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HIGHER-DIMENSIONAL ALGEBRA V: 2-GROUPS

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Abstract. A 2-group is a 'categorified' version of a group, in which the underlying set G has been replaced by a category and the multiplication map $m\colon G \times G \to G$ has been replaced by a functor. Various versions of this notion have already been explored; our goal here is to provide a detailed introduction to two, which we call 'weak' and 'coherent' 2-groups. A weak 2-group is a weak monoidal category in which every morphism has an inverse and every object x has a 'weak inverse': an object y such that $x \otimes y \cong 1 \cong y \otimes x$. A coherent 2-group is a weak 2-group in which every object x is equipped with a *specified* weak inverse $\bar{x}$ and isomorphisms $i_x\colon 1 \to x \otimes \bar{x}$, $e_x\colon \bar{x} \otimes x \to 1$ forming an adjunction. We describe 2-categories of weak and coherent 2-groups and an 'improvement' 2-functor that turns weak 2-groups into coherent ones, and prove that this 2-functor is a 2-equivalence of 2-categories. We internalize the concept of coherent 2-group, which gives a quick way to define Lie 2-groups. We give a tour of examples, including the 'fundamental 2-group' of a space and various Lie 2-groups. We also explain how coherent 2-groups can be classified in terms of 3rd cohomology classes in group cohomology. Finally, using this classification, we construct for any connected and simply-connected compact simple Lie group G a family of 2-groups $G_\hbar$ ($\hbar \in \mathbb{Z}$) having G as its group of objects and U(1) as the group of automorphisms of its identity object. These 2-groups are built using Chern–Simons theory, and are closely related to the Lie 2-algebras $\mathfrak{g}_\hbar$ ($\hbar \in \mathbb{R}$) described in a companion paper.

1. Introduction

Group theory is a powerful tool in all branches of science where symmetry plays a role. However, thanks in large part to the vision and persistence of Ronald Brown [14], it has become clear that group theory is just the tip of a larger subject that deserves to be called 'higher-dimensional group theory'. For example, in many contexts where we are tempted to use groups, it is actually more natural to use a richer sort of structure, where in addition to group elements describing symmetries, we also have isomorphisms between these, describing *symmetries between symmetries*. One might call this structure a 'categorified' group, since the underlying set G of a traditional group is replaced by a category, and the multiplication function $m\colon G \times G \to G$ is replaced by a functor. However, to hint at a sequence of further generalizations where we use n-categories and n-functors, we prefer the term '2-group'.

There are many different ways to make the notion of a 2-group precise, so the history of this idea is complex, and we can only briefly sketch it here. A crucial first step was

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J. H. C. Whitehead's [53] concept of 'crossed module', formulated around 1946 without the aid of category theory. In 1950, Mac Lane and Whitehead [41] proved that a crossed module captures in algebraic form all the homotopy-invariant information about what is now called a 'connected pointed homotopy 2-type'—roughly speaking, a nice connected space equipped with a basepoint and having homotopy groups that vanish above π_2. By the 1960s it was clear to Verdier and others that crossed modules are essentially the same as 'categorical groups'. In the present paper we call these 'strict 2-groups', since they are categorified groups in which the group laws hold strictly, as *equations*.

Brown and Spencer [15] published a proof that crossed modules are equivalent to categorical groups in 1976. However, Grothendieck was already familiar with these ideas, and in 1975 his student Hoang Xuan Sinh wrote her thesis [44] on a more general concept, namely 'gr-categories', in which the group laws hold only *up to isomorphism*. In the present paper we call these 'weak' or 'coherent' 2-groups, depending on the precise formulation.

While influential, Sinh's thesis was never published, and is now quite hard to find. Also, while the precise relation between 2-groups, crossed modules and group cohomology was greatly clarified in the 1986 draft of Joyal and Street's paper on braided tensor categories [33], this section was omitted from the final published version. So, while the basic facts about 2-groups are familiar to most experts in category theory, it is difficult for beginners to find an introduction to this material. This is becoming a real nuisance as 2-groups find their way into ever more branches of mathematics, and lately even physics. The first aim of the present paper is to fill this gap.

So, let us begin at the beginning. Whenever one categorifies a mathematical concept, there are some choices involved. For example, one might define a 2-group simply to be a category equipped with functors describing multiplication, inverses and the identity, satisfying the usual group axioms 'on the nose'—that is, as equations between functors. We call this a 'strict' 2-group. Part of the charm of strict 2-groups is that they can be defined in a large number of equivalent ways, including:

- a strict monoidal category in which all objects and morphisms are invertible,
- a strict 2-category with one object in which all 1-morphisms and 2-morphisms are invertible,
- a group object in Cat (also called a 'categorical group'),
- a category object in Grp,
- a crossed module.

There is an excellent review article by Forrester-Barker that explains most of these notions and why they are equivalent [26].

Strict 2-groups have been applied in a variety of contexts, from homotopy theory [13, 15] and topological quantum field theory [54] to nonabelian cohomology [8, 9, 27], the theory of nonabelian gerbes [9, 11], categorified gauge field theory [1, 2, 28, 43], and

even quantum gravity [21, 22]. However, the strict version of the 2-group concept is not the best for all applications. Rather than imposing the group axioms as equational laws, it is sometimes better to 'weaken' them: in other words, to require only that they hold up to specified isomorphisms satisfying certain laws of their own. This leads to the concept of a 'coherent 2-group'.

For example, given objects x, y, z in a strict 2-group we have

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

where we write multiplication as $\otimes$. In a coherent 2-group, we instead specify an isomorphism called the 'associator':

$$a_{x,y,z} \colon (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)\,.$$

Similarly, we replace the left and right unit laws

$$1 \otimes x = x, \qquad x \otimes 1 = x$$

by isomorphisms

$$\ell_x \colon 1 \otimes x \xrightarrow{\sim} x\,, \qquad r_x \colon x \otimes 1 \xrightarrow{\sim} x$$

and replace the equations

$$x \otimes x^{-1} = 1, \qquad x^{-1} \otimes x = 1$$

by isomorphisms called the 'unit' and 'counit'. Thus, instead of an inverse in the strict sense, the object x only has a specified 'weak inverse'. To emphasize this fact, we denote this weak inverse by $\bar{x}$.

Next, to manipulate all these isomorphisms with some of the same facility as equations, we require that they satisfy conditions known as 'coherence laws'. The coherence laws for the associator and the left and right unit laws were developed by Mac Lane [39] in his groundbreaking work on monoidal categories, while those for the unit and counit are familiar from the definition of an adjunction in a monoidal category [33]. Putting these ideas together, one obtains Ulbrich and Laplaza's definition of a 'category with group structure' [36, 50]. Finally, a 'coherent 2-group' is a category G with group structure in which all morphisms are invertible. This last condition ensures that there is a covariant functor

$$\mathrm{inv} \colon G \to G$$

sending each object $x \in G$ to its weak inverse $\bar{x}$; otherwise there will only be a contravariant functor of this sort.

In this paper we compare this sort of 2-group to a simpler sort, which we call a 'weak 2-group'. This is a weak monoidal category in which every morphism has an inverse and every object x has a weak inverse: an object y such that $y \otimes x \cong 1$ and $x \otimes y \cong 1$. Note that in this definition, we do not *specify* the weak inverse y or the isomorphisms from

$y \otimes x$ and $x \otimes y$ to 1, nor do we impose any coherence laws upon them. Instead, we merely demand that they *exist*. Nonetheless, it turns out that any weak 2-group can be improved to become a coherent one! While this follows from a theorem of Laplaza [36], it seems worthwhile to give an expository account here, and to formalize this process as a 2-functor

$$\mathrm{Imp}\colon \mathrm{W2G} \to \mathrm{C2G}$$

where W2G and C2G are suitable strict 2-categories of weak and coherent 2-groups, respectively.

On the other hand, there is also a forgetful 2-functor

$$\mathrm{F}\colon \mathrm{C2G} \to \mathrm{W2G}.$$

One of the goals of this paper is to show that Imp and F fit together to define a 2-equivalence of strict 2-categories. This means that the 2-category of weak 2-groups and the 2-category of coherent 2-groups are 'the same' in a suitably weakened sense. Thus there is ultimately not much difference between weak and coherent 2-groups.

To show this, we start in Section 2 by defining weak 2-groups and the 2-category W2G. In Section 3 we define coherent 2-groups and the 2-category C2G. To do calculations in 2-groups, it turns out that certain 2-dimensional pictures called 'string diagrams' can be helpful, so we explain these in Section 4. In Section 5 we use string diagrams to define the 'improvement' 2-functor $\mathrm{Imp}\colon \mathrm{W2G} \to \mathrm{C2G}$ and prove that it extends to a 2-equivalence of strict 2-categories. This result relies crucially on the fact that morphisms in C2G are just weak monoidal functors, with no requirement that they preserve weak inverses. In Section 6 we justify this choice, which may at first seem questionable, by showing that weak monoidal functors automatically preserve the specified weak inverses, up to a well-behaved isomorphism.

In applications of 2-groups to geometry and physics, we expect the concept of *Lie 2-group* to be particularly important. This is essentially just a 2-group where the set of objects and the set of morphisms are manifolds, and all relevant maps are smooth. Until now, only strict Lie 2-groups have been defined [2]. In section 7 we show that the concept of 'coherent 2-group' can be defined in any 2-category with finite products. This allows us to efficiently define coherent Lie 2-groups, topological 2-groups and the like.

In Section 8 we discuss examples of 2-groups. These include various sorts of 'automorphism 2-group' for an object in a 2-category, the 'fundamental 2-group' of a topological space, and a variety of strict Lie 2-groups. We also describe a way to classify 2-groups using group cohomology. As we explain, coherent 2-groups—and thus also weak 2-groups—can be classified up to equivalence in terms of a group G, an action α of G on an abelian group H, and an element $[a]$ of the 3rd cohomology group of G with coefficients in H. Here G is the group of objects in a 'skeletal' version of the 2-group in question: that is, an equivalent 2-group containing just one representative from each isomorphism class of objects. H is the group of automorphisms of the identity object, the action α is defined using conjugation, and the 3-cocycle a comes from the associator in the skeletal version.

Thus, $[a]$ can be thought of as the obstruction to making the 2-group simultaneously both skeletal and strict.

In a companion to this paper, called HDA6 [3] for short, Baez and Crans prove a Lie algebra analogue of this result: a classification of 'semistrict Lie 2-algebras'. These are categorified Lie algebras in which the antisymmetry of the Lie bracket holds on the nose, but the Jacobi identity holds only up to a natural isomorphism called the 'Jacobiator'. It turns out that semistrict Lie 2-algebras are classified up to equivalence by a Lie algebra $\mathfrak{g}$, a representation ρ of $\mathfrak{g}$ on an abelian Lie algebra $\mathfrak{h}$, and an element $[j]$ of the 3rd Lie algebra cohomology group of $\mathfrak{g}$ with coefficients in $\mathfrak{h}$. Here the cohomology class $[j]$ comes from the Jacobiator in a skeletal version of the Lie 2-algebra in question. A semistrict Lie 2-algebra in which the Jacobiator is the identity is called 'strict'. Thus, the class $[j]$ is the obstruction to making a Lie 2-algebra simultaneously skeletal and strict.

Interesting examples of Lie 2-algebras that cannot be made both skeletal and strict arise when $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over the real numbers. In this case we may assume without essential loss of generality that ρ is irreducible, since any representation is a direct sum of irreducibles. When ρ is irreducible, it turns out that $H^3(\mathfrak{g}, \rho) = \{0\}$ unless ρ is the trivial representation on the 1-dimensional abelian Lie algebra $\mathfrak{u}(1)$, in which case we have

$$H^3(\mathfrak{g}, \mathfrak{u}(1)) \cong \mathbb{R}.$$

This implies that for any value of $\hbar \in \mathbb{R}$ we obtain a skeletal Lie 2-algebra $\mathfrak{g}_\hbar$ with $\mathfrak{g}$ as its Lie algebra of objects, $\mathfrak{u}(1)$ as the endomorphisms of its zero object, and $[j]$ proportional to $\hbar \in \mathbb{R}$. When $\hbar = 0$, this Lie 2-algebra is just $\mathfrak{g}$ with identity morphisms adjoined to make it into a strict Lie 2-algebra. But when $\hbar \neq 0$, this Lie 2-algebra is not equivalent to a skeletal strict one.

In short, the Lie algebra $\mathfrak{g}$ sits inside a one-parameter family of skeletal Lie 2-algebras $\mathfrak{g}_\hbar$, which are strict only for $\hbar = 0$. This is strongly reminiscent of some other well-known deformation phenomena arising from the third cohomology of a simple Lie algebra. For example, the universal enveloping algebra of $\mathfrak{g}$ gives a one-parameter family of quasitriangular Hopf algebras $U_\hbar\mathfrak{g}$, called 'quantum groups'. These Hopf algebras are cocommutative only for $\hbar = 0$. The theory of 'affine Lie algebras' is based on a closely related phenomenon: the Lie algebra of smooth functions $C^\infty(S^1, \mathfrak{g})$ has a one-parameter family of central extensions, which only split for $\hbar = 0$. There is also a group version of this phenomenon, which involves an integrality condition: the loop group $C^\infty(S^1, G)$ has a one-parameter family of central extensions, one for each $\hbar \in \mathbb{Z}$. Again, these central extensions split only for $\hbar = 0$.

All these other phenomena are closely connected to Chern–Simons theory, a topological quantum field theory whose action is the secondary characteristic class associated to an element of $H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$. The relation to Lie algebra cohomology comes from the existence of an inclusion $H^4(BG, \mathbb{Z}) \hookrightarrow H^3(\mathfrak{g}, \mathfrak{u}(1)) \cong \mathbb{R}$.

Given all this, it is tempting to seek a 2-group analogue of the Lie 2-algebras $\mathfrak{g}_\hbar$. Indeed, such an analogue exists! Suppose that G is a connected and simply-connected

compact simple Lie group. In Section 8.5 we construct a family of skeletal 2-groups $G_\hbar$, one for each $\hbar \in \mathbb{Z}$, each having G as its group of objects and U(1) as the group of automorphisms of its identity object. The associator in these 2-groups depends on $\hbar$, and they are strict only for $\hbar = 0$.

Unfortunately, for reasons we shall explain, these 2-groups are not Lie 2-groups except for the trivial case $\hbar = 0$. However, the construction of these 2-groups uses Chern–Simons theory in an essential way, so we feel confident that they are related to all the other deformation phenomena listed above. Since the rest of these phenomena are important in mathematical physics, we hope these 2-groups $G_\hbar$ will be relevant as well. A full understanding of them may require a generalization of the concept of Lie 2-group presented in this paper.

Note: in all that follows, we write the composite of morphisms $f\colon x \to y$ and $g\colon y \to z$ as $fg\colon x \to z$. We use the term 'weak 2-category' to refer to a 'bicategory' in Bénabou's sense [5], and the term 'strict 2-category' to refer to what is often called simply a '2-category' [46].

2. Weak 2-groups

Before we define a weak 2-group, recall that a **weak monoidal category** consists of:

(i) a category M,

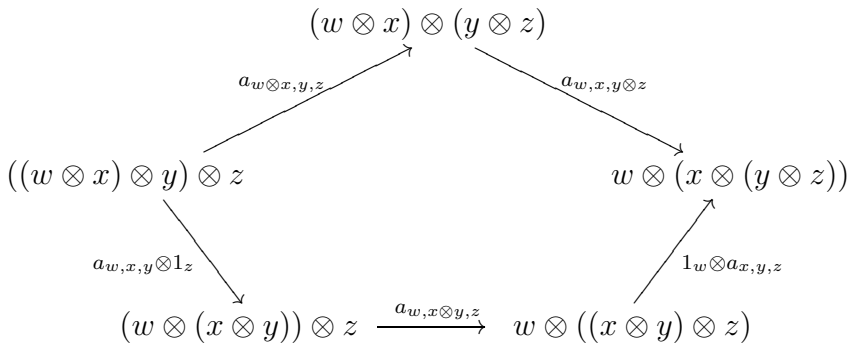
(ii) a functor $m\colon M \times M \to M$, where we write $m(x, y) = x \otimes y$ and $m(f, g) = f \otimes g$ for objects $x, y, \in M$ and morphisms f, g in M,

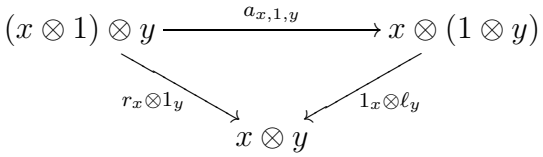
(iii) an 'identity object' $1 \in M$,

(iv) natural isomorphisms

$$a_{x,y,z}\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z),$$
$$\ell_x\colon 1 \otimes x \to x,$$
$$r_x\colon x \otimes 1 \to x,$$

such that the following diagrams commute for all objects $w, x, y, z \in M$:





A **strict monoidal category** is the special case where $a_{x,y,z}, \ell_x, r_x$ are all identity morphisms. In this case we have

$$(x \otimes y) \otimes z = x \otimes (y \otimes z),$$

$$1 \otimes x = x, \qquad x \otimes 1 = x.$$

As mentioned in the Introduction, a **strict 2-group** is a strict monoidal category where every morphism is invertible and every object x has an inverse x^{-1}, meaning that

$$x \otimes x^{-1} = 1, \qquad x^{-1} \otimes x = 1.$$

Following the principle that it is wrong to impose equations between objects in a category, we can instead start with a weak monoidal category and require that every object has a 'weak' inverse. With these changes we obtain the definition of 'weak 2-group':

2.1. Definition. *If x is an object in a weak monoidal category, a* **weak inverse** *for x is an object y such that $x \otimes y \cong 1$ and $y \otimes x \cong 1$. If x has a weak inverse, we call it* **weakly invertible**.

2.2. Definition. *A* **weak 2-group** *is a weak monoidal category where all objects are weakly invertible and all morphisms are invertible.*

In fact, Joyal and Street [33] point out that when every object in a weak monoidal category has a 'one-sided' weak inverse, every object is weakly invertible in the above sense. Suppose for example that every object x has an object y with $y \otimes x \cong 1$. Then y has an object z with $z \otimes y \cong 1$, and

$$z \cong z \otimes 1 \cong z \otimes (y \otimes x) \cong (z \otimes y) \otimes x \cong 1 \otimes x \cong x,$$

so we also have $x \otimes y \cong 1$.

Weak 2-groups are the objects of a strict 2-category W2G; now let us describe the morphisms and 2-morphisms in this 2-category. Notice that the only *structure* in a weak 2-group is that of its underlying weak monoidal category; the invertibility conditions on objects and morphisms are only *properties*. With this in mind, it is natural to define a morphism between weak 2-groups to be a weak monoidal functor. Recall that a **weak monoidal functor** $F\colon C \to C'$ between monoidal categories C and C' consists of:

(i) a functor $F\colon C \to C'$,

(ii) a natural isomorphism $F_2\colon F(x) \otimes F(y) \to F(x \otimes y)$, where for brevity we suppress the subscripts indicating the dependence of this isomorphism on x and y,

(iii) an isomorphism $F_0\colon 1' \to F(1)$, where 1 is the identity object of C and $1'$ is the identity object of C',

such that the following diagrams commute for all objects $x, y, z \in C$:

$$\begin{array}{ccccc}
(F(x) \otimes F(y)) \otimes F(z) & \xrightarrow{F_2 \otimes 1} & F(x \otimes y) \otimes F(z) & \xrightarrow{F_2} & F((x \otimes y) \otimes z) \\
\downarrow{\scriptstyle a_{F(x),F(y),F(z)}} & & & & \downarrow{\scriptstyle F(a_{x,y,z})} \\
F(x) \otimes (F(y) \otimes F(z)) & \xrightarrow{1 \otimes F_2} & F(x) \otimes F(y \otimes z) & \xrightarrow{F_2} & F(x \otimes (y \otimes z))
\end{array}$$

$$\begin{array}{ccc}
1' \otimes F(x) & \xrightarrow{\ell'_{F(x)}} & F(x) \\
\downarrow{\scriptstyle F_0 \otimes 1} & & \uparrow{\scriptstyle F(\ell_x)} \\
F(1) \otimes F(x) & \xrightarrow{F_2} & F(1 \otimes x)
\end{array}$$

$$\begin{array}{ccc}
F(x) \otimes 1' & \xrightarrow{r'_{F(x)}} & F(x) \\
\downarrow{\scriptstyle 1 \otimes F_0} & & \uparrow{\scriptstyle F(r_x)} \\
F(x) \otimes F(1) & \xrightarrow{F_2} & F(x \otimes 1)
\end{array}$$

A weak monoidal functor preserves tensor products and the identity object up to specified isomorphism. As a consequence, it also preserves weak inverses:

2.3. Proposition. *If $F\colon C \to C'$ is a weak monoidal functor and $y \in C$ is a weak inverse of $x \in C$, then $F(y)$ is a weak inverse of $F(x)$ in C'.*

Proof. Since y is a weak inverse of x, there exist isomorphisms $\gamma\colon x \otimes y \to 1$ and $\xi\colon y \otimes x \to 1$. The proposition is then established by composing the following isomorphisms:

$$\begin{array}{ccc}
F(y) \otimes F(x) & \xrightarrow{\sim} & 1' \\
\downarrow{\scriptstyle F_2} & & \uparrow{\scriptstyle F_0^{-1}} \\
F(y \otimes x) & \xrightarrow[F(\xi)]{} & F(1)
\end{array}
\qquad\qquad
\begin{array}{ccc}
F(x) \otimes F(y) & \xrightarrow{\sim} & 1' \\
\downarrow{\scriptstyle F_2} & & \uparrow{\scriptstyle F_0^{-1}} \\
F(x \otimes y) & \xrightarrow[F(\gamma)]{} & F(1)
\end{array}$$

∎

We thus make the following definition:

2.4. Definition. *A* **homomorphism** $F\colon C \to C'$ *between weak 2-groups is a weak monoidal functor.*

The composite of weak monoidal functors is again a weak monoidal functor [25], and composition satisfies associativity and the unit laws. Thus, 2-groups and the homomorphisms between them form a category.

