Theory and Applications of Categories, Vol. 16, No. 26, 2006, pp. 709–735.

QUOTIENT MODELS OF A CATEGORY UP TO DIRECTED HOMOTOPY

MARCO GRANDIS

ABSTRACT.

Directed Algebraic Topology is a recent field, deeply linked with ordinary and higher dimensional Category Theory. A 'directed space', e.g. an ordered topological space, has directed homotopies (which are generally non reversible) and a fundamental *category* (replacing the fundamental groupoid of the classical case). Finding a simple - possibly finite - model of the latter is a non-trivial problem, whose solution gives relevant information on the given 'space'; a problem which is of interest for applications as well as in general Category Theory.

Here we continue the work "The shape of a category up to directed homotopy", with a deeper analysis of 'surjective models', motivated by studying the singularities of 3-dimensional ordered spaces.

Introduction

Directed Algebraic Topology studies 'directed spaces' in some sense, where paths and homotopies cannot generally be reversed; for instance: ordered topological spaces, 'spaces with distinguished paths', 'inequilogical spaces', simplicial and cubical sets, etc. Its present applications deal mostly with the analysis of concurrent processes (see [4, 5, 6, 7, 8]) and rewrite systems, but its natural range covers non reversible phenomena, in any domain.

The study of invariance under directed homotopy is far richer and more complex than in the classical case, where homotopy equivalence between 'spaces' produces a plain equivalence of their fundamental groupoids, for which one can simply take - as a minimal model - the categorical skeleton. Our directed structures have a *fundamental category* $\uparrow\Pi_1(X)$; this must be studied up to appropriate notions of *directed homotopy equivalence of categories*, which are *more general* than ordinary categorical equivalence: the latter would often be of no use, since the fundamental category of an ordered topological space is always skeletal; the same situation shows that all the fundamental *monoids* $\uparrow\pi_1(X, x_0)$ can be trivial, without $\uparrow\Pi_1(X)$ being so (cf. 1.2). Such a study has been carried on in a previous work [11], which will be cited as Part I; the references I.2 or I.2.3 apply,

Work supported by MIUR Research Projects.

Received by the editors 2006-03-22 and, in revised form, 2006-10-05.

Transmitted by R. Brown. Published on 2006-10-12.

2000 Mathematics Subject Classification: 55Pxx, 18A40, 68Q85.

Key words and phrases: homotopy theory, adjunctions, reflective subcategories, directed algebraic topology, fundamental category, concurrent processes.

respectively, to its Section 2 or Subsection 2.3. Other references for Directed Algebraic Topology and its applications can be found there.

In Part I we have introduced two (dual) directed notions, which take care, respectively, of variation 'in the future' or 'from the past': *future equivalence* (a symmetric version of an adjunction, with two units) and its dual, a *past equivalence* (with two counits); and studied how to combine them. *Minimal models* of a category, up to these equivalences, have been introduced to better understand the 'shape' and properties of the category we are analysing, and of the process it represents.

The paper [5] has similar goals and results, based on a different categorical tool, categories of fractions. More recently, the thesis of E. Haucourt [15] has combined this tool with a more effective and more 'computable' one: the quotient of a category with respect to the generalised congruence generated by a set of arrows which are to become identities; or, in the present terminology, a *normal quotient* (with respect to the ideal of discrete functors, see 2.1 - 2.2).

Now, the analysis of Part I, captures essential facts of many *planar* ordered spaces (subspaces of the ordered plane $\uparrow\mathbf{R}^2$), but may say little about objects embedded in the ordered space $\uparrow\mathbf{R}^3$, much in the same way as the fundamental group does not recognise the singularity of a 2-sphere. There seem to be two ways of exploring such higher-dimensional singularities.

The 'obvious' one would be to use a higher dimensional study, based on the fundamental *2-category* of the directed space, in its *strict* version $\uparrow\Pi_2(X)$ - introduced and studied in [12] - or in some *lax* version, as the ones introduced in [13, 14]. Yet, 2-categories are complicated structures and their models, even when they are finite, can hardly be considered to be 'simple'.

Here we follow another way, closely linked with the study of [5] and [15], and based on analysing the fundamental category $\uparrow\Pi_1(X)$ with *quotient models*. It is interesting to note that - for the hollow cube - *such a finer analysis has no counterpart outside of* directed *homotopy*: the fundamental group (or groupoid) of the underlying topological space is trivial (see 1.9).

Notation. A homotopy φ between maps $f, g\colon X \to Y$ is written as $\varphi\colon f \to g\colon X \to Y$. A *preorder* relation is assumed to be reflexive and transitive; it is a (partial) *order* if it is also anti-symmetric. As usual, a preordered set is *identified* with a (small) category having at most one arrow between any two given objects. The ordered topological space $\uparrow\mathbf{R}$ is the euclidean line with the natural order. The classical properties of adjunctions and equivalences of categories are used without explicit reference (see [18]). **Cat** denotes the category of small categories; if C is a small category, $x \in C$ means that x is an object of C (also called a *point* of C, cf. 1.3).

1. An analysis of directed spaces up to directed homotopy

We begin with a brief review of the basic ideas and results of Part I. To make this outline lighter, some definitions and results are deferred to the end, in Section 5.

1.1. Homotopy for preordered spaces. The simplest topological setting where one can study directed paths and directed homotopies is likely the category p**Top** of *preordered topological spaces* and *preorder-preserving continuous mappings*; the latter will be simply called *morphisms* or *maps*, when it is understood we are in this category.

In this setting, a (directed) *path* in the preordered space X is a map $a\colon \uparrow[0,1] \to X$, defined on the standard directed interval $\uparrow\mathbf{I} = \uparrow[0,1]$ (with euclidean topology and natural order). A (directed) *homotopy* $\varphi\colon f \to g\colon X \to Y$, *from* f *to* g, is a map $\varphi\colon X \times \uparrow\mathbf{I} \to Y$ coinciding with f on the lower basis of the *cylinder* $X \times \uparrow\mathbf{I}$, with g on the upper one. Of course, this (directed) cylinder is a product in p**Top** : it is equipped with the product topology *and* with the product preorder, where $(x,t) \prec (x',t')$ if $x \prec x'$ in X and $t \leq t'$ in $\uparrow\mathbf{I}$.

The fundamental category $C = \uparrow\Pi_1(X)$ has, for arrows, the classes of directed paths up to the equivalence relation *generated* by directed homotopy with fixed endpoints; composition is induced by the concatenation of consecutive paths.

Note that, generally, the fundamental category of a preordered space X is *not* a preorder, i.e. can have different arrows $x \to x'$ between two given points (cf. 1.2); but any loop in X lives in a zone of equivalent points and is reversible, so that all endomorphisms of $\uparrow\Pi_1(X)$ are invertible. Moreover, if X is *ordered*, the fundamental category has no endomorphisms and no isomorphisms, except the identities, and is *skeletal*; therefore, *ordinary equivalence of categories cannot yield any simpler model.* Note also that, in this case, all the fundamental monoids $\uparrow\pi_1(X,x_0) = \uparrow\Pi_1(X)(x_0,x_0)$ are trivial. All these are crucial differences with the classical fundamental groupoid $\Pi_1(X)$ of a space, for which a model up to homotopy invariance is given by the skeleton: a family of fundamental groups $\pi_1(X,x_i)$, obtained by choosing precisely one point in each path-connected component of X.

The fundamental category of a preordered space can be computed by a van Kampen-type theorem, as proved in [10], Thm. 3.6, in a much more general setting (spaces with distinguished paths). A map of preordered spaces $f\colon X \to Y$ induces a functor $f_*\colon \uparrow\Pi_1(X) \to \uparrow\Pi_1(Y)$, and a homotopy $\varphi\colon f \to g$ induces a natural transformation $\varphi_*\colon f_* \to g_*$, generally *non* invertible. One can consider *various* notions of *directed* homotopy equivalence, for directed spaces [10] and categories (1.4 - 1.8).

The forgetful functor $U\colon$ p**Top** $\to$ **Top** with values in the category of topological spaces has both a left and a right adjoint, $D \dashv U \dashv C$, where DX (resp. CX) is the space X with the *discrete* order (resp. the *coarse* preorder). Therefore, U preserves limits and colimits. *The standard embedding of* **Top** *in* p**Top** *will be the coarse one*, so that all (ordinary) paths in X are directed in CX. Note that the category of *ordered* spaces does not allow for such an embedding, and has different colimits.

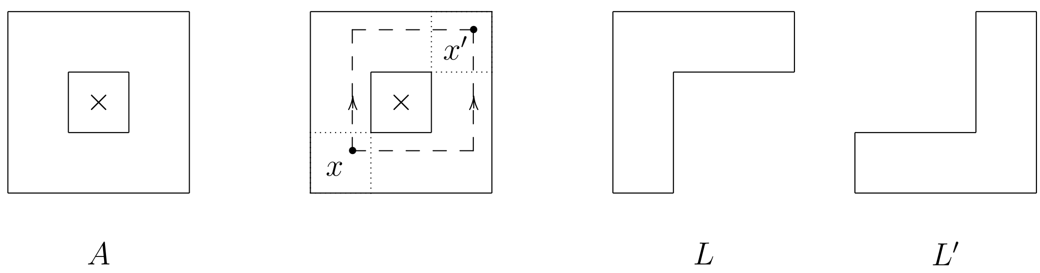
Preordered spaces are a *very basic* setting for Directed Algebraic Topology, which - for instance - does not contain a 'directed circle' nor a 'directed torus': in a preordered space, every loop is reversible. More complex settings, allowing for such structures, have been studied. For instance, *d-spaces* or *spaces with distinguished paths* [10]; or *inequilogical spaces*, introduced in other papers as a directed version of Dana Scott's equilogical spaces

[19, 1]. More recently, S. Krishnan [16] has proposed a 'convenient category of locally preordered spaces' which should allow for the usual constructions of homotopy theory, like mapping cones and suspension.

