles, where we shall derive some results confirming the interpretation of a 1–cocycle as a principal bundle over a poset. In the last part we discuss the connection between 1–cohomology and homotopy of posets.

The category of 1–cochains. Given a 1-cochain $v \in \mathrm{C}^1(K,G)$, we can and will extend v from 1-simplices to paths by defining for $p = \{b_n, \ldots, b_1\}$

$$v(p) \equiv v(b_n) \, \cdots \, v(b_2) \, v(b_1). \tag{29}$$

3.6. Definition. *Consider $v, v_1 \in \mathrm{C}^1(K,G)$. A* ***morphism*** *$\mathrm{f}$ from v_1 to v is a function $\mathrm{f} : \Sigma_0(K) \to G$ satisfying the equation*

$$\mathrm{f}_{\partial_0 p} \, v_1(p) = v(p) \, \mathrm{f}_{\partial_1 p},$$

for all paths p. We denote the set of morphisms from v_1 to v by (v_1, v).

There is an obvious composition law between morphisms given by pointwise multiplication and this makes $\mathrm{C}^1(K,G)$ into a category. The identity arrow $1_v \in (v,v)$ takes the constant value e, the identity of the group. Given a group homomorphism $\gamma : G_1 \to G$ and a morphism $\mathrm{f} \in (v_1, v)$ of 1–cochains with values in G_1 then $\gamma \circ v$, defined as

$$(\gamma \circ v)(b) \equiv \gamma(v(b)), \qquad b \in \Sigma_1(K), \tag{30}$$

is a 1–cochain with values in G, and $\gamma \circ \mathrm{f}$ defined as

$$(\gamma \circ \mathrm{f})_a \equiv \gamma(\mathrm{f}_a), \qquad a \in \Sigma_0(K), \tag{31}$$

is a morphism of $(\gamma \circ v_1, \gamma \circ v)$. One checks at once that $\gamma\circ$ is a functor from $\mathrm{C}^1(K,G_1)$ to $\mathrm{C}^1(K,G)$, and that if γ is a group isomorphism, then $\gamma\circ$ is an isomorphism of categories.

Note that $\mathrm{f} \in (v_1, v)$ implies $\mathrm{f}^{-1} \in (v, v_1)$, where f^{-1} here denotes the composition of f with the inverse of G. We say that v_1 and v are *equivalent*, written $v_1 \cong v$, whenever (v_1, v) is nonempty. Observe that a 1–cochain v is equivalent to the trivial 1–cochain $\imath$ if, and only if, it is a 1–coboundary. We will say that $v \in \mathrm{C}^1(K,G)$ is *reducible* if there exists a proper subgroup $G_1 \subset G$ and a 1–cochain $v_1 \in \mathrm{C}^1(K,G_1)$ with $\jmath \circ v_1$ equivalent to v, where $\jmath$ denotes the inclusion $G_1 \subset G$. If v is not reducible it will be said to be *irreducible.*

A 1–cochain v is said to be *path-independent* whenever given a pair of paths p, q, then

$$\partial p = \partial q \quad \Rightarrow \quad v(p) = v(q) \; . \tag{32}$$

Of course, if v is path-independent then so is any equivalent 1–cochain. It is worth observing that if γ is an injective homomorphism then v is path-independent if, and only if, $\gamma \circ v$ is path-independent.

3.7. Lemma. *Any 1–cochain is path-independent if, and only if, it is a 1–coboundary.*

PROOF. Assume that $v \in \mathrm{C}^1(K,G)$ is path-independent. Fix a 0–simplex a_0. For any 0–simplex a, choose a path p_a from a_0 to a and define $\mathrm{f}_a \equiv v(p_a)$. As v is path-independent, for any 1–simplex b we have

$$v(b)\,\mathrm{f}_{\partial_1 b} = v(b)\,v(p_{\partial_1 b}) = v(b * p_{\partial_1 b}) = v(p_{\partial_0 b}) = \mathrm{f}_{\partial_0 b}.$$

Hence v is a 1–coboundary, see (21) and Definition 3.4. The converse is obvious. ∎

1–COCYCLES AS PRINCIPAL BUNDLES. Recall that a 1–cocycle $z \in \mathrm{Z}^1(K,G)$ is a mapping $z : \Sigma_1(K) \to G$ satisfying the equation

$$z(\partial_0 c)\, z(\partial_2 c) = z(\partial_1 c), \qquad c \in \Sigma_2(K)$$

Some observations are in order. First, the trivial 1–cochain $\imath$ is a 1–cocycle (see Section 3.1). So, from now on, we will refer to $\imath$ as the *trivial* 1-*cocycle*. Secondly, if z is a 1–cocycle then so is any equivalent 1–cochain. In fact, let $v \in \mathrm{C}^1(K,G)$ and let $\mathrm{f} \in (v,z)$. Given a 2–simplex c we have

$$\begin{aligned} v(\partial_0 c)\, v(\partial_2 c) &= \mathrm{f}^{-1}_{\partial_{00} c}\, z(\partial_0 c)\, \mathrm{f}_{\partial_{10} c}\, \mathrm{f}^{-1}_{\partial_{02} c}\, z(\partial_2 c)\, \mathrm{f}_{\partial_{12} c} \\ &= \mathrm{f}^{-1}_{\partial_{00} c}\, z(\partial_0 c)\, z(\partial_2 c)\, \mathrm{f}_{\partial_{12} c} = \mathrm{f}^{-1}_{\partial_{00} c}\, z(\partial_1 c)\, \mathrm{f}_{\partial_{12} c} \\ &= \mathrm{f}^{-1}_{\partial_{01} c}\, z(\partial_1 c)\, \mathrm{f}_{\partial_{11} c} = v(\partial_1 c), \end{aligned}$$

where relations (1) have been used.

3.8. LEMMA. *Let $\gamma : G_1 \to G$ be a group homomorphism. Given $v \in \mathrm{C}^1(K,G_1)$ consider $\gamma \circ v \in \mathrm{C}^1(K,G)$. Then: if v is a 1–cocycle, then $\gamma \circ v$ is a 1–cocycle; the converse holds if γ is injective.*

PROOF. If v is a 1–cocycle, it is easy to see that $\gamma \circ v$ is a 1–cocycle too. Conversely, assume that γ is injective and that $\gamma \circ v$ is a 1–cocycle, then

$$\gamma\big(v(\partial_0 c)\ v(\partial_2 c)\big) = \gamma \circ v(\partial_0 c)\ \gamma \circ v(\partial_2 c) = \gamma \circ v(\partial_1 c) = \gamma\big(v(\partial_1 c)\big)$$

for any 2–simplex c. Since γ is injective, v is a 1–cocycle. ∎

Given a 1–cocycle $z \in \mathrm{Z}^1(K,G)$, a *cross section of* z is a function $\mathrm{s} : \Sigma_0(U) \to G$, where U is an open subset of K with respect to the Alexandroff topology [2], such that

$$z(b)\,\mathrm{s}_{\partial_1 b} = \mathrm{s}_{\partial_0 b}, \qquad b \in \Sigma_1(U)\ . \tag{33}$$

The cross section s is said to be *global* whenever $U = K$. A reason for the terminology cross section of a 1–cocycle is provided by the following

3.9. LEMMA. *A 1–cocycle is a 1–coboundary if, and only if, it admits a global cross section.*

PROOF. The proof follows straightforwardly from the definition of a global cross section and from the definition of a 1–coboundary. ∎

[2]The Alexandroff topology of a poset K is a T_0 topology in which a subset U of K is open whenever given $\mathcal{O} \in U$ and $\mathcal{O}_1 \in K$, with $\mathcal{O} \leq \mathcal{O}_1$, then $\mathcal{O}_1 \in U$.

3.10. REMARK. Given a group G, it is very easy to define 1–coboundaries of the poset K with values in G. It is enough to assign an element $\mathrm{s}_a \in G$ to any 0–simplex a and set

$$z(b) \equiv \mathrm{s}_{\partial_0 b}\, \mathrm{s}_{\partial_1 b}^{-1}, \qquad b \in \Sigma_1(K).$$

It is clear that z is a 1–cocycle. It is a 1–coboundary because the function $\mathrm{s} : \Sigma_0(K) \to G$ is a global cross section of z. As we shall see in the next section, the existence of 1–cocycles, which are not 1–coboundaries, with values in a group G is equivalent to the existence of nontrivial group homomorphisms from the first homotopy group of K into G.

We call *the category of* 1*–cocycles* with values in G, the full subcategory of $\mathrm{C}^1(K, G)$ whose set of objects is $\mathrm{Z}^1(K, G)$. We denote this category by the same symbol $\mathrm{Z}^1(K, G)$ as used to denote the corresponding set of objects. It is worth observing that, given a group homomorphism $\gamma : G_1 \to G$, by Lemma 3.8, the restriction of the functor $\gamma\circ$ to $\mathrm{Z}^1(K, G_1)$ defines a functor from $\mathrm{Z}^1(K, G_1)$ into $\mathrm{Z}^1(K, G)$.

We interpret 1–cocycles of $\mathrm{Z}^1(K, G)$ as principal bundles over the poset K, having G as a structure group. It is very easy to see which notion corresponds to that of an associated bundle in this framework. Assume that there is an action $A : G \times X \ni (g, x) \to A(g, x) \in X$ of G on a set X. Consider the group homomorphism $\alpha : G \ni g \to \alpha_g \in \mathrm{Aut}(X)$ defined by

$$\alpha_g(x) \equiv A(g, x), \qquad x \in X,$$

for any $g \in G$. Given a 1–cocycle $z \in \mathrm{Z}^1(K, G)$, we call the 1–cocycle

$$\alpha \circ z \in \mathrm{Z}^1(K, \mathrm{Aut}(X)), \tag{34}$$

associated with z, where $\alpha\circ$ is the functor, associated with the group homomorphism α, from the category $\mathrm{Z}^1(K, G)$ into $\mathrm{Z}^1(K, \mathrm{Aut}(X))$.

HOMOTOPY AND 1–COHOMOLOGY. The relation between the homotopy and the 1–cohomology of K has been established in [22]. Here we reformulate this result in the language of categories. We begin by recalling some basic properties of 1–cocycles. First, any 1–cocycle $z \in \mathrm{Z}^1(K, G)$ is *invariant under homotopy.* To be precise given a pair of paths p and q with the same endpoints, we have

$$p \sim q \quad \Rightarrow \quad z(p) = z(q). \tag{35}$$

Secondly, the following properties hold:

$$\begin{array}{llcll} (a) & z(\overline{p}) & = & z(p)^{-1}, & \text{for any path } p; \\ (b) & z(\sigma_0 a) & = & e, & \text{for any 0–simplex } a. \end{array} \tag{36}$$

Now in order to relate the homotopy of a poset to 1–cocycles, a preliminary definition is necessary.

Fix a group S. Given a group G we denote the set of group homomorphisms from S

into G by $H(S,G)$. For any pair $\sigma, \sigma_1 \in H(S,G)$ a *morphism* from σ_1 to σ is an element h of G such that

$$h\,\sigma_1(g) = \sigma(g)\,h, \qquad g \in S. \tag{37}$$

The set of morphisms from σ_1 to σ is denoted by (σ_1, σ) and there is an obvious composition rule between morphisms yielding a category again denoted by $H(S,G)$. Given a group homomorphism $\gamma : G_1 \to G$, there is a functor $\gamma\circ : H(S,G_1) \to H(S,G)$ defined as

$$\begin{array}{ll} \gamma \circ \sigma \equiv \gamma\sigma & \sigma \in H(S,G_1); \\ \gamma \circ h \equiv \gamma(h) & h \in (\sigma,\sigma_1),\ \sigma,\sigma_1 \in H(S,G_1). \end{array} \tag{38}$$

When γ is a group isomorphism, then $\gamma\circ$ is an isomorphism of categories, too. Similarly, let S_1 be a group and let $\rho : S_1 \to S$ be a group homomorphism. Then there is a functor $\circ\rho : H(S,G) \to H(S_1,G)$ defined by

$$\begin{array}{ll} \sigma \circ \rho \equiv \sigma\rho & \sigma \in H(S,G); \\ h \circ \rho \equiv h & h \in (\sigma,\sigma_1),\ \sigma,\sigma_1 \in H(S,G). \end{array} \tag{39}$$

When ρ is a group isomorphism, then $\circ\rho$ is an isomorphism of categories, too.

