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SYMMETRIC MONOIDAL COMPLETIONS AND THE EXPONENTIAL PRINCIPLE AMONG LABELED COMBINATORIAL STRUCTURES.

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ABSTRACT. We generalize Dress and Müller's main result in [5]. We observe that their result can be seen as a characterization of free algebras for certain monad on the category of species. This perspective allows to formulate a general *exponential principle* in a symmetric monoidal category. We show that for any groupoid $\mathbf{G}$, the category $\widehat{!\mathbf{G}}$ of presheaves on the symmetric monoidal completion $!\mathbf{G}$ of $\mathbf{G}$ satisfies the exponential principle. The main result in [5] reduces to the case $\mathbf{G} = 1$. We discuss two notions of functor between categories satisfying the exponential principle and express some well known combinatorial identities as instances of the preservation properties of these functors. Finally, we give a characterization of $\mathbf{G}$ as a subcategory of $\widehat{!\mathbf{G}}$.

1. Introduction

Let $\mathbb{F}$ be the category of finite sets and functions and let $\mathbb{B}$ be the subgroupoid of $\mathbb{F}$ induced by bijections. The category of species is introduced in Section 1.2 of [10] as the category of functors from $\mathbb{B}$ to $\mathbb{F}$. To each species F there is an associated formal power series

$$F = |F0| + |F1|x + \ldots + |Fn|\frac{x^n}{n!} + \ldots = \sum_{n\geq 0} F\mathbf{n}\frac{x^n}{\mathbf{Aut}(\mathbf{n})}$$

where $\mathbf{Aut}(\mathbf{n})$ is the cardinality of the set of endomorphisms on $[1..n]$ and $F\mathbf{n}$ denotes the cardinality of $F[1..n]$. The assignment of series to species maps combinatorial constructions between species to operations between power series. For example, the category of species (denoted by **Joy** from now on) has a symmetric monoidal structure denoted by $\cdot$ which, at the level of power series, corresponds with the usual product of series. There is also a "derivative" functor $\partial : \mathbf{Joy} \to \mathbf{Joy}$ and a (non-symmetric) monoidal structure that corresponds to composition. The theory obtained allows to calculate with combinatorial objects as though they were power series while keeping a clear intuition of what is the combinatorial meaning of the resulting identities in the calculation. In the notes to Chapter 5 of his book [23], Stanley refers to the theory of species as the "most sophisticated combinatorial theory of power series composition" and refers to the book [3] for further

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information. We assume from now on that the reader is familiar with [10]. Using the basic ideas behind the definition of species a number of variations have been proposed in order to deal with different problems in combinatorics. For example, linear species [10, 15, 3], partitionals [21], permutationals [2], colored species [18], Möbius species [19], tensorial species [11] and species on digraphs [20]. In all these variations, there is a groupoid $\mathbf{G}$ of "combinations of indeterminates" and a category $\mathcal{E}$ of "coefficients" so that the functor category $\mathcal{E}^{\mathbf{G}}$ can be seen as a category of combinatorial interpretations of some kind of power series. Let us look at an example. Consider the groupoid $\mathbb{S}$ whose objects are finite sets equipped with a permutation and whose maps are bijections between the underlying sets that preserve the permutation. Let $x_1, x_2, \ldots$ be a numerable set of formal variables. For any permutation σ of a set U we define $p\sigma = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ where n is the cardinality of U and d_i is the number of cycles of length i in σ. We also say that σ is of class $\mathbf{d} = (d_1, d_2, \ldots)$. For any functor $F : \mathbb{S} \to \mathbb{F}$ we can associate the power series

$$F(x_1, x_2, \ldots) = \sum_{n\geq 0} \sum_{\mathbf{d}\vdash n} F\mathbf{d} \frac{p\mathbf{d}}{\mathbf{Aut}(\mathbf{d})} = \sum_{n\geq 0} \sum_{\mathbf{d}\vdash n} F\mathbf{d} \frac{x_1^{d_1} x_2^{d_2} \ldots}{d_1! 1^{d_1} d_2! 2^{d_2} \ldots}$$

where $\mathbf{d} = (d_1, d_2, \ldots)$ is a sequence of non-negative integers, the notation $\mathbf{d} \vdash n$ indicates that $d_1 + 2d_2 + \ldots = n$ and $F\mathbf{d}$ is the cardinality of the set given by F applied to a permutation σ of class $\mathbf{d}$ (it is clear that this is well defined). Functors $F : \mathbb{S} \to \mathbb{F}$ are called *permutationals* and they were introduced in [2].

In this paper we will work mainly with the categories, mentioning the associated power series mainly in the examples. But before we go into the results of the present paper, let us discuss Dress and Müller's work [5]. For any species F seen as a functor $F : \mathbb{B} \to \mathbb{B}$ a *DM-decomposition* is a natural transformation t from the functor

$$F \times F : \mathbb{B} \times \mathbb{B} \xrightarrow{F\times F} \mathbb{B} \times \mathbb{B} \xrightarrow{\times} \mathbb{B} \longrightarrow \mathbb{F}$$

to the functor

$$F.+ : \mathbb{B} \times \mathbb{B} \xrightarrow{+} \mathbb{B} \xrightarrow{F} \mathbb{B} \longrightarrow \mathbb{F}$$

such that for all finite sets U and V, $t_{U,V} : FU \times FV \to F(U + V)$ is mono and such that

(D1) for each finite set U and two binary ordered partitions (U_0, U_1) and (V_0, V_1) of U it holds that

$$t(FU_0 \times FU_1) \cap t(FV_0 \times FV_1) = t(t(FW_{00} \times FW_{01}) \times t(FW_{10} \times FW_{11}))$$

where $W_{ij} = U_i \cap V_j$.

Dress and Müller call a functor $F : \mathbb{B} \to \mathbb{B}$ *weakly decomposable* if it is not initial and has a DM-decomposition. For a DM-decomposition t on a functor F, Dress and Müller define a functor F_t whose value at 0 is empty and at non-empty U is given by the formula below

$$F_t U = FU \setminus \bigcup_{I,J} t(FI \times FJ)$$

where the union is taken over the non-empty I, J's such that $I + J = U$. We write $F(x)$ for the formal power series associated to F. For a weakly decomposable functor F with DM-decomposition t define $(F_t^{(0)})U = FU$ if $U = \emptyset$ and $(F_t^{(0)})U = \emptyset$ otherwise. Then define $F_t^{(k+1)}$ by recursion:

$$F_t^{(k+1)}U = \bigcup_{(U_0,U_1)\in \mathrm{Prt}_b U} t(F_t U_0 \times F_t^k U_1)$$

and define a functor F to be *decomposable* if it has a DM-decomposition that satisfies the following condition

(D2) For each finite set U the sets $F_t^{(0)}, F_t^{(1)}, F_t^{(2)}, \ldots$ are pairwise disjoint.