1.2. The fundamental category of a square annulus. An elementary example will give some idea of the analysis developed below. Let us start from the standard *ordered* square $\uparrow[0,1]^2$, with the euclidean topology and the product order

$$(x, y) \leq (x', y') \qquad \text{if:} \qquad x \leq x',\ y \leq y',$$

and consider the 'square annulus' $X \subset \uparrow[0,1]^2$, the ordered compact subspace obtained by taking out the *open* square $]1/3, 2/3[^2$ (marked with a cross)

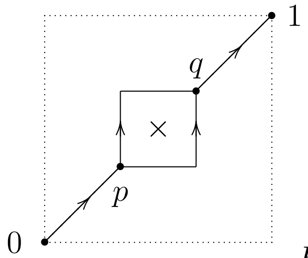


(1)

Its directed paths are, by definition, the continuous *order-preserving* maps $\uparrow[0,1] \to X$ defined on the standard ordered interval, and move 'rightward and upward' (in the weak sense). Directed homotopies of such paths are continuous order-preserving maps $\uparrow[0,1]^2 \to X$. The fundamental category $C = \uparrow\Pi_1(X)$ has, for arrows, the classes of directed paths up to the equivalence relation *generated* by directed homotopy (with fixed endpoints, of course).

In our example, the fundamental category C has *some* arrow $x \to x'$ provided that $x \leq x'$ and both points are in L or L' (the closed subspaces represented above). Precisely, there are *two* arrows when $x \leq p = (1/3, 1/3)$ and $x' \geq q = (2/3, 2/3)$ (as in the second figure above), and *one* otherwise. This evident fact can be easily proved with the 'van Kampen' theorem recalled above, using the subspaces L, L' (whose fundamental category is the induced order).

Thus, the whole category C is easy to visualise and 'essentially represented' by the full subcategory E on four vertices $0, p, q, 1$ (the central cell does not commute)



E

$$0 \longrightarrow p \rightrightarrows q \longrightarrow 1 \qquad\qquad (2)$$
$$E$$

But E is far from being equivalent to C, as a category, since C *is already a skeleton*, in the ordinary sense. The situation can be analysed as follows, in E :

- the action begins at 0, from where we move to the point p,
- p is an (effective) future branching point, where we have to choose between two paths,
- which join at q, an (effective) past branching point,
- from where we can only move to 1, where the process ends.

(Definitions and properties of *regular* and *branching* points can be found in 5.3, 5.4).

In order to make precise how E can 'model' the category C, we proved in Part I (and will recall below) that E is both *future equivalent* and *past equivalent* to C, and actually it is the 'join' of a minimal 'future model' with a minimal 'past model' of the latter.

1.3. Directed homotopy for categories. This analysis requires, to begin with, some notions of *directed homotopy in* **Cat**, the category of small categories. This elementary theory is based on the *directed interval* $\mathbf{2} = \{0 \to 1\}$, an order category on two objects, with the obvious *faces* $\partial^{\pm} \colon \mathbf{1} \to \mathbf{2}$ defined on the pointlike category $\mathbf{1} = \{*\}$, $\partial^-(*) = 0$ and $\partial^+(*) = 1$.

A *point* $x \colon \mathbf{1} \to C$ of a small category C is an object of the latter; we will also write $x \in C$. A (directed) *path* $a \colon \mathbf{2} \to C$ from x to x' is an arrow $a \colon x \to x'$ of C; concatenation of paths amounts to composition in C (strictly associative, with strict identities). The (directed) *cylinder* functor $I(C) = C \times \mathbf{2}$ and its right adjoint, $P(D) = D^{\mathbf{2}}$ (the category of morphisms of D) show that a (directed) *homotopy* $\varphi \colon f \to g \colon C \to D$ is the same as a natural transformation between functors; *their operations coincide with the 2-categorical structure of* **Cat**.

The existence of a map $x \to x'$ in C (a path) produces the *path preorder* $x \prec x'$ (x *reaches* x') on the points of C; the resulting *path equivalence* relation, meaning that there are maps $x \rightleftarrows x'$, will be written as $x \simeq x'$. For this preorder, a point x is

- *maximal* if it can only reach the points $\simeq x$,
- a *maximum* if it can be reached from every point of C;

(the latter is the same as a *weakly terminal* object, and is only determined up to path equivalence). If the category C 'is' a preorder, the path preorder coincides with the original relation.

For the fundamental category $C = \uparrow\Pi_1(X)$ of a preordered space X, note that the path preorder $x \prec x'$ in C means that there is some directed path from x to x' in X, and *implies* the original preorder in X, which is generally coarser (cf. 1.2).

1.4. Future equivalence of categories. A *future equivalence* $(f, g; \varphi, \psi)$ between the categories C, D (I.2.1) is a symmetric version of an adjunction, *with two units.* It consists of a pair of functors and a pair of natural transformations (i.e., directed homotopies), the units, satisfying two coherence conditions:

$$f\colon C \rightleftarrows D : g \qquad \varphi\colon 1_C \to gf, \qquad \psi\colon 1_D \to fg, \tag{3}$$

$$f\varphi = \psi f\colon f \to fgf, \qquad \varphi g = g\psi\colon g \to gfg \qquad (\textit{coherence}). \tag{4}$$

Note that the *directed homotopies* φ, ψ proceed *from* the identities *to* the composites gf, fg ('in the future'); and that f does not determine g, in general. Future equivalences compose (much in the same way as adjunctions), and yield an equivalence relation of categories.

It will be useful to recall (from I.2.7) that a functor $f\colon C \to D$ which is (part of) a future equivalence, always preserves a *terminal* object, any *weakly terminal* object and any *maximal* point for the path preorder (1.3). Other *future invariant* properties will be recalled in Section 5.

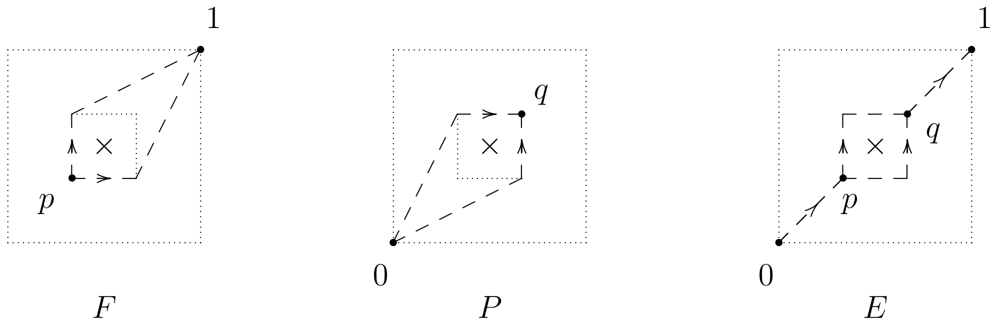
Dually, *past equivalences* have *co*units, in the opposite direction.

An adjunction $f \dashv g$ with *invertible* counit $\varepsilon\colon fg \cong 1$ amounts to a future equivalence with invertible $\psi = \varepsilon^{-1}$. In this case, a 'split' future equivalence, D can be identified with a full reflective subcategory of C, also called a *future retract.* Theorem I.2.5 shows that *two categories are future equivalent if and only if they can be embedded into a common one, as full reflective subcategories.*

A *pf-presentation* (I.4.2) of the category C will be a diagram consisting of a past retract P and a future retract F of C (which are thus a full coreflective and a full reflective subcategory, respectively) with adjunctions $i^- \dashv p^-$ and $p^+ \dashv i^+$

$$P \underset{p^-}{\overset{i^-}{\rightleftarrows}} C \underset{i^+}{\overset{p^+}{\rightleftarrows}} F \qquad \begin{array}{l} \varepsilon\colon i^-p^- \to 1_C \;\; (p^-i^- = 1,\; p^-\varepsilon = 1,\; \varepsilon i^- = 1), \\[2ex] \eta\colon 1_C \to i^+p^+ \;\; (p^+i^+ = 1,\; p^+\eta = 1,\; \eta i^+ = 1). \end{array} \tag{5}$$

1.5. Spectra. Reconsidering our basic example, the square annulus X (1.2), the category $C = \uparrow\Pi_1(X)$ has a least *full reflective* subcategory F, which is future equivalent to C and minimal as such. Its objects form the *future spectrum* $sp^+(C) = \{p, 1\}$ (whose definition is recalled in 5.5); also the full subcategory $F = Sp^+(C)$ on these objects is called a *future* *spectrum* of C

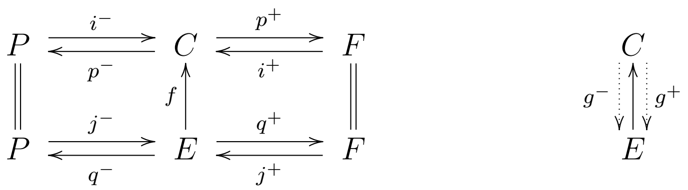


(6)

Dually, we have the least *full coreflective* subcategory $P = Sp^-(C)$, on the *past spectrum* $sp^-(C) = \{0, q\}$.

Together, they form a pf-presentation of C (5), called the *spectral* pf-presentation. Moreover the *pf-spectrum* $E = Sp(C)$ is the full subcategory of C on the set of objects $sp(C) = sp^-(C) \cup sp^+(C)$ (I.7.6). E is a *strongly minimal injective model* of the category C (Thm. I.8.4), in a sense which we recall now.

1.6. Injective models. Given a pf-presentation of the category C (5), be it the spectral one or not, let E be the full subcategory of C on $\mathrm{Ob}P \cup \mathrm{Ob}F$ and $f\colon E \subset C$ its embedding. One can form, in a canonical way, a diagram with four commutative squares (I.4.3)



(7)

Adding the two functors $g^\alpha = j^\alpha p^\alpha : C \to E$ (where $\alpha = \pm$), E becomes the *injective model* of the category C *associated* to the given pf-presentation.

This means that we have a *pf-injection*, or *pf-embedding*, $E \leftrightarrows C$ (I.3.4), formed of a *full embedding* $f\colon E \to C$ (full, faithful and injective on objects) together with a *past equivalence* $(f, g^-; \varepsilon_E, \varepsilon)$ and a *future* one $(f, g^+; \eta_E, \eta)$

$$\begin{array}{llll} f\colon E \leftrightarrows C : g^-, g^+, & & & \\ \varepsilon_E\colon g^- f \to 1_E, & \varepsilon\colon fg^- \to 1, & \eta_E\colon 1_E \to g^+ f, & \eta\colon 1 \to fg^+, \\ f\varepsilon_E = \varepsilon f\colon fg^- f \to f, & & \varepsilon_E g^- = g^-\varepsilon\colon g^- f g^- \to g^-, & \\ f\eta_E = \eta f\colon f \to fg^+ f, & & \eta_E g^+ = g^+\eta\colon g \to g^+ f g^+. & \end{array} \qquad (8)$$

(A coherence property between these two structures automatically holds, I.3.3. By I.3.4, it suffices to assign the three functors f, g^-, g^+ - the first being a full embedding - together with the natural transformations ε and η, under the conditions $fg^-\varepsilon = \varepsilon fg^-$, $fg^+\eta = \eta fg^+$.)

