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EVERY SMALL $\mathcal{S}l$-ENRICHED CATEGORY IS MORITA EQUIVALENT TO AN $\mathcal{S}l$-MONOID

Dedicated to Professor Aurelio Carboni on the occasion of his sixtieth birthday

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ABSTRACT. We show that every small category enriched over $\mathcal{S}l$—the symmetric monoidal closed category of sup-lattices and sup-preserving morphisms—is Morita equivalent to a $\mathcal{S}l$-monoid. As a corollary, we obtain a result of Borceux and Vitale [1] asserting that every separable $\mathcal{S}l$-category is Morita equivalent to a separable $\mathcal{S}l$-monoid.

1. Theorem

We use [2] as a reference for enriched category theory.

Let $\mathcal{S}l = (\mathcal{S}l_\circ, -\otimes-, I)$ denote the symmetric monoidal closed category of sup-lattices with sup-preserving maps.

It is well-known [3] that $\mathcal{S}l_\circ$ is a complete, cocomplete pointed category. Moreover, (small) coproducts are biproducts; that is, if $\{x_i\}$ is a (small) family of objects of $\mathcal{S}l_\circ$, the unique morphism

$$\delta : \coprod x_i \longrightarrow \prod x_i$$

with components $\delta_{ii} = 1_{x_i}$ and $\delta_{ij} = 0$ otherwise is an isomorphism. For any small $\mathcal{S}l$-category $\mathcal{A}$, this property transfers to the $\mathcal{S}l$-functor category $[\mathcal{A}^{op}, \mathcal{S}l]$, since limits and colimits there are computed pointwise.

We will write $\oplus x_i$ for the biproduct of the family $\{x_i\}$.

1.1. THEOREM. *Every small $\mathcal{S}l$-category is Morita equivalent to an $\mathcal{S}l$-monoid.*

PROOF. We first observe that for each pair of objects c, c' of an $\mathcal{S}l$-enriched category $\mathcal{C}$, the set $\mathcal{C}_\circ(c, c')$ is non-empty because every sup-lattice contains at least the element 0.

Now let $\mathcal{A}$ be a small $\mathcal{S}l$-category. Define the $\mathcal{S}l$-functor $P_\mathcal{A} : \mathcal{A}^{op} \longrightarrow \mathcal{S}l$ to be $\oplus_{a\in\mathcal{A}}\mathcal{A}(-, a)$. Since arbitrary (small) coproducts (=products) of representable $\mathcal{S}l$-functors are in the Cauchy completion $\overline{\mathcal{A}}$ of $\mathcal{A}$ (see [4]), $P_\mathcal{A}$ is in $\overline{\mathcal{A}}$. Thus the $\mathcal{S}l$-functor $P_\mathcal{A}$ is small projective in $[\mathcal{A}^{op}, \mathcal{S}l]$.

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Next, by the observation at the beginning of the proof, no $\big[\mathcal{A}^{op}, \mathcal{S}l\big]\big(\mathcal{A}(-,a), \mathcal{A}(-,a')\big)$ is empty. Moreover, the small $\mathcal{S}l$-full subcategory determined by the representable $\mathcal{S}l$-functors is dense in $[\mathcal{A}^{op}, \mathcal{S}l]$. So that, in view of Proposition 5.22 of [2], the $\mathcal{S}l$-full subcategory $[P_{\mathcal{A}}] \subseteq [\mathcal{A}^{op}, \mathcal{S}l]$ generated by the $\mathcal{S}l$-functor $P_{\mathcal{A}}$ is dense, and *a fortiori* strongly generating, in $[\mathcal{A}^{op}, \mathcal{S}l]$. Thus $P_{\mathcal{A}} : \mathcal{A}^{op} \longrightarrow \mathcal{S}l$, besides being small projective in $[\mathcal{A}^{op}, \mathcal{S}l]$, is a strong generator for $[\mathcal{A}^{op}, \mathcal{S}l]$. It now follows from Theorem 5.26 of [2] that the $\mathcal{S}l$-functor

$$\mathcal{N}_{\mathcal{A}} : [\mathcal{A}^{op}, \mathcal{S}l] \longrightarrow \big[[P_{\mathcal{A}}]^{op}, \mathcal{S}l\big]$$

given by

$$\mathcal{N}_{\mathcal{A}}(F) = \big[\mathcal{A}^{op}, \mathcal{S}l\big]\big(P_{\mathcal{A}}, F\big)$$

is an equivalence of $\mathcal{S}l$-categories. Whence $\mathcal{A}$ is Morita equivalent to the $\mathcal{S}l$-monoid $[\mathcal{A}^{op}, \mathcal{S}l](P_{\mathcal{A}}, P_{\mathcal{A}})$. ∎

