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A HOMOTOPY DOUBLE GROUPOID OF A HAUSDORFF SPACE II: A VAN KAMPEN THEOREM

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Abstract. This paper is the second in a series exploring the properties of a functor which assigns a homotopy double groupoid with connections to a Hausdorff space.

We show that this functor satisfies a version of the van Kampen theorem, and so is a suitable tool for nonabelian, 2-dimensional, local-to-global problems. The methods are analogous to those developed by Brown and Higgins for similar theorems for other higher homotopy groupoids.

An integral part of the proof is a detailed discussion of commutative cubes in a double category with connections, and a proof of the key result that any composition of commutative cubes is commutative. These results have recently been generalised to all dimensions by Philip Higgins.

1. Introduction

A classical and key example of a nonabelian local-to-global theorem in dimension 1 is the van Kampen theorem for the fundamental group of a space with base point: if a space is the union of two connected open sets with connected intersection, the theorem determines the fundamental group of the whole space, and so the 'global' information, in terms of the local information on the fundamental groups of the parts and the morphisms induced by inclusions. Such a theorem thus relates a particular pushout of spaces with base point to a pushout, or free product with amalgamation, of their fundamental groups.

van Kampen's 1935 paper, [18], gave, in fact, a formula for the case of *non connected intersection*, as required for the algebraic geometry applications he had in mind. His formula follows from the version of the theorem for the fundamental groupoid of a space [2] (for the deduction of the formula, see §8.4.9 of [3]). An outstanding feature of this version is that it involves a universal property which thus gives, in the non connected case, a *complete determination of this nonabelian invariant* which includes the fundamental groups based at various points. The basic reason for this success seems to be that groupoids have structure in dimensions 0 and 1, and so allow a better algebraic modelling of the geometry of the intersections of the components of the two open sets.

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General theorems of this type, as in for example [7], involve covers more general than by two open sets, and so require the notion of *coequaliser* rather than pushout. Recall that the following diagram is a basic example of a coequaliser in topology:

$$\bigsqcup_{(U,V)\in\mathcal{U}^2} U\cap V \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{U\in\mathcal{U}} U \xrightarrow{c} X. \tag{1}$$

Here $\mathcal{U}$ is an open cover of the space X; $\bigsqcup$ denotes coproduct in the category of spaces, which is given by disjoint union; the functions a, b, c are determined by the inclusions

$$a_{UV} : U\cap V \to U, \quad b_{UV} : U\cap V \to V, \quad c_U : U \to X$$

for each $(U,V)\in\mathcal{U}^2, U\in\mathcal{U}$. The coequaliser condition in this case says simply that any continuous function $f : X \to Y$ is entirely determined by continuous functions $f_U : U \to Y$ which agree on the intersections $U\cap V$. Thus a van Kampen/coequaliser theorem states that the fundamental groupoid functor applied to diagram (1) gives a coequaliser diagram of groupoids.

The paper [16], published in 2000, showed the existence of an *absolute* homotopy 2-groupoid, $\boldsymbol{\rho}^{\bigcirc}(X)$ of a space. By virtue of the equivalence between 2-groupoids and double groupoids with connection proved in [10, 21], this also gave a homotopy double groupoid with connection, $\boldsymbol{\rho}^{\square}(X)$, of a Hausdorff space X. An explicit description of $\boldsymbol{\rho}^{\square}(X)$ in terms of certain homotopy classes of paths and squares, i.e. maps of I and I^2, was given in the first paper of this series, [4]. The interest of these functors is that their 1-dimensional part $\boldsymbol{\rho}_1$, which is the same for both functors, has the fundamental groupoid of X as a quotient, and that they contain the second homotopy groups $\pi_2(X,x)$ for all base points x in X.

We adapt the methods of [5] to prove a van Kampen/coequaliser theorem, i.e. a 2-dimensional local-to-global property, for this functor $\boldsymbol{\rho}^{\square}$.

Theorem 4.1. [van Kampen/Coequaliser theorem for $\boldsymbol{\rho}^{\square}$] *If $\mathcal{U}$ is an open cover of the Hausdorff space X, then the following diagram of morphisms induced by inclusions*

$$\bigsqcup_{(U,V)\in\mathcal{U}^2} \boldsymbol{\rho}^{\square}(U\cap V) \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{U\in\mathcal{U}} \boldsymbol{\rho}^{\square}(U) \xrightarrow{c} \boldsymbol{\rho}^{\square}(X) \tag{2}$$

is a coequaliser diagram in the category of double groupoids with connections.

The morphisms a, b, c are determined by the inclusions a, b, c as above. Note also that $\bigsqcup$, the coproduct in the category of double groupoids with connections, is essentially given by disjoint union.

The validity of such a theorem is an argument for the utility of this particular algebraic structure. Note that the theorem yields a new result in dimension 1, since the functor $\boldsymbol{\rho}_1$ differs from the fundamental groupoid π_1. In view of the success of the 1-dimensional fundamental group(oid) in a variety of problems, as shown for example by a web search on "van Kampen theorem", we hope that this theorem will lead to wide investigations of this type of 2-dimensional homotopical algebra.

Here are some comments on the proof of the main theorem.

One key aspect of some proofs of a van Kampen type theorem is the realisation of *algebraic inverse to subdivision.* In dimension 1, this is more elegantly realised by groupoids rather than by groups. In dimension 2, this is more easily realised by squares, using $\boldsymbol{\rho}^{\square}$, rather than by globes, using $\boldsymbol{\rho}^{\bigcirc}$.

A second key aspect of a proof of the 1-dimensional van Kampen theorem is the notion of *commutative square* in a groupoid, and the fact that horizontal and vertical, and indeed any, compositions of such commutative squares are commutative. In this way, commutative squares in a groupoid form a double groupoid. Likewise, a key aspect in the proof for higher dimensions is the *definition of commutative cube* and the proof that any composition of commutative cubes is commutative. That is, the commutative cubes in a double category should form a triple category.

Our definition of commutative cubes requires the notion of connections on a cubical set, as first introduced for double groupoids in 1976 in [11], and for double categories in [21]. We explain these ideas in the early sections of this paper, and show their application to commutative cubes, here called 3-shells. The proofs for compositions are deferred to a penultimate section. These results have recently been generalised to all dimensions by Philip Higgins in [17], following on from work in [1, 10] and an earlier version of this paper. We also note the earlier work of [22] on what they call 'homotopy commutative cubes'; this work imposes more conditions on the underlying double category than here, but also uses connections.