Although they are not familiar from traditional group theory, it is natural in this categorified context to also consider '2-homomorphisms' between homomorphisms. Since a homomorphism between weak 2-groups is just a weak monoidal functor, it makes sense to define 2-homomorphisms to be monoidal natural transformations. Recall that if $F, G\colon C \to C'$ are weak monoidal functors, then a **monoidal natural transformation** $\theta\colon F \Rightarrow G$ is a natural transformation such that the following diagrams commute for all $x, y \in C$.

$$\begin{array}{ccc} F(x) \otimes F(y) & \xrightarrow{\theta_x \otimes \theta_y} & G(x) \otimes G(y) \\ {\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle G_2} \\ F(x \otimes y) & \xrightarrow{\theta_{x \otimes y}} & G(x \otimes y) \end{array}$$

$$\begin{array}{ccc} 1' & & \\ {\scriptstyle F_0}\downarrow & \searrow^{G_0} & \\ F(1) & \xrightarrow{\theta_1} & G(1) \end{array}$$

Thus we make the following definitions:

2.5. Definition. *A* **2-homomorphism** $\theta\colon F \Rightarrow G$ *between homomorphisms* $F, G\colon C \to C'$ *of weak 2-groups is a monoidal natural transformation.*

2.6. Definition. *Let* **W2G** *be the strict 2-category consisting of weak 2-groups, homomorphisms between these, and 2-homomorphisms between those.*

There is a strict 2-category MonCat with weak monoidal categories as objects, weak monoidal functors as 1-morphisms, and monoidal natural transformations as 2-morphisms [25]. W2G is a strict 2-category because it is a sub-2-category of MonCat.

3. Coherent 2-groups

In this section we explore another notion of 2-group. Rather than requiring that objects be weakly invertible, we will require that every object be equipped with a specified adjunction. Recall that an **adjunction** is a quadruple $(x, \bar{x}, i_x, e_x)$ where $i_x\colon 1 \to x \otimes \bar{x}$

(called the **unit**) and $e_x \colon \bar{x} \otimes x \to 1$ (called the **counit**) are morphisms such that the following diagrams commute:

$$\begin{array}{ccccc} 1 \otimes x & \xrightarrow{i_x \otimes 1} & (x \otimes \bar{x}) \otimes x & \xrightarrow{a_{x,\bar{x},x}} & x \otimes (\bar{x} \otimes x) \\ \downarrow{\scriptstyle \ell_x} & & & & \downarrow{\scriptstyle 1 \otimes e_x} \\ x & & \xrightarrow[r_x^{-1}]{} & & x \otimes 1 \end{array}$$

$$\begin{array}{ccccc} \bar{x} \otimes 1 & \xrightarrow{1 \otimes i_x} & \bar{x} \otimes (x \otimes \bar{x}) & \xrightarrow{a^{-1}_{\bar{x},x,\bar{x}}} & (\bar{x} \otimes x) \otimes \bar{x} \\ \downarrow{\scriptstyle r_{\bar{x}}} & & & & \downarrow{\scriptstyle e_x \otimes 1} \\ \bar{x} & & \xrightarrow[\ell_{\bar{x}}^{-1}]{} & & 1 \otimes \bar{x} \end{array}$$

When we express these laws using string diagrams in Section 4, we shall see that they give ways to 'straighten a zig-zag' in a piece of string. Thus, we refer to them as the first and second **zig-zag identities**, respectively.

An adjunction $(x, \bar{x}, i_x, e_x)$ for which the unit and counit are invertible is called an **adjoint equivalence**. In this case x and $\bar{x}$ are weak inverses. Thus, specifying an adjoint equivalence for x ensures that $\bar{x}$ is weakly invertible—but it does so by providing x with extra *structure*, rather than merely asserting a *property* of x. We now make the following definition:

3.1. Definition. *A* **coherent 2-group** *is a weak monoidal category C in which every morphism is invertible and every object $x \in C$ is equipped with an adjoint equivalence $(x, \bar{x}, i_x, e_x)$.*

Coherent 2-groups have been studied under many names. Sinh [44] called them 'gr-categories' when she initiated work on them in 1975, and this name is also used by Saavedra Rivano [47] and Breen [9]. As noted in the Introduction, a coherent 2-group is the same as one of Ulbrich and Laplaza's 'categories with group structure' [36, 50] in which all morphisms are invertible. It is also the same as an 'autonomous monoidal category' [33] with all morphisms invertible, or a 'bigroupoid' [29] with one object.

As we did with weak 2-groups, we can define a homomorphism between coherent 2-groups. As in the weak 2-group case we can begin by taking it to be a weak monoidal functor, but now we must consider what additional structure this must have to preserve each adjoint equivalence $(x, \bar{x}, i_x, e_x)$, at least up to a specified isomorphism. At first it may seem that an additional structural map is required. That is, given a weak monoidal functor F between 2-groups, it may seem that we must include a natural isomorphism

$$F_{-1} \colon \overline{F(x)} \to F(\bar{x})$$

relating the weak inverse of the image of x to the image of the weak inverse $\bar{x}$. In Section 6 we shall show this is not the case: F_{-1} can be constructed from the data already present! Moreover, it automatically satisfies the appropriate coherence laws. Thus we make the following definitions:

3.2. Definition. *A* **homomorphism** $F\colon C \to C'$ *between coherent 2-groups is a weak monoidal functor.*

3.3. Definition. *A* **2-homomorphism** $\theta\colon F \Rightarrow G$ *between homomorphisms* $F, G\colon C \to C'$ *of coherent 2-groups is a monoidal natural transformation.*

3.4. Definition. *Let* **C2G** *be the strict 2-category consisting of coherent 2-groups, homomorphisms between these, and 2-homomorphisms between those.*

It is clear that C2G forms a strict 2-category since it is a sub-2-category of MonCat.

We conclude this section by stating the theorem that justifies the term 'coherent 2-group'. This result is analogous to Mac Lane's coherence theorem for monoidal categories. A version of this result was proved by Ulbrich [50] and Laplaza [36] for a structure called a **category with group structure**: a weak monoidal category equipped with an adjoint equivalence for every object. Through a series of lemmas, Laplaza establishes that there can be at most one morphism between any two objects in the free category with group structure on a set of objects. Here we translate this into the language of 2-groups and explain the significance of this result.

Let $\mathsf{c2g}$ be the category of coherent 2-groups where the morphisms are the functors that *strictly* preserve the monoidal structure and specified adjoint equivalences for each object. Clearly there exists a forgetful functor $U\colon \mathsf{c2g} \to \mathrm{Set}$ sending any coherent 2-group to its underlying set. The interesting part is:

3.5. Proposition. *The functor* $U\colon \mathsf{c2g} \to \mathrm{Set}$ *has a left adjoint* $F\colon \mathrm{Set} \to \mathsf{c2g}$.

Since a, ℓ, r, i and e are all isomorphism, the free category with group structure on a set S is the same as the free coherent 2-group on S, so Laplaza's construction of $F(S)$ provides most of what we need for the proof of this theorem. In Laplaza's words, the construction of $F(S)$ for a set S is "long, straightforward, and rather deceptive", because it hides the essential simplicity of the ideas involved. For this reason, we omit the proof of this theorem and refer the interested reader to Laplaza's paper.

It follows that for any coherent 2-group G there exists a homomorphism of 2-groups $e_G\colon F(U(G)) \to G$ that *strictly* preserves the monoidal structure and chosen adjoint equivalences. This map allows us to interpret formal expressions in the free coherent 2-group $F(U(G))$ as actual objects and morphisms in G. We now state the coherence theorem:

3.6. Theorem. *There exists at most one morphism between any pair of objects in* $F(U(G))$.

This theorem, together with the homomorphism e_G, makes precise the rough idea that there is at most one way to build an isomorphism between two tensor products of objects and their weak inverses in G using a, ℓ, r, i, and e.

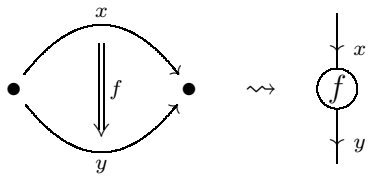
4. String diagrams

Just as calculations in group theory are often done using 1-dimensional symbolic expressions such as

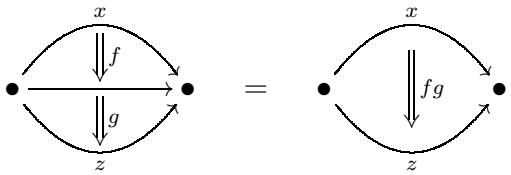
$$x(yz)x^{-1} = (xyx^{-1})(xzx^{-1}),$$

calculations in 2-groups are often done using 2-dimensional pictures called string diagrams. This is one of the reasons for the term 'higher-dimensional algebra'. String diagrams for 2-categories [45] are Poincaré dual to the more traditional globular diagrams in which objects are represented as dots, 1-morphisms as arrows and 2-morphisms as 2-dimensional globes. In other words, in a string diagram one draws objects in a 2-category as 2-dimensional regions in the plane, 1-morphisms as 1-dimensional 'strings' separating regions, and 2-morphisms as 0-dimensional points (or small discs, if we wish to label them).

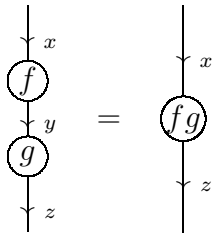
To apply these diagrams to 2-groups, first let us assume our 2-group is a strict monoidal category, which we may think of as a strict 2-category with a single object, say $\bullet$. A morphism $f\colon x \to y$ in the monoidal category corresponds to a 2-morphism in the 2-category, and we convert the globular picture of this into a string diagram as follows:



We can use this idea to draw the composite or tensor product of morphisms. Composition of morphisms $f\colon x \to y$ and $g\colon y \to z$ in the strict monoidal category corresponds to vertical composition of 2-morphisms in the strict 2-category with one object. The globular picture of this is:

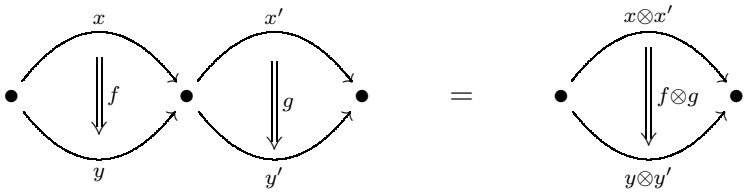


and the Poincaré dual string diagram is:

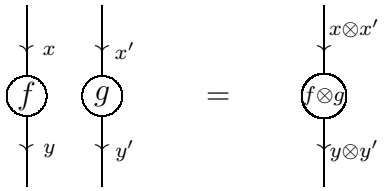


Similarly, the tensor product of morphisms $f\colon x \to y$ and $g\colon x' \to y'$ corresponds to

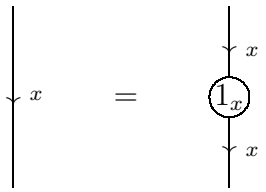
horizontal composition of 2-morphisms in the 2-category. The globular picture is:



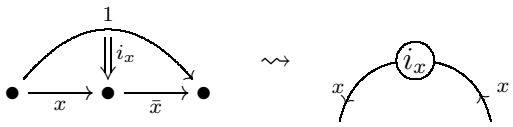
and the Poincaré dual string diagram is:



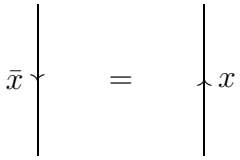
We also introduce abbreviations for identity morphisms and the identity object. We draw the identity morphism $1_x \colon x \to x$ as a straight vertical line:



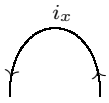
The identity object will not be drawn in the diagrams, but merely implied. As an example of this, consider how we obtain the string diagram for $i_x \colon 1 \to x \otimes \bar{x}$:



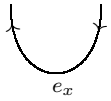
Note that we omit the incoming string corresponding to the identity object 1. Also, we indicate weak inverse objects with arrows 'going backwards in time', following this rule:



In calculations, it is handy to draw the unit i_x in an even more abbreviated form:



where we omit the disc surrounding the morphism label 'i_x', and it is understood that the downward pointing arrow corresponds to x and the upward pointing arrow to $\bar{x}$. Similarly, we draw the morphism e_x as



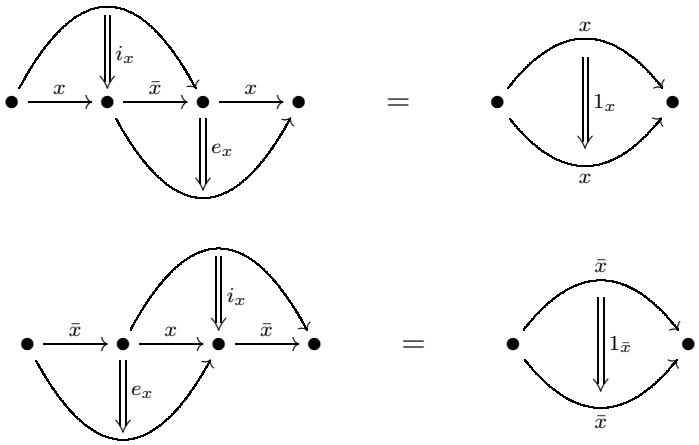
In a strict monoidal category, where the associator and the left and right unit laws are identity morphisms, one can interpret any string diagram as a morphism in a unique way. In fact, Joyal and Street have proved some rigorous theorems to this effect [32]. With the help of Mac Lane's coherence theorem [39] we can also do this in a weak monoidal category. To do this, we interpret any string of objects and 1's as a tensor product of objects where all parentheses start in front and all 1's are removed. Using the associator and left/right unit laws to do any necessary reparenthesization and introduction or elimination of 1's, any string diagram then describes a morphism between tensor products of this sort. The fact that this morphism is unambiguously defined follows from Mac Lane's coherence theorem.

For a simple example of string diagram technology in action, consider the zig-zag identities. To begin with, these say that the following diagrams commute:

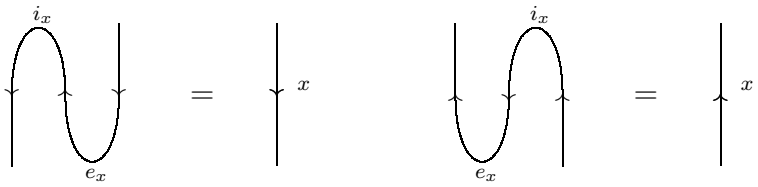
$$\begin{array}{ccccc} 1 \otimes x & \xrightarrow{i_x \otimes 1} & (x \otimes \bar{x}) \otimes x & \xrightarrow{a_{x,\bar{x},x}} & x \otimes (\bar{x} \otimes x) \\ \downarrow{\scriptstyle \ell_x} & & & & \downarrow{\scriptstyle 1 \otimes e_x} \\ x & & \xrightarrow[r_x^{-1}]{} & & x \otimes 1 \end{array}$$

$$\begin{array}{ccccc} \bar{x} \otimes 1 & \xrightarrow{1 \otimes i_x} & \bar{x} \otimes (x \otimes \bar{x}) & \xrightarrow{a^{-1}_{\bar{x},x,\bar{x}}} & (\bar{x} \otimes x) \otimes \bar{x} \\ \downarrow{\scriptstyle r_{\bar{x}}} & & & & \downarrow{\scriptstyle e_x \otimes 1} \\ \bar{x} & & \xrightarrow[\ell_{\bar{x}}^{-1}]{} & & 1 \otimes x \end{array}$$

In globular notation these diagrams become:



Taking Poincaré duals, we obtain the zig-zag identities in string diagram form:



This picture explains their name! The zig-zag identities simply allow us to straighten a piece of string.

In most of our calculations we only need string diagrams where all strings are labelled by x and $\bar{x}$. In this case we can omit these labels and just use downwards or upwards arrows to distinguish between x and $\bar{x}$. We draw i_x as

and draw e_x as

The zig-zag identities become just:

We also obtain some rules for manipulating string diagrams just from the fact that i_x and e_x have inverses. For these, we draw i_x^{-1} as

and e_x^{-1} as

The equations $i_x i_x^{-1} = 1_1$ and $e_x^{-1} e_x = 1_1$ give the rules

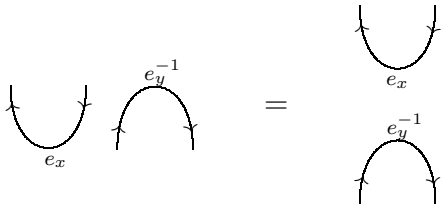
which mean that in a string diagram, a loop of either form may be removed or inserted without changing the morphism described by the diagram. Similarly, the equations $e_x e_x^{-1} = 1_{\bar{x}\otimes x}$ and $i_x^{-1} i_x = 1_{x\otimes\bar{x}}$ give the rules

Again, these rules mean that in a string diagram we can modify any portion as above without changing the morphism in question.

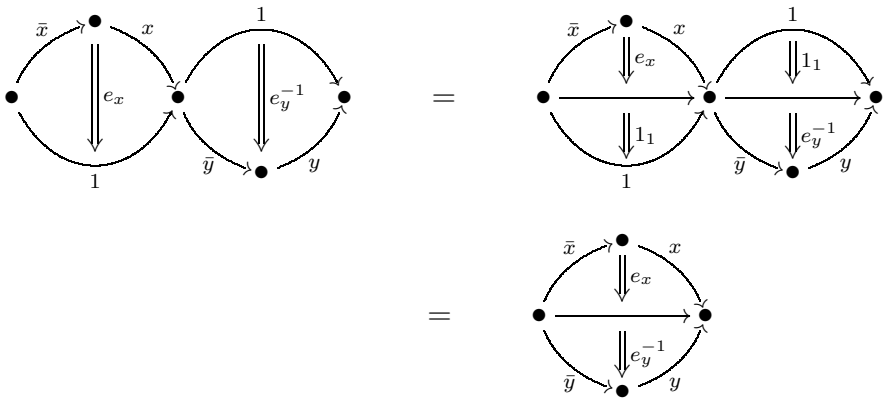
By taking the inverse of both sides in the zig-zag identities, we obtain extra zig-zag identities involving i_x^{-1} and e_x^{-1}:

Conceptually, this means that whenever $(x, \bar{x}, i_x, e_x)$ is an adjoint equivalence, so is $(\bar{x}, x, e_x^{-1}, i_x^{-1})$.

In the calculations to come we shall also use another rule, the 'horizontal slide':



This follows from general results on the isotopy-invariance of the morphisms described by string diagrams [33], but it also follows directly from the interchange law relating vertical and horizontal composition in a 2-category:

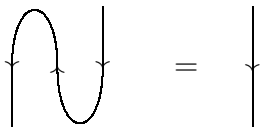


We will also be using other slightly different versions of the horizontal slide, which can be proved the same way.