If we write $[n]$ for $\{1, \ldots, n\}$ and $F(x, y)$ for the formal power series below:

$$\sum_{n,k\geq 0} (F_t^{(k)}[n]) \frac{y^k x^n}{k!n!}$$

then we can state the main result (the *exponential principle*) in [5] as follows.

1.1. Theorem. *(Dress-Müller) If $F : \mathbb{B} \to \mathbb{B}$ is a weakly decomposable functor with decomposition t then:*

1. *the equation $F(x) = \exp(F_t(x))$ holds and*
2. *if t satisfies (D2) then $F(x, y) = \exp(yF_t(x))$.*

Dress and Müller apply their result to some enumeration problems arising in group theory and they show that their result implies that of Wilf (see [24], where more examples of enumeration problems solved by the exponential principle can be found). The purpose of the present paper is to generalize Theorem 1.1 to other variants of the theory of species. In order to do this we will first formulate an exponential principle in an arbitrary symmetric monoidal category and then prove that this principle holds in categories of presheaves over certain groupoids. The first item of Theorem 1.1 will be the instance of our result given by the terminal groupoid 1 while the second will follow from the case $1 + 1$. We now define the general exponential principle and outline the contents of the paper.

Let $(\mathbf{D}, \cdot, \mathtt{I})$ be a symmetric monoidal category with tensor $\cdot$, unit $\mathtt{I}$ and natural isos $r : F \cdot \mathtt{I} \to F$, $l : \mathtt{I} \cdot F \to F$, $a : (F \cdot G) \cdot H \to F \cdot (G \cdot H)$ and $c : F \cdot G \to G \cdot F$ satisfying the usual coherence conditions [6]. We denote by **CMon** the category of commutative monoids in $\mathbf{D}$. We say that an object F of $\mathbf{D}$ is *simple* if there exists a unique map $\mathtt{I} \to F$. We denote by **SCMon** the full subcategory of **CMon** given by those commutative monoids whose underlying object is simple (notice that the unit of a simple monoid is uniquely determined). We say that $(\mathbf{D}, \cdot, \mathtt{I})$ has *algebraic families* if the forgetful functor $\mathbf{SCMon} \to \mathbf{D}$ is monadic. In this case we will denote the relevant monad by $(\mathtt{E}, \mu, \eta)$.

Let $\delta : (F_0 \cdot F_1) \cdot (F_2 \cdot F_3) \to (F_0 \cdot F_2) \cdot (F_1 \cdot F_3)$ be some fixed composition of c's and a's that (naturally) swaps F_1 and F_2.

1.2. Definition. A *decomposition* is a simple commutative monoid $m : F \cdot F \to F$ such that the following diagram is a pullback.

$$\begin{tikzcd}
(F\cdot F)\cdot(F\cdot F) \arrow[r, "(m\cdot m).\delta"] \arrow[d, "m\cdot m"'] & F\cdot F \arrow[d, "m"] \\
F\cdot F \arrow[r, "m"'] & F
\end{tikzcd}$$

Let **Dec** be the full subcategory of **SCMon** induced by decompositions.

1.3. Definition. A symmetric monoidal category $(\mathbf{D}, \cdot, \mathtt{I})$ is said to *satisfy the exponential principle* if it has algebraic families and the equivalence $\mathbf{SCMon} \to \mathbf{Alg}_{\mathtt{E}}$ restricts to one $\mathbf{Dec} \to \mathbf{Kl}_{\mathtt{E}}$.

Let us introduce some examples. For any essentially small category $\mathbf{C}$, its symmetric monoidal completion $!\mathbf{C}$ can be described as follows. The objects of $!\mathbf{C}$ are collections $\{g_i\}_{i\in I}$ with I a finite set and $g_i \in \mathbf{C}$. A morphism $\{g_i\}_{i\in I} \to \{h_j\}_{j\in J}$ is given by a bijection $f : I \to J$ together with maps $\alpha_i : g_i \to h_{fi}$ in $\mathbf{C}$. Its monoidal structure $\sqcup$ is given by $\{g_i\}_{i\in I} \sqcup \{h_j\}_{j\in J} = \{t_k\}_{k\in I+J}$ where $k_{in_0 i} = g_i$ and $t_{in_1 j} = h_j$. The empty collection $\varnothing$ is the unit of this monoidal structure. (The obvious embedding $\mathbf{C} \to !\mathbf{C}$ induces, for every symmetric monoidal category $\mathbf{D}$, an equivalence between the category of symmetric monoidal functors $!\mathbf{C} \to \mathbf{D}$ and the category of functors $\mathbf{C} \to \mathbf{D}$.) Fix now an essentially small groupoid $\mathbf{G}$. It is clear that $!\mathbf{G}$ is also a groupoid. Consider now the presheaf topos $\widehat{!\mathbf{G}} = \mathbf{Set}^{(!\mathbf{G})^{\mathbf{op}}}$. The monoidal structure $(!\mathbf{G}, \sqcup, \varnothing)$ induces, via Day's construction [7], a symmetric monoidal (closed) structure on $\widehat{!\mathbf{G}}$ that we denote by $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$. Intuitively, $\widehat{!\mathbf{G}}$ is a category of formal power series and the monoidal structure corresponds to the product of series.

1.4. Proposition. *For any groupoid $\mathbf{G}$, the symmetric monoidal category $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$ satisfies the exponential principle. Moreover, the full subcategory of $\widehat{!\mathbf{G}}$ induced by the functors that take values in finite sets inherits the monoidal structure and the resulting symmetric monoidal category also satisfies the exponential principle.*

In Section 2 we prove Proposition 1.4 and in Section 3 we show that many variations of the theory of species in the literature (e.g. [21, 2, 18, 20]) appear as instances of this proposition. In particular, below Example 3.2 we show that item 1 of Theorem 1.1 follows from the case $\mathbf{G} = 1$. Below Example 3.5 we show that item 2 follows from the case $\mathbf{G} = 1 + 1$. Also in Section 3 we show that for every symmetric monoidal category $(\mathbf{D}, \cdot, \mathtt{I})$ satisfying the exponential principle and every C in $\mathbf{D}$, the slice $\mathbf{D}/(\mathtt{E}C)$ also satisfies the exponential principle. This provides some insight into our main examples and also an abstract tool to start discussing morphisms between categories satisfying the exponential principle.