E is called a *minimal injective model* of C (I.5.2) if:

(i) E is an injective model of every injective model E' of C,

(ii) every injective model E' of E is isomorphic to E.

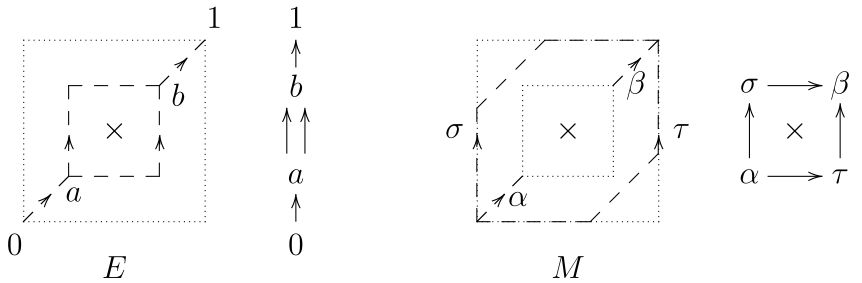
Plainly, E is determined up to isomorphism. We also say that E is a *strongly minimal injective model* if it satisfies the stronger condition (i′), together with (ii):

(i′) E is an injective model of every category injectively equivalent to C,

where two categories are said to be *injectively equivalent* if they can be linked by a finite chain of pf-embeddings, forward or backward (I.4.1). We have already recalled that the *spectral injective model* (associated to the spectral pf-presentation) is always the strongly minimal injective model of C (I.8.4).

1.7. Surjective and projective models. An alternative 'description' of a category can be obtained with the *spectral projective model* $p\colon C \to M$. Now, M is the full subcategory of the category $C^{\mathbf{2}}$ containing the morphisms of C of type $p(x) = \eta x.\varepsilon x\colon i^-p^-x \to i^+p^+x$, obtained from the adjunctions $i^- \dashv p^-$, $p^+ \dashv i^+$ of the past and future spectrum of C.

For the square annulus, we get thus the full subcategory of $C^{\mathbf{2}}$ on the four maps $\alpha, \beta, \sigma, \tau$



(9)

Note that our functor $p\colon C \to M$ is surjective on objects but *not* full (it identifies points of C which are not comparable in the path order). Interestingly, this model M *is isomorphic to the 'category of components' of* C constructed in [5] as a category of fractions of C.

To recall these definitions, a category M is made a *surjective model* of C by assigning a *pf-surjection* (I.3.4) $p\colon C \rightleftarrows M : g^\alpha$. This consists of functor $p\colon C \to M$ which is *surjective on objects* and (again) belongs to a past equivalence $(p, g^-; \varepsilon, \varepsilon_M)$ and a future equivalence $(p, g^+; \eta, \eta_M)$

$$\begin{array}{ll}
p\colon C \rightleftarrows M : g^-, g^+, & \\
\varepsilon\colon g^-p \to 1_C, \qquad \varepsilon_M\colon pg^- \to 1_M, & \eta\colon 1_C \to g^+p, \qquad \eta_M\colon 1_M \to pg^+, \\
p\varepsilon = \varepsilon_M p\colon pg^-p \to p, & \varepsilon g^- = g^-\varepsilon_M\colon g^-pg^- \to g^-, \\
p\eta = \eta_M p\colon p \to pg^+p, & \eta g^+ = g^+\eta_M\colon g^+ \to g^+pg^+.
\end{array} \tag{10}$$

(Also here, a coherence property between these two structures automatically holds, by I.3.3.) We speak of a *pf-projection* (and a *projective* model) when, moreover, the associated *comparison* functor $g\colon M \to C^{\mathbf{2}}$ (corresponding to the natural transformation $\varepsilon g^+.g^-\eta_M\colon g^- \to g^+$) is a full embedding. Again, *projective equivalence* of categories is defined by a finite chain of pf-projections, forward or backward.

Given a (spectral) pf-presentation of the category C (1.4), there is an associated *spectral projective model* M of C, constructed as follows (Thm. I.4.6)

$$\begin{array}{ccccc} P & \underset{p^-}{\overset{i^-}{\rightleftarrows}} & C & \underset{i^+}{\overset{p^+}{\rightleftarrows}} & F \\ \| & & \downarrow p & & \| \\ P & \underset{q^-}{\overset{j^-}{\rightleftarrows}} & W & \underset{j^+}{\overset{q^+}{\rightleftarrows}} & F \end{array} \qquad \begin{array}{c} C \\ g^- \uparrow \downarrow \uparrow g^+ \\ W \end{array} \qquad \begin{array}{c} C \\ \uparrow \downarrow \uparrow \\ M \end{array} \tag{11}$$

The lower row is the canonical factorisation of the composed adjunction $P \rightleftarrows F$ (I.4.4), through its graph, the category W, which (here) can be embedded as a full subcategory of $C^{\mathbf{2}}$

$$W = (P \mid p^-i^+) = (p^+i^- \mid F) = (i^- \mid i^+) \subset C^{\mathbf{2}}. \tag{12}$$

Then, there is a pf-equivalence $p\colon C \leftrightarrows W : g^\alpha$, with

$$\begin{array}{lll} p(x) = (p^-x, p^+x; \eta x.\varepsilon x), & g^-(x,y;w) = i^-x, & g^+(x,y;w) = i^+y, \\ g^\alpha p = i^\alpha p^\alpha, & j^\alpha = pi^\alpha, & \end{array} \tag{13}$$

which inherits the counit ε from the adjunction $i^- \dashv p^-$ and the unit η from $p^+ \dashv i^+$; its comparison $g\colon W \to C^{\mathbf{2}}$ coincides with the full embedding $(i^- \mid i^+) \subset C^{\mathbf{2}}$.

Finally, replacing W with the full subcategory M of objects of type px (for $x \in C$) we have a projective model $p\colon C \leftrightarrows M$. The adjunctions of the lower row can be restricted to M (since $j^\alpha = pi^\alpha$), so that P and F are also, canonically, a past and a future retract of M.

1.8. Directed equivalence and contractibility. The basic notions of future and past equivalence of categories yield various notions of 'directed equivalence' of categories. First, their conjunction is called *past and future equivalence*, while *coarse equivalence* is the equivalence relation generated by them; the latter coincides with the equivalence relation generated by the existence of an adjunction between two categories (by I.2.5, I.4.4). More complex combinations give *injective equivalence* (1.6) and *projective equivalence* (1.7).

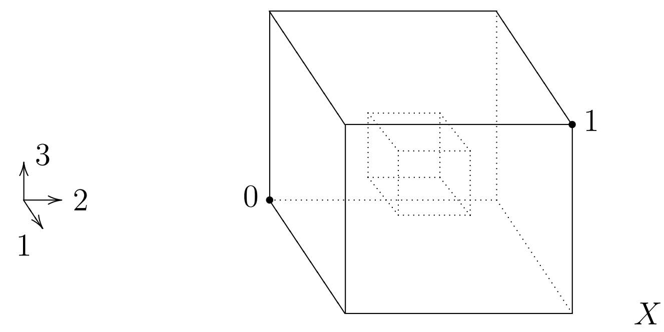
Correspondingly, we have various notions of contractibility in **Cat**. First, we say that a category X is *future contractible* if it is future equivalent to $\mathbf{1}$ (1.3). It is easy to see that this happens if and only if X has a terminal object (I.2.6). Dually, a category is *past contractible* if it is past equivalent to $\mathbf{1}$, which amounts to saying that it has an initial object.

Further, a category X is said to be *injectively contractible* if it is injectively equivalent to $\mathbf{1}$ (1.6), which is true if and only if X has a zero object (I.5.4). On the other hand, a category X with *non-isomorphic* initial and terminal object is injectively modelled by the ordinal $\mathbf{2} = \{0 \to 1\}$, with an obvious pf-embedding $i\colon \mathbf{2} \leftrightarrows X : g^\alpha$ (*not* split) which is actually *the spectral injective model* of X (1.6). Notice that the category $\mathbf{2}$ itself, which is the *standard interval* of $\mathbf{Cat}$ (1.3), is *not* contractible in this sense.

Finally, the existence of the initial and terminal objects is sufficient (and necessary) to make a category X *projectively equivalent* to $\mathbf{1}$, via the pf-projection $p\colon X \rightleftarrows \mathbf{1} : i^\alpha$, with $i^-(*) = 0$ and $i^+(*) = 1$. In other words, this condition is equivalent to being past contractible and future contractible. (While, in general, injective equivalence is stronger than past and future equivalence.)

1.9. The hollow cube. The analysis recalled above, based on the fundamental category, gives relevant information for *planar* ordered spaces (subspaces of $\uparrow\mathbf{R}^2$), also in complicated examples (see I.9 and 4.2-4.3). But it may be insufficient for higher dimensional singularities.

The simplest case is a 3-dimensional analogue of our previous example, the 'hollow cube' $X \subset \uparrow[0,1]^3$ represented below (again an ordered compact space):



(14)

$$X = \uparrow[0,1]^3 \setminus A, \qquad\qquad A =]1/3,\, 2/3[^3.$$

The fundamental category $C = \uparrow\Pi_1(X)$ seems to say little about this space: C has an initial and a terminal object ($0 = (0,0,0)$ and $1 = (1,1,1)$), whence it is future contractible (to its object 1) and past contractible as well (to 0); its minimal injective model is the category $\mathbf{2} = \{0 \to 1\}$ (1.6).

Now, this injective model is *not* faithful, in the sense that, of the three functors $\mathbf{2} \rightleftarrows C$, those starting at C are not faithful. In fact, the category C is not a preorder, since $C(x, y)$ contains two arrows when x, y are suitably placed 'around' the obstruction; a phenomenon which only appears within *directed* homotopy theory: the fundamental group of the underlying topological space is trivial, and the fundamental groupoid is codiscrete (one arrow between any two given points). We shall therefore try to extract a better information from C, using a *partially faithful* surjective model $C \to M$.