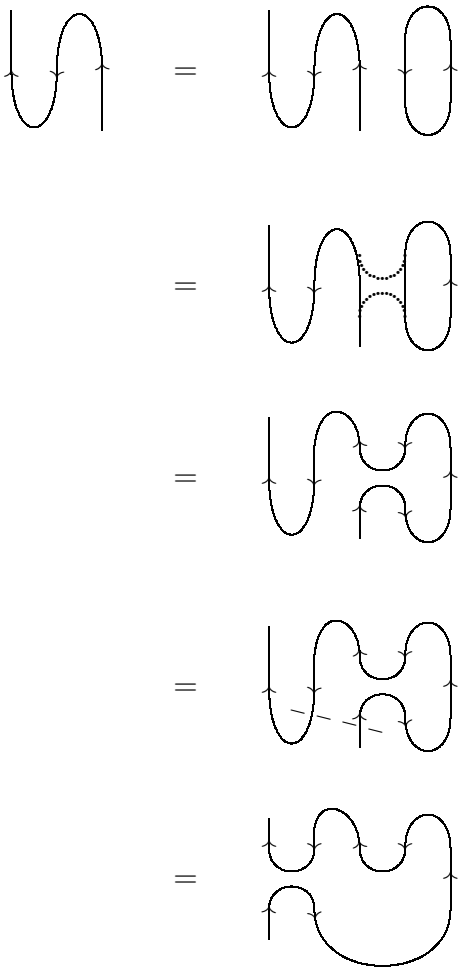
As an illustration of how these rules are used, we give a string diagram proof of a result due to Saavedra Rivano [47], which allows a certain simplification in the definition of 'coherent 2-group':

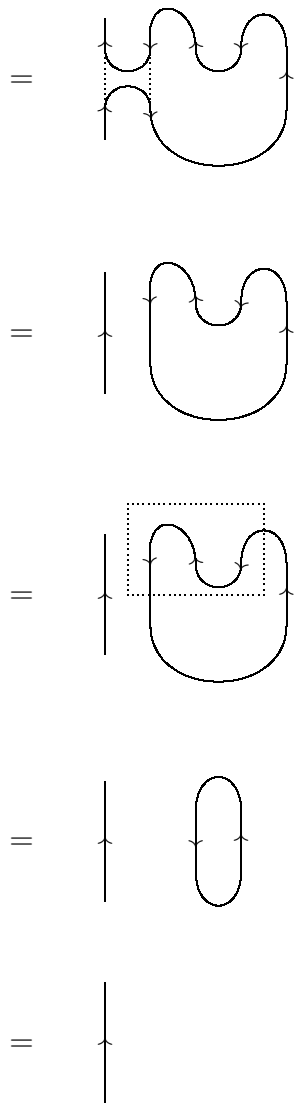
4.1. PROPOSITION. *Let C be a weak monoidal category, and let $x, \bar{x} \in C$ be objects equipped with isomorphisms $i_x\colon 1 \to x \otimes \bar{x}$ and $e_x\colon \bar{x} \otimes x \to 1$. If the quadruple $(x, \bar{x}, i_x, e_x)$ satisfies either one of the zig-zag identities, it automatically satisfies the other as well.*

PROOF. Suppose the first zig-zag identity holds:



Then the second zig-zag identity may be shown as follows:





In this calculation, we indicate an application of the 'horizontal slide' rule by a dashed line. Dotted curves or lines indicate applications of the rule $e_x e_x^{-1} = 1_{\bar{x}\otimes x}$. A box indicates an application of the first zig-zag identity. The converse can be proven similarly. ∎

5. Improvement

We now use string diagrams to show that any weak 2-group can be improved to a coherent one. There are shorter proofs, but none quite so pretty—at least in a purely visual sense. Given a weak 2-group C and any object $x \in C$, we can choose a weak inverse $\bar{x}$ for x together with isomorphisms $i_x \colon 1 \to x \otimes \bar{x}$, $e_x \colon \bar{x} \otimes x \to 1$. From this data we shall construct an adjoint equivalence $(x, \bar{x}, i'_x, e'_x)$. By doing this for every object of C, we make C into a coherent 2-group.

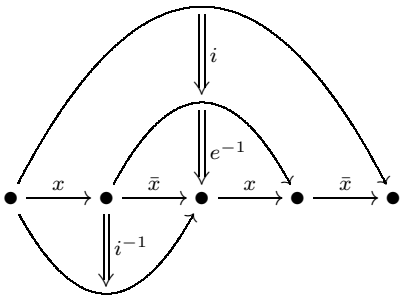
5.1. THEOREM. *Any weak 2-group C can be given the structure of a coherent 2-group* $\mathrm{Imp}(C)$ *by equipping each object with an adjoint equivalence.*

PROOF. First, for each object x we choose a weak inverse $\bar{x}$ and isomorphisms $i_x\colon 1 \to x \otimes \bar{x}$, $e_x\colon \bar{x} \otimes x \to 1$. From this data we construct an adjoint equivalence $(x, \bar{x}, i'_x, e'_x)$. To do this, we set $e'_x = e_x$ and define i'_x as the following composite morphism:

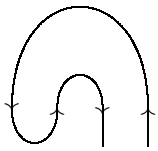
$$1 \xrightarrow{i_x} x\bar{x} \xrightarrow{x\ell_{\bar{x}}^{-1}} x(1\bar{x}) \xrightarrow{xe_x^{-1}\bar{x}} x((\bar{x}x)\bar{x}) \xrightarrow{xa_{\bar{x},x,\bar{x}}} x(\bar{x}(x\bar{x})) \xrightarrow{a^{-1}_{x,\bar{x},x\bar{x}}} (x\bar{x})(x\bar{x}) \xrightarrow{i_x^{-1}(x\bar{x})} 1(x\bar{x}) \xrightarrow{a^{-1}_{1,x,\bar{x}}} (1x)\bar{x} \xrightarrow{\ell_x\bar{x}} x\bar{x}.$$

where we omit tensor product symbols for brevity.

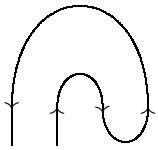
The above rather cryptic formula for i'_x becomes much clearer if we use pictures. If we think of a weak 2-group as a one-object 2-category and write this formula in globular notation it becomes:



where we have suppressed associators and the left unit law for clarity. If we write it as a string diagram it looks even simpler:



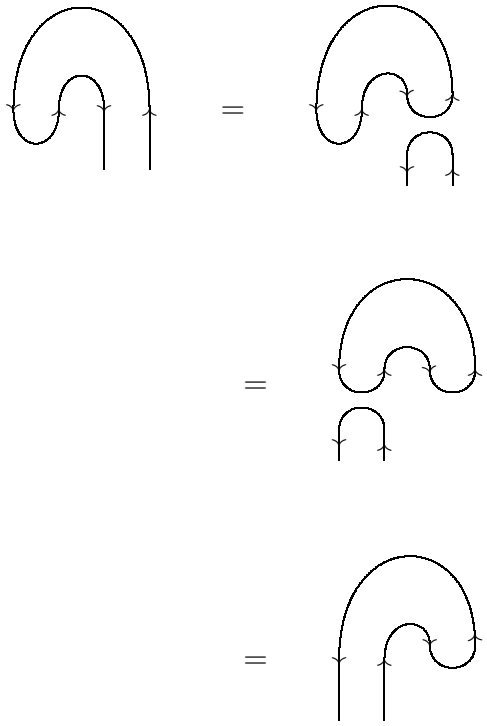
At this point one may wonder why we did not choose some other isomorphism going from the identity to $x \otimes \bar{x}$. For instance:



is another morphism with the desired properties. In fact, these two morphisms are equal, as the following lemma shows.

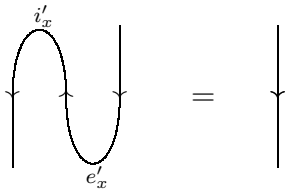
5.2. LEMMA.

PROOF.

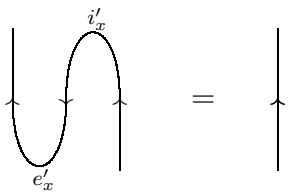


■

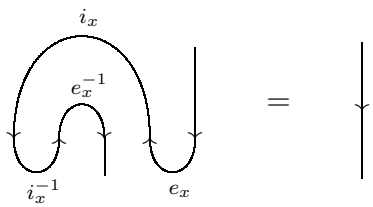
Now let us show that $(x, \bar{x}, i'_x, e'_x)$ satisfies the zig-zag identities. Recall that these identities say that:



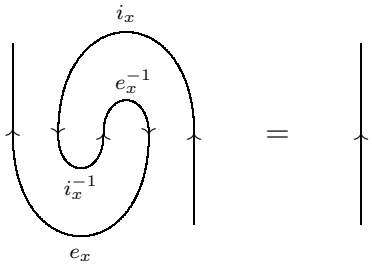
and



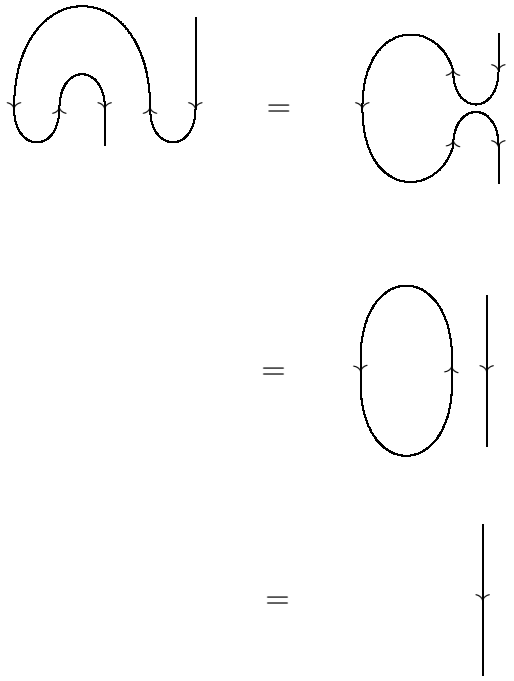
If we express i'_x and e'_x in terms of i_x and e_x, these equations become



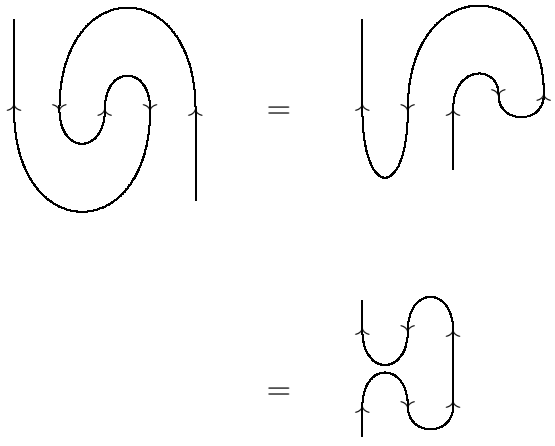
and

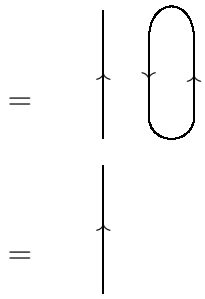


To verify these two equations we use string diagrams. The first equation can be shown as follows:



The second equation can be shown with the help of Lemma 5.2:





■

The 'improvement' process of Theorem 5.1 can be made into a 2-functor $\mathrm{Imp}\colon \mathrm{W2G} \to \mathrm{C2G}$:

5.3. Corollary. *There exists a 2-functor* $\mathrm{Imp}\colon \mathrm{W2G} \to \mathrm{C2G}$ *which sends any object* $C \in \mathrm{W2G}$ *to* $\mathrm{Imp}(C) \in \mathrm{C2G}$ *and acts as the identity on morphisms and 2-morphisms.*

Proof. This is a trivial consequence of Theorem 5.1. Obviously all domains, codomains, identities and composites are preserved, since the 1-morphisms and 2-morphisms are unchanged as a result of Definitions 3.2 and 3.3. ■

On the other hand, there is also a forgetful 2-functor $\mathrm{F}\colon \mathrm{C2G} \to \mathrm{W2G}$, which forgets the extra structure on objects and acts as the identity on morphisms and 2-morphisms.

5.4. Theorem. *The 2-functors* $\mathrm{Imp}\colon \mathrm{W2G} \to \mathrm{C2G}$, $\mathrm{F}\colon \mathrm{C2G} \to \mathrm{W2G}$ *extend to define a 2-equivalence between the 2-categories* W2G *and* C2G.

Proof. The 2-functor Imp equips each object of W2G with the structure of a coherent 2-group, while F forgets this extra structure. Both act as the identity on morphisms and 2-morphisms. As a consequence, Imp followed by F acts as the identity on W2G:

$$\mathrm{Imp} \circ \mathrm{F} = 1_{\mathrm{W2G}}$$

(where we write the functors in order of application). To prove the theorem, it therefore suffices to construct a natural isomorphism

$$e\colon \mathrm{F} \circ \mathrm{Imp} \Rightarrow 1_{\mathrm{C2G}}.$$

To do this, note that applying F and then Imp to a coherent 2-group C amounts to forgetting the choice of adjoint equivalence for each object of C and then making a new such choice. We obtain a new coherent 2-group $\mathrm{Imp}(\mathrm{F}(C))$, but it has the same underlying weak monoidal category, so the identity functor on C defines a coherent 2-group isomorphism from $\mathrm{Imp}(\mathrm{F}(C))$ to C. We take this as $e_C\colon \mathrm{Imp}(\mathrm{F}(C)) \to C$.

To see that this defines a natural isomorphism between 2-functors, note that for every coherent 2-group homomorphism $f\colon C \to C'$ we have a commutative square:

$$\begin{array}{ccc} \mathrm{Imp}(F(C)) & \xrightarrow{\mathrm{Imp}(F(f))} & \mathrm{Imp}(F(C')) \\ {\scriptstyle e_C}\downarrow & & \downarrow{\scriptstyle e_{C'}} \\ C & \xrightarrow{\quad f \quad} & C' \end{array}$$

This commutes because $\mathrm{Imp}(F(f)) = f$ as weak monoidal functors, while e_C and $e_{C'}$ are the identity as weak monoidal functors. ∎

The significance of this theorem is that while we have been carefully distinguishing between weak and coherent 2-groups, the difference is really not so great. Since the 2-category of weak 2-groups is 2-equivalent to the 2-category of coherent ones, one can use whichever sort of 2-group happens to be more convenient at the time, freely translating results back and forth as desired. So, except when one is trying to be precise, one can relax and use the term **2-group** for either sort.

Of course, we made heavy use of the axiom of choice in proving the existence of the improvement 2-functor $\mathrm{Imp}\colon \mathrm{W2G} \to \mathrm{C2G}$, so constructivists will not consider weak and coherent 2-groups to be equivalent. Mathematicians of this ilk are urged to use coherent 2-groups. Indeed, even pro-choice mathematicians will find it preferable to use coherent 2-groups when working in contexts where the axiom of choice fails. These are not at all exotic. For example, the theory of 'Lie 2-groups' works well with coherent 2-groups, but not very well with weak 2-groups, as we shall see in Section 7.

To conclude, let us summarize *why* weak and coherent 2-groups are not really so different. At first, the choice of a specified adjoint equivalence for each object seems like a substantial extra structure to put on a weak 2-group. However, Theorem 5.1 shows that we can always succeed in putting this extra structure on any weak 2-group. Furthermore, while there are many ways to equip a weak 2-group with this extra structure, there is 'essentially' just one way, since all coherent 2-groups with the same underlying weak 2-group are isomorphic. It is thus an example of what Kelly and Lack [35] call a 'property-like structure'.

Of course, the observant reader will note that this fact has simply been built into our definitions! The reason all coherent 2-groups with the same underlying weak 2-group are isomorphic is that we have defined a homomorphism of coherent 2-groups to be a weak monoidal functor, not requiring it to preserve the choice of adjoint equivalence for each object. This may seem like 'cheating', but in the next section we justify it by showing that this choice is *automatically preserved up to coherent isomorphism* by any weak monoidal functor.

6. Preservation of weak inverses

Suppose that $F\colon C \to C'$ is a weak monoidal functor between coherent 2-groups. To show that F automatically preserves the specified weak inverses up to isomorphism, we now construct an isomorphism

$$(F_{-1})_x\colon \overline{F(x)} \to F(\bar{x})$$

for each object $x \in C$. This isomorphism is uniquely determined if we require the following coherence laws:

H1

$$\begin{array}{ccccc} F(x)\otimes\overline{F(x)} & \xrightarrow{1\otimes F_{-1}} & F(x)\otimes F(\bar{x}) & \xrightarrow{F_2} & F(x\otimes\bar{x}) \\ {\scriptstyle i_{F(x)}}\uparrow & & & & \uparrow{\scriptstyle F(i_x)} \\ 1' & & \xrightarrow[F_0]{} & & F(1) \end{array}$$

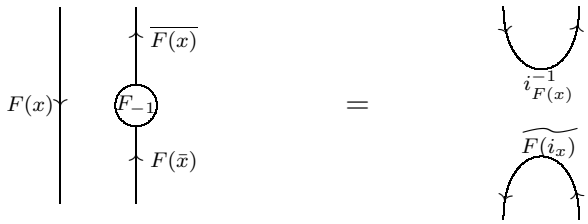
H2

$$\begin{array}{ccccc} \overline{F(x)}\otimes F(x) & \xrightarrow{F_{-1}\otimes 1} & F(\bar{x})\otimes F(x) & \xrightarrow{F_2} & F(\bar{x}\otimes x) \\ {\scriptstyle e_{F(x)}}\downarrow & & & & \downarrow{\scriptstyle F(e_x)} \\ 1' & & \xrightarrow[F_0]{} & & F(1) \end{array}$$

These say that F_{-1} is compatible with units and counits. In the above diagrams and in what follows, we suppress the subscript on F_{-1}, just as we are already doing for F_2.

6.1. THEOREM. *Suppose that $F\colon C \to C'$ is a homomorphism of coherent 2-groups. Then for any object $x \in C$ there exists a unique isomorphism $F_{-1}\colon \overline{F(x)} \to F(\bar{x})$ satisfying the coherence laws* **H1** *and* **H2**.

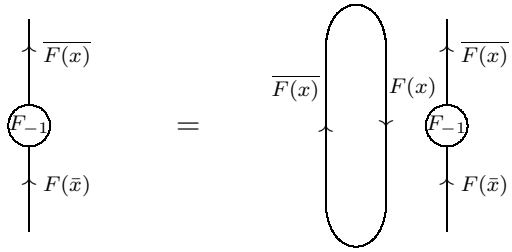
PROOF. This follows from the general fact that pseudofunctors between bicategories preserve adjunctions. However, to illustrate the use of string diagrams we prefer to simply take one of these laws, solve it for F_{-1}, and show that the result also satisfies the other law. We begin by solving equation **H1** for $1 \otimes F_{-1}$. In string notation the result is:

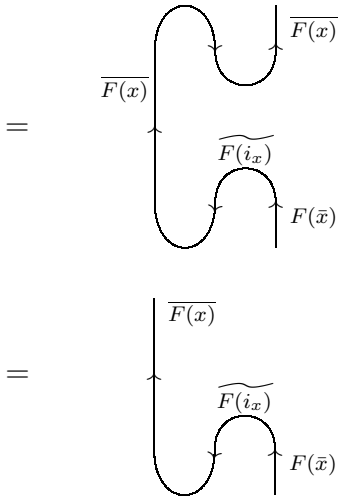


where we set

$$\widetilde{F(i_x)} = F_0 \circ F(i_x) \circ F_2^{-1}\colon 1' \to F(x)\otimes F(\bar{x}).$$

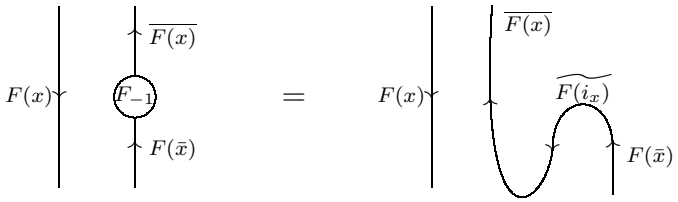
This equation can in turn be solved for F_{-1}, as follows:

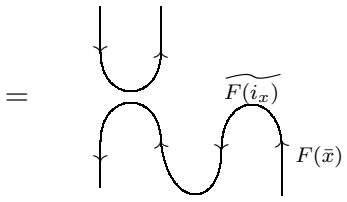


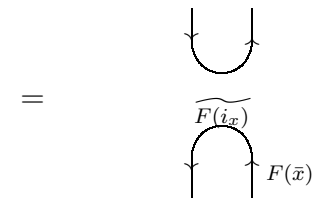


Here and in the arguments to come we omit the labels $i_{F(x)}$, $e_{F(x)}$, $i^{-1}_{F(x)}$, $e^{-1}_{F(x)}$.

Since we have solved for F_{-1} starting from **H1**, we have already shown the morphism satisfying this law is unique. We also know it is an isomorphism, since all morphisms in C' are invertible. However, we should check that it exists—that is, it really does satisfy this coherence law. The proof is a string diagram calculation:

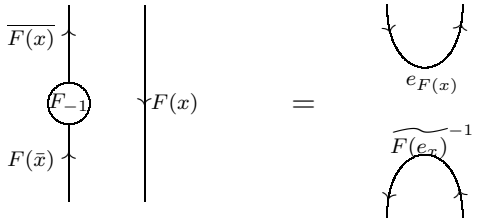






To conclude, we must show that F_{-1} also satisfies the coherence law **H2**. In string

notation, this law says:



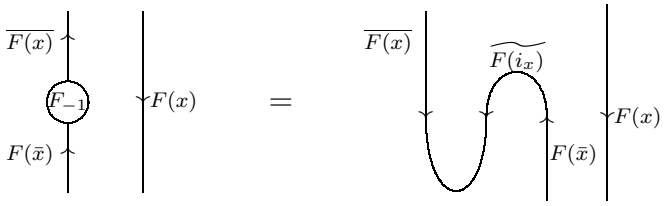
where we set

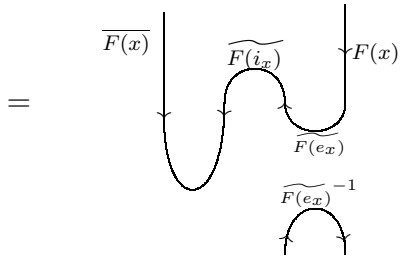
$$\widetilde{F(e_x)} = F_2 \circ F(e_x) \circ F_0^{-1} \colon F(\bar{x}) \otimes F(x) \to 1'.$$

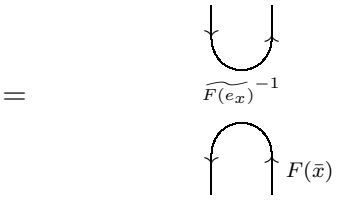
Again, the proof is a string diagram calculation. Here we need the fact that

$$(F(x), F(\bar{x}), \widetilde{F(i_x)}, \widetilde{F(e_x)})$$

is an adjunction. This allows us to use a zig-zag identity for $\widetilde{F(i_x)}$ and $\widetilde{F(e_x)}$ below:







■

In short, we do not need to include F_{-1} and its coherence laws in the definition of a coherent 2-group homomorphism; we get them 'for free'.