The proof in section 4 of the main coequaliser theorem follows the model for the 2-dimensional version for pairs of spaces given in [5], with some extra twists. Given the main machinery, the proof is quite short. It does not need the deformations based on connectivity assumptions used in [5]. The proof in [5] extends to a coequaliser theorem for the homotopy ω-groupoid of a filtered space, as in [7]. This suggests that the results of this paper should extend to higher dimensions, but no construction is yet available of suitable *absolute* higher homotopy groupoids.

In the final section, we explain intended future directions for this work.

We would like to thank for helpful comments on an earlier draft: Philip Higgins, who has developed higher dimensional work in [17]; and also Uli Fahrenberg, who has used some of these ideas in directed homotopy theory, [13].

2. Double categories, connections and thin structures: definition and notational conventions

By a *double category* K, we will mean what has been called an *edge symmetric double category* in [10].

We briefly recall some of the basic facts about double categories in the above sense. In the first place, a double category, K, consists of a triple of category structures

$$(K_2, K_1, \partial_1^-, \partial_1^+, +_1, \varepsilon_1), \quad (K_2, K_1, \partial_2^-, \partial_2^+, +_2, \varepsilon_2)$$
$$(K_1, K_0, \partial^-, \partial^+, +, \varepsilon)$$

as partly shown in the diagram

$$\begin{tikzcd}
K_2 \arrow[r, shift left, "\partial_2^-"] \arrow[r, shift right, "\partial_2^+"'] \arrow[d, shift right, "\partial_1^-"'] \arrow[d, shift left, "\partial_1^+"] & K_1 \arrow[d, shift right, "\partial^-"'] \arrow[d, shift left, "\partial^+"] \\
K_1 \arrow[r, shift left, "\partial^-"] \arrow[r, shift right, "\partial^+"'] & K_0 \ .
\end{tikzcd}$$

The elements of K_0, K_1, K_2 will be called respectively *points or objects, edges, squares.* The maps $\partial^\pm$, $\partial_i^\pm$, $i = 1, 2$, will be called *face maps*, the maps $\varepsilon_i : K_1 \longrightarrow K_2$, $i = 1, 2$, resp. $\varepsilon : K_0 \longrightarrow K_1$ will be called *degeneracies.*

The compositions, $+_1$, resp. $+_2$, are referred to as *vertical* resp. *horizontal composition* of squares. The axioms for a double category include the usual relations of a 2-cubical set and the *interchange law.* We use matrix notation for compositions as

$$\begin{bmatrix} a \\ c \end{bmatrix} = a +_1 c, \quad \begin{bmatrix} a & b \end{bmatrix} = a +_2 b,$$

and the interchange law allows one to use matrix notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for double composites of squares, as in [6]. We also allow as in [6] the multiple composition $[a_{ij}]$ of an array (a_{ij}) whenever for all appropriate i, j we have $\partial_1^+ a_{ij} = \partial_1^- a_{i+1,j}$, $\partial_2^+ a_{ij} = \partial_2^- a_{i,j+1}$.

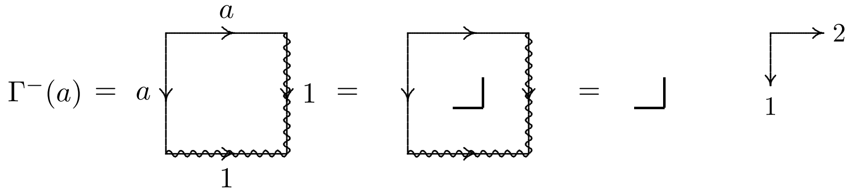
The identities with respect to $+_1$ (*vertical identities*) are given by ε_1 and will be denoted by $\mathsf{I}\;\mathsf{I}$. Similarly, we have *horizontal identities* denoted by $\overline{\underline{}}$. Elements of the form $\varepsilon_1\varepsilon(a) = \varepsilon_2\varepsilon(a)$ for $a \in K_0$ are called *double degeneracies* and will be denoted by $\odot$.

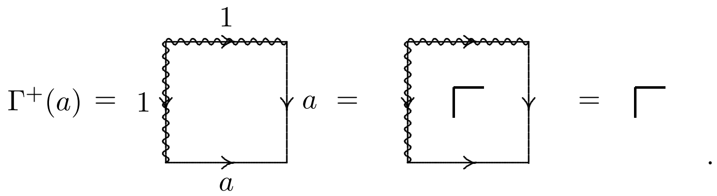
A *morphism of double categories* $f : \mathsf{K} \to \mathsf{L}$ consists of a triple of maps $f_i : K_i \to L_i$, $(i = 0, 1, 2)$, respecting the cubical structure, compositions and identities.

A *connection pair* on a double category K is given by a pair of maps

$$\Gamma^-, \Gamma^+ : K_1 \longrightarrow K_2$$

whose edges are given by the following diagrams for $a \in K_1$:





The axioms for the connection pair are listed in [10], Section 4. In particular, the *transport laws*

$$\begin{bmatrix} \ulcorner & = \\ \| & \ulcorner \end{bmatrix} = \ulcorner, \qquad \begin{bmatrix} \lrcorner & \| \\ = & \lrcorner \end{bmatrix} = \lrcorner \tag{3}$$

describe the connections of the composition of two elements, while the laws

$$\begin{bmatrix} \ulcorner \\ \lrcorner \end{bmatrix} = \, =, \qquad \begin{bmatrix} \ulcorner & \lrcorner \end{bmatrix} = \|, \tag{4}$$

allow cancellation of connections.

A *morphism of double categories with connections* is a morphism of double categories respecting connections.

Brown and Mosa in [10] have shown that a pair of connections on a double category K is equivalent to a *thin structure* on K, whose definition we recall.

We recall the standard language for 'commutative squares'; we will later move to 'commutative cubes'.

First, if α is a square in K, then the *boundary* (2*-shell*) of α is the quadruple

$$(\partial_2^- \alpha, \partial_1^+ \alpha, \partial_1^- \alpha, \partial_2^+ \alpha).$$

We also say α is a *filler* of its boundary. This boundary *commutes* if $\partial_2^- \alpha + \partial_1^+ \alpha = \partial_1^- \alpha + \partial_2^+ \alpha$. More generally, a 2*-shell* is defined to be a quadruple (a, b, c, d) of edges such that $\partial^- a = \partial^- c, \partial^+ a = \partial^- b, \partial^+ c = \partial^- d, \partial^+ b = \partial^+ d$, and this 2-shell *commutes* if $a + b = c + d$.