Now, fix a base 0–simplex a_0 and consider the category $H(\pi_1(K,a_0),G)$ associated with the first homotopy group of the poset. Then

3.11. PROPOSITION. *Given a group G and any 0–simplex a_0 the categories $\mathrm{Z}^1(K,G)$ and $H(\pi_1(K,a_0),G)$ are equivalent.*

PROOF. Let us start by defining a functor from $\mathrm{Z}^1(K,G)$ to $H(\pi_1(K,a_0),G)$. Given $z, z_1 \in \mathrm{Z}^1(K,G)$ and $\mathrm{f} \in (z_1,z)$, define

$$\begin{aligned} F(z)([p]) &\equiv z(p), \quad [p] \in \pi_1(K,a_0); \\ F(\mathrm{f}) &\equiv \mathrm{f}_{a_0}. \end{aligned}$$

$F(z)$ is well defined since 1–cocycles are homotopy invariant. Moreover, it is easy to see by (36) that $F(z)$ is a group homomorphism from $\pi_1(K,a_0)$ into G. Note that

$$\mathrm{f}_{a_0}\,F(z_1)([p]) = \mathrm{f}_{a_0}\,z_1(p) = z(p)\,\mathrm{f}_{a_0} = F(z)([p])\,\mathrm{f}_{a_0},$$

hence $F(\mathrm{f}) \in (F(z_1),F(z))$. So F is well defined and easily shown to be a covariant functor. To define a functor C in the other direction, let us choose a path p_a from a_0 to a, for any $a \in \Sigma_0(K)$. In particular we set $p_{a_0} = \sigma_0(a_0)$. Given $\sigma \in H(\pi_1(K,a_0),G)$ and $h \in (\sigma_1,\sigma)$, define

$$\begin{aligned} C(\sigma)(b) &\equiv \sigma([\overline{p}_{\partial_0 b} * b * p_{\partial_1 b}])\ , \quad b \in \Sigma_1(K); \\ C(h) &\equiv \mathrm{c}(h), \end{aligned}$$

where $\mathrm{c}(h) : \Sigma_0(K) \to G$ is the constant function taking the value h for any $a \in \Sigma_0(K)$. It is easily seen that C is a covariant functor. Concerning the equivalence, note that

$$(F \cdot C)(\sigma)([p]) = C(\sigma)(p) = \sigma([\overline{\sigma_0(a_0)} * p * \sigma_0(a_0)]) = \sigma([p]),$$

and that $(F \cdot C)(h) = F(\mathrm{c}(h)) = h$. Hence $F \cdot C = \mathrm{id}_{H(\pi_1(K,a_0),G)}$. Conversely, given a 1–simplex b we have

$$(C \cdot F)(z)(b) = F(z)([\overline{p_{\partial_0 b}} * b * p_{\partial_1 b}]) = z(p_{\partial_0 b})^{-1}\ z(b)\ z(p_{\partial_1 b}),$$

and given a 0–simplex a we have $(C \cdot F)(\mathrm{f}) = C(\mathrm{f}_{a_0}) = \mathrm{c}(\mathrm{f}_{a_0})$. Define $u(z)_a \equiv z(p_a)$ for $a \in \Sigma_0(K)$. It can be easily seen that the mapping $\mathrm{Z}^1(K,G) \ni z \to u(z)$ defines a natural isomorphism between $C \cdot F$ and $\mathrm{id}_{\mathrm{Z}^1(K,G)}$. See details in [22]. ∎

Observe that the group homomorphism corresponding to the trivial 1–cocycle $\imath$ is the trivial one, namely $\sigma([p]) = e$ for any $[p] \in \pi_1(K,a_0)$. Hence, a 1–cocycle of $\mathrm{Z}^1(K,G)$ is a 1–coboundary if, and only if, the corresponding group homomorphism $F(z)$ is equivalent to the trivial one. In particular if K is simply connected, then $\mathrm{Z}^1(K,G) = \mathrm{B}^1(K,G)$.

The existence of 1–cocycles, which are not 1–coboundaries, relies, in particular, on the following corollary

3.12. Corollary. *Let M be a nonempty, Hausdorff and arcwise connected topological space which admits a base for the topology consisting of arcwise and simply connected subsets of M. Let K denote the poset formed by such a base ordered under inclusion $\subseteq$. Then*

$$H(\pi_1(M,x_0),G) \cong H(\pi_1(K,a_0),G) \cong \mathrm{Z}^1(K,G)\ ,$$

for any $x_0 \in M$ and $a_0 \in \Sigma_0(K)$ with $x_0 \in a_0$, where $\cong$ means equivalence of categories.

Proof. $\pi_1(M,x_0)$ is isomorphic to $\pi_1(K,a_0)$ [22, Theorem 2.18] (see also Section 2). As observed at the beginning of this section, this entails that the categories $H(\pi_1(M,x_0),G)$ and $H(\pi_1(K,a_0),G)$ are isomorphic. Therefore the proof follows by Proposition 3.11. ∎

Let M be a nonsimply connected topological space and let K be a base for the topology of M as in the statement of Corollary 3.12. Then to any nontrivial group homomorphism in $H(\pi_1(M,x_0),G)$ there corresponds a 1–cocycle of $\mathrm{Z}^1(K,G)$ which is not a 1–coboundary.

4. Connections 1–cochains

This section is entirely devoted to studying the connection 1–cochains of a poset K and related notions like curvature, holonomy group and central connection 1–cochains. The inflationary structure of $\Sigma_*(K)$ enters the theory at this point. We shall show how connection 1–cochains and 1–cocycles are related, thus allowing one to interpret a 1–cocycle as a principal bundle and a connection 1-cochain as a connection on this principal bundle. We shall prove the existence of nonflat connection 1–cochains, a "poset" version of the Ambrose-Singer Theorem, and that to any flat connection 1-cochain with values in G, there corresponds a homomorphism from the fundamental group of the poset into G. All the definitions admit an obvious generalization to symmetric simplicial sets having an inflationary structure. However, the main result, the relation between connection and cocycles, holds only in a particular case, as we shall see.

4.1. Connections and curvature. We now give the definition of a connection 1–cochain of a poset with values in a group. To this end, recall the definition of the set $\Sigma_n^{\mathrm{inf}}(K)$ of inflating n–simplices (see Section 2).

4.2. Definition. *A 1–cochain u of $\mathrm{C}^1(K,G)$ is said to be a* ***connection* 1–*cochain***, *or, simply, a* ***connection***, *if it satisfies the following properties:*

(i) $u(\overline{b}) = u(b)^{-1}$ for any $b \in \Sigma_1(K)$;

(ii) $u(\partial_0 c)\, u(\partial_2 c) = u(\partial_1 c)$, for any $c \in \Sigma_2^{\mathrm{inf}}(K)$.

We denote the set of connection with values in G by $\mathrm{U}^1(K,G)$.

This definition of a connection is related to the notion of the link operator in a lattice gauge theory ([6]) and to the notion of a generalized connection in loop quantum gravity ([1, 13]). Both the link operator and the generalized connection can be seen as a mapping A which associates an element $A(e)$ of a group G to any oriented edge e of a graph α, and enjoys the following properties

$$A(\overline{e}) = A(e)^{-1}, \quad A(e_2 * e_1) = A(e_2)\, A(e_1), \tag{40}$$

where, $\overline{e}$ is the reverse of the edge e; $e_2 * e_1$ is the composition of the edges e_1, e_2 obtained by composing the end of e_1 with the beginning of e_2. Now, observe that to any poset K there corresponds an oriented graph $\alpha(K)$ whose set of vertices is $\Sigma_0(K)$, and whose set of edges is $\Sigma_1(K)$. Then, by property (i) of the above definition and property (29), any connection $u \in \mathrm{U}^1(K,G)$ defines a mapping from the edges of $\alpha(K)$ to G satisfying (40). The new feature of our definition of connection, is to require property (ii) in Definition 4.2, thus involving the poset structure. The motivation for this property will become clear in the next section: thanks to this property any connection u can be seen as a connection on the principal bundle described by a 1-cocycle (see Theorem 4.12).

Let us now observe that any 1–cocycle is a connection. Furthermore, if u is a connection then so is any equivalent 1–cochain (the proof is similar to the proof of the corresponding property for 1–cocycles, see Section 3.2).

4.3. Lemma. *Let $\gamma : G_1 \to G$ be a group homomorphism. Given $v \in \mathrm{C}^1(K,G_1)$ consider $\gamma \circ v \in \mathrm{C}^1(K,G)$. Then: if v is a connection then $\gamma \circ v$ is a connection; the converse holds if γ is injective.*

Proof. Clearly, if v is a connection so is $\gamma \circ v$. Conversely, assume that γ is injective and that $\gamma \circ v$ is a connection. If $c \in \Sigma_2^{\mathrm{inf}}(K)$, then

$$\gamma\big(v(\partial_0 c)\, v(\partial_2 c)\big) = \gamma \circ v(\partial_0 c)\, \gamma \circ v(\partial_2 c) = \gamma \circ v(\partial_1 c) = \gamma\big(v(\partial_1 c)\big),$$

hence $v(\partial_0 c)\, v(\partial_2 c) = v(\partial_1 c)$, since γ is injective. Furthermore, for any 1–simplex b we have

$$\gamma(v(\overline{b})) = \gamma \circ v(\overline{b}) = (\gamma \circ v(b))^{-1} = \gamma(v(b)^{-1}).$$

So, as γ is injective, we have $v(\overline{b}) = v(b)^{-1}$, and this entails that v is a connection. ∎

4.4. Lemma. *Given $u \in \mathrm{U}^1(K,G)$, then $u(\sigma_0 a) = e$ for any $a \in \Sigma_0(K)$.*

Proof. Note that $c \equiv \sigma_1\sigma_0 a$ is an inflating 2–simplex with $\partial_i c = \sigma_0 a$ for $i = 0,1,2$. By Definition 4.2(ii) we have $u(\sigma_0 a)\, u(\sigma_0 a) = u(\sigma_0 a)$ which is equivalent to $u(\sigma_0 a) = e$. ■

We call the full subcategory of $\mathrm{C}^1(K,G)$ whose set of objects is $\mathrm{U}^1(K,G)$ *the category of connection* 1*–cochains* with values in G. It will be denoted by the same symbol $\mathrm{U}^1(K,G)$ as used to denote the corresponding set of objects. Note that $\mathrm{Z}^1(K,G)$ is a full subcategory of $\mathrm{U}^1(K,G)$. Furthermore, if $\gamma : G_1 \to G$ is a group homomorphism, by Lemma 4.3, the restriction of the functor $\gamma\circ$ to $\mathrm{U}^1(K,G_1)$ defines a functor from $\mathrm{U}^1(K,G_1)$ into $\mathrm{U}^1(K,G)$.

As observed, any 1–cocycle is a connection. The converse does not hold, in general, and the obstruction is a 2–coboundary.