7. Internalization

The concept of 'group' was born in the category Set, but groups can live in other categories too. This vastly enhances the power of group theory: for example, we have 'topological groups', 'Lie groups', 'affine group schemes', and so on—each with their own special features, but all related.

The theory of 2-groups has a similar flexibility. Since 2-groups are categories, we have implicitly defined the concept of 2-group in the 2-category Cat. However, as noted by Joyal and Street, this concept can generalized to other 2-categories as well [33]. This makes it possible to define 'topological 2-groups', 'Lie 2-groups', 'affine 2-group schemes' and the like. In this section we describe how this generalization works. In the next section, we give many examples of Lie 2-groups.

'Internalization' is an efficient method of generalizing concepts from the category of sets to other categories. To internalize a concept, we need to express it in a purely diagrammatic form. Mac Lane illustrates this in his classic text [40] by internalizing the concept of a 'group'. We can define this notion using commutative diagrams by specifying:

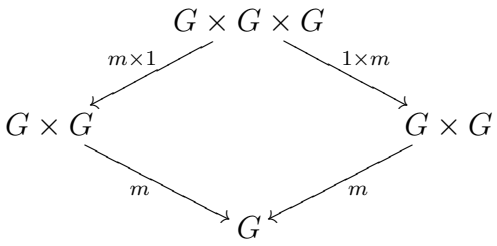
- a set G,

together with

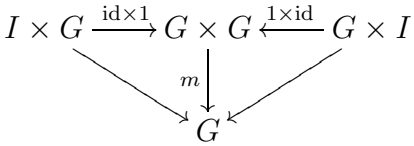
- a multiplication function $m\colon G \times G \to G$,
- an identity element for the multiplication given by the function $\mathrm{id}\colon I \to G$ where I is the terminal object in Set,
- a function $\mathrm{inv}\colon G \to G$,

such that the following diagrams commute:

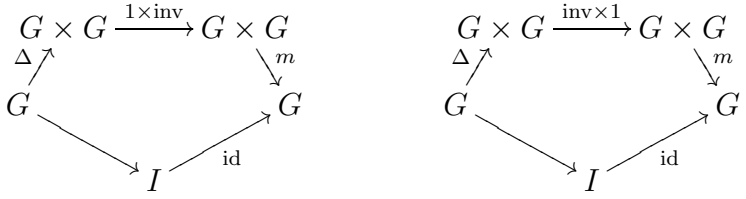
- the associative law:



- the right and left unit laws:



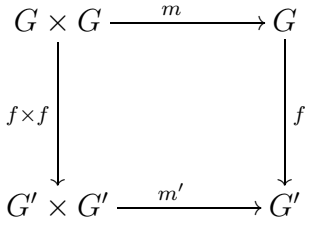
- the right and left inverse laws:



where $\Delta\colon G \to G \times G$ is the diagonal map.

To internalize the concept of group we simply replace the set G by an *object* in some category K and replace the functions m, id, and inv by *morphisms* in this category. Since the definition makes use of the Cartesian product $\times$, the terminal object I, and the diagonal Δ, the category K should have finite products. Making these substitutions in the above definition, we arrive at the definition of a **group object in** K. We shall usually call this simply a **group in** K.

In the special case where $K = \mathrm{Set}$, a group in K reduces to an ordinary group. A topological group is a group in Top, a Lie group is a group in Diff, and a affine group scheme is a group in $\mathrm{CommRing}^{\mathrm{op}}$, usually called the category of 'affine schemes'. Indeed, for any category K with finite products, there is a category $K\mathrm{Grp}$ consisting of groups in K and homomorphisms between these, where a **homomorphism** $f\colon G \to G'$ is a morphism in K that preserves multiplication, meaning that this diagram commutes:

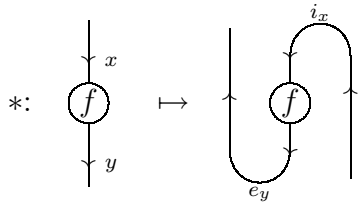


As usual, this implies that f also preserves the identity and inverses.

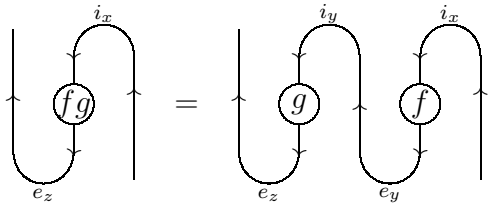
Following Joyal and Street [33], let us now internalize the concept of coherent 2-group and define a 2-category of 'coherent 2-groups in K' in a similar manner. For this, one must first define a coherent 2-group using only commutative diagrams. However, since the usual group axioms hold only up to natural isomorphism in a coherent 2-group, these will be 2-categorical rather than 1-categorical diagrams. As a result, the concept of coherent 2-group will make sense in any 2-category with finite products, K. For simplicity we shall limit ourselves to the case where K is a strict 2-category.

To define the concept of coherent 2-group using commutative diagrams, we start with a category C and equip it with a multiplication functor $m\colon C \times C \to C$ together with an identity object for multiplication given by the functor $\mathrm{id}\colon I \to C$, where I is the terminal category. The functor mapping each object to its specified weak inverse is a bit more subtle! One can try to define a functor $*\colon C \to C$ sending each object $x \in C$ to its

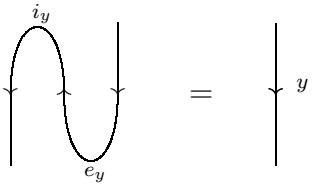
specified weak inverse $\bar{x}$, and acting on morphisms as follows:



However, $*$ is actually a *contravariant* functor. To see this, we consider composable morphisms $f\colon x \to y$ and $g\colon y \to z$ and check that $(fg)^* = g^* f^*$. In string diagram form, this equation says:

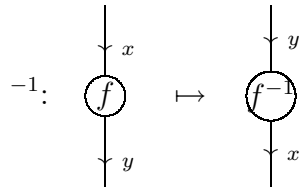


This equation holds if and only if

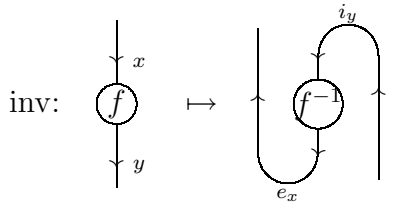


But this is merely the first zig-zag identity!

Contravariant functors are a bit annoying since they are not really morphisms in Cat. Luckily, there is also another contravariant functor $^{-1}\colon C \to C$ sending each morphism to its inverse, expressed diagrammatically as



If we compose the contravariant functor $*$ with this, we obtain a covariant functor $\mathrm{inv}\colon C \to C$ given by



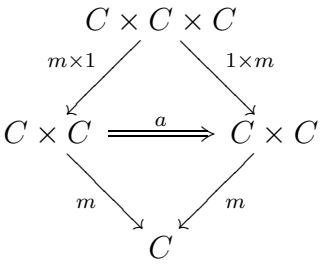
Thus, we can try to write the definition of a coherent 2-group in terms of:

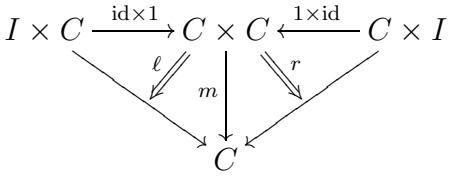
- the category C,

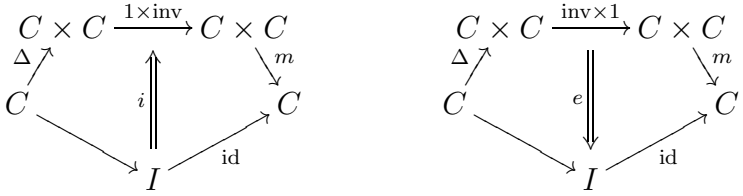
together with

- the functor $m\colon C \times C \to C$, where we write $m(x, y) = x \otimes y$ and $m(f, g) = f \otimes g$ for objects $x, y, \in C$ and morphisms f, g in C,
- the functor $\mathrm{id}\colon I \to C$ where I is the terminal category, and the object in the range of this functor is $1 \in C$,
- the functor $\mathrm{inv}\colon C \to C$,

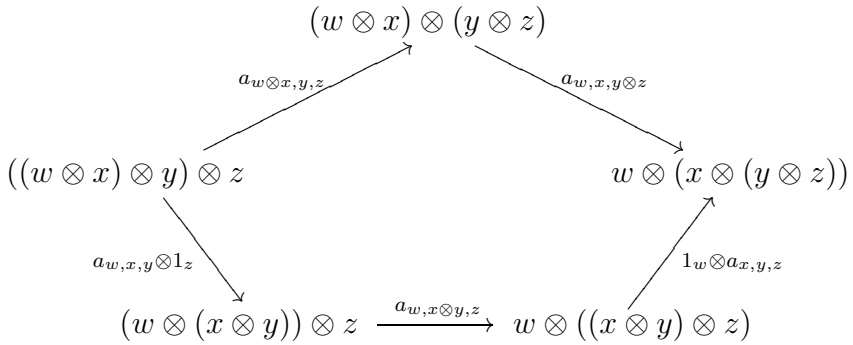
together with the following natural isomorphisms:



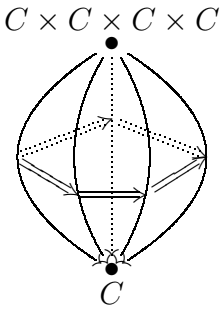




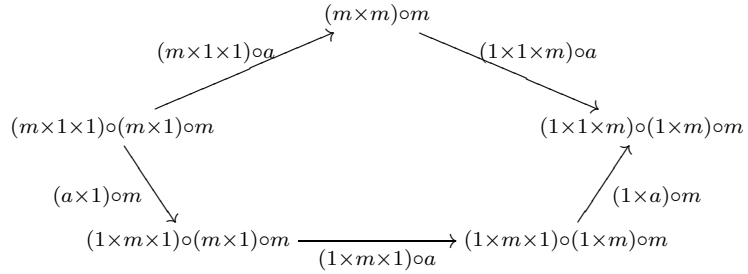
and finally the coherence laws satisfied by these isomorphisms. But to do this, we must write the coherence laws in a way that does not explicitly mention objects of C. For example, we must write the pentagon identity



without mentioning the objects $w, x, y, z \in C$. We can do this by working with (for example) the functor $(1 \times 1 \times m) \circ (1 \times m) \circ m$ instead of its value on the object $(x, y, z, w) \in C^4$, namely $x \otimes (y \otimes (z \otimes w))$. If we do this, we see that the diagram becomes 3-dimensional! It is a bit difficult to draw, but it looks something like this:



where the downwards-pointing single arrows are functors from C^4 to C, while the horizontal double arrows are natural transformations between these functors, forming a commutative pentagon. Luckily we can also draw this pentagon in a 2-dimensional way, as follows:



Using this idea we can write the definition of 'coherent 2-group' using only the structure of Cat as a 2-category with finite products. We can then internalize this definition, as follows:

7.1. Definition. *Given a 2-category K with finite products, a* **coherent 2-group in** *K consists of:*

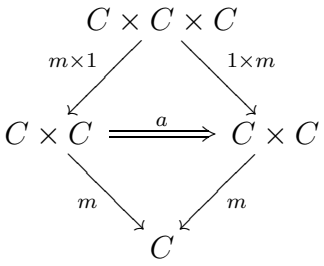
- *an object $C \in K$,*

together with:

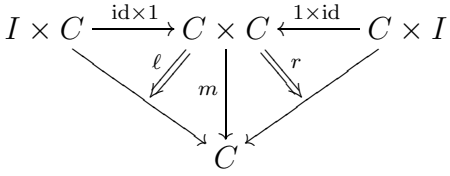
- *a* **multiplication** *morphism $m\colon C \times C \to C$,*
- *an* **identity-assigning** *morphism $\mathrm{id}\colon I \to C$ where I is the terminal object of K,*
- *an* **inverse** *morphism $\mathrm{inv}\colon C \to C$,*

together with the following 2-isomorphisms:

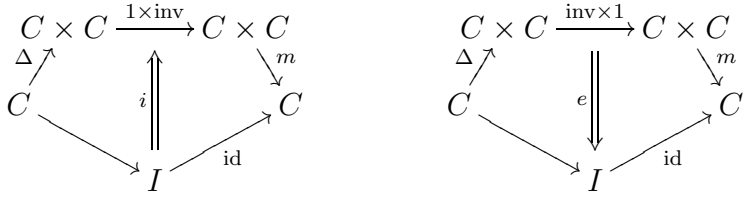
- *the* **associator***:*



- *the* **left and right unit laws***:*

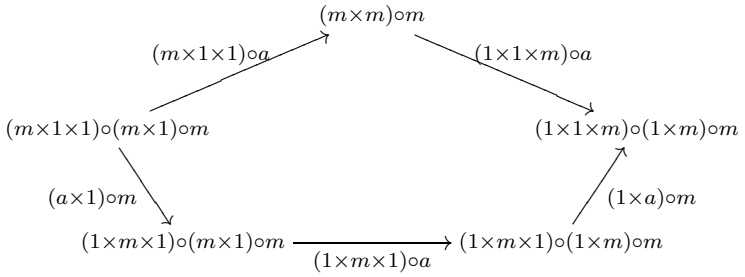


- *the* **unit** *and* **counit***:*

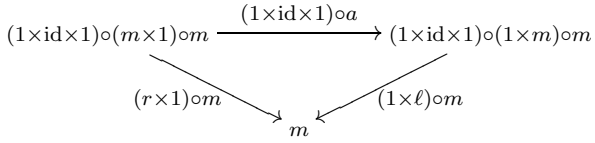


such that the following diagrams commute:

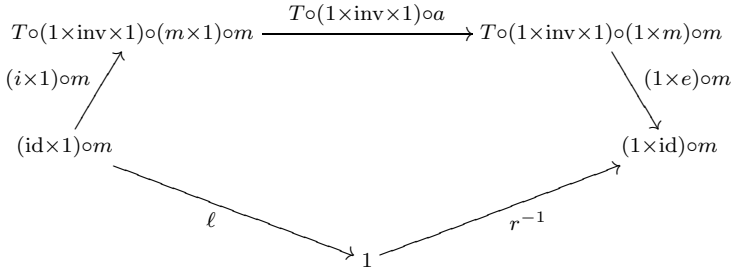
- *the* **pentagon identity** *for the associator:*



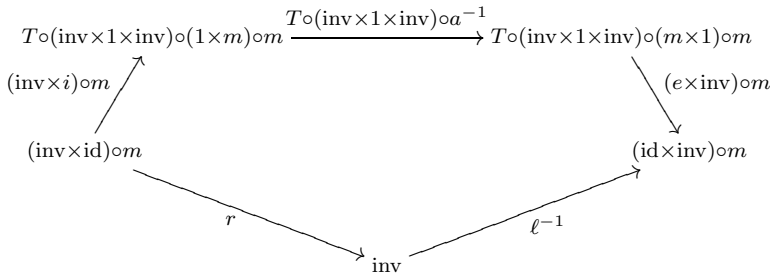
- *the* **triangle identity** *for the left and right unit laws:*



- *the* **first zig-zag identity***:*



- *the* **second zig-zag identity***:*

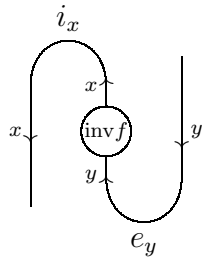


where $T\colon C \to C^3$ *is built using the diagonal functor.*

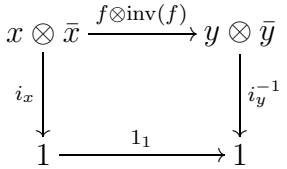
7.2. Proposition. *A coherent 2-group in* Cat *is the same as a coherent 2-group.*

Proof. Clearly any coherent 2-group gives a coherent 2-group in Cat. Conversely, suppose C is a coherent 2-group in Cat. It is easy to check that C is a weak monoidal category and that for each object $x \in C$ there is an adjoint equivalence $(x, \bar{x}, i_x, e_x)$ where $\bar{x} = \mathrm{inv}(x)$. This permits the use of string diagrams to verify the one remaining point, which is that all morphisms in C are invertible.

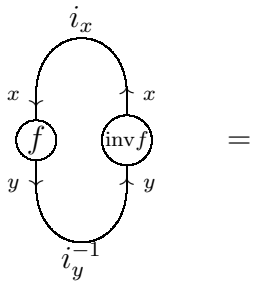
To do this, for any morphism $f\colon x \to y$ we define a morphism $f^{-1}\colon y \to x$ by



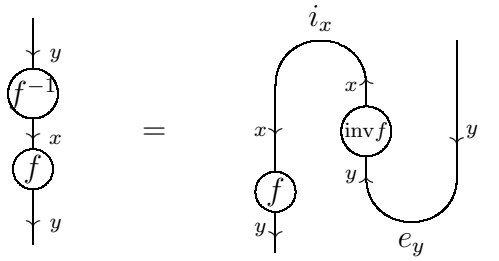
To check that $f^{-1}f$ is the identity, we use the fact that i is a natural isomorphism to note that this square commutes:

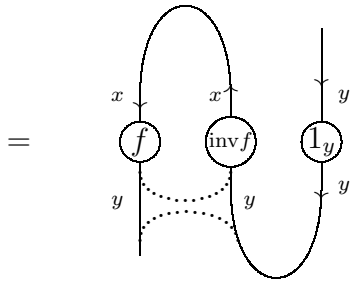


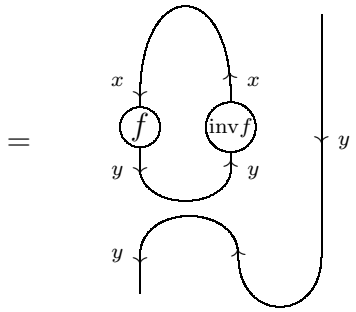
In string notation this says that:

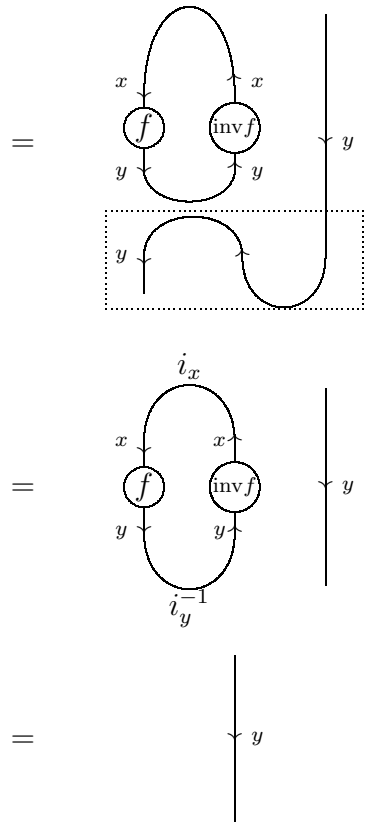


and we can use this equation to verify that $f^{-1}f = 1_y$. In the calculation that follows, the dotted curves connecting two parallel strings indicate an application of the rule $i_y^{-1}i_y = 1_{y\otimes\bar{y}}$. The box indicates an application of the first zig-zag identity.