If C is a category, then by $\square C$ we denote the double category of commuting squares (2-shells) in C with the obvious double category structure. Then a *thin structure* Θ on a double category K is a morphism of double categories

$$\Theta : \square K_1 \to \mathsf{K},$$

which is the identity on K_1. The elements of K_2 lying in $\Theta(\square K_1)$ are called *thin squares*. The definition implies immediately:

(T0) *A thin square has commuting boundary.*

(T1) *Any commutative* 2*-shell has a unique thin filler.*

(T2) *Any identity square, and any composition of thin squares, is thin.*

In the thin structure induced on a double category by a pair of connections, any thin square is a certain composite of identities, both horizontal and vertical, and connections. For an explicit formula, we refer to [10], Theorem 4.3, and we will refer to these thin squares as being *algebraically thin*. Of course, a morphism of double categories with connections will preserve (algebraic) thinness.

3. Commutative cubes

We need to extend the domain of discourse from double categories to triple categories in order to explain the notion of *commutative cube (3-shell) in a double category with connections*. This notion has been defined for n-shells in the case of cubical ω-groupoids with connections in [6, Section 5], and for the more difficult category case in [1, Section 9]. For the convenience of the reader we set up the theory in our low dimensional case.

Let $\mathbb{D}$, $\mathbb{T}$ be respectively the categories of double categories and of triple categories with connections, in the sense of [1]. There is a natural and obvious truncation functor $tr : \mathbb{T} \to \mathbb{D}$, which forgets the 3-dimensional structure. This functor has a right adjoint $cosk : \mathbb{D} \to \mathbb{T}$ which may be constructed in terms of *cubes* or 3-*shells*, which we now define.

3.1. DEFINITION. *Let* K *be a double category. A cube* (3-shell) *in* K*,*

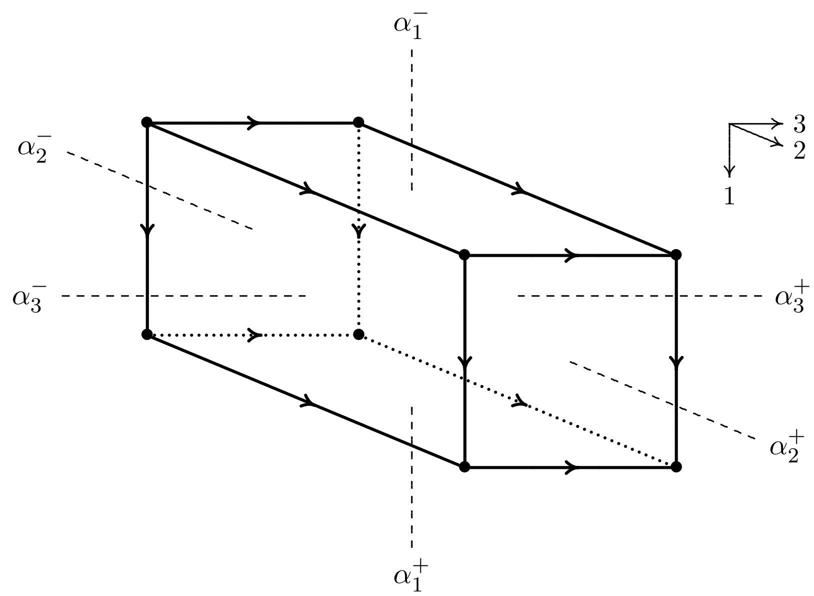
$$\alpha = (\alpha_1^-, \alpha_1^+, \alpha_2^-, \alpha_2^+, \alpha_3^-, \alpha_3^+)$$

consists of squares $\alpha_i^\pm \in K_2 \quad (i = 1, 2, 3)$ *such that*

$$\partial_i^\sigma(\alpha_j^\tau) = \partial_{j-1}^\tau(\alpha_i^\sigma)$$

for $\sigma, \tau = \pm$ *and* $1 \leqslant i < j \leqslant 3$. ■

A cube may be illustrated by the following picture:



3.2. DEFINITION. *We define three partial compositions,* $+_1, +_2, +_3$*, of cubes. Let* α, β *be cubes in* K*.*

(i) *If* $\alpha_1^+ = \beta_1^-$*, then we define*

$$\alpha +_1 \beta = (\alpha_1^-, \beta_1^+, \alpha_2^- +_1 \beta_2^-, \alpha_2^+ +_1 \beta_2^+, \alpha_3^- +_1 \beta_3^-, \alpha_3^+ +_1 \beta_3^+).$$

(ii) If $\alpha_2^+ = \beta_2^-$, *then we define*

$$\alpha +_2 \beta = (\alpha_1^- +_1 \beta_1^-, \alpha_1^+ +_1 \beta_1^+, \alpha_2^-, \beta_2^+, \alpha_3^- +_2 \beta_3^-, \alpha_3^+ +_2 \beta_3^+).$$

(iii) If $\alpha_3^+ = \beta_3^-$, *then we define*

$$\alpha +_3 \beta = (\alpha_1^- +_2 \beta_1^-, \alpha_1^+ +_2 \beta_1^+, \alpha_2^- +_2 \beta_2^-, \alpha_2^+ +_2 \beta_2^+, \alpha_3^-, \beta_3^+).$$

This is a special case of general definitions in dimension n given in [6], see also [17].

As explained earlier, we need the notion of a 3-shell α being 'commutative'. Intuitively, this says that the composition of the odd faces $\alpha_1^-, \alpha_3^-, \alpha_2^+$ of α is equal to the composition of the even faces $\alpha_1^+, \alpha_3^+, \alpha_2^-$ (think of -,+ as 0,1). The problem of how to make sense of such compositions of three squares in a double groupoid, and obtain the right boundaries, is solved using the connections.

3.3. DEFINITION. *Suppose given, in a double category with connections* K, *a cube (3-shell)*

$$\alpha = (\alpha_1^-, \alpha_1^+, \alpha_2^-, \alpha_2^+, \alpha_3^-, \alpha_3^+).$$

We define the composition of the odd faces of α *to be*

$$\boldsymbol{\partial}^{\mathrm{odd}}\alpha = \begin{bmatrix} \ulcorner & \alpha_1^- & \lrcorner \\ \alpha_3^- & \alpha_2^+ & = \end{bmatrix} \tag{5}$$

and the composition of the even faces of α *to be*

$$\boldsymbol{\partial}^{\mathrm{even}}\alpha = \begin{bmatrix} = & \alpha_2^- & \alpha_3^+ \\ \ulcorner & \alpha_1^+ & \lrcorner \end{bmatrix} \tag{6}$$

This definition can be regarded as a cubical, categorical (rather than groupoid) form of the Homotopy Addition Lemma (HAL) in dimension 3. We refer to the discussion below