Another approach, followed in [12], is based on studying the fundamental 2-category $C_2 = \uparrow\Pi_2(X)$, trying to reproduce one dimension up the previous study of $\uparrow\Pi_1(X)$, for the 'square annulus'. This can also be done with more interesting *lax* versions [13, 14].

2. Normal quotients of categories

Generalised quotients of categories, examined here, will be useful to construct surjective models. For the equivalence relations $\sim^+$, $\sim^-$ and $\sim^\pm$, see 5.1.

2.1. GENERALISED CONGRUENCES OF CATEGORIES. First, let us recall that a very general notion of *generalised congruence* in a category can be found in a paper by Bednarczyk, Borzyszkowski and Pawlowski [2]: like the generalised congruence associated to an arbitrary functor, it involves objects and sequences of arrows which are just 'composable up to equivalence of objects'. As shown there, the generalised congruences of a category form a complete lattice, and we can always take the intersection of all the generalised congruences containing a certain relation.

A *quotient* of categories will be viewed with respect to this notion. As in [15], we only use a particular case (still involving objects), *determined by the maps which we want to become identities*. More precisely, given a category X and a set A of its arrows, X/A denotes the quotient of X modulo the generalised congruence generated by declaring every arrow in A to be equivalent to the identity of its domain.

The quotient $p\colon X \to X/A$ is determined by the obvious universal property:

(i) for every functor $f\colon X \to Y$ which takes all the morphisms of A to identities, there is a unique functor $f'\colon X/A \to Y$ such that $f = f'p$.

These quotients of the category X also form a complete lattice, with the same arbitrary meets as the general quotients of X (in the previous sense): in fact, the meet of a family $p_i\colon X \to X/A_i$ is the quotient of X modulo the set-theoretical union of the sets A_i.

2.2. KERNELS AND NORMAL QUOTIENTS. The particular case we are interested in can be made clearer when viewed at the light of general considerations on kernels and cokernels with respect to an *assigned ideal* of 'null' arrows, studied in [9] - independently of the existence of a zero object. (See also Ehresmann [3], including the Comments in the same volume, p. 845-847; and Lavendhomme [17].)

Take, in **Cat**, the ideal of *discrete* functors, i.e. those functors which send every map to an identity; or, equivalently, consider as *null objects* in **Cat** the discrete categories and say that a functor is null if it factors through such a category (we have thus a *closed* ideal, according to an obvious Galois connection between sets of objects and sets of maps in a category, see [9]).

This ideal produces - by the usual universal properties *formulated with respect to null functors* - a notion of kernels and cokernels in **Cat**. Precisely, given a functor $f\colon X \to Y$, its kernel is the wide subcategory of all morphisms of X which f sends to identities of Y, while its cokernel is the quotient $Y \to Y/B$, produced by the set B of arrows of Y reached by f.

A *normal subcategory* $X_0 \subset X$, by definition, is a kernel of some functor starting at X, or, equivalently, the kernel of the cokernel of its embedding. It is necessarily a wide subcategory, and must satisfy the 'two-out-of-three property': if, in a composite $c = ba$ two maps are in X_0, also the third is.

Dually, a *normal quotient* $p\colon X \to X'$ is the cokernel of some functor with values in X (or, equivalently, the cokernel of its kernel). A normal quotient (or, more generally, any quotient modulo a generalised congruence) is always surjective on objects, as it follows easily using its factorisation through its full image, $p = jq\colon X \to X'' \subset X'$. (The functors p and q have clearly the same kernel, whence also q factors through p, as $q = hp$; moreover, $jh = 1$, by the universal property of p; it follows that $jhj = j$ and, cancelling the embedding, $hj = 1$.)

Now, the normal quotients of X are precisely those we are interested in. First, we already know that a normal quotient is always of the type $X \to X/A$. Conversely, given a set A of arrows of X, the quotient of $X \to X/A$ is the cokernel of some functor with values in X; for instance, we can take the free category A' on the graph A and the resulting functor $A' \to X$.

The normal subcategories of a category X and its normal quotients form thus two complete lattices, anti-isomorphic via kernels and cokernels

(Similarly, the ideal in **Cat** of those functors which send all maps to isomorphisms would give, as normal quotients, the categories of fractions.)

2.3. Lemma. [The 2-dimensional universal property] *The normal quotient* $p\colon X \to X/A$ *satisfies (after (i) in 2.1), a 2-dimensional universal property:*

(ii) for every natural transformation $\varphi\colon f \to g\colon X \to Y$*, where* f *and* g *take all the morphisms of* A *to identities of* Y*, there is a unique natural transformation* $\varphi'\colon f' \to g'\colon X/A \to Y$ *such that* $\varphi = \varphi' p$*.*

(More generally, all quotients of categories in the sense recalled above satisfy a similar 2-dimensional universal property, with a similar proof.)

Proof. We have already recalled (in 1.3) that a natural transformation $\varphi\colon f \to g\colon X \to Y$ can be viewed as a functor $\varphi\colon X \to Y^{\mathbf{2}}$, which, when composed with the functors $\mathrm{Dom}, \mathrm{Cod}\colon Y^{\mathbf{2}} \to Y$ gives f and g, respectively. This functor sends the object $x \in X$ to the arrow $\varphi(x)\colon f(x) \to g(x)$ and the map $a\colon x \to x'$ to its naturality square

$$
\begin{array}{ccc}
f(x) & \xrightarrow{\ \varphi z\ } & g(x) \\
{\scriptstyle fa}\downarrow & & \downarrow{\scriptstyle ga} \\
f(x') & \xrightarrow[\ \varphi x'\ ]{} & g(x')
\end{array}
\qquad (15)
$$

Therefore, if $a \in A$, $f(a)$ and $g(a)$ are identities and $\varphi(a) = \mathrm{id}\varphi(x)$. Then, by the original universal property 2.1(i), the functor φ factors uniquely through p, by a functor $\varphi'\colon X/A \to Y^{\mathbf{2}}$; this is the natural transformation that we want. ∎

2.4. Theorem. *A normal quotient* $p\colon X \to X'$ *is given.*

(a) For two objects x, x' *in* X*, we have* $p(x) = p(x')$ *if and only if there exists a finite zig-zag* $(a_1, \ldots, a_n)\colon x \dashrightarrow x'$ *of morphisms* a_i *in* $Ker(p)$*, as below (the dashed arrow recalls*

that this sequence is not a map of X)

$$x = x_0 \xrightarrow{a_1} x_1 \xleftarrow{a_2} x_2 \quad \cdots \quad x_{2n-2} \xrightarrow{a_{2n-1}} x_{2n-1} \xleftarrow{a_{2n}} x_{2n} \qquad (16)$$

Since p *is surjective on objects (2.2),* $\mathrm{Ob}X'$ *can be identified with the quotient of* $\mathrm{Ob}X$ *modulo the equivalence relation of connection in* $Ker(p)$. *(Recall that two objects in a category are said to be connected if they are linked by a zig-zag of morphisms.)*

(b) A morphism $z\colon p(x) \to p(x')$ *in the quotient* X' *comes from a finite zig-zag as above where the backward arrows* a_{2i} *are in* $Ker(p)$, *and* $z = p(a_{2n-1})....p(a_3).p(a_1)$. *Thus, every arrow of* X' *is a composite of arrows in the graph-image of* p.

PROOF. (a) Let R be the generalised congruence of X generated by $Ker(p)$, and R_0 the equivalence relation which it induces on the objects; plainly, the equivalence relation R'_0 described above (by the existence of a diagram (16)) is contained in R_0. To show that it coincides with the latter, let R' be the relation given by R'_0 on the objects and as 'chaotic' on morphisms as possible:

$$a\, R'_1\, b \qquad \text{if} \qquad (\mathrm{Dom}(a)\, R'_0\, \mathrm{Dom}(b) \text{ and } \mathrm{Cod}(a)\, R'_0\, \mathrm{Cod}(b)). \qquad (17)$$

R' is a generalised congruence of X, whose quotient is the preorder category on $\mathrm{Ob}(X)/R'_0$ having one morphism from $[x]$ to $[x']$ whenever there is a chain of maps of X, composable up to R'_0, from x to x'.

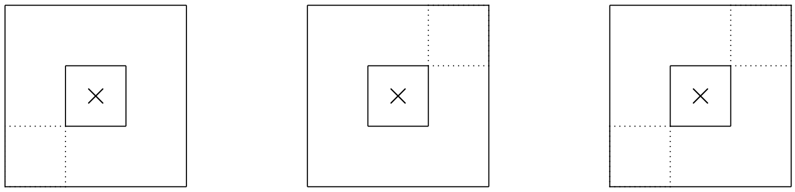
Therefore, the intersection $R \cap R'$ is again a congruence of categories. But $X/(R \cap R')$ plainly satisfies the universal property of R, whence $R = R \cap R'$, which means that $R_0 \subset R'_0$.

Finally, by the general theory of [2], a morphism $z\colon p(x) \to p(x')$ of the quotient category is the equivalence class of a finite sequence $(b_0, \dots, b_p)$ of maps in X which are 'composable up to the previous equivalence relation on objects'; inserting a zig-zag as above between all pairs $\mathrm{Cod}(b_{i-1}), \mathrm{Dom}(b_i)$, and identities where convenient, we can always form a global zig-zag (16), from x to x', where all 'backward' arrows are in A; and then $p(a_{2n-1}) \dots p(a_3).p(a_1) = z$. ■

2.5. PF-EQUIVALENCE OF OBJECTS. In Part I, we have defined the *future regularity equivalence* relation $x \sim^+ x'$ between two objects x, x' in a category X, meaning that there is a finite zig-zag $x \dashrightarrow x'$ (as in (16)), made of *future regular maps* (see 5.1). Its dual is written $x \sim^- x'$, and we will write $x \sim^\pm x'$ to mean that both these relations hold.

But we are more interested in a stronger equivalence relation (introduced here): we say that the objects x, x' are *pf-equivalent* if there is a finite zig-zag $x \dashrightarrow x'$ whose maps are both *past and future regular*.

For instance, in the fundamental category of the square annulus, there are two equivalence classes for $\sim^-$ (in the left picture below) and two classes for $\sim^+$ (in the central picture)



(18)

which produce 3 classes for $\sim^{\pm}$ (their intersections) but 4 classes of pf-equivalence, since the points of the two L-shaped zones in the right picture above (open in X) are $\sim^{\pm}$-equivalent, but not connected. We shall see that, if the category X has a past and a future spectrum, its pf-equivalence classes coincide with the connected components of the $\sim^{\pm}$-equivalence classes (2.8(c)), as in the example above.