4.5. Definition. *The **curvature** of a connection $u \in \mathrm{U}^1(K,G)$ is the 2–coboundary $W_u \equiv \mathrm{d}u \in \mathrm{B}^2(K,G)$. Explicitly, by using relation (22) we have*

$$\begin{aligned} (W_u)_1(b) &= \big(e, \mathrm{ad}(u(b))\big), & b \in \Sigma_1(K), \\ (W_u)_2(c) &= \big(w_u(c), \mathrm{ad}(u(\partial_1 c))\big), & c \in \Sigma_2(K), \end{aligned}$$

where $w_u : \Sigma_2(K) \to G$ defined as

$$w_u(c) \equiv u(\partial_0 c)\ u(\partial_2 c)\ u(\partial_1 c)^{-1}, \qquad c \in \Sigma_2(K).$$

*A connection $u \in \mathrm{U}^1(K,G)$ is said to be **flat** whenever its curvature is trivial i.e. $W_u \in (2G)_1$ or, equivalently, if $w_u(c) = e$ for any 2–simplex c.*

We now draw some consequences of our definition of the curvature of a connection and point out the relations of this notion to the corresponding one in the theory of principal bundles.

First, note that a connection u is flat if, and only if, u is a 1–cocycle. Then, as an immediate consequence of Proposition 3.11, we have a poset version of a classical result of the theory of principal bundles [11, 7].

4.6. Corollary. *There is, up to equivalence, a 1-1 correspondence between flat connections of K with values in G and group homomorphisms from $\pi_1(K)$ into G.*

The existence of nonflat connections will be shown in Section 4.22 where examples will be given.

Secondly, in a principal bundle the curvature form is the covariant exterior derivative of a connection form, namely the 2–form with values in the Lie algebra of the group, obtained by taking the exterior derivative of the connection form and evaluating this on the horizontal components of pairs of vectors of the tangent space (see [11]). Although, no differential structure is present in our approach, W_u encodes this type of information. In fact, given a connection u, if we interpret $u(p)$ as the horizontal lift of a path p, then the equation

$$w_u(c)\, u(\partial_1 c) = u(\partial_0 c * \partial_2 c)\, w_u(c), \qquad c \in \Sigma_2(K), \tag{41}$$

may be understood as saying that $w_u(c)$ intertwines the horizontal lift of the path $\partial_1 c$ and that of the path $\partial_0 c * \partial_2 c$.

Thirdly, the structure equation of the curvature form (see [11]) says that the curvature equals the exterior derivative of the connection form plus the commutator of the connection form. Notice that the second component $(W_u)_2$ of the curvature can be rewritten as

$$(W_u)_2(c) = \big(w_u(c), \mathrm{ad}(w_u(c))^{-1}\big) \times \big(e, \mathrm{ad}(u(\partial_0 c)u(\partial_2 c))\big), \qquad c \in \Sigma_2(K), \tag{42}$$

where $\times$ is the composition (11) of the 2–category $2G$. This equation represents, in our formalism, the structure equation of the curvature with $\big(w_u(c), \mathrm{ad}(w_u(c)^{-1})\big)$ in place of the exterior derivative, and $\big(\iota, \mathrm{ad}(u(\partial_0 c)u(\partial_2 c))\big)$ in place of the commutator of the connection form.

Fourthly, as a consequence of Lemma 3.3 we have that W_u is a 2–cocycle. The 2–cocycle identity, $\mathrm{d}W_u \in (3G)_2$, or, equivalently,

$$w_u(\partial_0 d)\ w_u(\partial_2 d) = \mathrm{ad}(u(\partial_{01} d))\big(w_u(\partial_3 d)\big)\ w_u(\partial_1 d), \qquad d \in \Sigma_3(K), \tag{43}$$

corresponds to the Bianchi identity in our framework.

We conclude with the following result.

4.7. Lemma. *For any connection u the following assertions hold:*

(a) $w_u(\tau_0 c) = w_u(c)^{-1}$ for any 2–simplex c;

(b) $w_u(c) = e$ if c is either a degenerate or an inflating 2-simplex.

Proof. (a) follows directly from the definition of $\tau_0 c$, see Section 2. (b) If c is an inflating 2–simplex, then $w_u(c) = e$ because of Definition 4.2(ii). Given a 1–simplex b, then

$$w_u(\sigma_0(b)) = u(\partial_0\sigma_0(b))\ u(\partial_2\sigma_0(b))\ u(\partial_1\sigma_0(b))^{-1} = u(b)\ u(\sigma_0(\partial_1 b))\ u(b)^{-1} = e,$$

because $u(\sigma_0(\partial_1 b)) = e$ (Lemma 4.4). Analogously we have that $w_u(\sigma_1(b)) = e$. ∎

In words, statement (b) asserts that the curvature of a connection is trivial when restricted to inflating simplices.

4.8. Remark. It is worth pointing out some analogies between the theory of connections, as presented in this paper, and that developed in synthetic geometry by A. Kock [12], and in algebraic geometry by L. Breen and W. Messing [4]. The contact point with our approach resides in the fact that both of the other approaches make use of a combinatorial notion of differential forms taking values in a group G. So in both cases connections turn out to be combinatorial 1–forms. Concerning the curvature, the definition of W_u is formally the same as the definition of curving data given in [4], since this is the 2–coboundary of a connection, taking values in a 2–category associated with G. Whereas, in [12] the curvature is the 2–coboundary of a connection, taking values in G, and is formally the same as w_u. A significant difference from these other two approaches is that in our case w_u is not invariant under oriented equivalence of 2-simplices (examples of connections having this feature will be studied in Section 4.19).

4.9. The cocycle induced by a connection. We analyze the relation between connections and 1–cocycles more deeply. The main result is that to any connection there corresponds a unique 1–cocycle. This, on the one hand, confirms the interpretation of 1–cocycles as principal bundles. On the other hand this result will allow us to construct examples of nonflat connections in Section 4.22.

To begin with, we need a preliminary definition. We call the collection $\{V_a,\ a \in \Sigma_0(K)\}$ of open subsets of K defined by

$$V_a \equiv \{\mathcal{O} \in K \mid a \leq \mathcal{O}\}, \qquad a \in \Sigma_0(K), \tag{44}$$

the *minimal open covering* of K. It is an open covering because any element of K belongs to some V_a, and because any V_a is an open subset of K for the Alexandroff topology (see Subsection 3.5). It is minimal because if $\mathcal{U}$ is an open covering of K, then any V_a is contained in some element of $\mathcal{U}$.

We now turn to study the relation between connections and 1–cocycles.

4.10. Lemma. *For any $u \in \mathrm{U}^1(K,G)$, there exists a unique family $\{z_{a_0,a_1}\}$ of functions $z_{a_0,a_1} : V_{a_0} \cap V_{a_1} \to G$ satisfying the following two properties:*

(i) $z_{\partial_0 b,\partial_1 b}(\partial_0 b) = u(b)$ if $b \in \Sigma_1^{\inf}(K)$;

(ii) $z_{a_2,a_1}(a)\, z_{a_1,a_0}(a) = z_{a_2,a_0}(a)$ if $a \in V_{a_0} \cap V_{a_1} \cap V_{a_2}$.

Moreover, (i) and (ii) imply that

(iii) $z_{a_0,a_1}(a) = z_{a_0,a_1}(a')$, if $a \leq a'$ and $a \in V_{a_0} \cap V_{a_1}$.

Proof. Recall that (a_0, a_1) denotes the inflating 1–simplex b such that $\partial_0 b = a_0$ and $\partial_1 b = a_1$ (see Subsection 2.4). Given $u \in \mathrm{U}^1(K,G)$, define

$$z_{a_0,a_1}(a) \equiv u(a,a_0)^{-1}\, u(a,a_1), \qquad a \in V_{a_0} \cap V_{a_1}, \tag{45}$$

(i) Let $b \in \Sigma_1^{\inf}(K)$. By Lemma 4.4 we have

$$z_{\partial_0 b,\partial_1 b}(\partial_0 b) = u(\partial_0 b, \partial_0 b)^{-1}\, u(\partial_0 b, \partial_1 b) = u(\sigma_0\partial_0 b)^{-1}\, u(b) = u(b).$$

(ii) Let $a \in V_{a_0} \cap V_{a_1} \cap V_{a_2}$. Then

$$\begin{aligned} z_{a_2,a_1}(a)\, z_{a_1,a_0}(a) &= u(a,a_2)^{-1}\, u(a,a_1)\, u(a,a_1)^{-1}\, u(a,a_0) \\ &= u(a,a_2)^{-1}\, u(a,a_0) \\ &= z_{a_2,a_0}(a). \end{aligned}$$

Before proving uniqueness, we observe that by (ii), $z_{a_0,a_1}(a) = z_{a_1,a_0}(a)^{-1}$ for any $a \in V_{a_0} \cap V_{a_1}$. Now, let $\{z'_{a_0,a_1}\}$ be another family of functions satisfying the above relations.

By (i), $\{z'_{a_0,a_1}\}$ and $\{z_{a_0,a_1}\}$ must agree on inflating 1–simplices. Using this observation and (ii), we have

$$z'_{a_0,a_1}(a) = z'_{a_0,a}(a)\, z'_{a,a_1}(a) = z'_{a,a_0}(a)^{-1}\, z'_{a,a_1}(a) = z_{a,a_0}(a)^{-1}\, z_{a,a_1}(a) = z_{a_0,a_1}(a)$$

for any $a \in V_{a_0} \cap V_{a_1}$. Finally, we prove that (i) and (ii) imply (iii). Let $a \leq a'$ and $a \in V_{a_0} \cap V_{a_1}$. By (i) and (ii) we have

$$z_{a_0,a_1}(a) = z_{a_0,a}(a)\, z_{a,a_1}(a) = u(a,a_0)^{-1}\, u(a,a_1).$$

As observed after Definition 2.6, there are $c, c' \in \Sigma_2^{\mathrm{inf}}(K)$ such that $\partial_2 c = (a, a_1)$, $\partial_0 c = (a', a)$, $\partial_1 c = (a', a_1)$, and $\partial_2 c' = (a, a_0)$, $\partial_0 c' = (a', a)$, $\partial_1 c' = (a', a_0)$. Then

$$z_{a_0,a_1}(a) = \big(u(\partial_0 c')^{-1}\, u(\partial_1 c')\big)^{-1}\, u(\partial_0 c)^{-1}\, u(\partial_1 c) = u(a', a_0)\, u(a', a_1) = z_{a_0,a_1}(a'),$$

and this completes the proof. ∎

4.11. Remark. Lemma 4.10 is the contact point with Čech cohomology. In fact, one can understand the family of functions $\{z_{a_0,a_1}\}$, defined by (45), as a 1–cocycle of a poset, in the sense of the Čech cohomology, with respect to the minimal covering of the poset. In a forthcoming paper [21], we shall see that such functions are nothing but the transition functions of a principal bundle over a poset. We shall also see that such bundles can be mapped in to locally constant bundles over M when the poset K is a base for the topology of a topological space M.

We now prove the main result of this section.

4.12. Theorem. *For any* $u \in \mathrm{U}^1(K, G)$*, there exists a unique* 1*–cocycle* $z \in \mathrm{Z}^1(K, G)$ *such that*

$$u(b) = z(b), \qquad b \in \Sigma_1^{\mathrm{inf}}(K).$$

Proof. Consider the family of function $\{z_{a_0,a_1}\}$ associated with u by Lemma 4.10. Note that for any 1–simplex b, since $\partial_0 b, \partial_1 b \leq |b|$, we have that $|b| \in V_{\partial_0 b} \cap V_{\partial_1 b}$. Hence we can define

$$z(b) \equiv z_{\partial_0 b, \partial_1 b}(|b|), \qquad b \in \Sigma_1(K). \tag{46}$$

Given a 2–simplex c and using properties (ii) and (iii) of $\{z_{a_0,a_1}\}$, we have

$$\begin{aligned} z(\partial_0 c)\, z(\partial_2 c) &= z_{\partial_{00}c, \partial_{10}c}(|\partial_0 c|)\, z_{\partial_{02}c, \partial_{12}c}(|\partial_2 c|) = z_{\partial_{00}c, \partial_{10}c}(|c|)\, z_{\partial_{02}c, \partial_{12}c}(|c|) \\ &= z_{\partial_{01}c, \partial_{10}c}(|c|)\, z_{\partial_{10}c, \partial_{11}c}(|c|) = z_{\partial_{01}c, \partial_{11}c}(|c|) = z_{\partial_{01}c, \partial_{11}c}(|\partial_1 c|) \\ &= z(\partial_1 c). \end{aligned}$$

If $b \in \Sigma_1^{\mathrm{inf}}(K)$, by property (i) of $\{z_{a_0,a_1}\}$ we have $z(b) = z_{\partial_0 b, \partial_1 b}(\partial_0 b) = u(b)$. Uniqueness is obvious. ∎

On the basis of Theorem 4.12 we can introduce the following definition.