The proof that $ff^{-1} = 1_x$ is similar, based on the fact that e is a natural isomorphism. ∎

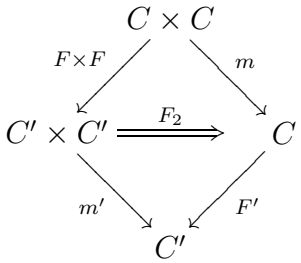
Given a 2-category K with finite products, we can also define homomorphisms between coherent 2-groups in K, and 2-homomorphisms between these, by internalizing the definitions of 'weak monoidal functor' and 'monoidal natural transformation':

7.3. Definition. *Given coherent 2-groups C, C' in K, a* **homomorphism** $F\colon C \to C'$ *consists of:*

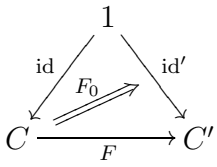
- *a morphism $F\colon C \to C'$*

equipped with:

- *a 2-isomorphism*



- *a 2-isomorphism*



such that diagrams commute expressing these laws:

- *compatibility of F_2 with the associator:*

$$\begin{array}{ccccc}
(F\times F\times F)(m'\times 1)m' & \xrightarrow{(F_2\times F)\circ m'} & (m\times 1)(F\times F)m' & \xrightarrow{(m\times 1)\circ F_2} & (m\times 1)mF \\
\downarrow{\scriptstyle (F\times F\times F)\circ a} & & & & \downarrow{\scriptstyle a\circ F} \\
(F\times F\times F)(1\times m')m' & \xrightarrow[(F\times F_2)\circ m']{} & (1\times m)(F\times F)m' & \xrightarrow[(1\times m)\circ F_2]{} & (1\times m)mF
\end{array}$$

- *compatibility of F_0 with the left unit law:*

$$\begin{array}{ccc}
(\mathrm{id}'\times F)m' & \xrightarrow{F\circ \ell'} & F \\
\downarrow{\scriptstyle (F_0\times F)\circ m'} & & \uparrow{\scriptstyle \ell\circ F} \\
(\mathrm{id}\times 1)(F\times F)m' & \xrightarrow[(\mathrm{id}\times 1)\circ F_2]{} & (\mathrm{id}\times 1)mF
\end{array}$$

- *compatibility of F_0 with the right unit law:*

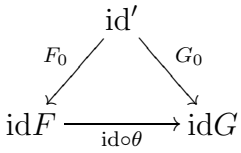
$$\begin{array}{ccc}
(F\times \mathrm{id}')m' & \xrightarrow{F\circ r'} & F \\
\downarrow{\scriptstyle (F\times F_0)\circ m'} & & \uparrow{\scriptstyle r\circ F} \\
(1\times \mathrm{id})(F\times F)m' & \xrightarrow[(1\times \mathrm{id})\circ F_2]{} & (1\times \mathrm{id})mF
\end{array}$$

7.4. Definition. *Given homomorphisms $F, G\colon C \to C'$ between coherent 2-groups C, C' in K, a* **2-homomorphism** *$\theta\colon F \Rightarrow G$ is a 2-morphism such that the following diagrams commute:*

- *compatibility with F_2 and G_2:*

$$\begin{array}{ccc}
(F\times F)m' & \xrightarrow{(\theta\times\theta)\circ m'} & (G\times G)m' \\
\downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle G_2} \\
mF & \xrightarrow{m\circ\theta} & mG
\end{array}$$

- *compatibility with F_0 and G_0:*



It is straightforward to define a 2-category ***K*C2G** of coherent 2-groups in K, homomorphisms between these, and 2-homomomorphisms between those. We leave this to the reader, who can also check that when $K = \mathrm{Cat}$, this 2-category KC2G reduces to C2G as already defined.

To define concepts such as 'topological 2-group', 'Lie 2-group' and 'affine 2-group scheme' we need to consider coherent 2-group objects in a special sort of 2-category which is defined by a further process of internalization. This is the 2-category of 'categories in K', where K itself is a category. A category in K is usually called an 'internal category'. This concept goes back to Ehresmann [24], but a more accessible treatment can be found in Borceux's handbook [7]. For completeness, we recall the definition here:

7.5. Definition. *Let K be a category. An* **internal category** *or* **category in** *K, say X, consists of:*

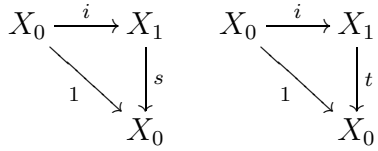
- *an* **object of objects** $X_0 \in K$,
- *an* **object of morphisms** $X_1 \in K$,

together with

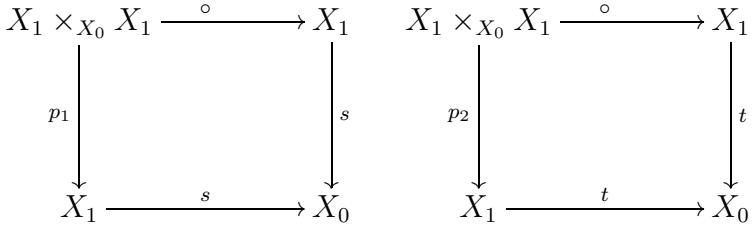
- **source** *and* **target** *morphisms* $s, t\colon X_1 \to X_0$,
- *a* **identity-assigning** *morphism* $i\colon X_0 \to X_1$,
- *a* **composition** *morphism* $\circ\colon X_1 \times_{X_0} X_1 \to X_1$

such that the following diagrams commute, expressing the usual category laws:

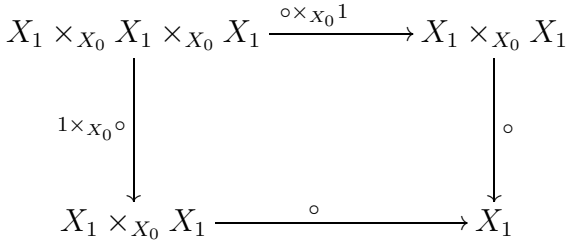
- *laws specifying the source and target of identity morphisms:*



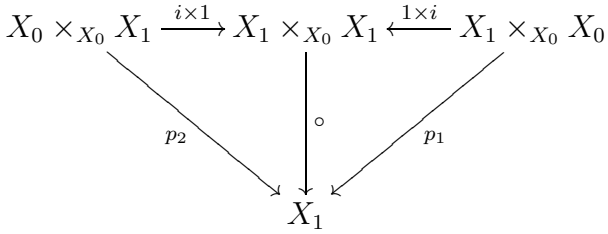
- *laws specifying the source and target of composite morphisms:*



- *the associative law for composition of morphisms:*



- *the left and right unit laws for composition of morphisms:*



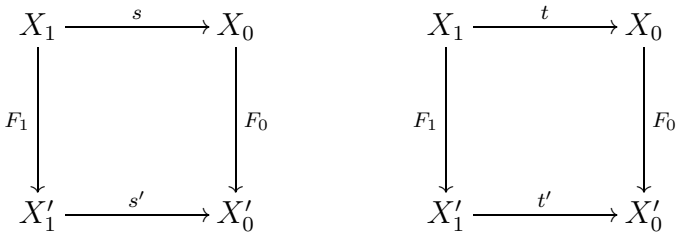
The pullbacks used in this definition should be obvious from the usual definition of category; for example, composition should be defined on pairs of morphisms such that the target of one is the source of the next, and the object of such pairs is the pullback $X_0 \times_{X_0} X_1$. Notice that inherent to the definition is the assumption that the pullbacks involved actually exist. This automatically holds if K is a category with finite limits, but there are some important examples like $K = \mathrm{Diff}$ where this is not the case.

7.6. Definition. *Let K be a category. Given categories X and X' in K, an* **internal functor** *or* **functor in** *K, say $F\colon X \to X'$, consists of:*

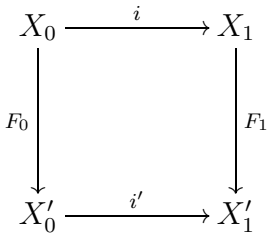
- *a morphism $F_0\colon X_0 \to X'_0$,*
- *a morphism $F_1\colon X_1 \to X'_1$*

such that the following diagrams commute, corresponding to the usual laws satisfied by a functor:

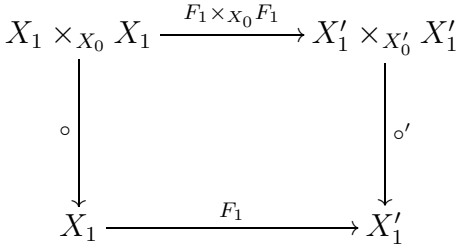
- *preservation of source and target:*



- *preservation of identity morphisms:*

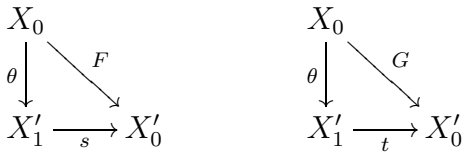


- *preservation of composite morphisms:*

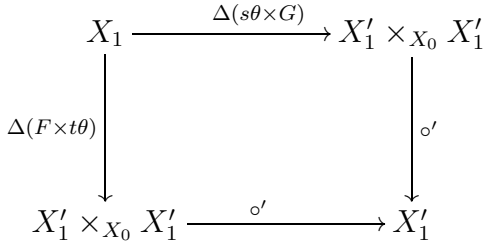


7.7. Definition. *Let K be a category. Given categories X, X' in K and functors $F, G\colon X \to X'$, an* **internal natural transformation** *or* **natural transformation in** *K, say $\theta\colon F \Rightarrow G$, is a morphism $\theta\colon X_0 \to X'_1$ for which the following diagrams commute, expressing the usual laws satisfied by a natural transformation:*

- *laws specifying the source and target of the natural transformation:*



- *the commutative square law:*



Given any category K, there is a strict 2-category $K\mathrm{Cat}$ whose objects are categories in K, whose morphisms are functors in K, and whose 2-morphisms are natural transformations in K. Of course, a full statement of this result requires defining how to compose

functors in K, how to vertically and horizontally compose natural transformations in K, and so on. We shall not do this here; the details can be found in Borceux's handbook [7] or HDA6 [3].

One can show that if K is a category with finite products, $K\mathrm{Cat}$ also has finite products. This allows us to define coherent 2-groups in $K\mathrm{Cat}$. For example:

7.8. Definition. *A* **topological category** *is a category in* Top*, the category of topological spaces and continuous maps. A* **topological 2-group** *is a coherent 2-group in* TopCat*.*

7.9. Definition. *A* **smooth category** *is a category in* Diff*, the category of smooth manifolds and smooth maps. A* **Lie 2-group** *is a coherent 2-group in* DiffCat*.*

7.10. Definition. *An* **affine category scheme** *is a category in* $\mathrm{CommRing}^{\mathrm{op}}$*, the opposite of the category of commutative rings and ring homomorphisms. An* **affine 2-group scheme** *is a coherent 2-group in* $\mathrm{CommRing}^{\mathrm{op}}\mathrm{Cat}$*.*

In the next section we shall give some examples of these things. For this, it sometimes handy to use an internalized version of the theory of crossed modules.

As mentioned in the Introduction, a strict 2-group is essentially the same thing as a **crossed module**: a quadruple (G, H, t, α) where G and H are groups, $t\colon H \to G$ is a homomorphism, and $\alpha\colon G \times H \to H$ is an action of G as automorphisms of H such that t is G-equivariant:

$$t(\alpha(g,h)) = g\, t(h)\, g^{-1}$$

and t satisfies the so-called **Peiffer identity**:

$$\alpha(t(h), h') = hh'h^{-1}.$$

To obtain a crossed module from a strict 2-group C we let $G = C_0$, let $H = \ker s \subseteq C_1$, let $t\colon H \to G$ be the restriction of the target map $t\colon C_1 \to C_0$ to H, and set

$$\alpha(g,h) = i(g)\, h\, i(g)^{-1}$$

for all $g \in G$ and $h \in H$. (In this formula multiplication and inverses refer to the group structure of H, not composition of morphisms in Conversely, we can build a strict 2-group from a crossed module (G, H, t, α) as follows. First we let $C_0 = G$ and let C_1 be the semidirect product $H \rtimes G$ in which multiplication is given by

$$(h,g)(h',g') = (h\alpha(g,h'), gg').$$

Then, we define source and target maps $s, t\colon C_1 \to C_0$ by:

$$s(h,g) = g, \qquad t(h,g) = t(h)g,$$

define the identity-assigning map $i\colon C_0 \to C_1$ by:

$$i(g) = (1,g),$$

and define the composite of morphisms

$$(h, g)\colon g \to g', \qquad (h', g')\colon g' \to g''$$

to be:

$$(hh', g)\colon g \to g''.$$

For a proof that these constructions really work, see the expository paper by Forrester-Barker [26].

Here we would like to internalize these constructions in order to build 'strict 2-groups in KCat' from 'crossed modules in K' whenever K is any category satisfying suitable conditions. Since the details are similar to the usual case where $K = \mathrm{Set}$, we shall be brief.

7.11. Definition. *A* **strict 2-group** *in a 2-category with finite products is a coherent 2-group in this 2-category such that a, i, e, l, r are all identity 2-morphisms—or equivalently, a group in the underlying category of this 2-category.*

7.12. Definition. *Given a category K with finite products and a group G in K, an* **action** *of G on an object $X \in K$ is a morphism $\alpha\colon G \times X \to X$ such that the following diagrams commute:*

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times 1_X} & G \times X \\ {\scriptstyle 1_G \times \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G \times X & \xrightarrow{\alpha} & X \end{array}$$

$$\begin{array}{ccc} I \times X & \xrightarrow{\mathrm{id} \times 1_X} & G \times X \\ & \searrow^{\cong} & \downarrow{\scriptstyle \alpha} \\ & & X \end{array}$$

If X is a group in K, we say α is an action of G **as automorphisms of** *X if this diagram also commutes:*

$$\begin{array}{ccccc} G \times X \times X & \xrightarrow{1_G \times m} & G \times X & \xrightarrow{\alpha} & X \\ {\scriptstyle (\Delta_G \times 1_{X \times X})}\downarrow & & & & \uparrow{\scriptstyle m} \\ G \times G \times X \times X & \xrightarrow{(1_G \times S_{G,X} \times 1_X)} & G \times X \times G \times X & \xrightarrow{\alpha \times \alpha} & X \times X \end{array}$$

where $S_{G,X}$ stands for the 'switch map' from $G \times X$ to $X \times G$.

7.13. Definition. *Given a category K with finite products, a* **crossed module in** *K is a quadruple (G, H, t, α) with G and H being groups in K, $t\colon H \to G$ a homomorphism, and $\alpha\colon G \times H \to H$ an action of G as automorphisms of H, such that diagrams commute*

expressing the G-equivariance of t:

$$\begin{array}{ccccc} G \times H & \xrightarrow{\alpha} & H & \xrightarrow{t} & G \\ {\scriptstyle (\Delta_G \times 1_H)(1_G \times S_{G,H})}\downarrow & & & & \uparrow{\scriptstyle m} \\ G \times H \times G & \xrightarrow{1_G \times t \times 1_G} & G \times G \times G & \xrightarrow{m \times 1} & G \times G \end{array}$$

and the Peiffer identity:

$$\begin{array}{ccccc} H \times H & & \xrightarrow{t \times 1_H} & & G \times H \\ {\scriptstyle (\Delta_H \times 1_H)(1_H \times S_{H,H})}\downarrow & & & & \downarrow{\scriptstyle \alpha} \\ H \times H \times H & \xrightarrow{m \times \mathrm{inv}} & H \times H & \xrightarrow{m} & H \end{array}$$

Next, consider a strict 2-group C in the 2-category $K\mathrm{Cat}$, where K is a category with finite products. This is the same as a group in the underlying category of $K\mathrm{Cat}$. By 'commutativity of internalization', this is the same as a category in $K\mathrm{Grp}$. So, C consists of:

- a group C_0 in K,
- a group C_1 in K,
- **source** and **target** homomorphisms $s, t \colon C_1 \to C_0$,
- an **identity-assigning** homomorphism $i \colon C_0 \to C_1$,
- a **composition** homomorphism $\circ \colon C_1 \times_{C_0} C_1 \to C_1$

such that the usual laws for a category hold:

- laws specifying the source and target of identity morphisms,
- laws specifying the source and target of composite morphisms,
- the associative law for composition,
- the left and right unit laws for composition of morphisms.

We shall use this viewpoint in the following:

7.14. PROPOSITION. *Let K be a category with finite products such that $K\mathrm{Grp}$ has finite limits. Given a strict 2-group C in $K\mathrm{Cat}$, there is a crossed module (G, H, t, α) in K such that*

$$G = C_0, \qquad H = \ker s,$$

such that

$$t\colon H \to G,$$

is the restriction of $t\colon C_1 \to C_0$ to the subobject H, and such that

$$\alpha\colon G \times H \to H$$

makes this diagram commute:

$$\begin{array}{ccc}
G \times H & \xrightarrow{\quad\alpha\quad} & H \\
{\scriptstyle (\Delta\times 1_H)(1_G\times S_{H,G})}\downarrow & & \uparrow{\scriptstyle m} \\
G \times H \times G & & H \times H \\
{\scriptstyle i\times 1_H\times i}\downarrow & & \uparrow{\scriptstyle m\times 1_H} \\
H \times H \times H & \xrightarrow{1_G\times 1_H\times \mathrm{inv}_H} & H \times H \times H
\end{array}$$

Conversely, given a crossed module (G, H, t, α) in K, there is a strict 2-group C in K for which

$$C_0 = G$$

and for which

$$C_1 = H \times G$$

is made into a group in K by taking the semidirect product using the action α of G as automorphisms on H. In this strict 2-group we define source and target maps $s, t\colon C_1 \to C_0$ so that these diagrams commute:

$$\begin{array}{ccc}
H \times G & \xrightarrow{s} & G \\
{\scriptstyle 1_{H\times G}}\downarrow & \nearrow{\scriptstyle \pi_2} & \\
H \times G & &
\end{array}$$

$$\begin{array}{ccc}
H \times G & \xrightarrow{t} & G \\
{\scriptstyle t\times 1_G}\downarrow & \nearrow{\scriptstyle m} & \\
G \times G & &
\end{array}$$

define the identity-assigning map $\mathrm{id}\colon C_0 \to C_1$ *so that this diagram commutes:*

$$\begin{array}{ccc}
G & \xrightarrow{\mathrm{id}} & H \times G \\
{\scriptstyle \cong}\downarrow & \nearrow{\scriptstyle i_H\times 1_G} & \\
I \times G & &
\end{array}$$

and define composition $\circ\colon C_1 \times_{C_0} C_1 \to C_1$ *such that this commutes:*

$$\begin{array}{ccc} (H \times G) \times_G (H \times G) & \xrightarrow{\circ} & H \times G \\ \downarrow{\scriptstyle \pi_{123}} & & \uparrow{\scriptstyle m \times 1_G} \\ H \times G \times H & \xrightarrow[1_H \times S_{G,H}]{} & H \times H \times G \end{array}$$

where π_{123} *projects onto the product of the first, second and third factors.*

PROOF. The proof is modeled directly after the case $K = \mathrm{Set}$; in particular, the rather longwinded formula for α reduces to

$$\alpha(g, h) = i(g)\, h\, i(g)^{-1}$$

in this case. Note that to define $\ker s$ we need $K\mathrm{Grp}$ to have finite limits, while to define C_1 and make it into a group in K, we need K to have finite products. ∎

When the category K satisfies the hypotheses of this proposition, one can go further and show that strict 2-groups in $K\mathrm{Cat}$ are indeed 'the same' as crossed modules in K. To do this, one should first construct a 2-category of strict 2-groups in $K\mathrm{Cat}$ and a 2-category of crossed modules in K, and then prove these 2-categories are equivalent. We leave this as an exercise for the diligent reader.

8. Examples

8.1. AUTOMORPHISM 2-GROUPS. Just as groups arise most naturally from the consideration of *symmetries*, so do 2-groups. The most basic example of a group is a permutation group, or in other words, the automorphism group of a set. Similarly, the most basic example of a 2-group consists of the automorphism group of a category. More generally, we can talk about the automorphism group of an object in any category. Likewise, we can talk about the 'automorphism 2-group' of an object in any 2-category.

We can make this idea precise in somewhat different ways depending on whether we want a strict, weak, or coherent 2-group. So, let us consider various sorts of 'automorphism 2-group' for an object x in a 2-category K.

The simplest sort of automorphism 2-group is a strict one:

8.1.1. EXAMPLE. For any strict 2-category K and object $x \in K$ there is a strict 2-group $\mathrm{Aut}_s(x)$, the **strict automorphism 2-group** of x. The objects of this 2-group are isomorphisms $f\colon x \to x$, while the morphisms are 2-isomorphisms between these. Multiplication in this 2-group comes from composition of morphisms and horizontal composition of 2-morphisms. The identity object $1 \in \mathrm{Aut}_s(x)$ is the identity morphism $1_x\colon x \to x$.

To see what this construction really amounts to, take $K = \mathrm{Cat}$ and let $M \in K$ be a category with one object. A category with one object is secretly just a monoid, with the morphisms of the category corresponding to the elements of the monoid. An isomorphism

$f\colon M \to M$ is just an automorphism of this monoid. Given isomorphisms $f, f'\colon M \to M$, a 2-isomorphism from f to f' is just an invertible element of the monoid, say α, with the property that f conjugated by α gives f':

$$f'(m) = \alpha^{-1} f(m) \alpha$$

for all elements $m \in M$. This is just the usual commuting square law in the definition of a natural isomorphism, applied to a category with one object. So, $\mathrm{Aut}_s(M)$ is a strict 2-group that has automorphisms of x as its objects and 'conjugations' as its morphisms.