If we think of $\mathtt{E}$ as a "categorified" exponential function $e^{(\text{-})}$ then we can think of monoidal categories with algebraic families as categorified exponential rings (see [17] for the achievements and prospects of exponential algebra). This perspective provides a reasonable notion of morphism between categories with algebraic families. For the sake of the following discussion fix symmetric monoidal categories $(\mathbf{D}, \cdot, \mathtt{I})$ and $(\mathbf{D}', \cdot', \mathtt{I}')$ and a symmetric strict (i.e. structure maps are isos) monoidal functor $\Theta : (\mathbf{D}, \cdot, \mathtt{I}) \to (\mathbf{D}', \cdot', \mathtt{I}')$ preserving simple objects. (From now on, every monoidal functor is symmetric and strict.) The functor Θ induces a functor $\mathbf{SCMon}(\mathbf{D}) \to \mathbf{SCMon}(\mathbf{D}')$ that commutes with the "underlying object" functors. Assume from now on that $\mathbf{D}$ and $\mathbf{D}'$ have algebraic families. So that there is a functor $\Theta_0 : \mathbf{Alg}_{\mathbf{E}} \to \mathbf{Alg}_{\mathbf{E}'}$ commuting with the forgetful functors of the monads $\mathtt{E}$ and $\mathtt{E}'$. By results in [9], there is a natural transformation $\lambda : \mathtt{E}'\Theta \to \Theta\mathtt{E}$.

1.5. Definition. We say that $\Theta : (\mathbf{D}, \cdot, \mathtt{I}) \to (\mathbf{D}', \cdot', \mathtt{I}')$ is an $\mathtt{E}$-*functor* if the associated $\lambda : \mathtt{E}'\Theta \to \Theta\mathtt{E}$ is an iso.

If we let $\mathtt{E}_0 : \mathbf{D} \to \mathbf{Alg}_{\mathtt{E}}$ be the free functor associated with $\mathtt{E}$ and $\mathtt{E}'_0$ that associated with $\mathtt{E}'$ then (again by results in [9]) Θ as above is an $\mathtt{E}$-functor if λ induces and iso $\Theta_0\mathtt{E}_0 \cong \mathtt{E}'_0\Theta$ in $\mathbf{Alg}_{\mathtt{E}}$.

Every map $f : C \to \mathtt{E}D$ induces, by post-composition with $\mu.(\mathtt{E}f)$, a functor $\mathbf{D}/(\mathtt{E}C)$ to $\mathbf{D}/(\mathtt{E}D)$. In Section 4 we show that the induced functor between slices is an $\mathtt{E}$-functor and we present some examples of well-known combinatorial identities arising as applications of $\mathtt{E}$-functors. This is very natural but it is useful to consider also a weaker notion.

1.6. Definition. We say that Θ is a *weak* $\mathtt{E}$-*functor* if the functor $\mathbf{Alg}_{\mathtt{E}} \to \mathbf{Alg}_{\mathtt{E}'}$ induced by Θ maps free algebras to free algebras.

If the exponential principle holds in $\mathbf{D}$ and $\mathbf{D}'$ then there is a simple sufficient condition to recognize weak $\mathtt{E}$-functors.

1.7. Corollary. *If* $(\mathbf{D}, \cdot, \mathtt{I})$ *and* $(\mathbf{D}', \cdot', \mathtt{I}')$ *satisfy the exponential principle and the functor* $\Theta : \mathbf{D} \to \mathbf{D}'$ *is monoidal and preserves simple objects and pullbacks then* Θ *is a weak* $\mathtt{E}$-*functor.*

Proof. The functor Θ preserves the pullback diagram defining decompositions. ∎

Let $\mathbf{D}$ as above satisfy the exponential principle and let $f : C \to \mathtt{E}D$ be a map. If $\mathbf{D}$ has pullbacks, the monoidal functor $\mathbf{D}/(\mathtt{E}C) \to \mathbf{D}/(\mathtt{E}D)$ induced by f has a right adjoint. It is well known (see [12]) that the right adjoint must be monoidal and it is easy to see that it preserves simple objects. Of course, it preserves pullbacks so it is a weak $\mathtt{E}$-functor. We discuss these examples in more detail also in Section 4.

Finally, in Section 5, we give a purely categorical characterization of the image of the embedding $\mathbf{y} : \mathbf{G} \to \widehat{!\mathbf{G}}$. This will have no application in the present work but we believe it is relevant for further developments. Recall the notion of *strength* of a functor as reviewed, for example, in [8]. For any strong functor T with strength $\mathrm{str}_l : TF \cdot G \to T(F \cdot G)$ let $\mathrm{str}_r : F \cdot TG \to T(F \cdot G)$ be $(Tc).\mathrm{str}_l.c$.

1.8. Definition. Assume that $\mathbf{D}$ has finite coproducts. We say that a strong functor $\partial : \mathbf{D} \to \mathbf{D}$ *satisfies the Leibniz rule* if $[\mathrm{str}_l, \mathrm{str}_r] : (\partial F \cdot G) + (F \cdot \partial G) \to \partial(F \cdot G)$ and the unique $! : 0 \to \partial I$ are isomorphisms.

For example, the functor $\partial : \mathbf{Joy} \to \mathbf{Joy}$ defined by $(\partial F)U = F(1 + U)$ has an evident strength that satisfies the Leibniz rule. Another example is given by the pointing operation $(\_)^\bullet : \mathbf{Joy} \to \mathbf{Joy}$ defined by $F^\bullet U = U \times FU$. The first example motivates the following. Let D be an object in $\mathbf{D}$ and assume that the functor $D \cdot (\_)$ has a right adjoint. Then the right adjoint has a canonical strength induced, essentially, by the counit of the adjunction.

1.9. Definition. Assume that $\mathbf{D}$ has finite colimits. An object $\mathbf{x}$ of $\mathbf{D}$ is called an *infinitesimal* if the functor $\mathbf{x} \cdot (\_)$ has a right adjoint $\partial_{\mathbf{x}}$ that preserves finite colimits and satisfies Leibniz rule with respect to the canonical strength. We say that the infinitesimal $\mathbf{x}$ is *amazing* if $\partial_{\mathbf{x}}$ has a further right adjoint.