4.13. DEFINITION. *A connection* $u \in \mathrm{U}^1(K,G)$ *is said to* ***induce*** *the* 1*–cocycle* $z \in \mathrm{Z}^1(K,G)$ *whenever*

$$u(b) = z(b), \qquad b \in \Sigma_1^{\mathrm{inf}}(K) \ .$$

We denote the set of connections of $\mathrm{U}^1(K,G)$ *inducing the* 1*–cocycle* z *by* $\mathrm{U}^1(K,z)$.

The geometrical meaning of $\mathrm{U}^1(K,z)$ is the following: just as a 1–cocycle z stands for a principal bundle over K so the set of connections $\mathrm{U}^1(K,z)$ stands for the set of connections on that principal bundle. Theorem 4.12 says that the set of connections with values in G is partitioned as

$$\mathrm{U}^1(K,G) = \dot{\cup}\big\{\mathrm{U}^1(K,z) \mid z \in \mathrm{Z}^1(K,G)\big\} \tag{47}$$

where the symbol $\dot{\cup}$ means disjoint union.

4.14. LEMMA. *Given* $z_1, z \in \mathrm{Z}^1(K,G)$*, let* $u_1 \in \mathrm{U}^1(K,z_1)$ *and* $u \in \mathrm{U}^1(K,z)$*. Then* $(u_1,u) \subseteq (z_1,z)$*. In particular if* $u_1 \cong u$*, then* $z_1 \cong z$*.*

PROOF. Given a 1–simplex b, consider the inflating 1–simplices $(|b|, \partial_1 b)$ and $(|b|, \partial_0 b)$. By equations (45) and (46), we have

$$z(b) = u(|b|, \partial_0 b)^{-1}\, u(|b|, \partial_1 b).$$

The same holds for z_1 and u_1. Given $\mathrm{f} \in (u_1, u)$, we have

$$\begin{aligned} \mathrm{f}_{\partial_0 b}\, z_1(b) &= \mathrm{f}_{\partial_0 b}\, u_1(\overline{|b|, \partial_0 b})\, u_1(|b|, \partial_1 b) \\ &= u_1(\overline{|b|, \partial_0 b})\, \mathrm{f}_{|b|}\, u_1(|b|, \partial_1 b) = u(\overline{|b|, \partial_0 b})\, u(|b|, \partial_1 b)\, \mathrm{f}_{\partial_1 b} \\ &= z(b)\, \mathrm{f}_{\partial_1 b}, \end{aligned}$$

where $(\overline{|b|, \partial_0 b})$ is the reverse of $(|b|, \partial_0 b)$. Hence $\mathrm{f} \in (z_1, z)$. ∎

Now, given a 1-cocycle $z \in \mathrm{Z}^1(K,G)$, we call the category of *connections inducing* z, the full subcategory of $\mathrm{U}^1(K,G)$ whose objects belong to $\mathrm{U}^1(K,z)$. As it is customary in this paper, we denote this category by the same symbol $\mathrm{U}^1(K,z)$ as used to denote the corresponding set of objects.

4.15. LEMMA. *Let* $z \in \mathrm{Z}^1(K,G_1)$ *and let* $\gamma : G_1 \to G$ *be an injective group homomorphism. Then, the functor* $\gamma\circ : \mathrm{U}^1(K,z) \to \mathrm{U}^1(K,\gamma \circ z)$ *is injective and faithful.*

PROOF. Given $u \in \mathrm{U}^1(K,z)$, it is easy to see that $\gamma \circ u \in \mathrm{U}^1(K,\gamma \circ z)$. Clearly, as γ is injective, the functor $\gamma\circ$ is injective and faithful. ∎

We note the following simple result.

4.16. LEMMA. *If* $z_1 \cong z$*, then the categories* $\mathrm{U}^1(K, z_1)$ *and* $\mathrm{U}^1(K, z)$ *are equivalent.*

Assume that K is simply connected. In this case any 1-cocycle is a 1–coboundary (see Section 3.5). Then the category $\mathrm{U}^1(K, z)$ is equivalent to $\mathrm{U}^1(K, \imath)$ for any $z \in \mathrm{Z}^1(K, G)$.

THE CASE OF INFLATIONARY SYMMETRIC SIMPLICIAL SETS. The notions of a connection and curvature admit a straightforward generalization to a symmetric simplicial set Σ_* which is pathwise connected and has an inflationary structure Σ'_*. We now show that both Lemma 4.10 and Theorem 4.12 admit a generalization to this kind of simplicial sets. The latter, however, with an additional assumption.

Denote the set 1–cocycles of Σ_* with values in G by $\mathrm{Z}^1(\Sigma_*, G)$, and the set of connections of Σ_* with values in G by $\mathrm{U}^1(\Sigma_*, G)$. Moreover, let $V_a = \{\widehat{a} \in \Sigma_0 \mid a \leq \widehat{a}\}$, where $\leq$ is the order relation on Σ_0 induced by the inflationary structure of Σ_* (Lemma 2.7).

4.17. LEMMA. *For any* $u \in \mathrm{U}^1(\Sigma_*, G)$*, there exists a unique family* $\{z_{a_0,a_1}\}$ *of functions* $z_{a_0,a_1} : V_{a_0} \cap V_{a_1} \to G$ *satisfying the properties* (i)*–*(iii) *of Lemma 4.10.*

PROOF. Given $a \in V_{a_0} \cap V_{a_1}$, define

$$z_{a_0,a_1}(a) \equiv u(a, a_0)^{-1}\, u(a, a_1), \qquad a \in V_{a_0} \cap V_{a_1}.$$

The proof of the properties (i) and (ii), uniqueness, and property (iii) follows like the proof of Lemma 4.10. ∎

It is worth observing that in Theorem 4.12 we have used the support $|b|$ of a 1–simplex b to define the cocycle induced by a connection. Actually, $|b|$ is a preferred element of $V_{\partial_0 b} \cap V_{\partial_1 b}$, because of the property that $|b| \leq |c|$ if $b = \partial_i c$ for some $c \in \Sigma_2(K)$ and for some i. Such an element is, in general, missing in an arbitrary inflationary simplicial set Σ_*. Hence in order to define the 1–cocycle induced by a connection we shall need an additional assumption on Σ_*.

4.18. THEOREM. *Let* $u \in \mathrm{U}^1(\Sigma_*, G)$*. Assume that for any* 1*–simplices* b *the set* $V_{\partial_0 b} \cap V_{\partial_1 b}$ *is pathwise connected. Then, there exists a unique* 1*–cocycle* $z \in \mathrm{Z}^1(\Sigma_*, G)$ *such that* $z(b) = u(b)$ *for any* $b \in \Sigma'_1$*.*

PROOF. Given a 1–simplex b, observe that by property (v) of inflating simplices, the set $V_{\partial_0 b} \cap V_{\partial_1 b}$ is never empty. Define,

$$z(b) \equiv z_{\partial_0 b, \partial_1 b}(a), \qquad a \in V_{\partial_0 b} \cap V_{\partial_1 b}.$$

We show that this definition does not depend on the choice of a. Let $\widehat{a} \in V_{\partial_0 b} \cap V_{\partial_1 b}$. Assume that there is $\widehat{b} \in \Sigma_1$ such that $\partial_1 \widehat{b} = \widehat{a}$ and $\partial_0 \widehat{b} = a$. By property (v) of inflating simplices, there is a 0–simplex a' such that $a, \widehat{a} \leq a'$. By property (iii) of $\{z_{a_0,a_1}\}$ we have

$$z_{\partial_0 b, \partial_1 b}(a) = z_{\partial_0 b, \partial_1 b}(a') = z_{\partial_0 b, \partial_1 b}(\widehat{a}).$$

As $V_{\partial_0 b} \cap V_{\partial_1 b}$ is pathwise connected the proof follows. Given a 2–simplex c, property (v) of inflating n–simplices implies that there is a 0–simplex a greater than $\partial_{01}c$, $\partial_{02}c$, $\partial_{12}c$. Using properties (ii) and (iii) of $\{z_{a_0,a_1}\}$ we have

$$\begin{aligned} z(\partial_0 c)\, z(\partial_2 c) &= z_{\partial_{00}c,\partial_{10}c}(a)\, z_{\partial_{02}c,\partial_{12}c}(a) \\ &= z_{\partial_{00}c,\partial_{02}c}(a)\, z_{\partial_{02}c,\partial_{12}c}(a) = z_{\partial_{00}c,\partial_{12}c}(a) \\ &= z_{\partial_{01}c,\partial_{11}c}(a) = z(\partial_1 c). \end{aligned}$$

Hence z is a 1–cocycle. The rest of the proof follows straightforwardly. ■

4.19. Central connections. We now briefly study the family of central connections, whose main feature, as we shall show below, is that any such connection can be uniquely decomposed as the product of the induced cocycle by a suitable connection taking values in the centre of the group.

4.20. Definition. *A connection $u \in \mathrm{U}^1(K,G)$ is said to be a **central connection** whenever the component w_u of the curvature W_u takes values in the centre $\mathcal{Z}(G)$. We denote the set of central connections by $\mathrm{U}^1_{\mathcal{Z}}(K,G)$.*

Let us start to analyze the properties of central connections. Clearly 1–cocycles are central connections. However, the main property that can be directly deduced from the above definition is that the component w_u of the curvature W_u of a central connection u is invariant under oriented equivalence of 2–simplices. In fact by the definition of w_u, it is easily seen that

$$w_u(c) = w_u(c_1) = w_u(c_2)^{-1}, \qquad c_1 \in [c], c_2 \in \overline{[c]}, \tag{48}$$

for any 2–simplex c, where $[c]$ and $\overline{[c]}$ are, respectively, the classes of 2–simplices having the same and the reversed orientation of c.

4.21. Proposition. *A connection u of $\mathrm{U}^1(K,G)$ is central if, and only if, it can be uniquely decomposed as*

$$u(b) = z_u(b)\, \chi_u(b), \qquad b \in \Sigma_1(K),$$

where $z_u \in \mathrm{Z}^1(K,G)$, and $\chi_u \in \mathrm{U}^1(K,\imath)$ with values in $\mathcal{Z}(G)$.