One can note that, in the example above and in many planar examples, pf-equivalence coincides with a relation considered in [4], Def. 4.5, and called 'homotopy history equivalence'.

2.6. Lemma. *If $p\colon X \to M$ is a quotient model, then $p(x) = p(x')$ implies that x, x' are pf-equivalent.*

Proof. By 2.4, if $p(x) = p(x')$ there is a finite zig-zag $x \dashrightarrow x'$ of morphisms a_i which p takes to identities. But p, as a past and future equivalence, reflects past and future regular morphisms (cf. 5.2), whence all a_i are past and future regular. ■

2.7. Lemma. [Quotients and retracts] *(a) Given a retract $i\colon M \rightleftarrows X : p$ (with $pi = 1_M$), the following conditions are sufficient to ensure that p is a normal quotient:*

(i) for every $x \in X$ there exists a zig-zag $(a_1, \dots, a_n)\colon ip(x) \dashrightarrow x$ in $Ker(p)$,

(ii) for every map $u\colon x \to y$ in X there exists a zig-zag $((a_1, b_1) \dots, (a_n, b_n))\colon ip(u) \dashrightarrow u$ in the category of morphisms $X^{\mathbf{2}}$, whose arrows a_i, b_i belong to $Ker(p)$

$$\begin{array}{ccccccccc}
x_0 & \xrightarrow{a_1} & x_1 & \xleftarrow{a_2} & x_2 & \xrightarrow{a_3} & \cdots & \xleftarrow{a_{2n}} & x \\
{\scriptstyle ip(u)}\downarrow & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_2} & & & & \downarrow{\scriptstyle u} \\
y_0 & \xrightarrow[b_1]{} & y_1 & \xleftarrow[b_2]{} & y_2 & \xrightarrow[b_3]{} & \cdots & \xleftarrow[b_{2n}]{} & y
\end{array} \qquad (19)$$

(The condition (i) is a consequence of (ii); nevertheless, it makes things clearer.)

(b) In a future retract $i\colon F \rightleftarrows X : p$, the functor p is always a normal quotient.

Proof. (a) Let $f\colon X \to Y$ be any functor with $Ker(p) \subset Ker(f)$. Plainly, $p(x) = p(x')$ implies $f(x) = f(x')$ and $p(u) = p(u')$ implies $f(u) = f(u')$, for all objects x, x' and all maps u, u' in X. Since $p(ip) = p$, it follows that $f = f(ip) = (fi)p$. Therefore, f factors through p, obviously in a unique way.

(b) Follows easily from (a), since for every $x \in X$, the unit $\eta x\colon ip(x) \to x$ belongs to $Ker(p)$ (because of the coherence condition $p\eta = 1$). For (ii), just apply the naturality of η. ■

2.8. LEMMA. [Quotients and spectra] *(a) If the category X has a future spectrum, the retraction $p^+\colon X \to Sp^+(X)$ is a normal quotient. Its kernel is characterised by the following equivalent conditions, on a map $a\colon x \to x'$*

(i) $p^+(a)$ is an identity,

(ii) a is future regular,

(iii) $x \sim^+ x'$.

(b) Dually, the projection $p^-\colon X \to Sp^-(X)$ on the past spectrum (if it exists) is a normal quotient of X, whose kernel is the set of past regular morphisms.

(c) If the category X has a past and a future spectrum, then its pf-equivalence classes coincide with the connected components of $\sim^\pm$-equivalence classes (connected by zig-zags of maps in the class).

PROOF. (a) After 2.7(b), we have only to prove the characterisation of $Ker(p^+)$. In fact, if $p^+(a)$ is an identity, then a is future regular (since p^+ reflects such morphisms), which implies (iii). Finally, if $x \sim {}^+x'$, then $\eta x'.a = \eta x\colon x \to i^+p^+(x)$ (because $i^+p^+(x)$ is terminal in the full subcategory on $[x]^+$); but $p^+(\eta x)$ and $p^+(\eta x')$ are identities, and so is $p^+(a)$.

(c) Follows from (a) and its dual, which prove that a map $a\colon x \to x'$ is past and future regular if and only if $x \sim^\pm x'$. ■

3. Quotient models

We study now surjective models of categories which are normal quotients.

3.1. QUOTIENT MODELS AND MINIMALITY. A surjective model $p\colon X \leftrightarrows M : g^\alpha$ (1.7) will be said to be a *quotient model* of X if p is a normal quotient; and a *semi-faithful* quotient model if, moreover, p is faithful.

Also here, a quotient model will often be represented as $p\colon X \to M$, leaving the remaining structure understood:

$$\begin{array}{lll} p\colon C \leftrightarrows M : g^-, g^+, & & \\ \varepsilon\colon g^-p \to 1_C, & \varepsilon_M\colon pg^- \to 1_M & (p\varepsilon = \varepsilon_M p, \ \ \varepsilon g^- = g^-\varepsilon_M), \\ \eta\colon 1_C \to g^+p, & \eta_M\colon 1_M \to pg^+ & (p\eta = \eta_M, \ \ \eta g^+ = g^+\eta_M). \end{array} \tag{20}$$

A *minimal quotient model* is defined (up to isomorphism of categories) by the following properties (similar to the ones in 1.6):

(i) M is a quotient model of every quotient model M' of X,

(ii) every quotient model M' of M is isomorphic to M.

3.2. Pf-presentations and quotient models. Let us start from a pf-presentation (5) of the category X (not necessarily produced by spectra)

$$P \underset{p^-}{\overset{i^-}{\rightleftarrows}} X \underset{i^+}{\overset{p^+}{\rightleftarrows}} F \qquad \begin{array}{l} \varepsilon\colon i^-p^- \to 1_X \ \ (p^-i^- = 1,\ p^-\varepsilon = 1,\ \varepsilon i^- = 1), \\ \\ \eta\colon 1_X \to i^+p^+ \ \ (p^+i^+ = 1,\ p^+\eta = 1,\ \eta i^+ = 1). \end{array} \tag{21}$$

We say that a normal quotient $p\colon X \to M$ is *consistent* with this pf-presentation if the functors $p^-\colon X \to P$ and $p^+\colon X \to F$ factorise (uniquely) through p

$$q^\alpha\colon M \to P, \qquad q^-p = p^-, \qquad q^+p = p^+, \tag{22}$$

or, equivalently, if the kernel of p (i.e. the set of morphisms which p takes to identities, cf. 2.2) is contained in the set of past and future regular maps.

The next theorem shows that M is then a quotient model of X, admitting the same past and future retracts P and F. This fact will be of interest when the projection p is faithful, *while the injective and projective models associated to the given presentation* (1.6, 1.7) *are not faithful*.

3.3. Theorem. [Presentations and quotients] *Let us be given a pf-presentation of the category X and a consistent normal quotient $p\colon M \to X$ (as above, 3.2).*

(a) There is a (uniquely determined) induced pf-presentation of M, so that the four squares of the following left diagram commute

$$\begin{array}{ccccc} P & \underset{p^-}{\overset{i^-}{\rightleftarrows}} & X & \underset{i^+}{\overset{p^+}{\rightleftarrows}} & F \\ \| & & \downarrow p & & \| \\ P & \underset{q^-}{\overset{j^-}{\rightleftarrows}} & M & \underset{j^+}{\overset{q^+}{\rightleftarrows}} & F \end{array} \qquad\qquad \begin{array}{c} X \\ g^- \uparrow\ \downarrow\ \uparrow g^+ \\ M \end{array} \tag{23}$$

(b) There is an associated quotient model $p\colon X \rightleftarrows M : g^\alpha$, with

$$g^\alpha = i^\alpha q^\alpha \qquad (\alpha = \pm), \tag{24}$$

$$g^\alpha p = i^\alpha p^\alpha, \qquad pg^\alpha = j^\alpha q^\alpha, \qquad g^\alpha p g^\alpha = g^\alpha \tag{25}$$

so that, in particular, p is a past and future equivalence. (This model need not be projective, in the sense of 1.7).

Proof. (a) The new retractions are (and must be) defined taking $j^\alpha = pi^\alpha$ and the (uniquely determined) functors q^α such that $q^\alpha p = p^\alpha$ (cf. (22)).

Now (according to 2.3(ii)), the natural transformations $p\varepsilon\colon (j^-q^-)p \to 1_X p$ and $p\eta\colon 1_X p \to (j^+q^+)p$ induce two natural transformations, characterised by the following relations:

$$\begin{array}{ll} \varepsilon'\colon j^-q^- \to 1_M, & \qquad \varepsilon' p = p\varepsilon, \\ \eta'\colon 1_M \to j^+q^+, & \qquad \eta' p = p\eta. \end{array} \tag{26}$$

Thus, P becomes a past retract of M (and F a future retract):

$$\varepsilon' j^- = \varepsilon' p i^- = p\varepsilon i^- = 1, \qquad (q^-\varepsilon')p = q^- p\varepsilon = p^-\varepsilon = 1_{p^-} = (1_{q^-}).p. \tag{27}$$

(b) Let us define the functors g^α as in (24), which implies (25). The functor p is surjective on objects. The pair (p, g^-) becomes a past equivalence with the original ε and the previous ε'

$$\varepsilon\colon g^- p \to 1_X, \qquad \varepsilon'\colon pg^- \to 1_M. \tag{28}$$

As to coherence, we already know that $p\varepsilon = \varepsilon' p$; moreover:

$$\varepsilon g^- = (\varepsilon i^-)q^- = 1, \qquad g^-\varepsilon' = i^-(q^-\varepsilon') = 1. \tag{29}$$

Symmetrically, one shows that (p, g^+, η, η') is a future equivalence between X and M, noting that $\eta\colon 1_X \to i^+p^+ = g^+p$ and $\eta'\colon 1_X \to j^+q^+ = pg^+$.