The characterization can then be stated as follows.

1.10. Proposition. *For any groupoid $\mathbf{G}$, the embedding $\mathbf{G} \to !\mathbf{G} \to \widehat{!\mathbf{G}}$ is equivalent to the full subcategory of $\widehat{!\mathbf{G}}$ induced by the infinitesimals in $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$. Moreover, every infinitesimal in $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$ is amazing.*

If $\mathbf{D}$ satisfies the exponential principle and $\partial : \mathbf{D} \to \mathbf{D}$ is a strong functor then we say that ∂ is $\mathtt{E}$-*strong* if the map

$$\partial F \cdot \mathtt{E}F \xrightarrow{\mathrm{str}_l} \partial(F \cdot \mathtt{E}F) \xrightarrow{\partial(\eta \cdot id)} \partial(\mathtt{E}F \cdot \mathtt{E}F) \longrightarrow \partial \mathtt{E}F$$

is an isomorphism. Again, lifting ideas from exponential algebra (see Definition 4 in [17]) we can define an infinitesimal $\mathbf{x}$ to be an $\mathtt{E}$-*infinitesimal* if the associated strong functor $\partial_{\mathbf{x}}$ is $\mathtt{E}$-strong. We will not use this idea but it seems relevant to mention that after the proof of Proposition 1.10 it will be clear that the infinitesimals in the statement are $\mathtt{E}$-infinitesimals.

Other ways of generalizing the exponential formula are Stanley's *exponential structures* [23] and Bender and Goldman's *prefabs* [1]. See also [13, 25]. The precise connections between these and the approach of this paper will have to be treated elsewhere.

2. Proof of Proposition 1.4

The fact that $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$ has algebraic families is essentially an instance of a very general result involving categories of algebras for monads induced by (symmetric) operads in cocomplete symmetric monoidal closed categories (see e.g. [14]). We give an indication of this in the second paragraph. We are not aware of general results involving Kleisli categories for these monads so we give a concrete proof that the exponential principle holds. We do this with the help of explicit descriptions of the tensor $\cdot$ on $\widehat{!\mathbf{G}}$ and of the monad $\mathtt{E}$. With these descriptions, readers not familiar with operads will be able to convince themselves that $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$ has algebraic families. An extra advantage of the explicit presentation is that it will then be clear that the exponential principle holds also on the subcategory of $\widehat{!\mathbf{G}}$ induced by the functors valued in finite sets.

Let $(\mathcal{E}, \otimes, I)$ be a cocomplete symmetric monoidal closed category and let $\mathbb{B}$ be the groupoid of finite sets and bijections. The category of functors $\mathcal{E}^{\mathbb{B}}$ can be equipped with a (non-symmetric) monoidal structure with tensor denoted by $\circ$ and unit J. A monoid for this tensor is called an *operad* and each operad induces a monad on $\mathcal{E}$ (see [14]). The (functor underlying the) monad M induced by the 'trivial' operad $J \circ J \to J$ can be described by the coend

$$M(E \in \mathcal{E}) = \int^{n \in \mathbb{B}} E^{\otimes n}$$

and the category of algebras for M is equivalent to the category $\mathbf{CMon}(\mathcal{E}, \otimes, I)$ of commutative monoids in $\mathcal{E}$. As $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$ is cocomplete symmetric monoidal closed, the forgetful $\mathbf{CMon}(\widehat{!\mathbf{G}}) \to \widehat{!\mathbf{G}}$ is monadic. The fact that $\mathbf{SCMon}(\widehat{!\mathbf{G}}) \to \widehat{!\mathbf{G}}$ is monadic will follow easily once we have described the monads involved more explicitely.

2.1. Algebraic families. The monoidal structure $(!\mathbf{G}, \sqcup, \varnothing)$ is lifted to $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$ by defining

$$(F \cdot G)U = \int^{c,d \in !\mathbf{G}} !\mathbf{G}(U, c \sqcup d) \times Fc \times Gd$$

and $\mathtt{I} = !\mathbf{G}(\_, \varnothing)$. (See [7] for details.) It will be convenient to simplify this formula. Before we do that, notice that from the explicit description of the unit, it is easy to characterize the simple objects as those $F \in \widehat{!\mathbf{G}}$ such that $F\varnothing = 1$.

2.2. Definition. Let $U = \{g_i\}_{i \in I}$ be an object of $!\mathbf{G}$. For $n \geq 0$, an *n-partition* of U is a sequence $(I_1, \ldots, I_n)$ of subsets of I such that its components are pairwise disjoint and whose union equals I.

For $n \geq 0$, each n-partition $(I_1, \ldots, I_n)$ determines an n-tuple of objects $(U_1, \ldots, U_n)$ where $U_k = \{g_i\}_{i \in I_k}$. We will generally confuse an n-partition with the induced sequence of objects. Only $\varnothing$ has a 0-partition. It is the empty sequence $()$ and it determines the object $\varnothing$. Let us denote by $\mathrm{Prt}_b U$ the set of binary partitions of U.

2.3. Lemma. *The map*

$$\sum_{(U_1, U_2)} FU_1 \times GU_2 \to (F \cdot G)U$$

(where the sum ranges over $\mathrm{Prt}_b U$*) injecting the coproduct into the coend is actually an isomorphism.*

PROOF. This can be proved by the usual description of the coend $(F \cdot G)U$ as a colimit (see [16]) and using that $!\mathbf{G}$ is a groupoid. ∎

2.4. DEFINITION. Let $U = \{g_i\}_{i \in I}$ be an object of $!\mathbf{G}$. A *partition* π of U is just a partition of the set I in the usual sense, that is, a set of non-empty subsets of I that are pairwise disjoint and whose union is I.