Of course the automorphisms of a monoid are its symmetries in the classic sense, and these form a traditional group. The new feature of the automorphism 2-group is that it keeps track of the *symmetries between symmetries*: the conjugations carrying one automorphism to another. More generally, in an n-group, we would keep track of symmetries between symmetries between symmetries between... and so on to the nth degree.

The example we are discussing here is especially well-known when the monoid is actually a group, in which case its automorphism 2-group plays an important role in nonabelian cohomology and the theory of nonabelian gerbes [8, 9, 27]. In fact, given a group G, people often prefer to work, not with $\mathrm{Aut}_s(G)$, but with a certain weak 2-group that is equivalent to $\mathrm{Aut}_s(G)$ as an object of W2G. The objects of this group are called 'G-bitorsors'. They are worth understanding, in part because they illustrate how quite different-looking weak 2-groups can actually be equivalent.

Given a group G, a ***G*-bitorsor** X is a set with commuting left and right actions of G, both of which are free and transitive. We write these actions as $g \cdot x$ and $x \cdot g$, respectively. A morphism between G-bitorsors $f\colon X \to Y$ is a map which is equivariant with respect to both these actions. The tensor product of G-bitorsors X and Y is defined to be the space

$$X \otimes Y = X \times Y / ((x \cdot g, y) \sim (x, g \cdot y)),$$

which inherits a left G-action from X and a right G-action from Y. It is easy to check that $X \otimes Y$ is again a bitorsor. Accompanying this tensor product of bitorsors there is an obvious notion of the tensor product of morphisms between bitorsors, making G-bitorsors into a weak monoidal category which we call ***G*-Bitors**.

The identity object of G-Bitors is just G, with its standard left and right action on itself. This is the most obvious example of a G-bitorsor, but we can get others as follows. Suppose that $f\colon G \to G$ is any automorphism. Then we can define a G-bitorsor G_f whose underlying set is G, equipped with the standard left action of G:

$$g \cdot h = gh, \qquad g \in G, h \in G_f$$

but with the right action twisted by the automorphism f:

$$h \cdot g = hf(g), \qquad g \in G, h \in G_f.$$

The following facts are easy to check. First, every G-bitorsor is isomorphic to one of the form G_f. Second, every morphism from G_f to $G_{f'}$ is of the form

$$h \mapsto h\alpha$$

for some $\alpha \in G$ with

$$f'(g) = \alpha^{-1} f(g) \alpha$$

for all $g \in G$. Third, the tensor product of G_f and $G_{f'}$ is isomorphic to $G_{ff'}$.

With the help of these facts, one can show that G-Bitors is equivalent as a weak monoidal category to $\mathrm{Aut}_s(G)$. The point is that the objects of G-Bitors all correspond, up to isomorphism, to the objects of $\mathrm{Aut}_s(G)$: namely, automorphisms of G. Similarly, the morphisms of G-Bitors all correspond to the morphisms of $\mathrm{Aut}_s(G)$: namely, 'conjugations'. The tensor products agree as well, up to isomorphism.

Since $\mathrm{Aut}_s(G)$ is a strict 2-group, it is certainly a weak one as well. Since G-Bitors is equivalent to $\mathrm{Aut}_s(G)$ as a weak monoidal category, it too is a weak 2-group, and it is equivalent to $\mathrm{Aut}_s(G)$ as an object of the 2-category W2G.

In this particular example, the 'strict automorphism 2-group' construction seems quite useful. But for some applications, this construction is overly strict. First, we may be interested in automorphism 2-group of an object in a weak 2-category (bicategory), rather than a strict one. Second, given objects x, y in a weak 2-category K, it is often unwise to focus attention on the isomorphisms $f\colon x \to y$. A more robust concept is that of a **weakly invertible morphism**: a morphism $f\colon x \to y$ for which there exists a morphism $\bar{f}\colon y \to x$ and 2-isomorphisms $\iota\colon 1_x \Rightarrow f\bar{f}$, $\epsilon\colon \bar{f}f \Rightarrow 1_y$. Using weakly invertible morphisms as a substitute for isomorphisms gives a weak version of the automorphism 2-group:

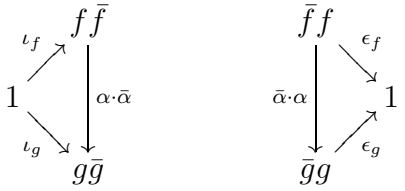
8.1.2. Example. For any weak 2-category K and object $x \in K$ there is a weak 2-group $\mathrm{Aut}_w(x)$, the **weak automorphism 2-group** of x. The objects of this 2-group are weakly invertible morphisms $f\colon x \to x$, while the morphisms are 2-isomorphisms between these. Multiplication in this 2-group comes from composition of morphisms and horizontal composition of 2-morphisms. The identity object $1 \in \mathrm{Aut}_w(C)$ is the identity functor.

A weakly invertible morphism $f\colon x \to y$ is sometimes called an 'equivalence'. Here we prefer to define an **equivalence** from x to y to be a morphism $f\colon x \to y$ with a *specified* weak inverse $\bar{f}\colon y \to x$ and *specified* 2-isomorphisms $\iota_f\colon 1_x \Rightarrow f\bar{f}$, $\epsilon_f\colon \bar{f}f \Rightarrow 1_y$. An equivalence from x to y is thus a quadruple $(f, \bar{f}, \iota_f, \epsilon_f)$. We can make a coherent 2-group whose objects are equivalences from x to itself:

8.1.3. Example. For any weak 2-category K and object $x \in K$ there is a coherent 2-group $\mathrm{Aut}_{eq}(x)$, the **autoequivalence 2-group** of x. The objects of $\mathrm{Aut}_{eq}(x)$ are equivalences from x to x. A morphism in $\mathrm{Aut}_w(x)$ from $(f, \bar{f}, \iota_f, \epsilon_f)$ to $(g, \bar{g}, \iota_g, \epsilon_g)$ consists of 2-isomorphisms

$$\alpha\colon f \Rightarrow g, \qquad \bar{\alpha}\colon \bar{f} \Rightarrow \bar{g}$$

such that the following diagrams commute:



Multiplication in this 2-group comes from the standard way of composing equivalences, together with horizontal composition of 2-morphisms. The identity object $1 \in \mathrm{Aut}_{eq}(x)$ is the equivalence $(1_x, 1_x, 1_{1_x}, 1_{1_x})$.

One can check that $\mathrm{Aut}_{eq}(x)$ is a weak 2-group, because every object $F = (f, \bar{f}, \iota_f, \epsilon_f)$ of $\mathrm{Aut}_{eq}(x)$ has the weak inverse $\bar{F} = (\bar{f}, f, \epsilon_f^{-1}, \iota_f^{-1})$. But in fact, the proof of this involves constructing isomorphisms

$$i_F \colon 1_x \Rightarrow F\bar{F}, \qquad e_F \colon \bar{F}F \Rightarrow 1_x$$

from the data at hand, and these isomorphisms can easily be chosen to satisfy the zig-zag identities, so $\mathrm{Aut}_{eq}(x)$ actually becomes a coherent 2-group.

An equivalence $(f, \bar{f}, \iota_f, \epsilon_f)$ is an **adjoint equivalence** if it satisfies the zig-zag identities. We can also construct a coherent 2-group whose objects are adjoint equivalences from x to itself:

8.1.4. Example. For any weak 2-category K and object $x \in K$ there is a coherent 2-group $\mathrm{Aut}_{ad}(x)$, the **adjoint autoequivalence group** of x. The objects of this 2-group are adjoint equivalences from x to x, while the morphisms are defined as in $\mathrm{Aut}_{eq}(x)$. Multiplication in this 2-group comes from the usual way of composing equivalences (using the fact that composite of adjoint equivalences is again an adjoint equivalence) together with horizontal composition of 2-morphisms. The identity object $1 \in \mathrm{Aut}_{ad}(x)$ is the adjoint equivalence $(1_x, 1_x, 1_{1_x}, 1_{1_x})$. $\mathrm{Aut}_{ad}(x)$ becomes a coherent 2-group using the fact that every object F of $\mathrm{Aut}_{ad}(x)$ becomes part of an adjunction $(F, \bar{F}, i_F, e_F)$ as in Example 8.1.3.

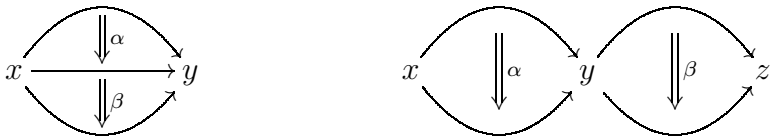
8.2. The fundamental 2-group. Another source of 2-groups is topology: for any topological space X and any point $x \in X$ there is a coherent 2-group $\Pi_2(X, x)$ called the 'fundamental 2-group' of X based at x. The fundamental 2-group is actually a watered-down version of what Hardie, Kamps and Kieboom [29] call the 'homotopy bigroupoid' of X, denoted by $\Pi_2(X)$. This is a weak 2-category whose objects are the points of X, whose morphisms are paths in X, and whose 2-morphisms are homotopy classes of paths-of-paths. More precisely, a morphism $f \colon x \to y$ is a map $f \colon [0,1] \to X$ with $f(0) = x$ and $f(1) = y$, while a 2-morphism $\alpha \colon f \Rightarrow g$ is an equivalence class of maps $\alpha \colon [0,1]^2 \to X$ with

$$\begin{aligned} \alpha(s,0) &= f(s) \\ \alpha(s,1) &= g(s) \\ \alpha(0,t) &= x \\ \alpha(1,t) &= y \end{aligned}$$

for all $s, t \in [0,1]$, where the equivalence relation is that $\alpha \sim \alpha'$ if there is a map $H\colon [0,1]^3 \to X$ with

$$\begin{aligned} H(s,t,0) &= \alpha(s,t) \\ H(s,t,1) &= \alpha'(s,t) \\ H(s,0,u) &= f(s) \\ H(s,1,u) &= g(s) \\ H(0,t,u) &= x \\ H(1,t,u) &= y \end{aligned}$$

for all $s, t, u \in [0,1]$. This becomes a weak 2-category in a natural way, with composition of paths giving composition of morphisms, and two ways of composing paths-of-paths giving vertical and horizontal composition of 2-morphisms:



Hardie, Kamps and Kieboom show that every 2-morphism in $\Pi_2(X)$ is invertible, and they construct an adjoint equivalence $(f, \bar{f}, \iota_f, \epsilon_f)$ for every morphism f in $\Pi_2(X)$. This is why they call $\Pi_2(X)$ a 'bigroupoid'. One might also call this a 'coherent 2-groupoid', since such a thing with one object is precisely a coherent 2-group. Regardless of the terminology, this implies that for any point $x \in X$ there is a coherent 2-group whose objects are morphisms $f\colon x \to x$ in $\Pi_2(X)$, and whose morphisms are those 2-morphisms $\alpha\colon f \Rightarrow g$ in $\Pi_2(X)$ for which $f, g\colon x \to x$. We call this coherent 2-group the **fundamental 2-group** $\Pi_2(X, x)$.

In fact, a fundamental 2-group is a special case of an 'autoequivalence 2-group', as defined in Example 8.1.3. A point $x \in X$ is an object of the weak 2-category $\Pi_2(X)$, and the autoequivalence 2-group of this object is precisely the fundamental 2-group $\Pi_2(X, x)$. Even better, we can turn this idea around: there is a way to see any autoequivalence 2-group as the fundamental 2-group of some space, at least up to equivalence! Unfortunately, proving this fact would take us too far out of our way here. However, the relation between 2-groups and topology is so important that we should at least sketch the basic idea.

Suppose K is a weak 2-category, and let K_0 be its underlying coherent 2-groupoid—that is, the weak 2-category with the same objects as K, but with the adjoint equivalences in K as its morphisms and the invertible 2-morphisms of K as its 2-morphisms. Let $|K_0|$ be the geometric realization of the nerve of K_0 as defined by Duskin [23]. Then any object $x \in K$ gives a point $x \in |K_0|$, and the autoequivalence 2-group $\mathrm{Aut}_{eq}(x)$ is equivalent to $\Pi_2(|K_0|, x)$.

In fact, something much stronger than this should be true. According to current thinking on n-categories and homotopy theory [4], 2-groups should really be 'the same' as connected pointed homotopy 2-types. For example, we should be able to construct a 2-category $\mathrm{Conn2Typ}_*$ having connected pointed CW complexes with $\pi_n = 0$ for $n > 2$ as objects, continuous basepoint-preserving maps as morphisms, and homotopy classes of

basepoint-preserving homotopies between such maps as 2-morphisms. The fundamental 2-group construction should give a 2-functor:

$$\begin{array}{rcl} \Pi_2 \colon \mathrm{Conn2Typ}_* & \to & \mathrm{C2G} \\ (X, x) & \mapsto & \Pi_2(X, x) \end{array}$$

while the geometric realization of the nerve should give a 2-functor going the other way:

$$\begin{array}{rcl} \Pi_2^{-1} \colon \mathrm{C2G} & \to & \mathrm{Conn2Typ}_* \\ C & \mapsto & (|C|, 1) \end{array}$$

and these should extend to a biequivalence of 2-categories. To the best of our knowledge, nobody has yet written up a proof of this result. However, a closely related higher–dimensional result has been shown by C. Berger [6]: the model category of homotopy 3-types is Quillen equivalent to a suitably defined model category of weak 3-groupoids.

8.3. Classifying 2-groups using group cohomology. In this section we sketch how a coherent 2-group is determined, up to equivalence, by four pieces of data:

- a group G,
- an abelian group H,
- an action α of G as automorphisms of H,
- an element $[a]$ of the cohomology group $H^3(G, H)$,

where the last item comes from the associator. Various versions of this result have been known to experts at least since Sinh's thesis [44], but since this thesis was unpublished they seem to have spread largely in the form of 'folk theorems'. A very elegant treatment can be found in the 1986 draft of Joyal and Street's paper on braided tensor categories [33], but not in the version that was finally published in 1993. So, it seems worthwhile to provide a precise statement and proof here.

One way to prove this result would be to take a detour through topology. Using the ideas sketched at end of the previous section, equivalence classes of coherent 2-groups should be in one-to-one correspondence with homotopy equivalence classes of connected pointed CW complexes having homotopy groups that vanish above π_2. The latter, in turn, can be classified in terms of their 'Postnikov data': the group $G = \pi_1$, the abelian group $H = \pi_2$, the action of π_1 on π_2, and the Postnikov k-invariant, which is an element of $H^3(\pi_1, \pi_2)$. The advantage of this approach is that it generalizes to n-groups for higher n, and clarifies their relation to topology. The disadvantage is that it is indirect and relies on results that themselves take some work to prove. Besides the relation between coherent 2-groups and homotopy 2-types, one needs the theory of Postnikov towers in the case where π_1 acts nontrivially on the higher homotopy groups [48].

To avoid all this, we take a more self-contained approach. First we show that every coherent 2-group is equivalent to a 'special' one:

8.3.1. Definition. *A coherent 2-group is* **skeletal** *if its underlying category is skeletal: that is, if any pair of isomorphic objects in this category are equal.*

8.3.2. Definition. *A* **special 2-group** *is a skeletal coherent 2-group such that the left unit law ℓ, the right unit law r, the unit i and the counit e are identity natural transformations.*

We then show that any special 2-group determines a quadruple (G, H, α, a). The objects of a special 2-group form a group G. The automorphisms of the unit object form an abelian group H. There is an action α of G on H, defined just as in the construction of a crossed module from a strict 2-group. The associator gives rise to a map $a\colon G^3 \to H$. Furthermore, the pentagon identity and other properties of monoidal categories imply that a is a 'normalized 3-cocycle' on G with values in the G-module H. When we work through this in detail, it will also become clear that conversely, any quadruple (G, H, α, a) of this sort determines a special 2-group.

Following Joyal and Street, we exploit these results by constructing a 2-category of special 2-groups that is biequivalent to C2G, for which not only the objects but also the morphisms and 2-morphisms can be described using group cohomology. As a corollary, it will follow that coherent 2-groups are classified up to equivalence by quadruples $(G, H, \alpha, [a])$, where $[a] \in H^3(G, H)$ is the cohomology class of the 3-cocycle a.

We begin by proving the following fact:

8.3.3. Proposition. *Every coherent 2-group is equivalent in* C2G *to a special 2-group.*

Proof. First suppose that C is a coherent 2-group. Note that C is equivalent, as an object of C2G, to a skeletal coherent 2-group. To see this, recall that every category is equivalent to a skeletal one: we can take this to be any full subcategory whose objects include precisely one representative from each isomorphism class. Using such an equivalence of categories, we can transfer the coherent 2-group structure from C to a skeletal category C_0. It is then routine to check that C and C_0 are equivalent as objects of C2G.

Next suppose C is a skeletal coherent 2-group. We shall construct a special 2-group $\tilde{C}$ that is equivalent to C. As a category, $\tilde{C}$ will be precisely the same as C, so it will still be skeletal. However, we shall adjust the tensor product, left and right unit laws, unit and counit, and associator to ensure that $\tilde{\ell}, \tilde{r}, \tilde{\imath}$ and $\tilde{e}$ are identity natural transformations. We do this using the following lemma:

8.3.4. Lemma. *If C is a coherent 2-group, and for each $x, y \in C$ we choose an isomorphism $\gamma_{x,y}\colon x \tilde{\otimes} y \to x \otimes y$ for some object $x \tilde{\otimes} y \in C$, then there exists a unique way to make the underlying category of C into a coherent 2-group $\tilde{C}$ such that:*

1. *the tensor product of any pair of objects x, y in $\tilde{C}$ is $x \tilde{\otimes} y$,*
2. *there is a homomorphism of coherent 2-groups $F\colon C \to \tilde{C}$ whose underlying functor is the identity, for which F_0 and F_{-1} are the identity, and for which*

$$(F_2)_{x,y} = \gamma_{x,y}$$

for every $x, y \in C$.