For the last remark, it suffices to note that the associated functor $g\colon M \to X^{\mathbf{2}}$ (1.7)

$$g(x) = \varepsilon g^+(x).g^-\eta'(x) = g^+\varepsilon'(x).\eta g^-(x)\colon g^-(x) \to g^+(x), \tag{30}$$

can even be constant (on objects and morphisms). For instance, take the fundamental category of the hollow cube (1.9): the past spectrum P reduces to the initial object, the future spectrum F reduces to the terminal object, and one *can* take as p the functor $X \to \mathbf{1}$. (Of course, the interesting quotient model is not this one; see 4.4.) ∎

3.4. THEOREM. [Spectral presentations and quotients] *Assume now that (23) is the spectral presentation of X.*

(a) If $p\colon X \to M$ is a normal quotient, consistent with the spectral pf-presentation of X, and surjective on maps, then the induced pf-presentation of M (3.3(a)) is also a spectral presentation. (We already know, by 3.3(b), that p is a quotient model of X, in a canonical way.)

(b) Let $p\colon X \to M$ be a normal quotient whose kernel is precisely the set of morphisms which are both past and future regular in X. Then, p is a quotient model of X, consistent with the spectral presentation of X. Moreover, if p is surjective on maps, it is the minimal quotient model of X.

(c) The projective model $p\colon X \to M \subset X^{\mathbf{2}}$ associated to the spectral pf-presentation (1.7) has for kernel the set of past and future regular morphisms of X. If this functor p is a normal quotient of X surjective on maps, then it is the minimal quotient model of X.

PROOF. (a) Note that p is a past and future equivalence (3.3). We use the characterisation of future spectra recalled in 5.5(a), to show that, since $i^+\colon F \rightleftarrows X : p^+$ is a future spectrum, also $j^+\colon F \rightleftarrows M : q^+$ is. (The dual fact holds for P.)

First, the category F has precisely one object in each future regularity class of M (since $j^+ = pi^+$ and p, as a future equivalence, gives a bijective correspondence between future regularity classes of X and M, cf. 5.2). Second, we already know that $j^+\colon F \rightleftarrows M : q^+$ is a future retract (3.3).

Third, for $y \in M$, we have to prove that the unit-component $\eta' y \colon y \to j^+q^+(y)$ is the unique M-morphism with these endpoints. By hypothesis, if $b \colon y \to j^+q^+(y)$ is an M-morphism, there is some $a \colon x \to x'$ such that $p(a) = b$; note that the objects x, x' are $\sim^+$-equivalent, because p reflects this relation. Now, consider the composite

$$a' = \eta x'.a \colon x \to i^+p^+(x'), \qquad i^+p^+(x') = i^+p^+(x), \tag{31}$$

where $p(\eta x') = \eta' p(x') = \eta' j^+q^+(y)$ is an identity and $p(a') = p(a)$. This a' must be the only X-morphism from x to $i^+p^+(x)$, i.e. $a' = \eta x$ (because F is a future retract of X). Finally

$$b = p(a) = p(a') = p(\eta x) = \eta' p(x) = \eta' y. \tag{32}$$

(b) The consistency of p is obvious, by definition (3.2), and we only have to prove that p is a *minimal* quotient model (assuming it is surjective on maps).

First, if a quotient model $p' \colon X \to M'$ sends the morphism a to an identity, a must be past and future regular in X (since this property is reflected by all functors which are both past and future equivalences, see 5.2), which means that a is sent to an identity in M; it follows that p' factors through p. Second, given a quotient model $q \colon M \to M'$ which sends the morphism b to an identity, then b must be past and future regular in M; but $b = p(a)$ for some a in X, which must be past and future regular, whence b is an identity; thus, the kernel of q is discrete and q is an isomorphism.

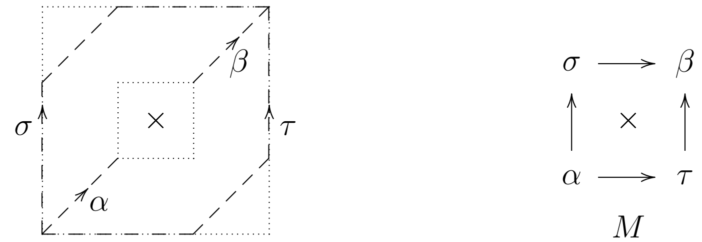
(c) Our functor p sends an object $x \in X$ to the arrow $p(x) = \eta x.\varepsilon x \colon i^-p^-(x) \to i^+p^+(x)$, and the arrow a to the pair $(i^-p^-(a), i^+p^+(a))$. The latter is an identity if and only if a is past and future regular in X (by 2.8). The last assertion follows from (b). ■

4. Applications

We apply here the previous results. First, we discuss some projective models of planar ordered spaces already considered in Part I, showing that they are *minimal quotient models*. Then, we deal with a semi-faithful quotient model of the hollow cube, consistent with the analysis of this ordered space in [5] and [15]. And we end with considering the *open* square annulus.

4.1. Reviewing the square annulus. Let us reconsider the fundamental category $C = \uparrow\Pi_1(X)$ of the square annulus X (1.2), and its spectral projective model $p \colon C \leftrightarrows M : g^\alpha$ (1.7)

$$p \colon C \leftrightarrows M : g^\alpha, \qquad p(x) = \eta x.\varepsilon x \colon i^-p^-x \to i^+p^+x, \tag{33}$$



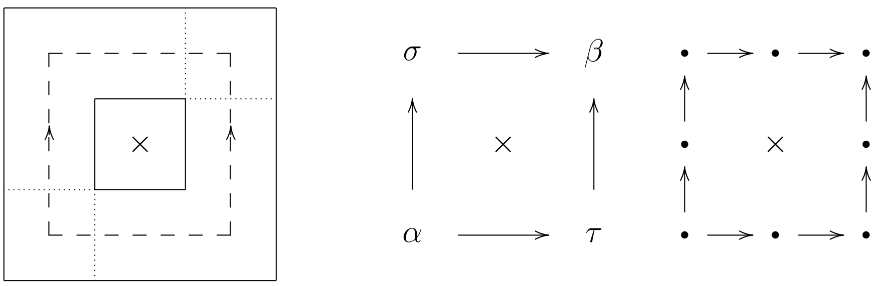
M (34)

The space X is decomposed into four classes, the pf-equivalence classes of C (2.5)

$$\begin{array}{llr} p^{-1}(\alpha) = [0,1/3]^2, \qquad p^{-1}(\beta) = [2/3,1]^2 & & \text{(closed in } X\text{)}, \\ p^{-1}(\sigma) = X \cap ([0,2/3[\times]1/3,1]), & & \text{(open in } X\text{)}, \\ p^{-1}(\tau) = X \cap (]1/3,1] \times [0,2/3[) & & \text{(open in } X\text{)}. \end{array} \qquad (35)$$

This projective model can be obtained as above, in Thm. 3.3, starting from the spectral presentation of C and a retract $i\colon M \rightleftarrows C : p$, where i is any section of p, i.e. any choice of points in the four classes satisfying the following inequalities

$$i(\alpha) \leq i(\sigma) \leq i(\beta), \qquad i(\alpha) \leq i(\tau) \leq i(\beta), \qquad (36)$$

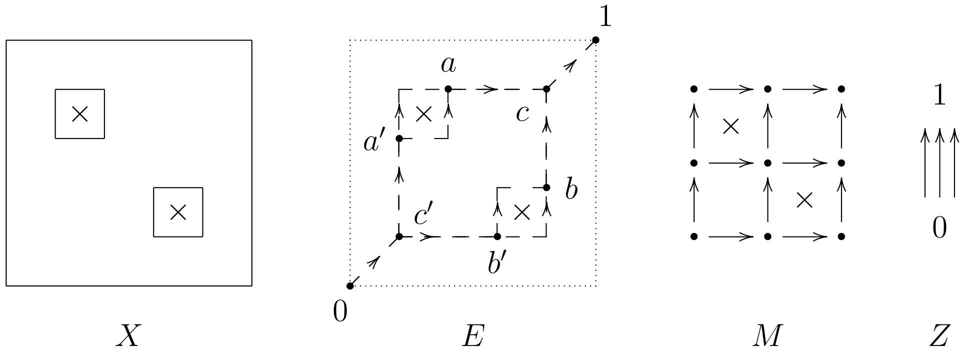


X M M' (37)

Now, the functor p is a normal quotient by 2.7(a), and satisfies the hypothesis of 3.4 (c) (its kernel is precisely the set of past and future regular maps of C) so that p is *the* minimal quotient model of C.

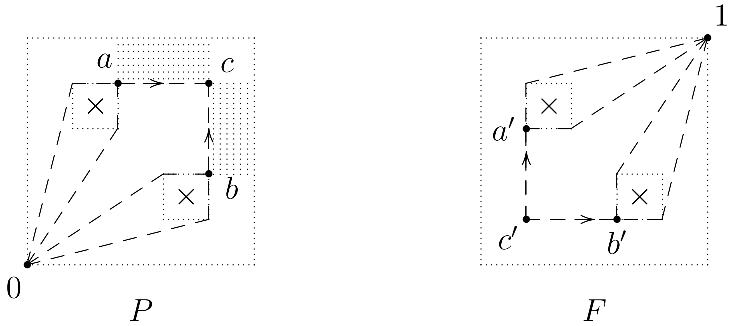
The procedure of Thm. 3.3 can also be applied to a larger retract $i'\colon M' \rightleftarrows C : p'$ (see the right figure above), taking for instance the middle points of all the eight 'subsquares' of X of edge 1/3, 'around the obstruction' (with some arbitrary choice in defining the eight equivalence classes of p'). Plainly, there would be no advantage in enlarging the model; but similar enlargements can be of interest *when the spectral projective model is unsatisfactory* - e.g. not faithful, as it happens with the hollow cube (see 4.4).

4.2. A SECOND EXAMPLE. Consider, in the category p**Top**, the (compact) ordered space X: a subspace of the standard ordered square $\uparrow[0,1]^2$ obtained by taking out two *open* squares (marked with a cross), as in the left figure below



(38)

The fundamental category $C = \uparrow\Pi_1(X)$ is easy to determine. Its past spectrum (I.9.2) is the full subcategory P on $sp^-(C) = \{0, a, b, c\}$; the points a, b, c are effective V^--branching points (5.3), while 0 is the global minimum, weakly initial in C



(39)

Symmetrically, we have the future spectrum: the full subcategory $F \subset C$ in the right figure above, on four objects: 1 (the global maximum, *weakly* terminal) and a', b', c' (V^+-branching points).