The object $\varnothing$ has exactly one partition: the empty set. Each partition of U determines an obvious set of objects in $!\mathbf{G}$. We may sometimes join and intersect such objects (using symbols $\sqcup$ and $\cap$). The meaning of this is clear when we understand the operations as acting on subsets of the indexing set underlying U. If π is a partition of U then for each p in π we may denote the induced object of $!\mathbf{G}$ by U_p. Let $\mathrm{Part}\, U$ be the set of partitions of U. Define a functor $\mathtt{E} : \widehat{!\mathbf{G}} \to \widehat{!\mathbf{G}}$ by the formula

$$(\mathtt{E}F)U = \sum_{\pi} \prod_{p \in \pi} FU_p$$

where the sum ranges over $\mathrm{Part}\, U$. Notice that $(\mathtt{E}F)\varnothing = 1$. There is a natural transformation $\eta : Id \to \mathtt{E}$ that for every F is defined as follows. When U is $\varnothing$, η is the unique $F\varnothing \to 1$. When U is not $\varnothing$, $\eta : FU \to (\mathtt{E}F)U$ maps an element x of FU to the unique partition of U with one element together with x. There is also a transformation $\mu : \mathtt{EE} \to \mathtt{E}$ whose explicit definition relies on the operation that takes a partition π of U together with a partition σ_p of p for each p in π and builds the evident, finer, partition of U given by the union of the σ_p's. We leave the details for the reader.

2.5. LEMMA. *The category of algebras for* $(\mathtt{E}, \eta, \mu)$ *is equivalent (over* $\widehat{!\mathbf{G}}$*) to the category* **SCMon**.

PROOF. It is easy to give a concrete proof using the explicit description of the monoidal structure given in Lemma 2.3. Alternatively, readers familiar with operads can derive most of the proof by comparing $\mathtt{E}$ with M. ∎

This finishes the sketch of the proof that $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$ has algebraic families. Consider now the full subcategory of those functors that take values in finite sets. From the explicit description of the tensor $\cdot$ it is easy to see that the subcategory inherits the tensor. Moreover, from the explicit description of $\mathtt{E}$ it is clear that the sub-monoidal-category has algebraic families.

2.6. The exponential principle. In this section we show that the category of decompositions of $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$ is equivalent (over $\widehat{!\mathbf{G}}$) to the category of free algebras for $\mathtt{E}$. For any algebra $\zeta : \mathtt{E}F \to F$ we define the subobject $\mathtt{L}(F, \zeta)$ of F by $(\mathtt{L}(F, \zeta))\varnothing = \emptyset$ and

$$(\mathtt{L}(F, \zeta))U = \{x \in FU \mid (\forall z \in (\mathtt{E}F)U)(\zeta z = x \to z = \eta x)\} \to FU$$

where U is not $\varnothing$. It is easy to see that $\mathtt{L}(F, \zeta)$ is a an object of $\widehat{!\mathbf{G}}$ and that we have a monomorphism $\mathtt{L}(F, \zeta) \to F$. We will denote this map by α_ζ or simply by α. Elements of $\mathtt{L}(F, \zeta)$ are called *connected.* Let x in FU and let $z \in (\mathtt{E}F)U$ be such that $\zeta z = x$. Let z be given by a partition π of U together with a family $\{x_p\}_{p \in \pi}$ with $x_p \in FU_p$. We say that z is a *splitting* of x if for every $p \in \pi$, x_p is connected. It is clear that every connected element has a unique splitting, namely, ηx. For the sake of the following statements let us call the unique element of $F\varnothing$ *trivial* and every other *non-trivial.*

2.7. Lemma. *For every algebra $\zeta : \mathtt{E}F \to F$, every non-trivial element has a splitting.*

Proof. If $U \in !\mathbf{G}$ is indexed by the singleton then the elements in FU are connected. Let $U = \{g_i\}_{i \in I}$ and let $x \in FU$. If x is connected then it has a unique splitting, if not, there exists an element z in $(\mathtt{E}F)U$, different from ηx, such that $\zeta z = x$. Let z be represented by a partition π of U together with a family $\{x_p\}_{p \in \pi}$. As π is a partition, each $U_p \in !\mathbf{G}$ is indexed by a set of cardinality less than that of the set indexing U. We can then use induction to obtain a splitting of x_p for each p. Joining these splittings we obtain a splitting of x. ∎

We can now characterize free algebras in terms of splittings.

2.8. Lemma. *An algebra is free if and only if every non-trivial element has a unique splitting.*

Proof. Let $\zeta : \mathtt{E}F \to F$ be an algebra. By Lemma 2.7, the maps

$$(\mathtt{E}\mathtt{L}(F, \zeta))U \xrightarrow{\mathtt{E}\alpha} (\mathtt{E}F)U \xrightarrow{\zeta} FU$$

are surjective. It is clear then that splittings are unique if and only if this map is an iso for each U. It is easy to show that this map is an algebra map from $\mu : \mathtt{E}\mathtt{E}\mathtt{L}(F, \zeta) \to \mathtt{E}\mathtt{L}(F, \zeta)$ to ζ. So, if splittings are unique then ζ is iso to a free algebra. On the other hand, it is easy to see that for any F and $U \neq \varnothing$, $\alpha : (\mathtt{L}(\mathtt{E}F, \mu))U \to (\mathtt{E}F)U$ is iso over $(\mathtt{E}F)U$ to $\eta : FU \to (\mathtt{E}F)U$. So $\mu.(\mathtt{E}\alpha) = \mu.(\mathtt{E}\eta) = id$ and hence splittings are unique in free algebras. ∎

Finally we need to relate decompositions and free algebras.

2.9. Lemma. *Let* $m : F \cdot F \to F$ *be a simple monoid and let* $\zeta : \mathtt{E}F \to F$ *be the associated algebra. Then* m *is a decomposition if and only if* ζ *is free.*

Proof. First assume that m is a decomposition. In order to prove that ζ is free it is enough, by Lemma 2.8, to prove that splittings are unique. So let $U = \{g_i\}_{i\in I}$ and x in FU be such that x has splittings given by a partition π and elements $\{x_p\}_{p\in\pi}$ and by a partition σ and elements $\{y_s\}_{s\in\sigma}$. Let U_k in π and V_l in σ be such that $U_k \cap V_l$ is non empty. We are going to show that $U_k = V_l$ and that $x_k = y_l$ which implies that the splittings are the same. Let U' be the complement of U_k in I. Similarly, let V' be the complement of V_l. The splittings induce an x' in FU' and a $y' \in FV'$ such that $m(x', x_k) = x = m(y', y_l)$. The condition defining decompositions then says that there is a unique 4-partition (W_0, W_1, W_2, W_3) of U together with elements $a \in FW_0$, $b \in FW_1$, $c \in FW_2$ and $d \in FW_3$ such that $m(a, b) = x'$, $m(a, c) = y'$, $m(c, d) = x_k$ and $m(b, d) = y_l$. It follows that $W_0 \sqcup W_1 = U'$, $W_2 \sqcup W_3 = U_k$, $W_0 \sqcup W_2 = V'$ and that $W_1 \sqcup W_4 = V_l$. In turn, this implies, that $W_0 = U' \cap V'$, $W_1 = U' \cap V_l$, $W_2 = U_k \cap V'$ and $W_3 = U_k \cap V_l$. As y_l is connected and $U_k \cap V_l$ is non-empty then it must be the case that $U' \cap V_l$ is empty and hence that $U_k = V_l$ and $d = y_l$. We then must have $U_k \cap V'$ empty and $x_k = d = y_l$.