Proof. $(\Leftarrow)$ Assume that a connection u admits a decomposition as in the statement. Since χ_u takes values in the centre, so does w_u. Furthermore, since $\chi_u \in \mathrm{U}^1(K,\imath)$ then $\chi_u(b) = e$ for any inflating 1–simplex b. This entails that z_u is nothing but the 1–cocycle induced by u. This is enough for uniqueness. $(\Rightarrow)$ Assume that u is central. For any 1–simplex b let c_b denote the 2–simplex defined as

$$\partial_1 c_b \equiv b, \quad \partial_0 c_b \equiv \overline{(|b|, \partial_0 b)}, \quad \partial_2 c_b \equiv (|b|, \partial_1 b), \quad |c_b| \equiv |b|.$$

Define $z_u(b) \equiv u(b)\, w_u(c_b)$, and observe that, as w_u takes values in $\mathcal{Z}(G)$, we have

$$z_u(b) = u(b)\, w_u(c_b) = u(\partial_1 c_b)\, w_u(c_b) = u(\partial_0 c_b)\, u(\partial_2 c_b) = u(|b|, \partial_0 b)^{-1}\, u(|b|, \partial_1 b).$$

Hence z_u is the 1–cocycle induced by u. Now, define

$$\chi_u(b) \equiv w_u(c_b), \qquad b \in \Sigma_1(K).$$

Since $\chi_u(b) = u(b)\, z_u(b)^{-1}$, one can easily deduce that $\chi_u \in \mathrm{U}^1(K, \imath)$, and this completes the proof. ∎

As a consequence of this result the set $\mathrm{U}^1_{\mathcal{Z}}(K, z)$ of central connections inducing the 1–cocycle z, has a the structure of an Abelian group. In fact, given $u, u_1 \in \mathrm{U}^1_{\mathcal{Z}}(K, z)$, define

$$u \star_z u_1(b) \equiv u(b)\ z(b)^{-1}\ u_1(b), \qquad b \in \Sigma_1(K). \tag{49}$$

By Proposition 4.21, we have $u \star_z u_1(b) = z(b)\ \chi_u(b)\ \chi_{u_1}(b)$ for any 1–simplex b. This entails that

$$u \star_z u_1 = u_1 \star_z u \ \text{ and } \ u \star_z u_1 \in \mathrm{U}^1_{\mathcal{Z}}(K, z).$$

By this relations, it is easily seen that $\mathrm{U}^1_{\mathcal{Z}}(K, z)$ with $\star_z$ is an Abelian group whose identity is z, and such that the inverse of a connection u is the connection defined as $z(b)\ \chi_u(b)^{-1}$ for any 1–simplex b.

Finally, in Section 4.1 we pointed out the analogy between equation (42) and the structure equation of the curvature of a connection in a principal bundle. This analogy is stronger for a central connection u since we have

$$\begin{aligned} (W_u)_2(c) &= \big(w_u(c), \imath\big) \times \big(e, \mathrm{ad}(u(\partial_0 c)u(\partial_2 c))\big) \\ &= \big(e, \mathrm{ad}(u(\partial_0 c)u(\partial_2 c))\big) \times \big(w_u(c), \imath\big), \end{aligned} \tag{50}$$

for any 2–simplex c. Hence, as for principal bundles, equation (42) for a central connection is symmetric with respect to the interchange of the two factors.

4.22. Existence of nonflat connections. We investigate the existence of nonflat connections. As a first step, we show that there is a very particular class of posets not admitting nonflat connections.

Recall that a poset K is said to be *totally ordered* whenever for any pair $\mathcal{O}, \mathcal{O}_1 \in K$ either $\mathcal{O} \leq \mathcal{O}_1$ or $\mathcal{O}_1 \leq \mathcal{O}$. Clearly, a totally ordered poset is directed and, consequently, pathwise connected (it is also simply connected, see Section 2).

4.23. Corollary. *If K is totally ordered, any connection is flat.*

Proof. If K is totally ordered and b is any 1–simplex either b or $\overline{b}$ is an inflating 1–simplex. Hence, by Theorem 4.12 any connection is equal to the induced 1–cocycle. ∎

Another obvious situation where nonflat connections do not exist is when the group of coefficients G is trivial, i.e. $G = e$. Two observations on these results are in order. First, Corollary 4.23 cannot be directly deduced from the definition of a connection. Secondly, as explained earlier, these two situations never arise in the applications we have in mind.

Now, our purpose is to show that, except when the poset is totally ordered or the group of coefficients is trivial, nonflat connections always exist. Let us start with the following

4.24. LEMMA. *Let $v \in \mathrm{C}^1(K, G)$ be such that $v(b) = e = v(\bar{b})$ for any inflating 1–simplex b. Then, for any 1–cocycle $z \in \mathrm{Z}^1(K, G)$ the 1–cochain $v(z)$ defined as*

$$v(z)(b) \equiv v(\bar{b})^{-1}\, z(b)\, v(b), \qquad b \in \Sigma_1(K), \tag{51}$$

is a connection inducing z.

PROOF. By the definition of v for any inflating 1–simplex b we have that

$$v(z)(b) = v(\bar{b})^{-1}\, z(b)\, v(b) = e\, z(b)\, e = z(b)\ .$$

This, in particular, entails that $v(z)$ satisfies property (ii) of the definition of connections. For any 1–simplex b we have

$$v(z)(\bar{b}) = v(\bar{\bar{b}})^{-1}\, z(\bar{b})\, v(\bar{b}) = v(b)^{-1}\, z(b)^{-1}\, v(\bar{b}) = \big(v(\bar{b})^{-1}\, z(b)\, v(b)\big)^{-1} = v(z)(b)^{-1}.$$

Hence $v(z) \in \mathrm{U}^1(K, z)$. ∎

It is very easy to prove the existence of 1–cochains satisfying the properties of the statement. For instance, given a 1–simplex b, define

$$v(b) \equiv \begin{cases} e & b \text{ or } \bar{b} \in \Sigma_1^{\mathrm{inf}}(K) \\ g(b) & \text{otherwise}\ , \end{cases} \tag{52}$$

where $g(b)$ is some element of the group G. So v is a 1–cochain satisfying the relation $v(b) = e = v(\bar{b})$ for any inflating 1–simplex b.

Now, assume that K is a pathwise connected but not totally ordered poset. Let G be a nontrival group. Let $v \in \mathrm{C}^1(K, G)$ be defined by (52), and let $z \in \mathrm{Z}^1(K, G)$. Consider the connection $v(z) \in \mathrm{U}^1(K, z)$. We want to find conditions on v implying that $v(z)$ is not flat.

As $v(z) \in \mathrm{U}^1(K, z)$, Theorem 4.12 says that if $v(z)$ is flat then $v(z) = z$. Hence, $v(z)$ is not flat if, and only if, it differs from z on a 1–simplex b such that both b and $\bar{b}$ are not inflating. Then

$$\begin{aligned} v(z)(b) \neq z(b) \iff\ & v(\bar{b})^{-1}\, z(b)\, v(b) \neq z(b) \\ \iff\ & z(b)\, v(b) \neq v(\bar{b})\, z(b) \iff z(b)\, g(b) \neq g(\bar{b})\, z(b) \end{aligned}$$

So, for instance, if we take

$$g(b) = z(b)^{-1} \quad \text{and} \quad g(\bar{b}) = g\, z(b)^{-1} \text{ with } g \neq e,$$

then $v(z)$ is not flat. Note that the above choice is always possible because G is nontrivial by assumption. In conclusion we have shown the following

4.25. THEOREM. *Let K be a pathwise connected but not totally ordered poset. Let G be a nontrivial group. Then for any 1–cocycle $z \in \mathrm{Z}^1(K, G)$ there are connections in $\mathrm{U}^1(K, z)$ which are not flat.*

Concerning central connections, in the case that K and G satisfy the hypotheses of the statement of Theorem 4.25, and the centre of the group G is nontrivial, then by using the above reasoning it is very easy to prove the existence of nonflat central connections.

4.26. Holonomy and reduction of connections. Keeping close to the theory of fibre bundles, we introduce the notion of holonomy group of a connection and link to it the property of reducibility of a connection.

Consider a connection u of $\mathrm{U}^1(K,G)$. Fix a base 0–simplex a and define

$$H_u(a) \equiv \{u(p) \in G \mid p : a \to a\} , \tag{53}$$

recalling that $p : a \to a$ denotes i a loop of K with endpoint a. By the defining properties of connections it is very easy to see that $H_u(a)$ is a subgroup of G. Furthermore let

$$H^0_u(a) \equiv \{u(p) \in G \mid p : a \to a,\ p \sim \sigma_0 a\} , \tag{54}$$

where $p \sim \sigma_0 a$ means that p is homotopic to the degenerate 1–simplex $\sigma_0 a$. In this case, too, it is easy to see that $H^0_u(a)$ is a subgroup of G. Moreover, since $p * q * \overline{p} \sim \sigma_0 a$ whenever $q, p \in (a, a)$ and $q \sim \sigma_0 a$, $H^0_u(a)$ is a normal subgroup of $H_u(a)$. $H_u(a)$ and $H^0_u(a)$ are called respectively *the holonomy and the restricted holonomy group of u based on a.*

As K is pathwise connected, we have the following

4.27. Lemma. *Given $u \in \mathrm{U}^1(K,G)$, let $\gamma : G_1 \to G$ be an injective homomorphism. The following assertions hold.*

(a) $H_u(a)$ and $H_u(a_1)$ are conjugate subgroups of G for any $a, a_1 \in \Sigma_0(K)$.

(b) Given $u_1 \in \mathrm{U}^1(K,G_1)$. If $\gamma \circ u_1$ is equivalent to u, then the holonomy groups $H_{u_1}(a)$ and $H_u(a)$ are isomorphic.

The same assertions hold for the restricted holonomy groups.

Proof. (a) Let p be a path from a to a_1. For any $g \in H_u(a)$, there is a loop $q : a \to a$ such that $g = u(q)$. Observe that $p*q*\overline{p} : a_1 \to a_1$, hence $u(p)\, g\, u(p)^{-1} = u(p*q*\overline{p}) \in H_u(a_1)$. By the symmetry of the reasoning, $H_u(a) \ni g \to u(p)\, g\, u(p)^{-1} \in H_u(a_1)$ is a group isomorphism. (b) Let $u_1 \in \mathrm{U}^1(K,G_1)$ and let $\mathrm{f} \in (\gamma \circ u_1, u)$. Since for any loop $p : a \to a$, $\mathrm{f}_a\, \gamma \circ u_1(p) = u(p)\, \mathrm{f}_a$, the map $H_{u_1}(a) \ni g \to \mathrm{f}_a\, \gamma(g)\, \mathrm{f}_a^{-1} \in H_u(a)$ is a group isomorphism. ∎

Now, according to the definition given in Section 3.5, a connection $u \in \mathrm{U}^1(K,G)$ is reducible if there is a proper subgroup G_1 of G and a connection $u_1 \in \mathrm{U}^1(K,G_1)$ such that $\jmath \circ u_1 \cong u$, where $\jmath : G_1 \to G$ is the inclusion mapping. The next result, an analogue of the Ambrose-Singer theorem [11] for connections of a poset, shows that u is reducible whenever its holonomy group is a proper subgroup of G.

4.28. Theorem. *Given $z \in \mathrm{Z}^1(K,G)$, let $u \in \mathrm{U}^1(K,z)$. Given $a_0 \in \Sigma_0(K)$, let $\jmath : H_u(a_0) \to G$ be the inclusion mapping. Then,*

(a) there exists $z_1 \in \mathrm{Z}^1(K, H_u(a_0))$ such that $\jmath \circ z_1 \cong z$,

(b) there exists $u_1 \in \mathrm{U}^1(K, z_1)$ such that $\jmath \circ u_1 \cong u$.