Globally, this is a spectral pf-presentation of C (1.5); it generates the spectral injective model E, which is the full subcategory of C on $sp(C) = \{0, a, b, c, a', b', c', 1\}$. The full subcategory $Z \subset E$ on the objects 0, 1 is isomorphic to the past spectrum of F, as well as to the future spectrum of P, hence coarse equivalent (1.8) to C and E.

Comments. The pf-spectrum E provides a category with the same past and future behaviour as C. This can be read as follows:

(a) the action begins at the 'starting point' 0, the minimum, from where we can only move to c';

(b) c' is an (effective) V^+-branching point, where we choose: either the upper/middle way or the lower/middle one;

(c) the first choice leads to a', a further V^+-branching point where we choose between the upper or the middle way; similarly, the second choice leads to the V^+-branching point b', where we choose between the lower or the middle way (the same as before);

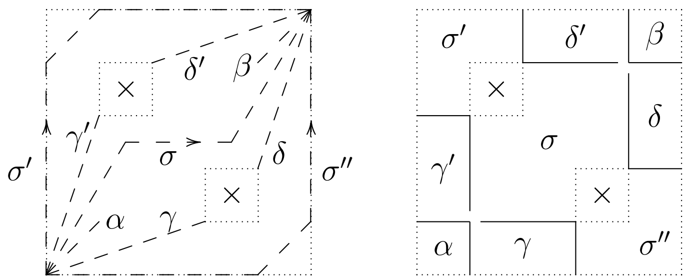
(d) the first bifurcation considered in (c) is 'joined' at a, the second at b (V^{-}-branching points);

(e) the resulting 'paths' come together at c (the last V^{-}-branching point);

(f) from where we can only move to the 'ending point' 1, the maximum.

The 'coarse model' Z only says that in C there are *three* homotopically distinct ways of going from 0 to 1, and looses relevant information on the branching structure of C. The projective model is studied below.

4.3. The projective model. For the same category $C = \uparrow\Pi_1(X)$, the spectral projective model M, represented in the right figure below, is the full subcategory of $C^{\mathbf{2}}$ on the 9 arrows displayed in the left figure



$$\begin{array}{ccccc} \sigma' & \longrightarrow & \delta' & \longrightarrow & \beta \\ \uparrow & \times & \uparrow & & \uparrow \\ \gamma' & \longrightarrow & \sigma & \longrightarrow & \delta \\ \uparrow & & \uparrow & \times & \uparrow \\ \alpha & \longrightarrow & \gamma & \longrightarrow & \sigma'' \end{array} \qquad (40)$$

M

The projection $p(x) = (p^{-}x, p^{+}x; \eta x.\varepsilon x)$ (3.4), from $X = \mathrm{Ob}C$ to $\mathrm{Ob}M \subset \mathrm{Mor}C$, has thus nine equivalence classes, analytically defined in (41) and 'sketched' in the middle figure above (the solid lines are meant to *suggest* that a certain boundary segment belongs to a certain region, as made precise below); in each of these regions, the morphism $p(x)$ is constant, and equal to $\alpha, \beta, \ldots$

$$\begin{array}{lll} p^{-1}(\alpha) = [0, 1/5]^2, & p^{-1}(\beta) = [4/5, 1]^2 & \text{(closed in X)}, \\ p^{-1}(\gamma) =]1/5, 3/5] \times [0, 1/5], & p^{-1}(\gamma') = [0, 1/5] \times]1/5, 3/5], & \\ p^{-1}(\delta) = [4/5, 1] \times [2/5, 4/5[, & p^{-1}(\delta') = [2/5, 4/5[\times [4/5, 1], & \\ p^{-1}(\sigma) = X \cap]1/5, 4/5[^2 & & \text{(open in X)}, \\ p^{-1}(\sigma') = X \cap ([0, 2/5[\times]3/5, 1]) & & \text{(open in X)}, \\ p^{-1}(\sigma'') = X \cap (]3/5, 1] \times [0, 2/5[) & & \text{(open in X)}. \end{array} \qquad (41)$$

The interpretation of the projective model M is practically the same as above, in 4.2, with some differences:

(i) in M there is no distinction between the starting point and the first future branching point, as well as between the ending point and the last past branching point;

(ii) the different paths produced by the obstructions are 'distinguished' in M by three new intermediate objects: $\sigma, \sigma', \sigma''$.

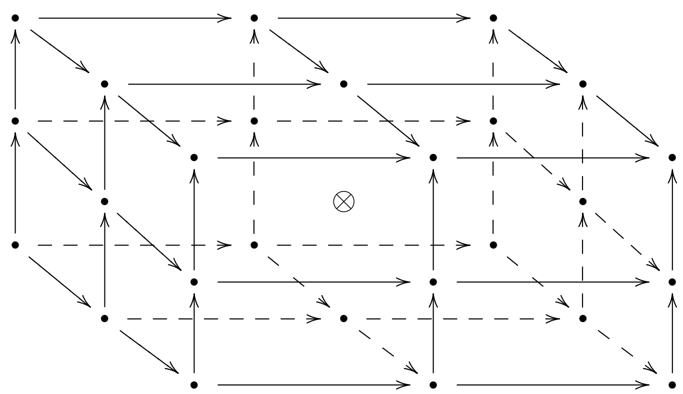
Note also that, again, one can embed M in C, by choosing a suitable point in each of the 9 regions above. Thus, the functor p is a normal quotient, by 2.7(a), and *the* minimal quotient model of C, by 3.4(c).

In order to compare the injective model E and the projective model M, various examples in I.9 show that distinguishing 0 from c' (or c from 1) carries *some* information (like distinguishing the initial from the terminal object, in the injective model **2** of a non-pointed category having both, cf. 1.8). According to applications, one may decide whether this information is useful or redundant.

4.4. A retractile model of the hollow cube. Similar facts hold for the fundamental category $C = \uparrow\Pi_1(X)$ of the hollow cube (1.9): the minimal quotient model of C is the category $\mathbf{1}$; but here the functor $C \to \mathbf{1}$ is not faithful.

On the other hand, the fundamental category C has a reasonably simple *semi-faithful* quotient model, the full subcategory M on the 26 middle points of the 'subcubes' of edge 1/3 placed 'around the obstruction' (as suggested by the analysis of [5], see figure 7).

M is generated by the following graph under the condition that all squares commute except the 3 ones around the obstruction $\otimes$



(42)

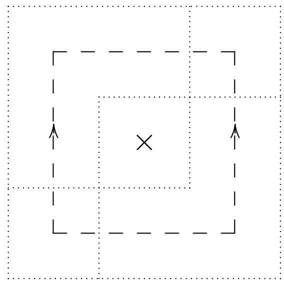
It is a retract of C, with a retraction $p\colon C \to M$ defined in the obvious way on the interior of the 26 *open* subcubes and *conveniently on their boundary* (with some arbitrary choices for that). This functor p is a normal quotient by 2.7(a), and a *semi-faithful* quotient model of C, by 3.3(b). It would be interesting to prove that this model is minimal *within the semi-faithful quotient models.*

Note also that this pf-surjection is *not* a pf-projection: the associated functor $g\colon M \to C^{\mathbf{2}}$ (1.7) is not even injective on objects.

4.5. The open square annulus. We end with considering the *open* square annulus

$$Y = \uparrow]0,1[^2 \setminus [1/3, 2/3]^2, \tag{43}$$

i.e. the interior of the square annulus X in $\uparrow\mathbf{R}^2$ (1.2, 4.1), and its fundamental category $D = \uparrow\Pi_1(Y)$, which is the full subcategory of $C = \uparrow\Pi_1(X)$ on the points of Y



$$\begin{array}{ccc} \sigma & \longrightarrow & \beta \\ \uparrow & \times & \uparrow \\ \alpha & \longrightarrow & \tau \end{array} \qquad (44)$$

Y $\qquad\qquad$ M

The quotient model M of C is still *a retract of* D, with one object in each pf-equivalence class, but it seems more difficult to say in what sense M can 'represent' the fundamental category D, as we discuss below. First, let us make clear that the pf-equivalence classes of D are *not* the traces on the subspace Y of the previous ones (written down in (35)), but differ from the latter 'at boundaries' (which is not important, but might lead to errors)

$$\begin{array}{ll} p^{-1}(\alpha) = \,]0,1/3[^2, \qquad p^{-1}(\beta) = \,]2/3,1[^2 & \text{(open in } Y\text{)}, \\ p^{-1}(\sigma) = Y \cap ([0,2/3]\times[1/3,1]), & \text{(closed in } Y\text{)}, \\ p^{-1}(\tau) = Y \cap ([1/3,1]\times[0,2/3]) & \text{(closed in } Y\text{)}. \end{array} \qquad (45)$$

Now, coming back to the main problem, notice that M is not a surjective model of D, nor an injective one. *It is not even future equivalent to D* (nor past equivalent), because M has a maximal point for the path preorder while D has none (and such objects are preserved by future equivalences, 1.4). By the same argument, one easily shows that D *cannot be future equivalent to any finite category*. Note also that the category D *has no spectra*: the $\sim^+$-equivalence class of $(1/6, 1/6)$ is now the open square $]0,1/3[^2$, which has no maximum; and symmetrically.

One could - perhaps - argue that there are two idempotent endofunctors $e^\alpha \colon D \to D$

$$e^-(y) = y \wedge ip(y), \qquad e^+(y) = y \vee ip(y), \qquad (46)$$

(since the four zones considered in (45) are lattices, for the induced order) and four natural transformations forming a commutative square

$$\begin{array}{ccc} e^- & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ ip & \longrightarrow & e^+ \end{array} \qquad (47)$$

so that *ip is linked to the identity by two 'elementary' zig-zags of homotopies in* **Cat** (1.3), a span and a cospan (which form a commutative square).

5. Future regularity and future spectra

We end with recalling some definitions and results of Part I, already used above.

5.1. Future regularity. A map $a\colon x \to x'$ in X is said to be *V^+-regular* if it satisfies condition (i), *O^+-regular* if it satisfies (ii), and *future regular* if it satisfies both (I.6.1):

(i) given $a'\colon x \to x''$, there is a commutative square $ha = ka'$ (*V^+-regularity*),

(ii) given $a_i\colon x' \to x''$ such that $a_1 a = a_2 a$, there is some h such that $ha_1 = ha_2$ (*O^+-regularity*),

$$\begin{array}{ccc} x & \xrightarrow{a} & x' \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle h} \\ x'' & \xrightarrow[k]{} & \bullet \end{array} \qquad\qquad x \xrightarrow{a} x' \overset{a_1}{\underset{a_2}{\rightrightarrows}} x'' \xrightarrow{h} \bullet \tag{48}$$

Future regular morphisms are closed under composition (I.6.2), but they are not invertible, in general. The equivalence relation $\sim^+$ in ObX generated by the existence of a future regular morphism between two objects is called *future regularity equivalence*. The future regularity class of an object x is written as $[x]^+$.