3.4. DEFINITION. *We define* α *to be commutative if it satisfies the Homotopy Commutativity Lemma (HCL), i.e.*

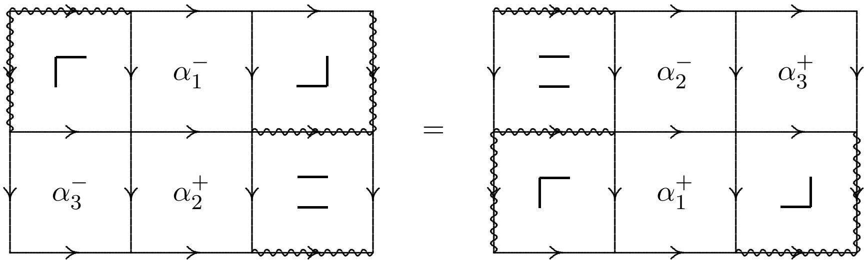
$$\boldsymbol{\partial}^{\mathrm{odd}}\alpha = \boldsymbol{\partial}^{\mathrm{even}}\alpha. \tag{HCL}$$

The reader should draw a 3-shell, label all the edges with letters, and see that this equation makes sense in that the 2-shells of each side of equation (HCL) coincide. Notice however that these 2-shells do not have coincident partitions along the edges: that is the edges of this 2-shell in direction 1 are formed from different compositions of the type $0 + a$ and $a + 0$. This formula should also be compared with the notion of 'homotopy commutative cube' defined in a more restricted kind of double category with connections in [22].

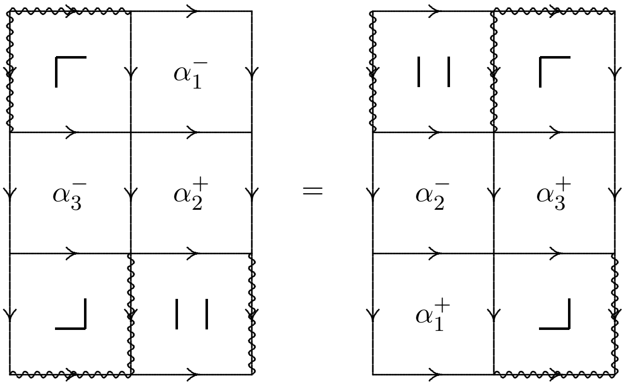
The formula in Definition 3.4 is unsymmetrical, and this seems in part a consequence of trying to express a 3-dimensional idea in a 2-dimensional formula. It also reflects the difficulty of the concept. A formula in dimension 4 is given in [14].

There are other pictures and forms of this which we now explore, and some of which will be used later.

First we give a picture in another format which also shows how the boundary of each side is made up:



This is in fact equivalent to:



The reader can check that in each case the two sides of the equation have the same 2-shells (but not as subdivided).

The second equation can also be written in matrix notation as:

$$\begin{bmatrix} \ulcorner & \alpha_1^- \\ \alpha_3^- & \alpha_2^+ \\ \lrcorner & \mid\mid \end{bmatrix} = \begin{bmatrix} \mid\mid & \ulcorner \\ \alpha_2^- & \alpha_3^+ \\ \alpha_1^+ & \lrcorner \end{bmatrix}. \qquad \text{(HCL}'\text{)}$$

The equivalence of (HCL) to (HCL′) can be seen by adding $[\odot \ \mid\mid \ \ulcorner]$, (respectively $[\lrcorner \ \mid\mid \ \odot]$) to the top (resp. bottom) of the two sides of (HCL), using equations (4) for further cancellation of connections, and then absorbing identities.

We note that a morphism of double categories with connections preserves commutativity of cubes.

The following theorem will be crucial for the proof of the van Kampen theorem.

3.5. THEOREM. *Let* K *be a double category with connections. Then any composition of commutative* 3*-shells is commutative.*

The proof is left to section 6.

3.6. Proposition. *The functor $tr : \mathbb{T} \to \mathbb{D}$ has a left adjoint $sk : \mathbb{D} \to \mathbb{T}$ which assigns to a double groupoid with connections D the triple groupoid $\square\mathsf{D}$ which agrees with D in dimensions $\leqslant 2$ and in dimension 3 consists of the commutative 3-shells in D.*

3.7. Remark. (i) Our homotopy commutativity lemma has been called also the 'homotopy addition lemma', and has been formulated in different equivalent forms in certain 2-dimensional groupoid type settings using inverses and reflections (cf. [5], Proposition 5, [22], Definition 3.8, [4], Proposition 5.5). Also [6], §5, 7, sets up, in the situation of cubical multiple groupoids with connections, truncation and skeleton functors tr^n and sk^n and an n-dimensional homotopy addition lemma as a formula for the boundary of an n-cube. The definition and use of thin elements in dimension n is a key part of the proof of the generalised van Kampen theorem in [7].

The homotopy addition lemma presented here applies to a general 2-dimensional *category* type setting. It is also a special case of results on cubical ω-categories with connections in [1, Section 9]. (The fact that any composition of commutative shells is commutative is not stated in the last reference, but does follow easily from the results stated there.)

(ii) Homotopy commutative cubes have been defined and investigated by Spencer and Wong in the more special setting of what they call *special double categories with connection* ([22], Definition 1.4). By this they mean a double category (with connections), K, such that the horizontal subdouble category of squares in K of the form

$$\begin{tikzcd} \cdot \arrow[r, squiggly, "1"] \arrow[d, "a"'] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, squiggly, "1"'] & \cdot \end{tikzcd}$$

is a groupoid under $+_2$. Spencer and Wong show that this is a suitable setting to deal with (homotopy) pullbacks/pushouts and gluing/cogluing theorems in homotopy theory.

It is not difficult to show that the definition of a homotopy commutative cube as given in [22], Definition 3.8, is equivalent to our definition of commutative cube.

As we mentioned before the fact that commutativity of cubes is preserved under composition will be crucial for our purposes. Spencer and Wong have a similar result in their setting ([22], Proposition 3.11).

The basic notions of relative homotopy carry over from topology to algebra. Let K be a double category with connections. A square u in K is called a *relative homotopy* if the two opposite edges $\partial_2^- u, \partial_2^+ u$ are identities. A *relative homotopy between squares* u, u' in K is a commutative cube α such that

$$\alpha_1^- = u, \quad \alpha_1^+ = u'$$

and the remaining faces are relative homotopies. If, in addition, the remaining faces are thin, then u and u' are called *thinly equivalent*

From (T0) and (T1), we conclude:

(T3) *A thin square which is a relative homotopy is an identity.*

From Definition 2.2, making use of [10], 3.1(i), we read off the following lemma:

3.8. LEMMA. *Squares which are thinly equivalent, coincide.*

This key lemma is used later to show that a construction is independent of the choices made.