On the other hand, if every element x has a unique splitting it is not difficult to show that the axiom for decompositions (Definition 1.2) follows. ∎

It follows that **Dec** is equivalent over $\widehat{!\mathbf{G}}$ to the Kleisli category of $\mathtt{E}$. So the exponential principle holds. It is clear that the proof works for functors valued in finite sets so the proof of Proposition 1.4 is finished. The relation with Dress and Müller's result will be discussed below Examples 3.2 and 3.5 below.

3. Some examples

The first example is a trivial case and the rest have been taken from the literature (in some cases, slightly modified to have sets as coefficients). The tensor $\cdot$ induced by Proposition 1.4 always corresponds, in each example, with the basic symmetric tensor that "explains" the combinatorics of the product of series modeled by the variant of species that motivates the example. This can be seen by checking the relevant reference and comparing with Lemma 2.3.

3.1. Example. (Trivial case with extensive coproducts.) Let $\mathbf{C}$ be an extensive category and consider the symmetric monoidal category $(\mathbf{C}, +, 0)$. In this case every object is simple and every monoid $F + F \to F$ is forced to be of the form $[id, id] : F + F \to F$ by the unit axioms. Also, every map is a map of monoids and hence the category of monoids is equivalent to $\mathbf{C}$. So in this case the forgetful functor $\mathbf{SCMon} \to \mathbf{C}$ is an isomorphism and hence trivially monadic. In other words, $(\mathbf{C}, +, 0)$ has algebraic families trivially. On the other hand, few objects are decompositions. Indeed, no finite coproduct of connected objects is a decomposition. So if $\mathbf{C}$ has a connected object then $(\mathbf{C}, +, 0)$ does not satisfy the exponential principle.

3.2. Example. (Joyal [10].) Let 1 be the trivial groupoid with one object and one map. Then $!1 = \mathbb{B}$ is equivalent to the groupoid of finite sets and bijections and $(\widehat{!1}, \cdot, \mathtt{I})$ is the monoidal category of species.

This is, of course, the case where Proposition 1.4 reduces to the first item of Theorem 1.1. For a weakly decomposable F, Dress and Müller show (using that t has monic components) that $F0 = 1$ and that a DM-decomposition t is commutative and associative. In order to relate DM-decompositions with decompositions as defined in this paper it seems more natural to start slightly differently. Consider natural transformations $t : F \times F \to F.+$ as above and notice that we can define a map $m : F \cdot F \to F$ as follows. For each binary ordered partition (U_0, U_1) of U we have the map

$$FU_0 \times FU_1 \xrightarrow{t} F(U_0 + U_1) \xrightarrow{\cong} FU$$

and by the universal property of coproducts we obtain a map $m : F \cdot F \to F$. If t is commutative and associative then m will be a commutative monoid. On the other hand, if we start with a commutative monoid m we can build a natural t as follows. For any pair U_0, U_1 of finite sets consider the binary ordered partition (U_0, U_1) of the finite set $U_0 + U_1$. If we define t at U_0, U_1 as the composition below

$$FU_0 \times FU_1 \longrightarrow (F \cdot F)(U_0 + U_1) \xrightarrow{m} F(U_0 + U_1)$$

then it is easy to see that t is a commutative and associative natural transformation. Moreover, it is easy to see that this establishes a correspondence between commutative monoids and such natural transformations. Simple objects are exactly the objects such that $F0 = 1$. Finally, it is easy to show that the commutative and associative natural transformations t satisfying (D1) on a simple object F are in correspondence with decompositions (in the sense of Definition 1.2) of F (the mono requirement of DM-decompositions is taken care of by the uniqueness of splittings stated in Lemma 2.8). In other words, DM-decompositions are exactly decompositions in the monoidal category (with tensor $\cdot$) of functors from $\mathbb{B}$ to finite sets. The functor $\mathtt{E}$ corresponds, at the level of power series, to the function $e^{(\text{-})}$ and this is why the first item of Theorem 1.1 is implied by the case $\mathbf{G} = 1$ of Proposition 1.4. (The assignment of F_t to $t \mapsto F_t$ in [5] corresponds to what we have called $\mathtt{L}$.)

3.3. Example. (Nava and Rota [21].) Consider the groupoid $\mathbb{U}$ of non-empty finite sets and bijections. It is easy to show that its completion $!\mathbb{U}$ is equivalent to the groupoid $\mathbb{P}$ whose objects are sets equipped with a partition and whose maps are bijections between the underlying sets that preserve components of the respective partitions. So that $(\widehat{!\mathbb{U}}, \cdot, \mathtt{I})$ is equivalent to the monoidal category of *partitionals*.

3.4. Example. (Joyal [10], Bergeron [2].) Let $\mathbb{C}$ be the groupoid of cyclic permutations and isomorphisms of permutations. Then $!\mathbb{C}$ is equivalent to the groupoid $\mathbb{S}$ of finite sets equipped with a permutation and isomorphisms between them. So the monoidal category $(\widehat{!\mathbb{C}}, \cdot, \mathtt{I})$ is equivalent to that of *permutationals*. Compare with Section 3 in [10].

3.5. Example. (Joyal [10], Méndez and Nava [18].) Let I be a non-empty set and think of it as a set of colors. If we take I as a discrete groupoid then the groupoid $!I$ is equivalent to the category $\mathbb{B}_I$ introduced in [18] whose objects are pairs (E, f) where E is a finite set and f is a function $E \to I$ (a coloration). Maps in $\mathbb{B}_I$ are bijections between the finite sets that preserve colorations. So $(\widehat{!I}, \cdot, \mathtt{I})$ is equivalent to the category of *I-colored species*. See also Section 5 in [10].