Proof. For any 0–simplex a, let p_a be a path from a_0 to a. Then define

$$u_1(b) \equiv u(\overline{p_{\partial_0 b}} * b * p_{\partial_1 b}), \qquad b \in \Sigma_1(K).$$

Note that $u_1(b) \in H_u(a_0)$ for any 1–simplex b because $\overline{p_{\partial_0 b}} * b * p_{\partial_1 b} : a_0 \to a_0$. Secondly, for any 1–simplex b we have

$$u_1(\overline{b}) = u(\overline{p_{\partial_1 b}} * \overline{b} * p_{\partial_0 b}) = u(\overline{\overline{p_{\partial_0 b}} * b * p_{\partial_1 b}}) = u_1(b)^{-1}.$$

Thirdly, let $c \in \Sigma_2^{\inf}(K)$. Then

$$\begin{aligned} u_1(\partial_0 c)\, u_1(\partial_2 c) &= u(\overline{p_{\partial_{00} c}} * \partial_0 c * p_{\partial_{10} c})\, u(\overline{p_{\partial_{02} c}} * \partial_2 c * p_{\partial_{12} c}) \\ &= u(\overline{p_{\partial_{00} c}})\, u(\partial_0 c)\, u(p_{\partial_{10} c})\, u(\overline{p_{\partial_{02} c}})\, u(\partial_2 c)\, u(p_{\partial_{12} c}) \\ &= u(\overline{p_{\partial_{01} c}})\, u(\partial_0 c)\, u(\partial_2 c)\, u(p_{\partial_{11} c}) \\ &= u(\overline{p_{\partial_{01} c}})\, u(\partial_1 c)\, u(p_{\partial_{11} c}) \\ &= u_1(\partial_1 c). \end{aligned}$$

Therefore we have that $u_1 \in \mathrm{U}^1(K, H_u(a_0))$. Now, for any 0–simplex a let $\mathrm{f}_a \equiv u(p_a)$. Then for any 1–simplex b we have

$$\mathrm{f}_{\partial_0 b}\, u_1(b) = u(p_{\partial_0 b})\, u(\overline{p_{\partial_0 b}} * b * p_{\partial_1 b}) = u(p_{\partial_0 b})\, u(\overline{p_{\partial_0 b}})\, u(b)\, u(p_{\partial_1 b}) = u(b)\, \mathrm{f}_{\partial_1 b},$$

namely $\mathrm{f} \in (\jmath \circ u_1, u)$. Thus $\jmath \circ u_1 \cong u$. Finally, let $z_1 \in \mathrm{Z}^1(K, H_u(a_0))$ be the 1–cocycle induced by u_1. Then $\jmath \circ z_1 \cong z$ because of Lemma 4.14. This completes the proof ∎

5. Gauge transformations

In the previous sections we have given several results to support the interpretation of 1–cocycles of a poset as principal bundles over the poset. As the final issue of the present paper, we now introduce what we mean by the group of gauge transformations of a 1–cocycle.

Given a 1–cocycle z of $\mathrm{Z}^1(K, G)$, define

$$\mathcal{G}(z) \equiv (z, z). \tag{55}$$

An element of $\mathcal{G}(z)$ will be denoted by g. The composition law between morphisms of 1–cochains endows $\mathcal{G}(z)$ with a structure of a group. The identity e of this group is given by $\mathrm{e}_a = e$ for any 0–simplex a. The inverse g^{-1} of an element $\mathrm{g} \in \mathcal{G}(z)$ is obtained by composing g with the inverse of G. We call $\mathcal{G}(z)$ the *group of gauge transformations of* z.

5.1. Lemma. *If* $z \in \mathrm{B}^1(K, G)$, *then* $\mathcal{G}(z) \cong G$.

Proof. Observe that, since K is connected, $\mathcal{G}(\imath)$ is the set of constant functions from $\Sigma_0(K)$ to G and hence is isomorphic to G. As z is a 1–coboundary, it is equivalent to the trivial 1–cocycle $\imath$, i.e. there exists an $\mathrm{f} \in (z, \imath)$. The mapping $\mathcal{G}(\imath) \ni \mathrm{g} \mapsto \mathrm{f}^{-1}\, \mathrm{g}\, \mathrm{f} \in \mathcal{G}(z)$ is a group isomorphism. ∎

As a consequence of this lemma and Proposition 3.11, if the poset is simply connected then $\mathcal{G}(z) \cong G$ for any 1–cocycle z. This is also the case when G is Abelian.

5.2. LEMMA. *If G is Abelian, then $\mathcal{G}(z) \cong G$ for any $z \in \mathrm{Z}^1(K, G)$.*

PROOF. For any $\mathrm{g} \in \mathcal{G}(z)$ and for any 1–simplex b we have

$$\mathrm{g}_{\partial_1 b}\, z(b) = z(b)\, \mathrm{g}_{\partial_0 b} = \mathrm{g}_{\partial_0 b}\, z(b) \ .$$

Hence $\mathrm{g}_{\partial_1 b} = \mathrm{g}_{\partial_0 b}$ for any 1–simplex b. Since K is pathwise connected, $\mathrm{g}_a = g$ for any 0–simplex a. ■

Thus, for Abelian groups, the action of the group of gauge transformations is always *global*, that is independent of the 0–simplex.

Given a 1–cocycle $z \in \mathrm{Z}^1(K, G)$ consider the group $\mathcal{G}(z)$ of gauge transformations of z. For any $\mathrm{u} \in \mathrm{U}^1(K, z)$ and $\mathrm{g} \in \mathcal{G}(z)$, define

$$\alpha_{\mathrm{g}}(u)(b) \equiv \mathrm{g}_{\partial_0 b}\, u(b)\, \mathrm{g}^{-1}_{\partial_1 b}, \qquad b \in \Sigma_1(K). \tag{56}$$

We have the following

5.3. PROPOSITION. *Given $z \in \mathrm{Z}^1(K, G)$, the following assertions hold:*

(a) given $\mathrm{g} \in \mathcal{G}(z)$, then $\alpha_{\mathrm{g}}(u) \in \mathrm{U}^1(K, z)$ for any $u \in \mathrm{U}^1(K, z)$;

(b) The mapping

$$\alpha : \mathcal{G}(z) \times \mathrm{U}^1(K, z) \ni (\mathrm{g}, u) \quad \longrightarrow \quad \alpha_{\mathrm{g}}(u) \in \mathrm{U}^1(K, z) \tag{57}$$

defines a left action, not free, of $\mathcal{G}(z)$ on $\mathrm{U}^1(K, z)$.

PROOF. (a) Clearly $\alpha_{\mathrm{g}}(u)(\overline{b}) = \alpha_{\mathrm{g}}(u)(b)^{-1}$ for any 1–simplex b. Moreover, if $b \in \Sigma^{\mathrm{inf}}_1(K)$, then $\alpha_{\mathrm{g}}(u)(b) = \mathrm{g}_{\partial_0 b}\, u(b)\, \mathrm{g}^{-1}_{\partial_1 b} = \mathrm{g}_{\partial_0 b}\, z(b)\, \mathrm{g}^{-1}_{\partial_1 b} = z(b)$. This entails that $\alpha_{\mathrm{g}}(u)$ satisfies property (ii) of the definition of connections. Hence $\alpha_{\mathrm{g}}(u) \in \mathrm{U}^1(K, z)$. (b) Clearly, α is a left action that is not free, because $z \in \mathrm{U}^1(K, z)$, hence $\alpha_{\mathrm{g}}(z) = z$ for any $\mathrm{g} \in \mathcal{G}(z)$. ■

6. Conclusions and outlook

We have developed a theory of bundles over posets from a cohomological standpoint, the analogue of describing the usual principal bundles in terms of their transition functions. In a sequel, we will introduce principal bundles over posets and their mappings directly and further develop such concepts as connection, curvature, holonomy, transition function, gauge group and gauge transformation. Although all these concepts are familiar from the usual theory of principal bundles, at this point it is worth stressing some of the differences from that theory. As we shall see in the sequel, the definition of principal bundle involves bijections between different fibres satisfying a 1–cocycle identity. An important rôle is played by the simplicial set of inflationary simplices. All principal bundles can be trivialized on the minimal covering. Finally, it should be stressed that the goal of these investigations is to develop gauge theories in the framework of algebraic quantum field theory. Our principal fibre bundles and the associated vector bundles are envisaged as stepping stones to the algebra of observables.

A. Some results on n–categories

In this appendix we explain the definition of the category $\mathcal{I}(\mathcal{C})$, and how to derive the categories nG and the form of a 3–cochain. References for this appendix are [16, 23, 2, 19].

THE CATEGORY $\mathcal{I}(\mathcal{C})$. Consider an n–category $\mathcal{C}$ and let $\diamond$ be a composition law of $\mathcal{C}$. Given an arrow t, the left and the right $\diamond$–units of t are the arrows $\mathrm{l}_\diamond(t)$ $\mathrm{r}_\diamond(t)$ of $\mathcal{C}$ satisfying the relations

$$\mathrm{l}_\diamond(t) \diamond t = t = t \diamond \mathrm{r}_\diamond(t).$$

Given another composition law $\times$, with $\times \prec \diamond$, the mappings $\mathrm{r}_\diamond : \mathcal{C} \ni t \to \mathrm{r}_\diamond(t) \in \mathcal{C}$ and $\mathrm{l}_\diamond : \mathcal{C} \ni t \to \mathrm{l}_\diamond(t) \in \mathcal{C}$ satisfy the following properties:

1. $\mathrm{r}_\diamond \mathrm{r}_\times = \mathrm{r}_\times = \mathrm{r}_\times \mathrm{r}_\diamond = r_\times \mathrm{l}_\diamond$ and $\mathrm{l}_\times = \mathrm{l}_\times \mathrm{r}_\diamond = \mathrm{l}_\times \mathrm{l}_\diamond$;
2. if $r_\times(s) = \mathrm{l}_\times(t)$, then
$$\mathrm{r}_\diamond(s \times t) = \mathrm{r}_\diamond(s) \times \mathrm{r}_\diamond(t) \quad \text{and} \quad \mathrm{l}_\diamond(s \times t) = \mathrm{l}_\diamond(s) \times \mathrm{l}_\diamond(t);$$
3. $\mathrm{r}_\diamond(s) = \mathrm{l}_\diamond(t)$, $\mathrm{r}_\diamond(s_1) = \mathrm{l}_\diamond(t_1)$ and $\mathrm{r}_\times(s \diamond t) = \mathrm{l}_\times(s_1 \diamond t_1)$ entail that
$$(s \diamond t) \times (s_1 \diamond t_1) = s \times s_1 \diamond t \times t_1;$$
that is the interchange law,

see for instance [23] where the mapping $\mathrm{r}_\diamond$ and $\mathrm{l}_\diamond$ are called, respectively, source and target of $\diamond$.

From now on we assume that $\mathcal{C}$ is an n–category satisfying the following property: *the arrows of $\mathcal{C}$ are invertible with respect to the greatest composition law.* Recall that an arrow t is said to be $\diamond$–invertible if there is an arrow $t^{-1}_\diamond$, called the $\diamond$–inverse of t, such that $t^{-1}_\diamond \diamond t = \mathrm{r}_\diamond(t)$, and $t \diamond t^{-1}_\diamond = \mathrm{l}_\diamond(t)$. Now, let $\diamond$ denote the greatest composition law of $\mathcal{C}$. Given an arrow t, define

$$\mathrm{R}_\diamond(t) \equiv \{m \in \mathcal{C} \mid \mathrm{r}_\diamond(m) = \mathrm{r}_\diamond(t) = \mathrm{l}_\diamond(m)\}, \tag{A.1}$$

and call the mapping $\beta_t : \mathrm{R}_\diamond(t) \to \mathcal{C}$, defined by

$$\beta_t(m) \equiv t \diamond m \diamond t^{-1}_\diamond, \qquad m \in \mathrm{R}_\diamond(t), \tag{A.2}$$

the *inner $\diamond$–automorphism* associated with t. Since all the arrows of $\mathcal{C}$ are $\diamond$–invertible, the set $\mathrm{R}_\diamond(t)$ is a group with respect to the composition law $\diamond$. We call this group, the *domain* of β_t. Note that the inner $\diamond$–automorphism β_t determines t only up to elements of the centre $\mathcal{Z}(\mathrm{R}_\diamond(t))$ of the group $\mathrm{R}_\diamond(t)$.