Moreover, $F\colon C \to \tilde{C}$ is an equivalence in C2G*.*

PROOF. First we show uniqueness. The tensor product of objects in $\tilde{C}$ is determined by item 1. For F as in item 2 to be a weak monoidal functor we need F_2 to be natural, so the tensor product $f\tilde{\otimes}g$ of morphisms $f\colon x \to x'$, $g\colon y \to y'$ in $\tilde{C}$ is determined by the requirement that

$$\begin{array}{ccc} x\tilde{\otimes}y & \xrightarrow{\gamma_{x,y}} & x \otimes y \\ {\scriptstyle f\tilde{\otimes}g}\downarrow & & \downarrow{\scriptstyle f\otimes g} \\ x'\tilde{\otimes}y' & \xrightarrow{\gamma_{x',y'}} & x' \otimes y' \end{array}$$

commute. The unit object of $\tilde{C}$ must be the same as that of C, since F_0 is the identity. The unit $\tilde{\imath}$ and counit $\tilde{e}$ of $\tilde{C}$ are determined by the coherence laws **H1** and **H2** in Section 6. The associator $\tilde{a}$ of $\tilde{C}$ is determined by this coherence law in the definition of 'weak monoidal functor':

$$\begin{array}{ccccc} (F(x)\tilde{\otimes}F(y))\tilde{\otimes}F(z) & \xrightarrow{F_2\tilde{\otimes}1} & F(x \otimes y)\tilde{\otimes}F(z) & \xrightarrow{F_2} & F((x \otimes y) \otimes z) \\ {\scriptstyle \tilde{a}_{F(x),F(y),F(z)}}\downarrow & & & & \downarrow{\scriptstyle F(a_{x,y,z})} \\ F(x)\tilde{\otimes}(F(y) \otimes F(z)) & \xrightarrow{1\tilde{\otimes}F_2} & F(x)\tilde{\otimes}F(y \otimes z) & \xrightarrow{F_2} & F(x \otimes (y \otimes z)) \end{array}$$

Similarly, the left and right unit laws $\tilde{l}, \tilde{r}$ of $\tilde{C}$ are determined by the other two coherence laws in this definition:

$$\begin{array}{ccc} 1\tilde{\otimes}F(x) & \xrightarrow{\tilde{\ell}_{F(x)}} & F(x) \\ {\scriptstyle F_0\tilde{\otimes}1}\downarrow & & \uparrow{\scriptstyle F(\ell_x)} \\ F(1)\tilde{\otimes}F(x) & \xrightarrow{F_2} & F(1 \otimes x) \end{array}$$

$$\begin{array}{ccc} F(x)\tilde{\otimes}1 & \xrightarrow{\tilde{r}_{F(x)}} & F(x) \\ {\scriptstyle 1\tilde{\otimes}F_0}\downarrow & & \uparrow{\scriptstyle F(r_x)} \\ F(x)\tilde{\otimes}F(1) & \xrightarrow{F_2} & F(x \otimes 1) \end{array}$$

It is then an exercise to check that with these choices, $\tilde{C}$ really does become a coherent 2-group, that $F\colon C \to \tilde{C}$ is a homomorphism, and that F is an equivalence of coherent 2-groups. ∎

We now apply this lemma, taking

$$x\tilde{\otimes}y = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 1 & \text{if } x = \bar{y} \\ 1 & \text{if } y = \bar{x} \\ x \otimes y & \text{otherwise} \end{cases}$$

and taking

$$\gamma_{x,y} = \begin{cases} \ell_y^{-1} & \text{if } x = 1 \\ r_x^{-1} & \text{if } y = 1 \\ e_y^{-1} & \text{if } x = \bar{y} \\ i_x & \text{if } y = \bar{x} \\ 1_{x \otimes y} & \text{otherwise} \end{cases}$$

Calculations then show that with these choices, $\tilde{\ell}, \tilde{r}, \tilde{\imath}$ and $\tilde{e}$ are identity natural transformations. For example, to show that $\tilde{\imath}$ is the identity we use coherence law **H1**, which says this diagram commutes:

$$\begin{array}{ccc} x \tilde{\otimes} \overline{F(x)} & \xrightarrow{1 \tilde{\otimes} F_{-1}} & F(x) \tilde{\otimes} F(\bar{x}) \xrightarrow{F_2} F(x \otimes \bar{x}) \\ {\scriptstyle \tilde{\imath}_{F(x)}} \uparrow & & \uparrow {\scriptstyle F(i_x)} \\ 1 & \xrightarrow{\qquad\qquad F_0 \qquad\qquad} & F(1) \end{array}$$

By the definition of F, F_0, and $F_2 = \gamma$ together with the fact that $x \tilde{\otimes} \bar{x} = 1$, this diagram reduces to

$$\begin{array}{ccccc} 1 & \xrightarrow{1_1} & 1 & \xrightarrow{i_x} & x \otimes \bar{x} \\ {\scriptstyle \tilde{\imath}_x} \uparrow & & & & \uparrow {\scriptstyle i_x} \\ 1 & & \xrightarrow{\qquad 1_1 \qquad} & & 1 \end{array}$$

which implies that $\tilde{\imath}_x = 1_1$. Similarly, to show that $\tilde{e}_x$ is the identity we use **H2**, and to show $\tilde{\ell}_x$ and $\tilde{r}_x$ are identities we use the coherence laws for the left and right unit laws in the definition of 'weak monoidal functor'. ∎

We now describe in a bit more detail how to get a quadruple (G, H, α, a) from a special 2-group C. In general, the objects of a 2-group need not form a group under multiplication, since we only have isomorphisms

$$\begin{aligned} &(x \otimes y) \otimes z \cong x \otimes (y \otimes z), \\ &1 \otimes x \cong x, \qquad x \otimes 1 \cong x, \\ &\bar{x} \otimes x \cong 1, \qquad x \otimes \bar{x} \cong 1. \end{aligned}$$

However, in a special 2-group, isomorphic objects are equal, so the objects form a group. This is our group G.

The Eckmann-Hilton argument shows that in any weak monoidal category, the endomorphisms of the unit object form a commutative monoid under tensor product or, what is the same, composition:

$$\begin{aligned} h \otimes h' &= (h 1_1) \otimes (1_1 h') \\ &= (h \otimes 1_1)(1_1 \otimes h') \\ &= hh' \\ &= (1_1 \otimes h)(h' \otimes 1_1) \\ &= (1_1 h') \otimes (h 1_1) \\ &= h' \otimes h \end{aligned}$$

for all $h, h' \colon 1 \to 1$. Applied to 2-groups this implies that the automorphisms of the object 1 form an abelian group. This is our abelian group H.

There is an action α of G as automorphisms of H given by

$$\alpha(g, h) = (1_g \otimes h) \otimes 1_{\overline{g}}.$$

This is the same formula for α as in the crossed module construction of Section 7; we are just writing it a bit differently now because a coherent 2-group is not a category object in Grp. Here we need to be a bit more careful to check that α is an action as automorphisms, since the associator is nontrivial.

Finally, since our 2-group is skeletal, we do not need to parenthesize tensor products of objects, and the associator gives an automorphism

$$a_{g_1,g_2,g_3} \colon g_1 \otimes g_2 \otimes g_3 \to g_1 \otimes g_2 \otimes g_3.$$

For any object $x \in G$ we identify $\mathrm{Aut}(x)$ with $\mathrm{Aut}(1) = H$ by tensoring with $\bar{x}$ on the right: if $f \colon x \to x$ then $f \otimes \bar{x} \colon 1 \to 1$, since $x \otimes \bar{x} = 1$. By this trick the associator can be thought of as a map from G^3 to H, and by abuse of language we denote this map by:

$$\begin{array}{rcl} a \colon G^3 & \to & H \\ (g_1, g_2, g_3) & \mapsto & a(g_1, g_2, g_3) := a_{g_1,g_2,g_3} \otimes \overline{g_1 \otimes g_2 \otimes g_3}. \end{array}$$

The pentagon identity implies that this map satisfies

$$g_0 a(g_1, g_2, g_3) - a(g_0 g_1, g_2, g_3) + a(g_0, g_1 g_2, g_3) - a(g_0, g_1, g_2 g_3) + a(g_0, g_1, g_2) = 0$$

for all $g_0, g_1, g_2, g_3 \in G$, where the first term is defined using the action of G on H, and we take advantage of the abelianness of H to write its group operation as addition. In the language of group cohomology [38], this says precisely that a is a '3-cocycle' on G with coefficients in the G-module H. Mac Lane's coherence theorem for monoidal categories also implies that a is a 'normalized' 3-cocycle, meaning that $a(g_1, g_2, g_3) = 1$ whenever g_1, g_2 or g_3 equals 1.

This completes the construction of a quadruple (G, H, α, a) from any special 2-group. Conversely, any such quadruple determines a unique 2-group of this sort. Since proving this is largely a matter of running the previous construction backwards, we leave this as an exercise for the reader.

Having shown that every coherent 2-group is equivalent to one that can be described using group cohomology, we now proceed to do the same thing for homomorphisms between coherent 2-groups.

8.3.5. Definition. *A* **special** *homomorphism $F \colon C \to C'$ is a homomorphism between special 2-groups such that F_0 is an identity morphism.*

8.3.6. Proposition. *Any homomorphism between special 2-groups is isomorphic in* C2G *to a special homomorphism.*

Proof. It suffices to show that for any weak monoidal functor $F \colon C \to C'$ between weak monoidal categories, there is a weak monoidal natural isomorphism $\theta \colon F \Rightarrow F'$ where $F'_0 \colon 1 \to F'(1)$ is an identity isomorphism. We leave this as an exercise for the reader. ∎

To give a cohomological description of special homomorphisms, let $F\colon C \to C'$ be a special homomorphism and let (G, H, α, a) and (C, H, α', a') be the quadruples corresponding to C and C', respectively. The functor F maps objects to objects and preserves tensor products up to isomorphism, so it gives a group homomorphism

$$\phi\colon G \to G'.$$

For similar reasons, F also gives a group homomorphism

$$\psi\colon H \to H',$$

and in fact this is a morphism of modules in the following sense:

$$\psi(\alpha(g)h) = \alpha'(g)\psi(h)$$

for all $g \in G$ and $h \in H$. As a weak monoidal functor, F also comes equipped with an natural isomorphism from $F(g_1) \otimes F(g_2)$ to $F(g_1 \otimes g_2)$ for all $g_1, g_2 \in G$. Since C' is skeletal, this is an automorphism:

$$(F_2)_{g_1,g_2}\colon F(g_1) \otimes F(g_2) \to F(g_1) \otimes F(g_2).$$

Copying what we did for the associator, we define a map

$$\begin{array}{rcl} k\colon G^2 & \to & H' \\ (g_1, g_2) & \mapsto & k(g_1, g_2) := (F_2)_{g_1,g_2} \otimes \overline{F(g_1) \otimes F(g_2)}. \end{array}$$

Using the fact that F_0 is the identity, the coherence laws for the left and right unit laws in the definition of a weak monoidal functor imply that $k(g_1, g_2) = 1$ whenever g_1 or g_2 equals 1. In the language of group cohomology, k is thus a 'normalized 2-cocycle' on G with values in H'. Furthermore, the coherence law for the associator in the definition of a weak monoidal functor implies that

$$\psi(a(g_0, g_1, g_2)) - a'(\phi(g_0), \phi(g_1), \phi(g_2)) =$$

$$\phi(g_0)k(g_1, g_2) - k(g_0 g_1, g_2) + k(g_0, g_1 g_2) - k(g_1, g_2)$$

for all $g_0, g_1, g_2 \in G$. This says precisely that ψa and $a'\phi^3$ differ by the coboundary of k:

$$\psi a - a'\phi^3 = dk.$$

In short, a special homomorphism $F\colon C \to C$ gives a triple (ϕ, ψ, k) where $\phi\colon G \to G'$ is a group homomorphism, $\psi\colon H \to H'$ is a module homomorphism, and k is a normalized 2-cocycle on G with values in H' such that $dk = \psi a - a'\phi^3$. Conversely, it is not hard to see that any such triple gives a special homomorphism from C to C'.

Finally, we give a cohomological description of 2-homomorphisms between special homomorphisms. Let $F, F'\colon C \to C'$ be special homomorphisms with corresponding triples

(ϕ, ψ, k) and (ϕ', ψ', k'), respectively. A 2-homomorphism $\theta\colon F \Rightarrow F'$ is just a monoidal natural transformation, so it gives a map

$$\begin{array}{rcl} p\colon G & \to & H' \\ g & \mapsto & k(g) := \theta_g \otimes \overline{F(g)}. \end{array}$$

The condition that θ be natural turns out to have no implications for p: it holds no matter what p is. However, the condition that θ be monoidal is equivalent to the equations $p(1) = 1$ and

$$k(g_1, g_2) - k'(g_1, g_2) = \phi'(g_1)p(g_2) - p(g_1 g_2) + p(g_1)$$

for all $g_1, g_2 \in G$. In the language of group cohomology, these equations say that p is a 1-cochain on G with values in H' such that $dp = k - k'$. So, 2-homomorphisms between special homomorphisms are in one-to-one correspondence with 1-cochains of this sort.

Summarizing all this, we obtain:

8.3.7. Theorem. *The 2-category* C2G *is biequivalent to the sub-2-category* **S2G** *for which the objects are special 2-groups, the morphisms are special homomorphisms between these, and the 2-morphisms are arbitrary 2-homomorphisms between those. Moreover:*

- *There is a one-to-one correspondence between special 2-groups C and quadruples (G, H, α, a) consisting of:*
  - *a group G,*
  - *an abelian group H,*
  - *an action α of G as automorphisms of H,*
  - *a normalized 3-cocycle $a\colon G^3 \to H$.*
- *Given special 2-groups C, C' with (G, H, α, a) and (G', H', α', a') the corresponding quadruples, there is a one-to-one correspondence between special homomorphisms $F\colon C \to C'$ and triples (ϕ, ψ, k) consisting of:*
  - *a homomorphism of groups $\phi\colon G \to G'$,*
  - *a homomorphism of modules $\psi\colon H \to H'$,*
  - *a normalized 2-cochain $k\colon G^2 \to H'$ such that $dk = \psi a - a'\phi^3$.*
- *Given special homomorphisms $F, F'\colon C \to C'$ with corresponding triples (ϕ, ψ, k) and (ϕ', ψ', k'), there is a one-to-one correspondence between 2-homomorphisms $\theta\colon F \Rightarrow F'$ and normalized 1-cochains $p\colon G \to H'$ with $dp = k - k'$.*

Proof. The fact that C2G is biequivalent to the sub-2-category S2G follows from the fact that every object of C2G is equivalent to an object in S2G (Proposition 8.3.3) and every morphism of C2G is isomorphic to a morphism in S2G (Proposition 8.3.6). The cohomological descriptions of objects, morphisms and 2-morphisms in S2G were deduced above. ∎

We could easily use this theorem to give a complete description of the 2-category S2G in terms of group cohomology, but we prefer to extract a simple corollary:

8.3.8. Corollary. *There is a 1-1 correspondence between equivalence classes of coherent 2-groups, where equivalence is as objects of the 2-category* C2G*, and isomorphism classes of quadruples* $(G, H, \alpha, [a])$ *consisting of:*

- *a group* G*,*
- *an abelian group* H*,*
- *an action* α *of* G *as automorphisms of* H*,*
- *an element* $[a]$ *of the cohomology group* $H^3(G, H)$*,*

where an isomorphism from $(G, H, \alpha, [a])$ *to* $(G', H', \alpha', [a'])$ *consists of an isomorphism from* G *to* G' *and an isomorphism from* H *to* H'*, carrying* α *to* α' *and* $[a]$ *to* $[a']$*.*

Proof. This follows directly from Theorem 8.3.7, together with the fact that group cohomology can be computed using normalized cochains. ∎

Though the main use of Proposition 8.3.3 is to help prove Theorem 8.3.7, it has some interest in its own right, because it clarifies the extent to which any coherent 2-group can be made simultaneously both skeletal and strict. Any coherent 2-group is equivalent to a skeletal one in which ℓ, r, i and e are identity natural transformations—but not the associator, unless the invariant $[a] \in H^3(G, H)$ vanishes. On the other hand, if we drop our insistence on making a 2-group skeletal, we can make it completely strict:

8.3.9. Proposition. *Every coherent 2-group is equivalent in* C2G *to a strict one—that is, one for which* ℓ, r, i, e *and* a *are identity natural transformations.*

Proof. Let C be a coherent 2-group. By a theorem of Mac Lane [39], there is a strict monoidal category C' that is equivalent to C' as a monoidal category. We can use this equivalence to transfer the coherent 2-group structure from C to C', making C' into a coherent 2-group for which ℓ, r, and a are identity natural transformations, but not yet i and e.

As a strict monoidal category, C' is an object of CatMon, the category of 'monoids in Cat'. There is a pair of adjoint functors consisting of the forgetful functor $U\colon \mathrm{CatGrp} \to \mathrm{CatMon}$ and its left adjoint $F\colon \mathrm{CatMon} \to \mathrm{CatGrp}$. Thus $C'' = F(C')$ is a group in Cat, or in other words a strict 2-group. It suffices to show that C' is equivalent to C'' as an object of C2G.

The unit of the adjunction between U and F gives a strict monoidal functor $i_{C'}\colon C' \to U(F(C'))$, which by Theorem 6.1 determines a 2-group homomorphism from C' to $C'' = F(C')$. One can check that this is extends to an equivalence in C2G; we leave this to the reader.

An alternative approach uses Proposition 8.3.3 to note that C is equivalent to a special 2-group C'. From the quadruple (G, H, α, a) corresponding to this special 2-group one

can construct a crossed module (see Mac Lane [38] or, for a more readable treatment, Ken Brown's text on group cohomology [12]). This crossed module in turn gives a strict 2-group C'', and one can check that C'' is equivalent to C in C2G. The details for this approach can be found in the 1986 draft of Joyal and Street's paper on braided tensor categories [33]. ∎

This result explains why Mac Lane and Whitehead [41] were able to use strict 2-groups (or actually crossed modules) to describe arbitrary connected pointed homotopy 2-types, instead of needing the more general coherent 2-groups.

8.4. Strict Lie 2-groups. It appears that just as Lie groups describe continuous symmetries in geometry, Lie 2-groups describe continuous symmetries in categorified geometry. In Definition 7.9 we said that Lie 2-groups are coherent 2-groups in DiffCat, the 2-category of smooth categories. In this section we shall give some examples, but only 'strict' ones, for which the associator, left and right unit laws, unit and counit are all identity 2-morphisms. We discuss the challenge of finding interesting nonstrict Lie 2-groups in the next section.

Strict Lie 2-groups make no use of the 2-morphisms in DiffCat, so they are really just groups in the underlying category of DiffCat. By 'commutativity of internalization', these are the same as categories in DiffGrp, the category of Lie groups. To see this, note that if C is a strict Lie 2-group, it is first of all an object in DiffCat. This means it is a category with a manifold of objects C_0 and a manifold of morphisms C_1, with its source, target, identity-assigning and composition maps all smooth. But since C is a *group* in DiffCat, C_0 and C_1 become Lie groups, and all these maps become Lie group homomorphisms. Thus, C is a category in DiffGrp. The converse can be shown by simply reversing this argument.

Treating strict Lie 2-groups as categories in DiffGrp leads naturally to yet another approach, where we treat them as 'Lie crossed modules'. Here we use the concept of 'crossed module in K', as described in Definition 7.13:

8.4.1. Definition. *A* **Lie crossed module** *is a crossed module in* Diff.

Concretely, a Lie crossed module is a quadruple (G, H, t, α) consisting of Lie groups G and H, a homomorphism $t\colon H \to G$, and an action α of G on H such that t is G-equivariant

$$t(\alpha(g, h) = g\, t(h)\, g^{-1}$$

and t satisfies the Peiffer identity

$$\alpha(t(h), h') = hh'h^{-1}$$

for all $g \in G$ and $h, h' \in H$. Proposition 7.14 shows how we can get a Lie crossed module from a strict Lie 2-group and vice versa. Using this, one can construct a 2-category of strict Lie 2-groups and a 2-category of Lie crossed modules and show that they are equivalent. This equivalence lets us efficiently construct many examples of strict Lie 2-groups:

8.4.2. Example. Given any Lie group G, abelian Lie group H, and homomorphism $\alpha\colon G \to \mathrm{Aut}(H)$, there is a Lie crossed module with $t\colon G \to H$ the trivial homomorphism and G acting on H via ρ. Because t is trivial, the corresponding strict Lie 2-group C is 'skeletal', meaning that any two isomorphic objects are equal. It is easy to see that conversely, all skeletal strict Lie 2-groups are of this form.

8.4.3. Example. Given any Lie group G, we can form a Lie crossed module as in Example 8.4.2 by taking $H = \mathfrak{g}$, thought of as an abelian Lie group, and letting α be the adjoint representation of G on $\mathfrak{g}$. If C is the corresponding strict Lie 2-group we have

$$C_1 = \mathfrak{g} \rtimes G \cong TG$$

where TG is the tangent bundle of G, which becomes a Lie group with product

$$dm\colon TG \times TG \to TG,$$

obtained by differentiating the product

$$m\colon G \times G \to G.$$

We call C the **tangent 2-group** of G and denote it as $\mathcal{T}G$.

Another route to the tangent 2-group is as follows. Given any smooth manifold M there is a smooth category $\mathcal{T}M$, the **tangent groupoid** of M, whose manifold of objects is M and whose manifold of morphisms is TM. The source and target maps $s, t\colon TM \to M$ are both the projection to the base space, the identity-assigning map $i\colon M \to TM$ is the zero section, and composition of morphisms is addition of tangent vectors. In this category the arrows are actually little arrows—that is, tangent vectors!

This construction extends to a functor

$$\mathcal{T}\colon \mathrm{Diff} \to \mathrm{DiffCat}$$

in an obvious way. This functor preserves products, so it sends group objects to group objects. Thus, if G is a Lie group, its tangent groupoid $\mathcal{T}G$ is a strict Lie 2-group.

8.4.4. Example. Similarly, given any Lie group G, we can form a Lie crossed module as in Example 8.4.2 by letting α be the coadjoint representation on $H = \mathfrak{g}^*$. If C is the corresponding Lie 2-group, we have

$$C_1 = \mathfrak{g}^* \rtimes G \cong T^*G$$

where T^*G is the cotangent bundle of G. We call C the **cotangent 2-group** of G and denote it as $\mathcal{T}^*G$.

8.4.5. Example. More generally, given any representation α of a Lie group G on a finite-dimensional vector space V, we can form a Lie crossed module and thus a strict Lie 2-group with this data, taking $H = V$. For example, if G is the Lorentz group $\mathrm{SO}(n,1)$, we can form a Lie crossed module by letting α be the defining representation of $\mathrm{SO}(n,1)$ on $H = \mathbb{R}^{n+1}$. If C is the corresponding strict Lie 2-group, we have

$$C_1 = \mathbb{R}^{n+1} \rtimes \mathrm{SO}(n,1) \cong \mathrm{ISO}(n,1)$$

where $\mathrm{ISO}(n,1)$ is the Poincaré group. We call C the **Poincaré 2-group**. After this example was introduced by one of the authors [2], it became the basis of an interesting new theory of quantum gravity [21, 22].

8.4.6. Example. Given any Lie group H, there is a Lie crossed module with $G = \mathrm{Aut}(H)$, $t\colon H \to G$ the homomorphism assigning to each element of H the corresponding inner automorphism, and the obvious action of G as automorphisms of H. We call the corresponding strict Lie 2-group the **strict automorphism 2-group** of H, $\mathrm{Aut}_s(H)$, because its underlying 2-group is just $\mathrm{Aut}_s(H)$ as defined previously.