Consider the special case when I is the set $1+1$ with two elements and denote $\widehat{!(1+1)}$ by $\mathbf{Joy}_2$. We think of $F \in \mathbf{Joy}_2$ as a formal power series

$$F(x,y) = \ldots + F([n],[m])\frac{x^n y^m}{n!m!} + \ldots$$

in two variables x and y. For any G in $\mathbf{Joy}$ let G_x in $\mathbf{Joy}_2$ be defined by $G_x(U,V) = GU$ if $V = 0$ and empty otherwise. So that G_x has the following representing series.

$$G_x(x,y) = G0 + (G1)x + \ldots + (Gn)\frac{x^n}{n!} + \ldots$$

Let Y be the object of $\mathbf{Joy}_2$ such that has value 0 at each (U,V) except for $Y(0,1) = 1$. So that the representing series $Y(x,y)$ is just y. For any species G the series

$$(G0)y + (G1)yx + (G2)\frac{yx^2}{2} + \ldots + (Gn)\frac{yx^n}{n!} + \ldots$$

represents $Y \cdot G_x$ since this functor is GU at stage $(U,1)$ and empty otherwise. We will explain item 2 of Theorem 1.1 as an application of the exponential principle in $\mathbf{Joy}_2$. For any G in $\mathbf{Joy}$ such that $G\varnothing = \emptyset$ define ΘG in $\mathbf{Joy}_2$ as follows:

$$(\Theta G)(U,V) = \{\{x_p\}_{p\in\pi} \in (\mathtt{E}G)U \mid |\pi| = |V|\}$$

where $|\_|$ denotes cardinality. Joining disjoint families induces a (simple) commutative monoid structure on ΘG and it is easy to show that it is a decomposition. To calculate the connected components notice that if $\{x_p\}_{p\in\pi}$ in $(\Theta G)(U,V)$ is connected then V must be 1 and in this case π is the trivial partition with one element. Indeed, we obtain that $\mathtt{L}(\Theta G) = Y \cdot G_x$, which is the combinatorial content of item 2 of Theorem 1.1. (The proof in [5] consists of a formal calculation using partial derivatives.) Notice that condition (D2) is not needed. Compare also with Example 39 in [10].

3.6. Example. (Mendez [20].) Let I be a set (this time, thought of as a set of vertices) and consider the set $J = I \times I$ as a discrete groupoid. The groupoid $!J$ can be thought of as the groupoid of directed graphs with nodes in I and labeled edges. The monoidal category $(\widehat{!I}, \cdot, \mathtt{I})$ of *species on digraphs* was introduced loc. cit. using the machinery of colored species (Example 3.5). Mendez also considers the case where K is the set of unordered pairs of elements of I so that $!K$ is the groupoid of undirected graphs with nodes in I.

3.7. Example. Also in the context of colored species Mendez and Nava introduce a colored generalization of Bergeron's permutationals (Example 3.4). For a fixed set I, let $\mathbb{C}_I$ be the groupoid whose objects are cyclic permutations (U, σ) together with a constant function $U \to I$. The monoidal category of I-permutationals described in Section 3.2 of [18] is equivalent to $(\widehat{!\mathbb{C}_I}, \cdot, \mathtt{I})$.

3.8. Example. (Leroux and Viennot [15].) Let $\mathbb{L}$ be the groupoid of finite sets equipped with a linear order and monotone bijections between them. Consider the functor category $\widehat{\mathbb{L}}$ and define the monoidal structure $\cdot$ as below

$$(F \cdot G)(U, \leq) = \sum F(U_0, \leq_0) \times G(U_1, \leq_1)$$

where the sum ranges over the binary partitions (U_0, U_1) of U and $\leq_0$ and $\leq_1$ are the total orders induced by $\leq$ on the components of the partition. The unit I is as in the case of ordinary species. A functor valued on finite sets is called an $\mathbb{L}$-species by Leroux and Viennot who use this category to develop a combinatorial theory of differential equations. It is clear that the groupoid $\mathbb{L}$ is not a symmetric monoidal completion so we can not apply Proposition 1.4. But using similar ideas one can show that $(\widehat{\mathbb{L}}, \cdot, \mathtt{I})$ satisfies the exponential principle. It is interesting to note that Joyal describes in Section 4 of [10] a different monoidal structure on the category $\widehat{\mathbb{L}}$.

In a sense, Proposition 1.4 can be seen as abstracting from the case of **Joy** by generalizing the nature of the indeterminates in the associated power series. Other general results can probably be obtained by generalizing the nature of coefficients. In our cases these are sets (cardinalities) or finite sets (natural numbers). But consider for example the tensorial species introduced in [11] where coefficients are vector spaces or the Möbius species introduced in [19] where coefficients are families of certain posets.

For many variants of the theory of species there is a monoidal structure $\circ$ that corresponds, at the level of formal power series, to composition. For species, colored-species and linear species (in the sense of [15]), $\mathtt{E} = 1 \circ (\_)$. (See Remark 1.2 in [18] for the case of colored species.) But this is not the case for permutationals and partitionals. Also, in [10] there is a different monoidal structure $\circ$ among linear species, which corresponds also to composition among certain type of power series, but which does not satisfy $\mathtt{E} = 1 \circ (\_)$.

3.9. Slices. It is well known that every commutative monoid X in a monoidal category $(\mathbf{D}, \cdot, \mathtt{I})$ induces a symmetric monoidal structure $(\mathbf{D}/X, \cdot, I)$ on the slice $\mathbf{D}/X$ and that the forgetful $\mathbf{D}/X \to \mathbf{D}$ is monoidal. Moreover, the category $\mathbf{CMon}(\mathbf{D}/X)$ is equivalent, over $\mathbf{CMon}(\mathbf{D})$, to $\mathbf{CMon}(\mathbf{D})/X$. If X is simple then the functor $\mathbf{D}/X \to \mathbf{D}$ creates simple objects and so, the equivalence $\mathbf{CMon}(\mathbf{D}/X) \to \mathbf{CMon}(\mathbf{D})/X$ restricts to an equivalence $\mathbf{SCMon}(\mathbf{D}/X) \to \mathbf{SCMon}(\mathbf{D})/X$. Analogously, for any monad (M, η, μ) on $\mathbf{D}$ and algebra $\zeta : MX \to X$ there is a monad M/ζ on $\mathbf{D}/X$ such that the category of algebras $\mathbf{Alg}(M/\zeta)$ is equivalent, over $\mathbf{Alg}(M)$, to $\mathbf{Alg}(M)/\zeta$. (The monad M/ζ assigns $\zeta.(Mt)$ to each object $t : A \to X$ in $\mathbf{D}/X$.) Moreover, if ζ is a free algebra $\mu : MMA \to MA$ then the Kleisli category $\mathbf{Kl}(M/\mu)$ is equivalent over $\mathbf{Alg}(M)$ to $\mathbf{Kl}(M)/\mu$. Using these well known facts it is easy to prove the following.