Consider now a pair of arrows s and t such that $s \times t$ is defined. By applying the exchange law we have

$$s \times t \ \diamond \ s_\diamond^{-1} \times t_\diamond^{-1} = (s \diamond s_\diamond^{-1}) \times (t \diamond t_\diamond^{-1}) = \mathrm{l}_\diamond(s) \times \mathrm{l}_\diamond(t) = \mathrm{l}_\diamond(s \times t)$$

that is $(s \times t)_\diamond^{-1} = s_\diamond^{-1} \times t_\diamond^{-1}$. So that the inner $\diamond$–automorphism $\beta_{s\times t}$ associated with $s \times t$ can be written as $\beta_{s\times t}(n) = s \times t \diamond n \diamond s_\diamond^{-1} \times t_\diamond^{-1}$ for $n \in \mathrm{R}_\diamond(s \times t)$. Now, for any pair of arrow t,s such that $s \times t$ is defined, we define

$$(\beta_s \times \beta_t)(n) \equiv \beta_{s\times t}(n), \qquad n \in \mathrm{R}_\diamond(s \times t). \tag{A.3}$$

We have seen that an inner $\diamond$–automorphism is uniquely determined only up central elements. Then, the above definition is well posed if we show that $\beta_s \times \beta_t$ depends only on β_s and β_t and not on the choice of the elements t, s which define these two $\diamond$–automorphisms. To this end, we first observe that given $z \in \mathcal{Z}(\mathrm{R}_\diamond(s))$ and $z' \in \mathcal{Z}(\mathrm{R}_\diamond(t))$ then $z \times z'$ is defined. This is easily seen by applying the relations between the source and target map, given at the beginning of this appendix, to the following identities

$$\mathrm{r}_\times(s) = \mathrm{l}_\times(t), \quad \mathrm{r}_\diamond(s) = \mathrm{r}_\diamond(z) = \mathrm{l}_\diamond(z), \quad \mathrm{r}_\diamond(t) = \mathrm{r}_\diamond(z') = \mathrm{l}_\diamond(z').$$

We now have the following

A.1. Lemma. *Let $\mathcal{C}$ be an n–category and let $\diamond$ be the greatest composition law of $\mathcal{C}$. Assume that:*

(i) the arrows of $\mathcal{C}$ are $\diamond$–invertible;

(ii) for any composition law $\times$, with $\times \prec \diamond$, we have

$$\mathcal{Z}(\mathrm{R}_\diamond(s)) \times \mathcal{Z}(\mathrm{R}_\diamond(t)) \subseteq \mathcal{Z}(\mathrm{R}_\diamond(s \times t)).$$

for any pair of arrows t, s such that $s \times t$ is defined.

Then, for any pair of arrows s, t such that $s \times t$ is defined the composition $\beta_s \times \beta_t$ given by (A.3), is well defined.

Proof. Let $z \in \mathcal{Z}(\mathrm{R}_\diamond(s))$, $z' \in \mathcal{Z}(\mathrm{R}_\diamond(t))$, and let $n \in \mathrm{R}_\diamond(s \times t)$. By property (ii) in the statement and by applying the exchange law we have

$$\begin{aligned}
(\beta_{s\diamond z} \times \beta_{t\diamond z'})(n) &= \beta_{(s\diamond z)\times(t\diamond z')}(n) \\
&= (s \diamond z) \times (t \diamond z') \diamond n \diamond \big((s \diamond z) \times (t \diamond z')\big)_\diamond^{-1} \\
&= s \times t \diamond z \times z' \diamond n \diamond \big(s \times t \diamond z \times z'\big)_\diamond^{-1} \\
&= s \times t \diamond z \times z' \diamond n \diamond \big(s \times t \diamond z \times z'\big)_\diamond^{-1} \\
&= s \times t \diamond z \times z' \diamond n \diamond \big(z \times z'\big)_\diamond^{-1} \diamond \big(s \times t\big)_\diamond^{-1} \\
&= s \times t \diamond n \diamond \big(s \times t\big)_\diamond^{-1} \\
&= \beta_{s\times t}(n) = (\beta_s \times \beta_t)(n),
\end{aligned}$$

Completing the proof. ■

Now, let $\mathcal{C}$ be an n–category $\mathcal{C}$ satisfying the hypotheses of Lemma A.1, and let $\diamond$ denote the greatest composition law. Following [19], we define the category $\mathcal{I}(\mathcal{C})$ as the set

$$\mathcal{I}(\mathcal{C}) \equiv \{(t,\mu) \in \mathcal{C} \times Inn_\diamond(\mathcal{C}) \mid t \diamond t \text{ and } \quad \beta_t \diamond \mu \text{ are defined}\}, \tag{A.4}$$

where $Inn_\diamond(\mathcal{C})$ is the set of inner $\diamond$–automorphisms of $\mathcal{C}$, with the following composition laws:

$$\begin{array}{rrcll} (i) & (t,\mu) \times (t',\mu') & \equiv & (t \times t', \mu \times \mu') & \text{if } \ \mu \times \mu' \ \text{ is defined;} \\ (ii) & (t,\mu) \diamond (t',\mu') & \equiv & (t \diamond \mu(t'), \mu \diamond \mu') & \text{if } \ \mu \diamond \mu' \ \text{ is defined;} \\ (iii) & (t,\mu) \cdot (t',\mu') & \equiv & (t \diamond t', \mu') & \text{if } \ \beta_{t'} \diamond \mu' = \mu, \end{array} \tag{A.5}$$

where $\times$ is any other composition law of $\mathcal{C}$ with $\times \prec \diamond$. Lemma A.1 entails that the composition law $\times$ in $\mathcal{I}(\mathcal{C})$ is well defined. It turns out that $\mathcal{I}(\mathcal{C})$ is an $(n+1)$–category with $\times \prec \diamond \prec \cdot$. The arrows of $\mathcal{I}(\mathcal{C})$ are invertible with respect the composition laws $\cdot$ and $\diamond$. However, in general, $\mathcal{I}(\mathcal{C})$ does not fulfill hypothesis (ii) of Lemma A.1.

The categories nG. Given a group G, consider the 1–category $1G$ whose arrows are the elements of the group G and whose composition law is the group composition. Clearly $1G$ satisfies the hypotheses of Lemma A.1, so we can define $2G \equiv \mathcal{I}(1G)$. Since the arrows of $1G$ are all composable, then any element of $2G$ is of the form (g,τ) with $g \in G$ and $\tau \in Inn(G)$, where $Inn(G)$ is the group of inner automorphisms of G. Furthermore, by A.5(ii) and A.5(iii), one can deduce the following composition laws

$$\begin{array}{rrcll} (i) & (g,\tau) \times (g',\tau') & = & (g\tau(g'), \tau\tau'), & \\ (ii) & (g,\tau) \diamond (g',\tau') & = & (gg', \tau'), & \quad \text{if } \ \sigma_{g'}\tau' = \tau. \end{array} \tag{A.6}$$

That is, the composition laws $\times$ and $\diamond$ are obtained from the composition law of $1G$ according to the definition (A.5)(ii) and (A.5)(iii) respectively. Notice that $\times$ is everywhere defined; $\times \prec \diamond$; the arrows of $2G$ are invertible with respect to any composition law. Moreover, given an arrow (g,τ) of $2G$, we have

$$\mathrm{l}_\diamond(g,\tau) = (e,\tau), \quad \mathrm{r}_\diamond(g,\tau) = (e,\sigma_g\tau), \quad (g,\tau)^{-1}_\diamond = (g^{-1},\sigma_g\tau).$$

while

$$\mathrm{l}_\times(g,\tau) = (e,\iota) = \mathrm{r}_\times(g,\tau), \quad (g,\tau)^{-1}_\times = (\tau^{-1}(g^{-1}),\iota).$$

Hence one can see that 1–arrows are the elements of $2G$ of the form (e,τ), and that the only 0–arrow is (e,ι), where ι is the identity automorphism.

Before studying the 3–category $\mathcal{I}(2G)$ we need some observations on the structure of the inner $\diamond$–automorphisms of $2G$. According to definition (A.1), the domain of an inner $\diamond$–automorphism $\beta_{(h,\tau)}$ of $2G$, is the set of those arrows (m,γ) of $2G$ such that the compositions

$$(m,\gamma) \diamond (h,\tau) \text{ and } \big(h,\tau) \diamond (m,\gamma)$$

are defined. By (A.6)(ii), these compositions are defined whenever $\tau = \gamma$ and $m \in \mathcal{Z}(G)$. Hence, the domain of $\beta_{(h,\tau)}$ is

$$\mathrm{R}_\diamond(h,\tau) = \{(m,\tau), \quad m \in \mathcal{Z}(G)\}. \tag{A.7}$$

Moreover, we have that

$$\beta_{(h,\tau)}(m,\tau) = (m, \sigma_h \tau), \qquad (m,\tau) \in \mathrm{R}_\diamond(h,\tau). \tag{A.8}$$

Note that $\mathrm{R}_\diamond(h,\tau)$ is an Abelian group and that

$$\mathrm{R}_\diamond(h,\tau) \times \mathrm{R}_\diamond(h',\tau') = \mathrm{R}_\diamond(h\tau(h'), \tau\tau'). \tag{A.9}$$

This equation implies that $2G$ satisfies the hypotheses of Lemma A.1, and we can define the 3–category $\mathcal{I}(2G)$.

By (A.4), an arrow of $\mathcal{I}(2G)$ is a 3-tuple $(g, \alpha, \beta_{(h,\tau)})$, where $(g,\alpha) \in 2G$, and $\beta_{(h,\tau)}$ is the inner $\diamond$–automorphism of $2G$ associated with (h,τ), such that the compositions

$$(g,\alpha) \diamond (g,\alpha) \text{ and } (g,\alpha) \diamond (h,\tau).$$

are defined. By (A.6)(ii), these two compositions are defined if, and only if, $\sigma_h \tau = \alpha$ and $\sigma_g \alpha = \alpha$. The latter, in particular, says that g is an element of the centre $\mathcal{Z}(G)$ of G. Hence

$$\mathcal{I}(2G) = \{(g, \sigma_h\tau, \beta_{(h,\tau)}) \mid g \in \mathcal{Z}(G), \quad h \in G, \quad \tau \in Inn(G)\}. \tag{A.10}$$

The purpose now is to provide an explicit form to the abstract composition laws (A.5) of $\mathcal{I}(2G)$. Let $\left(g, \sigma_h\tau, \beta_{(h,\tau)}\right)$, $\left(g', \sigma_{h'}\tau', \beta_{(h',\tau')}\right) \in \mathcal{I}(2G)$. First, we consider the composition law $\times$ as defined by (A.5)(i). Recalling definition (A.3), by (A.6) we have

$$\left(g, \sigma_h\tau, \beta_{(h,\tau)}\right) \times \left(g', \sigma_{h'}\tau', \beta_{(h',\tau')}\right) = \left(gg', \sigma_h\tau\sigma_{h'}\tau', \beta_{(h\tau(h'),\tau\tau')}\right). \tag{A.11}$$

This composition is everywhere defined because the composition law $\times$ in $2G$ is. Consider now the composition law $\diamond$ as defined by (A.5)(ii). Note that, by (A.6)(ii), $\diamond$ is defined in $\mathcal{I}(2G)$ whenever $\beta_{(h,\tau)} \diamond \beta_{(h',\tau')}$ is defined. This amounts to saying that $(h,\tau) \diamond (h',\tau')$ is defined, which is equivalent to the condition $\sigma_{h'}\tau' = \tau$. Hence, one can check that

$$\left(g, \sigma_h\tau, \beta_{(h,\tau)}\right) \diamond \left(g', \sigma_{h'}\tau', \beta_{(h',\tau')}\right) = \left(gg', \sigma_{hh'}\tau', \beta_{(hh',\tau')}\right), \quad \text{if} \quad \sigma_{h'}\tau' = \tau. \tag{A.12}$$

Finally, consider the composition $\cdot$ as defined by (A.5)(iii). Again, by (A.6)(ii), $\cdot$ is defined in $\mathcal{I}(2G)$ whenever

$$\beta_{(h,\tau)} = \beta_{(g',\sigma_{h'}\tau')} \diamond \beta_{(h',\tau')} = \beta_{(g',\sigma_{h'}\tau')\diamond(h',\tau')} = \beta_{(g'h',\tau')}.$$

By (A.7) and (A.8), this relation is equivalent to $\tau = \tau'$, and $\sigma_h\tau = \sigma_{g'h'}\tau'$. The latter, in particular, entails that $\sigma_h = \sigma_{h'}$ since g' is an element of the centre of G. In conclusion we have that

$$\left(g, \sigma_h\tau, \beta_{(h,\tau)}\right) \cdot \left(g', \sigma_{h'}\tau', \beta_{(h',\tau')}\right) = \left(gg', \sigma_{h'}\tau', \beta_{(h',\tau')}\right), \quad \text{if} \quad \sigma_h = \sigma_{h'}, \ \tau = \tau'. \tag{A.13}$$

Now, it is very easy to see that 2–arrows are those elements of $\mathcal{I}(2G)$ of the form $(e, \sigma_h\tau, \beta_{(h,\tau)})$; 1–arrows are of the form $(e, \tau, \beta_{(e,\tau)})$; the only 0–arrow is $(e, \iota, \beta_{(e,\iota)})$.