In a category with finite colimits or with terminal object, all morphisms are future regular. In a preordered set, all arrows are O^+-regular, and future regularity coincides with V^+-regularity.

On the other hand, we say that a is *V^+-branching* if it is not V^+-regular; that it is *O^+-branching* if it is not O^+-regular; that it is a *future branching morphism* if it falls in (at least) one of the previous cases, i.e. if it is not future regular. In the category represented below, at the left, the morphism a is V^+-branching and O^+-regular, while at the right a is O^+-branching and V^+-regular

$$\begin{array}{ccc} x & \xrightarrow{a} & x' \\ {\scriptstyle a'}\downarrow & & \\ x'' & & \end{array} \qquad\qquad \begin{array}{ccc} x & \xrightarrow{a} & x' \\ & {\scriptstyle b}\searrow & \downarrow\downarrow{\scriptstyle a_1, a_2} \\ & & x'' \end{array} \qquad\qquad (b = a_1 a = a_2 a). \tag{49}$$

Dually, we have *V^--regular*, *O^--regular*, *past regular* morphisms and the corresponding *branching* morphisms; the *past regularity equivalence* $\sim^-$ and its *past regularity classes* $[x]^-$.

We write $x \sim^{\pm} x'$ to mean that $x \sim^- x'$ *and* $x \sim^+ x'$.

Dually, we have *V^--regular*, *O^--regular*, *past regular* morphisms and the corresponding *branching* morphisms; the *past regularity equivalence* $\sim^-$ and its *past regularity classes* $[x]^-$.

We write $x \sim^{\pm} x'$ to mean that $x \sim^- x'$ *and* $x \sim^+ x'$.

5.2. Theorem. [Future equivalence and regular morphisms, I.6.3-6.4] *Given a future equivalence $f\colon X \rightleftarrows Y : g$, with natural transformations $\varphi\colon 1 \to gf$, $\psi\colon 1 \to fg$, we have:*

(a) all the components φx and ψy are future regular morphisms,

(b) the functors f and g preserve V^+-regular, O^+-regular and future regular morphisms,

(c) the functors f and g preserve V^+-branching, O^+-branching and future branching morphisms (i.e. reflect V^+-regular, O^+-regular and future regular morphisms),

It follows that a future equivalence $f\colon X \rightleftarrows Y : g$ induces a bijection

$$(\mathrm{Ob}X)/\!\sim^+ \ \rightleftarrows\ (\mathrm{Ob}Y)/\!\sim^+, \tag{50}$$

between the quotients of objects up to future regularity equivalence; f and g preserve and reflect the future regularity equivalence relations $\sim^+$.

5.3. Branching points. We consider now future invariant properties *of points* of a category X. We have already seen some of them, concerning maximal points (1.4).

A point x is said to be *V^+-regular* if it satisfies (i), *O^+-regular* if it satisfies (ii), *future regular* if it satisfies both:

(i) every arrow starting from x is V^+-regular (equivalently, two arrows starting from x can always be completed to a commutative square),

(ii) every arrow starting from x is O^+-regular (equivalently, given an arrow $a\colon x \to x'$ and two arrows $a_i\colon x' \to x''$ such that $a_1a = a_2a$, there exists an arrow h such that $ha_1 = ha_2$).

We say that x is a *V^+-branching point* in X if it is not V^+-regular (i.e., if there is some arrow starting from x which is V^+-branching); that x is an *O^+-branching point* if it is not O^+-regular; that x is a *future branching point* if it falls in at least one of the previous cases, i.e. if it is not future regular.

Note now that, in the fundamental category C considered in 1.2, the starting point 0 is V^+-branching, but the choice between the different paths starting from it can be deferred, while at the point p the choice must be made. To distinguish these situations, we say that a future branching point is *effective* when every future regular map starting from it is a split mono. (In the fundamental category of a preordered or ordered space, this amounts to an isomorphism or an identity, respectively.)

Dually, we have the notions of V^--, O^-- and *past regular* (resp. *branching*) *point* in X, and *effective past branching points.*

5.4. Theorem. [Future equivalence and branching points, I.6.6] *The following properties of a point are future invariant (i.e., invariant up to future equivalence):*

(a) being a V^+-regular, or an O^+-regular, or a future regular point,

(b) being a V^+-branching, or an O^+-branching, or a future branching point, or an effective one.

5.5. Future spectrum. A *future spectrum $sp^+(X)$* of the category X (I.7.2) is a subset of objects such that:

(sp$^+$.1) $sp^+(X)$ contains precisely one object, written $sp^+(x)$, in every future regularity class $[x]^+$,

(sp$^+$.2) for every $x \in X$ there is precisely one morphism $\eta x\colon x \to sp^+(x)$ in X, (sp$^+$.3) every morphism $a\colon x \to sp^+(x')$ factors as $a = h.\eta x$, for a unique $h\colon sp^+(x) \to sp^+(x')$.

The second condition can be equivalently written as:

(sp$^+$.2$'$) for every $x \in X$, $sp^+(x)$ is the terminal object of the full subcategory on $[x]^+$.

As proved in I.7.4, a functor $i\colon F \to X$ embeds F as a future retract of X if and only if:

(a) the category F has precisely one object in each future regularity class; the functor i is a future retract (i.e., it has a left adjoint $p\colon X \to F$ with $pi = 1_F$ as counit); moreover the unit-component $x \to ip(x)$ is the unique X-morphism with these endpoints.

Also the full subcategory $Sp^+(X)$ of X on this set of objects is called the *future spectrum*. The future spectrum (when it exists) is the least future retract of the given category (I.7.3). This full subcategory, as well as its embedding in X, is determined up to a *canonical* isomorphism (and is thus more strictly determined than the ordinary skeleton; cf. I.7.5).

Dually we have the *past spectrum* $sp^-(X)$ and its full subcategory $Sp^-(X)$.

A category has future spectrum **1** if and only if it is future equivalent to **1**, if and only if it has a terminal object.

References

[1] A. Bauer - L. Birkedal - D.S. Scott, Equilogical spaces, Theoretical Computer Science **315** (2004), 35-59.

[2] M.A. Bednarczyk - A.M. Borzyszkowski - W. Pawlowski, Generalized congruences-epimorphisms in Cat, Theory Appl. Categ. **5** (1999), No. 11, 266-280.

[3] C. Ehresmann, Cohomologie à valeurs dans une catégorie dominée, Extraits du Colloque de Topologie, Bruxelles 1964, in: C. Ehresmann, Oeuvres complètes et commentées, Partie III-2, 531-590, Amiens 1980.

[4] L. Fajstrup - E. Goubault - M. Raussen, Algebraic topology and concurrency, Theoret. Comp. Sci. **357** (2006), 241-278.

[5] L. Fajstrup, M. Raussen, E. Goubault, E. Haucourt, Components of the fundamental category, Appl. Categ. Structures **12** (2004), 81-108.

[6] P. Gaucher, A model category for the homotopy theory of concurrency, Homology Homotopy Appl. **5** (2003), 549-599.

[7] P. Gaucher - E. Goubault, Topological deformation of higher dimensional automata, Preprint 2001. `http://arXiv.org/abs/math.AT/0107060`

[8] E. Goubault, Geometry and concurrency: a user's guide, in: Geometry and concurrency, Math. Structures Comput. Sci. **10** (2000), no. 4, pp. 411-425.

[9] M. Grandis, On the categorical foundations of homological and homotopical algebra, Cah. Topol. Géom. Diff. Catég. **33** (1992), 135-175.

[10] M. Grandis, Directed homotopy theory, I. The fundamental category, Cah. Topol. Géom. Diff. Catég. **44** (2003), 281-316. http://www.dima.unige.it/∼grandis/Dht1.pdf

[11] M. Grandis, The shape of a category up to directed homotopy, Theory Appl. Categ. **15** (2005) (CT2004), No. 4, 95-146. http://tac.mta.ca/tac/

[12] M. Grandis, Modelling fundamental 2-categories for directed homotopy, Homology Homotopy Appl. **8** (2006), 31-70. http://intlpress.com/HHA/v8/n1/a2/

[13] M. Grandis, Lax 2-categories and directed homotopy, Cah. Topol. Géom. Diff. Catég., to appear. Preprint: Dip. Mat. Univ. Genova, Preprint **530** (2005). http://www.dima.unige.it/∼grandis/LCat.pdf

[14] M. Grandis, Absolute lax 2-categories, Appl. Categ. Struct. **14** (2006), 191-214. http://www.dima.unige.it/∼grandis/LCat2.pdf

[15] E. Haucourt, Topologie algébrique dirigée et concurrence, Thèse de Doctorat, Université Paris 7, UFR Informatique, 2005.

[16] S. Krishnan, A convenient category of locally ordered spaces, presented at GETCO 2005. Unpublished Manuscript. http://www.math.uchicago.edu/∼sanjeevi/

[17] R. Lavendhomme, Un plongement pleinement fidèle de la catégorie des groupes, Bull. Soc. Math. Belgique, **17** (1965), 153-185.

[18] S. Mac Lane, Categories for the working mathematician, Springer, Berlin 1971.

[19] D. Scott, A new category? Domains, spaces and equivalence relations, Unpublished manuscript (1996). http://www.cs.cmu.edu/Groups/LTC/

Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16146-Genova, Italy
Email: grandis@dima.unige.it

This article may be accessed via WWW at http://www.tac.mta.ca/tac/ or by anonymous ftp at ftp://ftp.tac.mta.ca/pub/tac/html/volumes/16/26/16-26.{dvi,ps,pdf}

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.
The method of distribution of the journal is via the Internet tools WWW/ftp. The journal is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi, Postscript and PDF. Details will be e-mailed to new subscribers. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, rrosebrugh@mta.ca.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX2e is the preferred flavour. TEX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at http://www.tac.mta.ca/tac/. You may also write to tac@mta.ca to receive details by e-mail.