4. The homotopy double groupoid, $\boldsymbol{\rho}^{\square}(X)$

This section is adapted from [4], and the reader should refer to that source for fuller details.

4.1. THE SINGULAR CUBICAL SET OF A TOPOLOGICAL SPACE. We shall be concerned with the low dimensional part (up to dimension 3) of the singular cubical set

$$R^{\square}(X) = (R^{\square}_n(X), \partial_i^-, \partial_i^+, \varepsilon_i)$$

of a topological space X. We recall the definition (cf. [6]).

For $n \geqslant 0$ let

$$R^{\square}_n(X) = \mathbf{Top}(I^n, X)$$

denote the set of *singular n–cubes* in X, i.e. continuous maps $I^n \longrightarrow X$, where $I = [0,1]$ is the unit interval of real numbers.

We shall identify $R^{\square}_0(X)$ with the set of points of X. For $n = 1, 2, 3$ a singular n–cube will be called a *path*, resp. *square*, resp. *cube*, in X.

The *face maps*

$$\partial_i^-, \partial_i^+ : R^{\square}_n(X) \longrightarrow R^{\square}_{n-1}(X) \quad (i = 1, \dots, n)$$

are given by inserting 0 resp. 1 at the i^{th} coordinate whereas the *degeneracy maps*

$$\varepsilon_i : R^{\square}_{n-1}(x) \longrightarrow R^{\square}_n(X) \quad (i = 1, \dots, n)$$

are given by omitting the i^{th} coordinate. The face and degeneracy maps satisfy the usual cubical relations (cf. [6], § 1.1; [19] , § 5.1). A path $a \in R^{\square}_1(X)$ has *initial point* $a(0)$ and *endpoint* $a(1)$. We will use the notation $a : a(0) \simeq a(1)$. If a, b are paths such that $a(1) = b(0)$, then we denote by $a + b : a(0) \simeq b(1)$ their *concatenation*, i.e.

$$(a+b)(s) = \begin{cases} a(2s), & 0 \leqslant s \leqslant \frac{1}{2} \\ b(2s-1), & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

If x is a point of X, then $\varepsilon_1(x) \in R^{\square}_1(X)$, denoted e_x, is the *constant path* at x, i.e.

$$e_x(s) = x \text{ for all } s \in I.$$

If $a : x \simeq y$ is a path in X, we denote by $-a : y \simeq x$ the *path reverse* to a, i.e. $(-a)(s) = a(1-s)$ for $s \in I$. In the set of squares $R_2^\square(X)$ we have two partial compositions $+_1$ (*vertical composition*) and $+_2$ (*horizontal composition*) given by concatenation in the first resp. second variable.

Similarly, in the set of cubes $R_3^\square(X)$ we have three partial compositions $+_1, +_2, +_3$.

The standard properties of vertical and horizontal composition of squares are listed in [6], §1. In particular we have the following *interchange law.* Let $u, u', w, w' \in R_2^\square(X)$ be squares, then

$$(u +_2 w) +_1 (u' +_2 w') = (u +_1 u') +_2 (w +_1 w'),$$

whenever both sides are defined. More generally, we have an interchange law for rectangular decomposition of squares. In more detail, for positive integers m, n let $\varphi_{m,n} : I^2 \longrightarrow [0, m] \times [0, n]$ be the homeomorphism $(s, t) \longmapsto (ms, nt)$. An $m \times n$ *subdivision* of a square $u : I^2 \longrightarrow X$ is a factorization $u = u' \circ \varphi_{m,n}$; its *parts* are the squares $u_{ij} : I^2 \longrightarrow X$ defined by

$$u_{ij}(s, t) = u'(s + i - 1, t + j - 1).$$

We then say that u is the *composite* of the array of squares (u_{ij}), and we use matrix notation $u = [u_{ij}]$. Note that as in §1, $u +_1 u'$, $u +_2 w$ and the two sides of the interchange law can be written respectively as

$$\begin{bmatrix} u \\ u' \end{bmatrix}, \qquad [u \ \ w], \qquad \begin{bmatrix} u & w \\ u' & w' \end{bmatrix}.$$

Finally, *connections*

$$\Gamma^-, \Gamma^+ : R_1^\square(X) \longrightarrow R_2^\square(X)$$

are defined as follows. If $a \in R_1^\square(X)$ is a path, $a : x \simeq y$, then let

$$\Gamma^-(a)(s, t) = a(\max(s, t)); \ \Gamma^+(a)(s, t) = a(\min(s, t)).$$

The full structure of $R^\square(X)$ as a *cubical complex with connections and compositions* has been exhibited in [1, 15].

4.2. Thin squares. In the setting of a geometrically defined double groupoid with connection, as in [6], (resp [4]), there is an appropriate notion of *geometrically thin* square. It is proved in [6], Theorem 5.2 (resp. [4], Proposition 4), that in the cases given there, geometrically and algebraically thin squares coincide. In our context the explicit definition is as follows:

4.3. Definition. *(1) A square $u : I^2 \longrightarrow X$ in a topological space X is* thin *if there is a factorisation of u*

$$u : I^2 \xrightarrow{\Phi_u} J_u \xrightarrow{p_u} X,$$

where J_u is a tree and Φ_u is piecewise linear (PWL, see below) on the boundary ∂I^2 of I^2.

Here, by a tree, we mean the underlying space $|K|$ of a finite 1*-connected* 1*-dimensional simplicial complex K.*

A map $\Phi : |K| \longrightarrow |L|$ where K and L are (finite) simplicial complexes is PWL (piecewise linear) if there exist subdivisions of K and L relative to which Φ is simplicial.

(2) Let u be as above, then the homotopy class of u relative to the boundary ∂I^2 of I is called a double track. A double track is thin if it has a thin representative. ■

We note that the class of thin squares in a topological space X is closed under vertical and horizontal composition of squares if X is assumed to be Hausdorff ([4, Proposition 3.5]). This is a consequence of appropriate pushout properties of trees (cf. [4, Section 5]).

4.4. The homotopy double groupoid of a Hausdorff space. The data for the homotopy double groupoid, $\boldsymbol{\rho}^{\square}(X)$, of a Hausdorff space X will be denoted by

$$(\boldsymbol{\rho}_2^{\square}(X), \boldsymbol{\rho}_1^{\square}(X), \partial_1^-, \partial_1^+, +_1, \varepsilon_1),\ (\boldsymbol{\rho}_2^{\square}(X), \boldsymbol{\rho}_1^{\square}(X), \partial_2^-, \partial_2^+, +_2, \varepsilon_2)$$
$$(\boldsymbol{\rho}_1^{\square}(X), X, \partial^-, \partial^+, +, \varepsilon).$$

Here $\boldsymbol{\rho}_1^{\square}(X)$ denotes the *path groupoid* of X of [16]. We recall the definition. The objects of $\boldsymbol{\rho}_1^{\square}(X)$ are the points of X. The morphisms of $\boldsymbol{\rho}_1^{\square}(X)$ are the equivalence classes of paths in X with respect to the following relation $\sim_T$.