1.2. Remark. There is precisely an analogous result wherein $\mathcal{S}l$ is replaced by the symmetric monoidal closed category of abelian groups, **Ab**, and $\mathcal{A}$ by an **Ab**-enriched category with finitely many objects (see, for example, [5]); and still another with $\mathcal{S}l$ replaced by the symmetric monoidal closed category of commutative monoids, once again in the finitely-many-object case.

2. Application

Let $\mathcal{V} = (\mathcal{V}_\circ, -\otimes-, I)$ be a symmetric monoidal closed category whose underlying ordinary category $\mathcal{V}_\circ$ is locally small, complete and cocomplete. Write $\mathcal{B}im(\mathcal{V})$ for the bicategory of small $\mathcal{V}$-categories, $\mathcal{V}$-bimodules and $\mathcal{V}$-natural transformation between them.

Recall [1] that a small $\mathcal{V}$-category $\mathcal{A}$ is separable when the canonical $\mathcal{V}$-functor

$$\mathcal{A}(-,-) : \mathcal{A}^{op} \otimes \mathcal{A} \longrightarrow \mathcal{V}$$

is small projective in the $\mathcal{V}$-functor category $[\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{V}]$.

2.1. Proposition. *Separability is invariant under Morita equivalence.*

Proof. Recall (for instance from [2]) that two small $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if and only if there exist bimodules $\phi : \mathcal{A} \multimap \mathcal{B}$ and $\psi : \mathcal{B} \multimap \mathcal{A}$ such that $\psi \circ \phi \simeq \mathcal{A}$ and $\phi \circ \psi \simeq \mathcal{B}$. Then

$$[\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{V}] \simeq \mathcal{B}im(\mathcal{V})(\mathcal{A}, \mathcal{A}) \xrightarrow[\simeq]{-\circ\psi} \mathcal{B}im(\mathcal{V})(\mathcal{B}, \mathcal{A}) \xrightarrow[\simeq]{\phi\circ-} \mathcal{B}im(\mathcal{V})(\mathcal{B}, \mathcal{B}) \simeq [\mathcal{B}^{op} \otimes \mathcal{B}, \mathcal{V}],$$

and (up to isomorphism) this composite takes the $\mathcal{V}$-functor $\mathcal{A}(-,-) : \mathcal{A}^{op} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ to the $\mathcal{V}$-functor $\mathcal{B}(-,-) : \mathcal{B}^{op} \otimes \mathcal{B} \longrightarrow \mathcal{V}$. The result now follows from the fact that any equivalence of cocomplete $\mathcal{V}$-categories preserves and reflects the property of objects of being small projective. ∎

It follows from Theorem 1.1 and Proposition 2.1 that

2.2. THEOREM. *[1] Every separable $\mathcal{S}l$-category is Morita equivalent to a separable $\mathcal{S}l$-monoid. In particular, every Azumaya $\mathcal{S}l$-category is Morita equivalent to an Azumaya $\mathcal{S}l$-monoid.*

References

[1] Borceux, F. and Vitale, E.: *Azumaya categories*, Applied Categorical Structures **10** (2002), 449–467.

[2] Kelly, G. M.: *Basic Concepts of Enriched Category Theory*, London Math. Soc. Lecture Notes Series **20**, Cambridge University Press, Cambridge, (1982).

[3] Joyal, A. and Tierney, M.: *An Extension of the Galois Theory of Grothendieck,* Mem. Amer. Math. Soc. **51,** no 309, (1984).

[4] Johnson, S. R.: *Small Cauchy Completions*, Journal of Pure and Applied Algebra **62** (1989), 35–45.

[5] Mitchell, B.: *Rings with Several Objects*, Advances in Math. **8** (1972), 1–161.

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