The 3–category $3G$ used in Section 3.2 as a set of coefficients for the cohomology of posets is different from $\mathcal{I}(2G)$. We now show that these two 3–categories are isomorphic.

A.2. LEMMA. *The mapping* $F : \mathcal{I}(2G) \to 3G$ *defined by*

$$F\big(g, \sigma_h\tau, \beta_{(h,\tau)}\big) \equiv (g, \tau, \sigma_h),$$

for any $\big(g, \sigma_h\tau, \beta_{(h,\tau)}\big) \in \mathcal{I}(2G)$*, is an isomorphism of* 3*–categories.*

PROOF. It is clear that the mapping F is injective and surjective. We check that F preserves the composition laws. To this end, fix a pair $\big(g, \sigma_h\tau, \beta_{(h,\tau)}\big)$, $\big(g', \sigma_{h'}\tau', \beta_{(h',\tau')}\big)$ of arrows of $\mathcal{I}(2G)$.

Concerning the composition law $\cdot$, by (13) we have

$$F\big(g, \sigma_h\tau, \beta_{(h,\tau)}\big) \cdot F\big(g', \sigma_{h'}\tau', \beta_{(h',\tau')}\big) = (g, \tau, \sigma_h) \cdot (g', \tau', \sigma_{h'}) = (gg', \tau', \sigma_{h'}),$$

if $\sigma_h = \sigma_{h'}$ and $\tau = \tau'$. On the other hand, by (A.13) we have

$$F\big((g, \sigma_h\tau, \beta_{(h,\tau)})\cdot(g', \sigma_{h'}\tau', \beta_{(h',\tau')})\big) = F\big(gg', \sigma_{h'}\tau', \beta_{(h',\tau')}\big) = (gg', \tau', \sigma_{h'}),$$

with $\sigma_h = \sigma_{h'}$ and $\tau = \tau'$. Hence F preserves the composition law $\cdot$.

Concerning the composition law $\diamond$, by (13) we have

$$F\big(g, \sigma_h\tau, \beta_{(h,\tau)}\big) \diamond F\big(g', \sigma_{h'}\tau', \beta_{(h',\tau')}\big) = (g, \tau, \sigma_h) \diamond (g', \tau', \sigma_{h'}) = (gg', \tau', \sigma_{hh'})$$

with the condition that $\sigma_{h'}\tau' = \tau$. On the other hand by (A.12) we have

$$F\big((g, \tau, \beta_{(h,\tau)}) \diamond (g', \tau', \beta_{(h',\tau')})\big) = F\big(gg', \sigma_{hh'}\tau', \beta_{(hh',\tau')}\big) = (gg', \tau', \sigma_{hh'}),$$

with the condition $\sigma_{h'}\tau' = \tau$. This implies that F preserves the composition law $\diamond$.

Finally, concerning the composition law $\times$, by (13) we have

$$F\big(g, \tau, \beta_{(h,\tau)}\big) \times F\big(g', \tau', \beta_{(h',\tau')}\big) = (g, \tau, \sigma_h) \times (g', \tau', \sigma_{h'}) = (gg', \tau\tau', \sigma_h\tau\sigma_{h'}\tau^{-1}),$$

while by (A.11) we have

$$\begin{aligned}
&F\big((g, \tau, \beta_{(h,\tau)}) \times (g', \tau', \beta_{(h',\tau')})\big) = \\
&\qquad = F\big(gg', \sigma_h\tau\sigma_{h'}\tau', \beta_{(h\tau(h'),\tau\tau')}\big) = F\big(gg', \sigma_{h\tau(h')}\tau\tau', \beta_{(h\tau(h'),\tau\tau')}\big) \\
&\qquad = (gg', \tau\tau', \sigma_{h\tau(h')}) = (gg', \tau\tau', \sigma_h\tau\sigma_{h'}\tau^{-1}).
\end{aligned}$$

Hence F preserves the composition law $\times$. Finally observe that

$$\begin{aligned}
&F\big(e, \iota, \beta_{(e,\iota)}\big) = (e, \iota, \iota), && \\
&F\big(e, \tau, \beta_{(e,\tau)}\big) = (e, \tau, \iota), && \tau \in Inn(G), \\
&F\big(e, \tau, \beta_{(h,\tau)}\big) = (e, \sigma_h\tau, \sigma_h), && h \in G, \tau \in Inn(G).
\end{aligned}$$

This means that F sends 0–, 1–, 2–arrows of $\mathcal{I}(2G)$ in 0–, 1–, 2–arrows of $3G$ respectively, completing the proof. ∎

Note that the composition law $\cdot$ of $3G$ is Abelian. So the only inner $\cdot$–automorphism of $3G$ is the identity automorphism. Hence the first two composition laws of the 4–category $\mathcal{I}(3G)$ coincide and are Abelian.

The form of a 3–cochain. We derive the formula (24) which defines a 3–cochain. Let x denote a 3–tuple (x_1, x_2, x_3) of mappings, where $x_i : \Sigma_1(K) \to (3G)_i$, for $i = 1, 2, 3$. This amounts to saying that

$$\begin{array}{ll} x_1(b) = (e,\ \tau_b,\ \iota), & b \in \Sigma_1(K), \\ x_2(c) = (e,\ \alpha_c,\ \gamma_c), & c \in \Sigma_2(K), \\ x_3(d) = (v(d),\ \sigma_d,\ \eta_d), & d \in \Sigma_3(K), \end{array}$$

where $\tau : \Sigma_1(K) \to Inn(G)$, $\alpha, \gamma : \Sigma_2(K) \to Inn(G)$, $\sigma, \eta : \Sigma_3(K) \to G$, and $v : \Sigma_3(K) \to Z(G)$. Such a map x is a 3–cochain whenever it satisfies equations (17) and (18). By imposing that x satisfies these two equations we will arrive at formula (24). In the proof we adopt the notation introduced in the proof of Lemma 3.3. Furthermore we will make use of the relations (1) and of the composition laws (13). Let us start by imposing that x satisfies equation (17), that is

$$x_2(c) \diamond x_1(\partial_1 c) = x_1(\partial_0 c) \times x_1(\partial_2 c) \diamond x_2(c),$$

for any 2–simplex c. The l.h.s. of this equation reads

$$x_2(c) \diamond x_1(\partial_1 c) = (e, \alpha, \gamma) \diamond (e,\ \tau_1,\ \iota) = (e, \tau_1, \gamma)$$

if $\tau_1 = \alpha$. Using this condition on the r.h.s. of equation (17) we have

$$\begin{aligned} x_1(\partial_0 c) \times x_1(\partial_2 c) \diamond x_2(c) &= (e, \tau_0, \iota) \times (e, \tau_2, \iota) \diamond (e, \tau_1, \gamma) \\ &= (e, \tau_0\tau_2, \iota) \diamond (e, \tau_1, \gamma) \\ &= (e, \tau_1, \gamma), \end{aligned}$$

if $\gamma\tau_1 = \tau_0\tau_2$, that is $\gamma = \tau_0\tau_2\tau_1^{-1}$. Hence x satisfies the equation (17) whenever

$$\begin{array}{ll} x_1(b) = (e,\ \tau_b,\ \iota), & b \in \Sigma_1(K), \\ x_2(c) = (e,\ \tau_{\partial_1 c},\ \tau_{\partial_0 c}\tau_{\partial_2 c}\tau^{-1}_{\partial_1 c}), & c \in \Sigma_2(K) \end{array}$$

In order to obtain the form of the component x_3, we now impose that x satisfies the equation (18), that is

$$x_3(d) \cdot x_1(\partial_{01} d) \times x_2(\partial_3 d) \diamond x_2(\partial_1 d) = x_2(\partial_0 d) \times x_1(\partial_{23} d) \diamond x_2(\partial_2 d) \cdot x_3(d),$$

for any 3–simplex d. The l.h.s. of this equation reads

$$\begin{aligned}
x_3(d)\cdot x_1(\partial_{01}d)\times x_2(\partial_3 d)\diamond x_2(\partial_1 d) &= \\
&= (v,\sigma,\eta)\cdot(e,\tau_{01},\iota)\times(e,\tau_{13},\tau_{03}\tau_{23}\tau_{13}^{-1})\diamond(e,\tau_{11},\tau_{01}\tau_{21}\tau_{11}^{-1})\\
&= (v,\sigma,\eta)\cdot(e,\tau_{01}\tau_{13},\tau_{01}\tau_{03}\tau_{23}\tau_{13}^{-1}\tau_{01}^{-1})\diamond(e,\tau_{11},\tau_{01}\tau_{21}\tau_{11}^{-1})\\
&= (v,\sigma,\eta)\cdot(e,\tau_{01}\tau_{13},\tau_{01}\tau_{03}\tau_{23}\tau_{13}^{-1}\tau_{01}^{-1})\diamond(e,\tau_{11},\tau_{01}\tau_{13}\tau_{11}^{-1})\\
&= (v,\sigma,\eta)\cdot(e,\tau_{11},\tau_{01}\tau_{03}\tau_{23}\tau_{13}^{-1}\tau_{01}^{-1}\tau_{01}\tau_{13}\tau_{11}^{-1})\\
&= (v,\sigma,\eta)\cdot(e,\tau_{12},\tau_{01}\tau_{03}\tau_{23}\tau_{11}^{-1})\\
&= (v,\sigma,\eta)\cdot(e,\tau_{12},\tau_{00}\tau_{20}\tau_{10}^{-1}\tau_{02}\tau_{22}\tau_{12}^{-1})\\
&= (v,\sigma,\eta)\cdot(e,\tau_{12},\gamma_0\gamma_2)\\
&= (v,\tau_{12},\gamma_0\gamma_2),
\end{aligned}$$

if $\sigma=\tau_{12}$ and $\eta=\gamma_0\gamma_2=\tau_{00}\tau_{20}\tau_{10}^{-1}\tau_{02}\tau_{22}\tau_{12}^{-1}$. Hence the l.h.s. of equation (18) determines the form of the third component of x, that is

$$x_3(d) = (v(d),\ \tau_{\partial_{12}d},\ \gamma_{\partial_0 d}\gamma_{\partial_2 d}), \qquad d\in\Sigma_3(K).$$

Hence x is equal to (24). What remains to be shown is that r.h.s. of (18) equals the l.h.s., that is, it is equal to $(v,\tau_{12},\gamma_0\gamma_2)$. Since $\gamma_2\tau_{12}=\tau_{02}\tau_{22}=\tau_{10}\tau_{23}$, we have

$$\begin{aligned}
x_2(\partial_0 d)\times x_1(\partial_{23}d)\diamond x_2(\partial_2 d)\cdot x_3(d) &= (e,\tau_{10},\gamma_0)\times(e,\tau_{23},\iota)\diamond(e,\tau_{12},\gamma_2)\cdot(v,\tau_{12},\gamma_0\gamma_2)\\
&= (e,\tau_{10}\tau_{23},\gamma_0)\diamond(e,\tau_{12},\gamma_2)\cdot(v,\tau_{12},\gamma_0\gamma_2)\\
&= (e,\tau_{12},\gamma_0\gamma_2)\cdot(v,\tau_{12},\gamma_0\gamma_2)\\
&= (v,\tau_{11},\gamma_0\gamma_2),
\end{aligned}$$

completing the proof.

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