4.5. Definition. *Let $a, a' : x \simeq y$ be paths in X. Then a is* thinly equivalent *to a', denoted $a \sim_T a'$, if there is a thin relative homotopy between a and a'.*

We note that $\sim_T$ is an equivalence relation, see [4]. We use $\langle a \rangle : x \simeq y$ to denote the $\sim_T$ class of a path $a : x \simeq y$ and call $\langle a \rangle$ the *semitrack* of a. The groupoid structure of $\boldsymbol{\rho}_1^{\square}(X)$ is induced by concatenation, $+$, of paths. Here one makes use of the fact that if $a : x \simeq x',\ a' : x' \simeq x'',\ a'' : x'' \simeq x'''$ are paths then there are canonical thin relative homotopies

$$(a + a') + a'' \simeq a + (a' + a'') : x \simeq x'''\ (\textit{rescale})$$
$$a + e_{x'} \simeq a : x \simeq x';\ e_x + a \simeq a : x \simeq x'\ (\textit{dilation})$$
$$a + (-a) \simeq e_x : x \simeq x\ (\textit{cancellation}).$$

The source and target maps of $\boldsymbol{\rho}_1^{\square}(X)$ are given by

$$\partial_1^- \langle a \rangle = x,\ \ \partial_1^+ \langle a \rangle = y,$$

if $\langle a \rangle : x \simeq y$ is a semitrack. Identities and inverses are given by

$$\varepsilon(x) = \langle e_x \rangle \quad \text{resp.} - \langle a \rangle = \langle -a \rangle.$$

In order to construct $\boldsymbol{\rho}_2^{\square}(X)$, we define a relation of cubically thin homotopy on the set $R_2^{\square}(X)$ of squares.

4.6. Definition. *Let u, u' be squares in X with common vertices. (1) A cubically thin homotopy $U : u \equiv_T^{\square} u'$ between u and u' is a cube $U \in R_3^{\square}(X)$ such that*

(i) U is a homotopy between u and u',

$$i.e.\ \partial_1^-(U) = u,\ \ \partial_1^+(U) = u',$$

(ii) U is rel. vertices of I^2,

$$i.e.\ \partial_2^-\partial_2^-(U),\ \ \partial_2^-\partial_2^+(U),\ \ \partial_2^+\partial_2^-(U),\ \ \partial_2^+\partial_2^+(U)\ \textit{are constant},$$

(iii) the faces $\partial_i^\alpha(U)$ are thin for $\alpha = \pm 1,\ i = 2, 3$.

(2) The square u is cubically T-equivalent to u', denoted $u \equiv_T^\square u'$ if there is a cubically thin homotopy between u and u'.

4.7. Proposition. *The relation $\equiv_T^\square$ is an equivalence relation on $R_2^\square(X)$.*

Proof The reader is referred to [4] for a proof. ■

If $u \in R_2^\square(X)$ we write $\{u\}_T^\square$, or simply $\{u\}_T$, for the equivalence class of u with respect to $\equiv_T^\square$. We denote the set of equivalence classes $R_2^\square(X) \equiv_T^\square$ by $\boldsymbol{\rho}_2^\square(X)$. This inherits the operations and the geometrically defined connections from $R_2^\square(X)$ and so becomes a double groupoid with connections. A proof of the final fine detail of the structure is given in [4].

4.8. Definition. *An element of $\boldsymbol{\rho}_2^\square(X)$ is* thin *if it has a thin representative (in the sense of Definition 4.3).*

¿From the remark at the beginning of this subsection we infer:

4.9. Lemma. *Let $f : \boldsymbol{\rho}^\square(X) \to \mathsf{D}$ be a morphism of double groupoids with connection. If $\alpha \in \boldsymbol{\rho}_2^\square(X)$ is thin, then $f(\alpha)$ is thin.* ■

4.10. The homotopy addition lemma. Let $u : I^3 \to X$ be a singular cube in a Hausdorff space X. Then by restricting u to the faces of I^3 and taking the corresponding elements in $\boldsymbol{\rho}_2^\square(X)$, we obtain a cube in $\boldsymbol{\rho}^\square(X)$ which is commutative by the homotopy addition lemma for $\boldsymbol{\rho}^\square(X)$ ([4], Proposition 5.5). Consequently, if $f : \boldsymbol{\rho}^\square(X) \to \mathsf{D}$ is a morphism of double groupoids with connections, any singular cube in X determines a commutative 3-shell in D.

5. The van Kampen/coequaliser Theorem

The general setting of the van Kampen/coequaliser theorem is that of a local-to global problem which can be explained as follows:

> Given an open covering $\mathcal{U}$ of X and knowledge of each $\boldsymbol{\rho}^\square(U)$ for U in $\mathcal{U}$, give a determination of $\boldsymbol{\rho}^\square(X)$.

Of course we need also to know the values of $\boldsymbol{\rho}^\square$ on intersections $U \cap V$ and on the inclusions from $U \cap V$ to U and V.

We first note that that the functor $\boldsymbol{\rho}^\square$ on the category **Haus** of Hausdorff spaces preserves coproducts $\bigsqcup$, since these are just disjoint union in Hausdorff spaces and in double groupoids. It is an advantage of the groupoid approach that the coproduct of such objects is so simple to describe.

Suppose we are given a cover $\mathcal{U}$ of X. Then the homotopy double groupoids in the following *$\boldsymbol{\rho}$-sequence of the cover* are well-defined:

$$\bigsqcup_{(U,V)\in\mathcal{U}^2} \boldsymbol{\rho}^{\square}(U\cap V) \ \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{U\in\mathcal{U}} \boldsymbol{\rho}^{\square}(U) \xrightarrow{c} \boldsymbol{\rho}^{\square}(X). \tag{7}$$

The morphisms a, b are determined by the inclusions

$$a_{UV} : U\cap V \to U, b_{UV} : U\cap V \to V$$

for each $(U,V)\in\mathcal{U}^2$ and c is determined by the inclusion $c_U : U \to X$ for each $U\in\mathcal{U}$.