8.4.7. Example. If we take $H = \mathrm{SU}(2)$ and form $\mathrm{Aut}_s(H)$, we get a strict Lie 2-group with $G = \mathrm{SO}(3)$. Similarly, if we take H to be the multiplicative group of nonzero quaternions, $\mathrm{Aut}_s(H)$ is again a strict Lie 2-group with $G = \mathrm{SO}(3)$. This latter example is implicit in Thompson's work on 'quaternionic gerbes' [49].

8.4.8. Example. Suppose that $t\colon G \to H$ is a surjective homomorphism of Lie groups. Then there exists a Lie crossed module (G, H, t, α) if and only if t is a central extension (that is, the kernel of t is contained in the center of G). Moreover, when this Lie crossed module exists it is unique.

8.4.9. Example. Suppose that V is a finite-dimensional real vector space equipped with an antisymmetric bilinear form $\omega\colon V \times V \to \mathbb{R}$. Make $H = V \oplus \mathbb{R}$ into a Lie group with the product

$$(v,\alpha)(w,\beta) = (v + w, \alpha + \beta + \omega(v,w)).$$

This Lie group is called the 'Heisenberg group'. Let G be V thought of as a Lie group, and let $t\colon H \to G$ be the surjective homomorphism given by

$$t(v,\alpha) = v.$$

Then t is a central extension, so by Example 8.4.8 we obtain a 2-group which we call the **Heisenberg 2-group** of (V,ω).

8.5. 2-Groups from Chern–Simons theory. We conclude by presenting some interesting examples of 2-groups built using Chern–Simons theory. Since the existence of these 2-groups was first predicted using an analogy between the classifications of 2-groups and Lie 2-algebras, we begin by sketching this analogy. We then describe some nonstrict Lie 2-algebras discussed in the companion paper HDA6, and use this analogy together with some results from Chern–Simons theory to build corresponding 2-groups. Naively,

one would expect these to be Lie 2-groups. However, we prove a 'no-go theorem' ruling out the simplest ways in which this could be true.

The paper HDA6 studies 'semistrict Lie 2-algebras'. These are categorified Lie algebras in which the Jacobi identity has been weakened, but not the antisymmetry of the bracket. A bit more precisely, a semistrict Lie 2-algebra is a category in Vect, say L, equipped with an antisymmetric bilinear functor called the 'bracket':

$$[\cdot,\cdot]\colon L \times L \to L,$$

together with a natural isomorphism called the 'Jacobiator':

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y]$$

satisfying certain coherence laws of its own.

HDA6 gives a classification of semistrict Lie 2-algebras that perfectly mirrors the classification of 2-groups summarized in Corollary 8.3.8 above, but with Lie algebras everywhere replacing groups. Namely, there is a 1-1 correspondence between equivalence classes of semistrict Lie 2-algebras L and isomorphism classes of quadruples $(\mathfrak{g}, \mathfrak{h}, \rho, [j])$ consisting of:

- a Lie algebra $\mathfrak{g}$,
- an abelian Lie algebra $\mathfrak{h}$,
- a representation ρ of $\mathfrak{g}$ as derivations of $\mathfrak{h}$,
- an element $[j]$ of the Lie algebra cohomology group $H^3(\mathfrak{g}, \mathfrak{h})$.

Here $\mathfrak{g}$ is the Lie algebra of objects in a skeletal version of L, $\mathfrak{h}$ is the Lie algebra of endomorphisms of the zero object of L, the representation ρ comes from the bracket in L, and the 3-cocycle j comes from the Jacobiator. Of course, an abelian Lie algebra is nothing but a vector space, so it adds nothing to say that in the representation ρ elements of $\mathfrak{g}$ act 'as derivations' of $\mathfrak{h}$. We say this merely to make the analogy to Corollary 8.3.8 as vivid as possible.

Recall that in the classification of 2-groups, the cohomology class $[a] \in H^3(G, H)$ comes from the associator in a skeletal version of the 2-group in question. In fact, this class is the only obstruction to finding an equivalent 2-group that is both skeletal and strict. The situation for Lie 2-algebras is analogous: the cohomology class $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$ comes from the Jacobiator, and gives the obstruction to finding an equivalent Lie 2-algebra that is both skeletal and strict.

Using this, in HDA6 we construct some Lie 2-algebras that are not equivalent to skeletal strict ones. Suppose G is a connected and simply-connected compact simple Lie group, and let $\mathfrak{g}$ be its Lie algebra. Let ρ be the trivial representation of $\mathfrak{g}$ on $\mathfrak{u}(1)$, the 1-dimensional abelian Lie algebra over the reals. Then

$$H^3(\mathfrak{g}, \mathfrak{u}(1)) \cong \mathbb{R}.$$

By the classification of Lie 2-algebras, for any value of $\hbar \in \mathbb{R}$ we obtain a skeletal Lie 2-algebra $\mathfrak{g}_\hbar$ having $\mathfrak{g}$ as its Lie algebra of objects and $\mathfrak{u}(1)$ as the endomorphisms of its zero object. When $\hbar = 0$ this Lie 2-algebra is just $\mathfrak{g}$ with identity morphisms adjoined to make it into a strict Lie 2-algebra. However, when $\hbar \neq 0$, this Lie 2-algebra is not equivalent to a skeletal strict one.

An interesting question is whether these Lie 2-algebras have corresponding Lie 2-groups. There is not a general construction of Lie 2-groups from Lie 2-algebras, but we can try to build them 'by hand'. We begin by seeking a skeletal 2-group $G_\hbar$ with G as its group of objects and $\mathrm{U}(1)$ as the automorphism group of its identity object, which is strict only at $\hbar = 0$. To define the associator in $G_\hbar$, we would like to somehow 'exponentiate' the element of $H^3(\mathfrak{g}, \mathfrak{u}(1))$ coming from the Jacobiator in $\mathfrak{g}_\hbar$ to obtain an element of $H^3(G, \mathrm{U}(1))$. However, from experience with affine Lie algebras and central extensions of loop groups, we expect this to be possible only for elements of $H^3(\mathfrak{g}, \mathfrak{u}(1))$ satisfying some sort of integrality condition.

Indeed this is the case: sitting inside the Lie algebra cohomology $H^3(\mathfrak{g}, \mathfrak{u}(1)) \cong \mathbb{R}$ there is a lattice Λ, which we can identify with $\mathbb{Z}$, that comes equipped with an inclusion

$$\iota \colon \Lambda \hookrightarrow H^3(G, \mathrm{U}(1)).$$

This is actually a key result from the papers of Chern–Simons [19] and Cheeger–Simons [18] on secondary characteristic classes. We describe how this inclusion is constructed below, but for now we record this:

8.5.1. Theorem. *Let G be a connected and simply-connected compact simple Lie group. Then for any $\hbar \in \mathbb{Z}$ there exists a special 2-group $G_\hbar$ having G as its group of objects, $\mathrm{U}(1)$ as the group of endomorphisms of its unit object, the trivial action of G on $\mathrm{U}(1)$, and $[a] \in H^3(G, \mathrm{U}(1))$ given by $\iota(\hbar)$. The 2-groups $G_\hbar$ are inequivalent for different values of $\hbar$, and strict only for $\hbar = 0$.*

To give more of a feeling for this result, let us sketch how the lattice Λ and the map ι can be constructed. Perhaps the most illuminating approach uses this commutative diagram:

$$\begin{array}{ccccc}
 & & & & H^{2n-1}(G, \mathrm{U}(1)) \\
 & & & & \uparrow{\scriptstyle \kappa} \\
 & & H^{2n}_{\mathrm{top}}(BG, \mathbb{R}) & \xleftarrow{\iota_{BG}} & H^{2n}_{\mathrm{top}}(BG, \mathbb{Z}) \\
 & & \downarrow{\scriptstyle \tau_{\mathbb{R}}} & & \downarrow{\scriptstyle \tau_{\mathbb{Z}}} \\
H^{2n-1}(\mathfrak{g}, \mathfrak{u}(1)) & \xleftarrow{\sim} & H^{2n-1}_{\mathrm{top}}(G, \mathbb{R}) & \xleftarrow{\iota_G} & H^{2n-1}_{\mathrm{top}}(G, \mathbb{Z})
\end{array}$$

In this diagram, the subscript 'top' refers to the cohomology of the compact simple Lie group G or its classifying space BG as a topological space. The integral cohomology $H^{2n-1}_{\mathrm{top}}(G, \mathbb{Z})$ maps to a lattice in the vector space $H^{2n-1}_{\mathrm{top}}(G, \mathbb{R})$, and thus defines a lattice Λ inside the isomorphic vector space $H^{2n-1}(\mathfrak{g}, \mathfrak{u}(1))$. In the case relevant here, namely

$n = 2$, the maps labelled τ are isomorphisms and the maps labelled ι and κ are injections. Thus, in this case the diagram serves to define an injection

$$\iota \colon \Lambda \hookrightarrow H^3(G, \mathrm{U}(1)).$$

Let us say a few words about the maps in this diagram. The isomorphism from $H^n_{\mathrm{top}}(G, \mathbb{R})$ to $H^n(\mathfrak{g}, \mathfrak{u}(1))$ is defined using deRham theory: there is a cochain map given by averaging differential forms on G to obtain left-invariant forms, which can be identified with cochains in Lie algebra cohomology [20]. Since the classifying space has $\Omega(BG) \simeq G$, there are 'transgression' maps $\tau_{\mathbb{Z}}$ and $\tau_{\mathbb{R}}$ from the $2n$th integral or real cohomology BG to the $(2n-1)$st cohomology of G. These are isomorphisms for $n = 2$, since in general the transgression map $\tau \colon H^{q+1}_{\mathrm{top}}(X, R) \to H^q_{\mathrm{top}}(\Omega X, R)$ is an isomorphism whenever X is k-connected, $q \leq 2k-1$ and the coefficient ring R is a principal ideal domain [52]. Finally, the change–of–coefficient maps ι_{BG} and ι_G map the integral cohomology of either of these spaces to a full lattice in its real cohomology. The universal coefficient theorem implies ι_G is an injection for $n = 2$ because the 3rd integral cohomology of a compact simple Lie group is torsion-free, in fact $\mathbb{Z}$. Similarly, ι_{BG} is an injection because $H^4_{\mathrm{top}}(BG, \mathbb{Z}) \cong \mathbb{Z}$.

The innovation of Chern, Cheeger and Simons was the homomorphism κ, which maps elements of $H^{2n}_{\mathrm{top}}(BG, \mathbb{Z})$ to certain elements of $H^{2n-1}(G, \mathrm{U}(1))$ called 'secondary characteristic classes'. This is where some differential geometry enters the story. For ease of exposition, we describe this map only in the case we need, namely $n = 2$. In this particular case we only need to say what κ does to the standard generator of $H^4_{\mathrm{top}}(BG, \mathbb{Z})$, which is called the 'second Chern class' c_2.

Since BG is the classifying space for principal G-bundles, any principal G-bundle P over a smooth manifold M gives a homotopy class of maps $M \to BG$, which we can use to pull back c_2 to an element of $H^4_{\mathrm{top}}(M, \mathbb{Z})$. Chern showed that the corresponding element of $H^4_{\mathrm{top}}(M, \mathbb{R})$ can be described using deRham theory by choosing an arbitrary connection A on P. We can think of this connection as $\mathfrak{g}$-valued 1-form on P, and its curvature

$$F = dA + A \wedge A$$

as an $\mathfrak{g}$-valued 2-form. This allows us to define a 4-form on P:

$$c_2(A) = \frac{1}{8\pi^2} \mathrm{tr}(F \wedge F).$$

where 'tr' is defined using a suitably normalized invariant bilinear form on $\mathfrak{g}$. The 4-form $c_2(A)$ is the pullback of a unique closed 4-form on M, which represents the image of $c_2(P)$ in $H^4_{\mathrm{top}}(M, \mathbb{R})$.

While the 4-form down on M is merely closed, Chern and Simons noted that $c_2(A)$ itself is actually exact, being the differential of this 3-form:

$$\mathrm{CS}_2(A) = \frac{1}{8\pi^2} \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

If the connection A is flat, meaning $F = 0$, then $\mathrm{CS}_2(A)$ is closed. It thus represents an element of $H^3_{\mathrm{top}}(P, \mathbb{R})$. This element is not canonically the pullback of an element of $H^3_{\mathrm{top}}(M, \mathbb{R})$, but it is up to an integral cohomology class.

As a consequence, $\mathrm{CS}_2(A)$ canonically gives rise to an element of $H^3_{\mathrm{top}}(M, \mathbb{R}/\mathbb{Z}) \cong H^3_{\mathrm{top}}(M, \mathrm{U}(1))$ for any principal G-bundle with flat connection over M. Note however that a principal G-bundle with flat connection is the same as a principal G_δ-bundle, where G_δ is the group G equipped with the discrete topology. Since our assignment of cohomology classes to manifolds equipped with principal G_δ-bundle is functorial, it must be a characteristic class: in other words, it must come from pulling back some element of $H^3_{\mathrm{top}}(BG_\delta, \mathrm{U}(1))$ along the classifying map $M \to BG_\delta$. But $H^3_{\mathrm{top}}(BG_\delta, \mathrm{U}(1))$ is just another way of talking about the group cohomology $H^3(G, \mathrm{U}(1))$. Thus we obtain an element $\mathrm{CS}_2 \in H^3(G, \mathrm{U}(1))$.

Since the second Chern class generates $H^4_{\mathrm{top}}(BG, \mathbb{Z})$, we can define

$$\kappa \colon H^4_{\mathrm{top}}(BG, \mathbb{Z}) \to H^3(G, \mathrm{U}(1))$$

by

$$\kappa(c_2) = \mathrm{CS}_2.$$

One can show that κ is an injection by explicit calculations [31].

It would be natural to hope the 2-groups $G_\hbar$ are Lie 2-groups and therefore topological 2-groups. However, we shall conclude with a 'no-go theorem' saying that $G_\hbar$ can be made into a topological 2-group with a reasonable topology only in the trivial case $\hbar = 0$. For this, we start by internalizing the cohomological classification of special 2-groups given in Theorem 8.3.7. Suppose K is any category with finite products such that $K\mathrm{Grp}$ has finite limits. We discussed the concept of 'coherent 2-group in $K\mathrm{Cat}$' in Section 7. We now say what it means for such a 2-group to be 'special':

8.5.2. Definition. *A* **special 2-group** *C in $K\mathrm{Cat}$ is a coherent 2-group in $K\mathrm{Cat}$ for which:*

1. *its underlying category in K is* **skeletal***, meaning that the source and target morphisms $s, t \colon C_1 \to C_0$ are equal,*

2. *the equalizer of the morphisms $s \colon C_1 \to C_0$ and $C_1 \longrightarrow I \xrightarrow{i} C_0$ exists,*

3. *the left unit law ℓ, the right unit law r, the unit i and the counit e are identity natural transformations.*

Given a special 2-group C in $K\mathrm{Cat}$, we can obtain a quadruple (G, H, α, a) by internalizing the construction described in Section 8.3. We merely sketch how this works. The multiplication in C makes C_0 into a group in K, even if the associator is nontrivial, since C is skeletal. Let G be this group in K. Composition of morphisms makes the equalizer in item 2 into an abelian group in K, thanks to the Eckmann–Hilton argument. Let H be this abelian group in K. Conjugation in C gives an action α of G as automorphisms of H,

and the associator of C gives a morphism $a\colon G^3 \to H$. This morphism a is a normalized 3-cocycle in the cochain complex for **internal group cohomology**:

$$\hom(G^0, H) \xrightarrow{d} \hom(G^1, H) \xrightarrow{d} \hom(G^2, H) \xrightarrow{d} \cdots$$

where the differential is defined as usual for group cohomology. It thus defines an element $[a] \in H^3(G, H)$ of internal group cohomology. Conversely, given a quadruple (G, H, α, a) of this form, we can obtain a special 2-group in KCat.

We have been unable to show that every coherent 2-group in KCat is equivalent to a special one, or even a skeletal one. After all, to show this for $K = \mathrm{Set}$, we used the axiom of choice to pick a representative for each isomorphism class of objects in a given 2-group C. This axiom is special to Set, and fails in many other categories. So, the above cohomological description of special 2-groups in KCat may not yield a complete classification of coherent 2-groups in KCat. Nonetheless we can use it to obtain some information about the problem of making the 2-groups $G_\hbar$ into topological or Lie 2-groups.

To do this, we also need the concept of 'special homomorphisms' between special 2-groups in KCat:

8.5.3. Definition. *A* **special** *homomorphism $F\colon C \to C'$ is a homomorphism between special 2-groups such that F_0 is an identity morphism.*

Recall that KCatC2G is the 2-category of coherent 2-groups in KCat. By a straightforward internalization of Theorem 8.3.7 we obtain:

8.5.4. Proposition. *Suppose that K is a category with finite products. The 2-category* KCatC2G *has a sub-2-category* ***K*CatS2G** *for which the objects are special 2-groups, the morphisms are special homomorphisms between these, and the 2-morphisms are arbitrary 2-homomorphisms between those. There is a 1-1 correspondence between equivalence classes of objects in* KCatS2G *and isomorphism classes of quadruples $(G, H, \alpha, [a])$ consisting of:*

- *a group G in K,*
- *an abelian group H,*
- *an action α of G as automorphisms of H,*
- *an element $[a] \in H^3(G, H)$.*

Now we consider $K = \mathrm{Top}$. In this case the internal group cohomology is usually called 'continuous cohomology', and we shall denote it by $H^n_{\mathrm{cont}}(G, H)$ to avoid confusion.

8.5.5. Theorem. *Let G be a connected compact Lie group and H a connected abelian Lie group. Suppose C is a special topological 2-group having G as its group of objects and H as the group of endomorphisms of its unit object. Then the associator a of C has $[a] = 0$. Thus C is equivalent in* TopCatS2G *to a special topological 2-group that is strict.*

PROOF. The work of Hu [30], van Est [51] and Mostow [42] on continuous cohomology implies that $H^3_{\mathrm{cont}}(G, H)$ is trivial. We thus have $[a] = 0$, and the rest follows from Proposition 8.5.4.

For the sake of completeness we sketch the proof that $H^3_{\mathrm{cont}}(G, H) \cong \{0\}$. First we consider the case where H is a real vector space equipped with an arbitrary representation of G. For any continuous cocycle $f\colon G^n \to H$ with $n \geq 1$ there is a continuous cochain $F\colon G^{n-1} \to H$ given by

$$F(g_1, \ldots, g_{n-1}) = \int_G f(g_1, \ldots g_n) dg_n,$$

where the integral is done using the normalized Haar measure on G. A simple calculation shows that $dF = \pm f$. This implies that $H^n_{\mathrm{cont}}(G, H) \cong \{0\}$ for all $n \geq 1$.

In general, any action of G on a connected abelian Lie group H lifts uniquely to an action on the universal cover $\tilde{H}$, which is a real vector space. Any normalized continuous cochain $f\colon G^n \to H$ lifts uniquely to a normalized continuous cochain $\tilde{f}\colon G^n \to \tilde{H}$ for $n \geq 2$, since the n-fold smash product of G with itself is simply-connected in this case. Since $d\tilde{f} = \widetilde{df}$, this implies that $H^n_{\mathrm{cont}}(G, H) \cong H^n_{\mathrm{cont}}(G, \tilde{H}) \cong \{0\}$ for $n \geq 3$. ∎

Now suppose C is a topological 2-group whose underlying 2-group is isomorphic to a 2-group of the form $G_\hbar$ for some $\hbar \in \mathbb{Z}$. Then the objects of C form a topological group which is isomorphic as a group to G, but possibly with some nonstandard topology, e.g. the discrete topology. Similarly, the endomorphisms of its identity object form a topological group which is isomorphic as a group to $\mathrm{U}(1)$, but possibly with some nonstandard topology.

8.5.6. COROLLARY. *Let G be a connected and simply-connected compact simple Lie group. Suppose C is a topological 2-group whose underlying 2-group is isomorphic to $G_\hbar$ for some $\hbar \in \mathbb{Z}$. If the topological group of objects of C is isomorphic to G with its usual topology, and the topological group of endomorphisms of its identity object is isomorphic to* $\mathrm{U}(1)$ *with its usual topology, then $\hbar = 0$.*

PROOF. Given the assumptions, C is a special topological 2-group which fulfills the hypotheses of Theorem 8.5.5. It is thus equivalent in TopCatS2G to a strict special topological 2-group, so its underlying 2-group $G_\hbar$ is equivalent in C2G to a strict skeletal 2-group. By Theorem 8.5.1 this happens only for $\hbar = 0$. ∎

In rough terms, this means that for $\hbar \neq 0$, the 2-group $G_\hbar$ cannot be made into a topological 2-group with a sensible topology. However, we have not ruled out the possibility that it is *equivalent* to the underlying 2-group of some interesting topological 2-group, or even of some Lie 2-group. Another possibility is that the concept of Lie 2-group needs to be broadened to handle this case—perhaps along lines suggested by Brylinksi's paper on multiplicative gerbes [16, 17].

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