5.1. Theorem. [van Kampen/coequaliser theorem] *If the interiors of the sets of $\mathcal{U}$ cover X, then in the above $\boldsymbol{\rho}$-sequence of the cover, c is the coequaliser of a, b in the category of double groupoids with connections.*

A special case of this result is when $\mathcal{U}$ has two elements. In this case the coequaliser reduces to a pushout.

The proof of the theorem is a direct verification of the universal property for the coequaliser. So suppose D is a double groupoid, and

$$f : \bigsqcup_{U\in\mathcal{U}} \boldsymbol{\rho}^{\square}(U) \to \mathsf{D}$$

is a morphism such that $fa = fb$. We require to construct uniquely a morphism $F : \boldsymbol{\rho}^{\square}(X) \to \mathsf{D}$ such that $Fc = f$. It is convenient to write f^U for the restriction of f to $\boldsymbol{\rho}^{\square}(U)$.

First, F is uniquely defined on objects, since if $x \in X$ then $x \in U$ for some $U\in\mathcal{U}$ and so $F(x) = Fc(x) = f^U(x)$, and the condition $fa = fb$ ensures this value is independent of U.

We next consider F on an element $\langle u\rangle \in \boldsymbol{\rho}_1^{\square}(X)$. By the Lebesgue covering lemma, we can write

$$\langle u\rangle = \langle u_1\rangle + \cdots + \langle u_n\rangle$$

where u_i is a path in a set U_i of the cover, and so determines an element $\langle u_i'\rangle$ in $\boldsymbol{\rho}_1^{\square}(U_i)$. The rule $fa = fb$ implies that the elements $f^{U_i}\langle u_i'\rangle$ are composable in D, and, since F is a morphism, and $Fc = f$, their sum must be $F\langle u\rangle$.

A similar argument applies to an element $\langle\alpha\rangle \in \boldsymbol{\rho}_2^{\square}(X)$, where this time we choose a subdivision of α as a multiple composition $[\alpha_{ij}]$ with α_{ij} lying in some U_{ij}. We must have $F\langle\alpha\rangle = [f^{ij}\langle\alpha_{ij}'\rangle]$.

Now we must prove that F can be well defined by choices of this kind. Independence of the subdivision chosen is easily verified by superimposing subdivisions. The hard part is to show the effect of a homotopy.

We start in dimension 1.

Suppose $h : u \sim_T v$ is a thin homotopy of paths u, v. Then there is a factorisation

$$h : I^2 \xrightarrow{\Phi} J \xrightarrow{p} X$$

where J is a tree, and Φ is PWL on the boundary ∂I^2 of I^2.

Now $p^{-1}(\mathcal{U})$ covers J. Choose a subdivision of J so that the open star of each edge of the subdivision is contained in a set of $p^{-1}(\mathcal{U})$.

By a *grid subdivision of* I^n we mean a subdivision determined by equally spaced hyperplanes of the form $x_i =$ constant, $i = 1, \dots, n$. Choose a grid subdivision of I^2 such that Φ maps each little square s of the subdivision into some open star $St(s)$ of an edge of J, and so $\Phi(s) \subseteq p^{-1}U(s)$ for some $U(s) \in \mathcal{U}$.

Assign extra vertices to J as Φ of the vertices of the grid subdivision.

Consider a non boundary edge e of the grid subdivision, with adjacent squares s, s'. Deform Φ on e, keeping its image in $St(s) \cap St(s')$ so that Φ is PWL on e. Doing this for each edge of a square s gives a homotopy of the restriction $\Phi \mid \partial s$ with image always in $St(s)$. The HEP allows this homotopy to be extended to a homotopy of $\Phi \mid s$ still with values in $St(s)$. These determine a homotopy $\Phi \simeq \Phi'$ rel ∂I^2 such that if $h' = p\Phi'$, then $h' \mid s$ is a thin square lying in $U(s) \in \mathcal{U}$ for each s of the grid subdivision. The class $\langle h' \mid s \rangle$ in $\boldsymbol{\rho}_2^{\square}(U(s))$ is mapped by $f^{U(s)}$ to a thin square in D. The condition $fa = fb$ implies these squares are composable, and by the property (T2) of thin squares, their composite is a thin square. By (T3) of §2, this composite is an identity, and so $F(\langle u \rangle) = F(\langle v \rangle)$. Thus F is well defined.

The proof that F in this dimension is a morphism of groupoids is immediate from the definition.

We apply a similar argument in dimension 2.

Suppose $W : \alpha \equiv_T \beta$ is a cubically thin homotopy. Make a grid subdivision of I^3 into subcubes c such that W maps c into a set $U(c) \in \mathcal{U}$. Then c determines a commutative 3-shell in $\boldsymbol{\rho}^{\square}(U(c))$ which is mapped by $f^{U(c)}$ to a commutative 3-shell in D. Again by $fa = fb$ but now in the next dimension, these commutative 3-shells are composable in D to give a commutative 3-shell C. But the faces of W not in direction 1 are given to be thin homotopies. The argument as above shows that the faces of C not in direction 1 are thin squares in D. By Lemma 2.6, $F(\langle \alpha \rangle) = F(\langle \beta \rangle)$, and so F is well defined.

It is easy to check that F is a morphism of double groupoids with connections. This completes the proof of our main result, apart from the proof of Theorem 3.5.

6. Proof of Theorem 3.5

The proof uses some 2-dimensional rewriting using connections of the type used since the 1970s. However there are some tricky points which we would like to emphasise.

We often have to rearrange some block subdivision of a multiple composition. The general validity of this process is discussed in [12], and its application to double categories with connections in [10]. Here we point out the following.

Let α, β be squares in a double category such that

$$\gamma = \begin{bmatrix} \alpha & \beta \end{bmatrix} = \alpha +_2 \beta$$

is defined. Suppose further that

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

If $\alpha_1 +_2 \beta_1, \quad \alpha_2 +_2 \beta_2$ are defined, then we can write

$$\gamma = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}.$$

However, if we rewrite

$$\alpha = \begin{bmatrix} \alpha_1' \\ \alpha_2' \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1' \\ \beta_2' \end{bmatrix}$$

then we cannot write

$$\gamma = \begin{bmatrix} \alpha_1' & \beta_1' \\ \alpha_2' & \beta_2' \end{bmatrix}$$

unless we are sure the compositions $\alpha_1' +_2 \beta_1', \quad \alpha_2' +_2 \beta_2'$ are defined. Thus care is needed in 2-dimensional rewriting.

We now proceed with the proof that all compositions of commutative 3-cubes are commutative. We first do the case of direction 2. This requires the transport laws.

Let α, β be 3-shells in K such that $\alpha_2^+ = \beta_2^-$. We assume that α, β are commutative, so that the HCL holds for each. Then

$$
\begin{aligned}
\boldsymbol{\partial}^{\mathrm{odd}}(\alpha +_2 \beta) &= \begin{bmatrix} \ulcorner & (\alpha +_2 \beta)_1^- & \lrcorner \\ (\alpha +_2 \beta)_3^- & (\alpha +_2 \beta)_2^+ & = \end{bmatrix} && \text{(by definition)} \\
&= \begin{bmatrix} \ulcorner & \alpha_1^- +_1 \beta_1^- & \lrcorner \\ \alpha_3^- +_2 \beta_3^- & \beta_2^+ & = \end{bmatrix} && \text{(by definition of } +_2 \text{ for 3-shells)} \\
&= \begin{bmatrix} \ulcorner & = & \alpha_1^- & \lrcorner & \mid\mid \\ \mid\mid & \ulcorner & \beta_1^- & = & \lrcorner \\ \alpha_3^- & \beta_3^- & \beta_2^+ & = & = \end{bmatrix} && \text{(by the transport laws)} \\
&= \begin{bmatrix} \ulcorner & = & \alpha_1^- & \lrcorner & \mid\mid \\ \alpha_3^- & = & \beta_2^- & = & \beta_3^+ \\ \mid\mid & \ulcorner & \beta_1^+ & = & \lrcorner \end{bmatrix} && \text{(by HCL for } \beta) \\
&= \begin{bmatrix} \ulcorner & = & \alpha_1^- & \lrcorner & \mid\mid \\ \alpha_3^- & = & \alpha_2^+ & = & \beta_3^+ \\ \mid\mid & \ulcorner & \beta_1^+ & = & \lrcorner \end{bmatrix} && \text{(since } \beta_2^- = \alpha_2^+ \text{)} \\
&= \begin{bmatrix} = & = & \alpha_2^- & \alpha_3^+ & \beta_3^+ \\ \ulcorner & = & \alpha_1^+ & \lrcorner & \mid\mid \\ \mid\mid & \ulcorner & \beta_1^+ & = & \lrcorner \end{bmatrix} && \text{(by HCL for } \alpha) \\
&= \begin{bmatrix} = & (\alpha +_2 \beta)_2^- & (\alpha +_2 \beta)_3^+ \\ \ulcorner & (\alpha +_2 \beta)_1^+ & \lrcorner \end{bmatrix} && \text{(by transport laws and composition rules)} \\
&= \boldsymbol{\partial}^{\mathrm{even}}(\alpha +_2 \beta) && \text{(as required.)}
\end{aligned}
$$

We now consider the case of $+_3$ and for this it turns out to be convenient to use (HCL$'$). We write the left hand side of this as $\boldsymbol{\partial}'^{\mathrm{odd}}$ and the right hand side as $\boldsymbol{\partial}'^{\mathrm{even}}$.

Suppose then $\alpha +_3 \beta$ is defined. Then

$$
\begin{aligned}
\boldsymbol{\partial}'^{\mathrm{odd}}(\alpha +_3 \beta) &= \begin{bmatrix} \ulcorner & (\alpha +_3 \beta)_1^- \\ (\alpha +_3 \beta)_3^- & (\alpha +_3 \beta)_2^+ \\ \lrcorner & \vert\,\vert \end{bmatrix} \\
&= \begin{bmatrix} \ulcorner & \alpha_1^- +_2 \beta_1^- \\ \alpha_3^- & \alpha_2^+ +_2 \beta_2^+ \\ \lrcorner & \vert\,\vert \end{bmatrix} \\
&= \begin{bmatrix} \ulcorner & \alpha_1^- & \beta_1^- \\ \alpha_3^- & \alpha_2^+ & \beta_2^+ \\ \lrcorner & \vert\,\vert & \vert\,\vert \end{bmatrix} \\
&= \begin{bmatrix} \vert\,\vert & \ulcorner & \beta_1^- \\ \alpha_2^- & \alpha_3^+ & \beta_2^+ \\ \alpha_1^+ & \lrcorner & \vert\,\vert \end{bmatrix} && \text{(by HCL}'\text{)} \\
&= \begin{bmatrix} \vert\,\vert & \ulcorner & \beta_1^- \\ \alpha_2^- & \beta_3^- & \beta_2^+ \\ \alpha_1^+ & \lrcorner & \vert\,\vert \end{bmatrix} && (\text{as } \alpha_3^+ = \beta_3^-) \\
&= \begin{bmatrix} \vert\,\vert & \vert\,\vert & \ulcorner \\ \alpha_2^- & \beta_2^- & \beta_3^+ \\ \alpha_1^+ & \beta_1^+ & \lrcorner \end{bmatrix} && \text{(by HCL}'\text{)} \\
&= \begin{bmatrix} \vert\,\vert & \ulcorner \\ (\alpha +_3 \beta)_2^- & (\alpha +_3 \beta)_3^+ \\ (\alpha +_3 \beta)_1^+ & \lrcorner \end{bmatrix} \\
&= \partial'^{\mathrm{even}}(\alpha +_3 \beta) && \text{(as required.)}
\end{aligned}
$$

The proof for $+_1$ is similar to the last one, but using (HCL). We leave it to the reader. ■

To see the complications of these ideas in higher dimensions, see [1, 17] and also p.361-362 of [14], which deals with the 4-cube.

Note also that these calculations, when applied to $\boldsymbol{\rho}^{\square}(X)$, imply the existence, even a construction, of certain homotopies which would otherwise be difficult to find. The possibility of calculating with such homotopies is indeed one of the aims of this theory.

7. Future directions

Our intention for future papers in this series is to investigate:

A) Does $\boldsymbol{\rho}^{\square}(X)$ capture the weak homotopy 2-type of X? This seems likely in view of the facts on $\boldsymbol{\rho}^{\square}(X)$ given in the Introduction, and since crossed modules of groupoids, and hence also double groupoids with connections, capture all weak homotopy 2-types (see, for example, [9]).

B) How does the functor $\boldsymbol{\rho}^{\square}$ behave on homotopies? This question requires for its answer notions of tensor product and homotopies for double groupoids with connections. These have been developed in the corresponding ω-groupoid case in [8], and in the ω-category case in [1].

C) It is also expected that the above developments will allow an enrichment of the category of Hausdorff spaces over a monoidal closed category of double groupoids with connection, analogous to the enrichment of the category of filtered spaces over the category of crossed complexes, given in [9]. Such an enrichment should be useful for calculations with some multiple compositions of homotopies.

D) Are there smooth analogues of $\boldsymbol{\rho}^{\square}(M)$ in the case when M is a smooth manifold? This possibility is suggested by the methods of [20].

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