3.10. Lemma. *Let $(\mathbf{D}, \cdot, \mathtt{I})$ be a symmetric monoidal category with algebraic families. For any simple commutative monoid $m : X \cdot X \to X$, the monoidal category $\mathbf{D}/m$ has algebraic families. Moreover, if $\mathbf{D}$ satisfies the exponential principle and m is a decomposition then $\mathbf{D}/m$ also satisfies the exponential principle.*

Proof. Let ζ be the $\mathtt{E}$-algebra related with m via the equivalence $\mathbf{SCMon} \cong \mathbf{Alg}_{\mathtt{E}}$. Then $\mathbf{SCMon}(\mathbf{D}/m) \cong \mathbf{SCMon}(\mathbf{D})/m \cong \mathbf{Alg}_{\mathtt{E}}/\zeta \cong \mathbf{Alg}_{\mathtt{E}/\zeta}$. Now, concerning decompositions, the functor $\mathbf{D}/X \to \mathbf{D}$ creates pullbacks (and recall that it is monoidal) so if m is a decomposition then a commutative monoid in $\mathbf{D}/X$ is a decomposition if and only if its image in $\mathbf{D}$ is a decomposition. So, if m is the decomposition corresponding to the free algebra $\mu : \mathtt{EE}A \to \mathtt{E}A$, we can calculate $\mathbf{Dec}(\mathbf{D}/m) \cong \mathbf{Dec}(\mathbf{D})/m \cong \mathbf{Kl}_{\mathtt{E}}/\mu \cong \mathbf{Kl}_{\mathtt{E}/\mu}$. In other words, $\mathbf{D}/(\mathtt{E}A)$ satisfies the exponential principle. ∎

When $\mathbf{D}$ is a category of the form $\widehat{!\mathbf{G}}$, it is possible to give a more explicit description of $\mathbf{D}/(\mathtt{E}A)$. Let us start with an arbitrary groupoid $\mathbf{H}$, and R in $\widehat{\mathbf{H}}$. Define an *R-structure* to be an element of RU for some U in $\mathbf{H}$ (see [10] 1.1). Let $u : V \to U$ be a map in $\mathbf{H}$ and let $r \in RU$, if $r' = (Ru)r$ then we say that u is a *morphism from r' to r*. We denote by $\mathrm{el}(R)$ the category (obviously a groupoid) of R-structures and morphisms between them. There is forgetful functor $\mathrm{el}(R) \to \mathbf{H}$ and every map $R \to R'$ in $\widehat{\mathbf{H}}$ induces a functor $\mathrm{el}(R) \to \mathrm{el}(R')$ that commutes with the forgetful functors. Proposition 2 in [10] also lifts to show that the slice category $\widehat{\mathbf{H}}/R$ is equivalent to $\widehat{\mathrm{el}(R)}$.

3.11. Lemma. *For any C in $\widehat{!\mathbf{G}}$, $\mathrm{el}(\mathtt{E}C) \cong !\mathrm{el}(C)$.*

Proof. An object of $\mathrm{el}(\mathtt{E}C)$ is given by an object $\{g_i\}_{i \in I}$ in $!\mathbf{G}$, a partition π of I and for every $p \in \pi$ an element $x_p \in Cp$. The collection $\{x_p\}_{p \in \pi}$ is an object in $!\mathrm{el}(C)$. It is not difficult to show that the induced functor $\mathrm{el}(\mathtt{E}C) \to !\mathrm{el}(C)$ is an equivalence. ∎

The equivalence $\mathrm{el}(\mathtt{E}C) \to !\mathrm{el}(C)$ induces an equivalence $\widehat{\mathrm{el}(\mathtt{E}C)} \to \widehat{!\mathrm{el}(C)}$ and so, an equivalence $\widehat{!\mathbf{G}}/(\mathtt{E}C) \to \widehat{!\mathrm{el}(C)}$. It is not difficult to show that it is a monoidal equivalence. We now look at some examples, slightly abusing notation in order to make clear the connection with the examples in Section 2.

3.12. Example. Let $\mathbb{U}$ in **Joy** be the subspecies of 1 obtained by requiring that $\mathbb{U}\varnothing = \emptyset$. The species $\mathtt{E}\mathbb{U}$ is the species $\mathbb{P}$ of partitions and the monoidal category $\mathbf{Joy}/\mathbb{P}$ is equivalent to that of partitionals (Example 3.3).

3.13. Example. Let $\mathbb{C}$ in **Joy** be the species of cyclic permutations. Then $\mathtt{E}\mathbb{C}$ is the species $\mathbb{S}$ of permutations and $\mathbf{Joy}/\mathbb{S}$ is equivalent to the category of permutationals (Example 3.4).

3.14. Example. For a fixed set I, let ε_I be the species such that $\varepsilon_I\varnothing = \emptyset$, $\varepsilon_I 1 = I$ and for every other finite set U, $\varepsilon_I U = \emptyset$. It is clear that $\mathbf{Joy}/(\mathtt{E}\varepsilon_I)$ is the monoidal category of I-colored species (Example 3.5). The categories of (di)graphical species (Example 3.6) are particular cases of this construction.

3.15. Example. Again for a fixed set I, let $\mathbb{C}_I U$ be the set of pairs (σ, f) such that σ is a cyclic permutation on U and $f : U \to I$ is a constant function. The monoidal category $\mathbf{Joy}/(\mathtt{E}\mathbb{C}_I)$ is that of I-permutationals (see Example 3.7).

3.16. Example. Let $\mathbb{L}$ be the species of total orders. Concatenation induces a (non-commutative) monoid $m : \mathbb{L} \cdot \mathbb{L} \to \mathbb{L}$. The slice category $\mathbf{Joy}/\mathbb{L}$ is the category described in Example 3.8. On the other hand, the monoidal structure described in that example is not the one induced by m.

Before we go on to discuss $\mathtt{E}$-functors let us mention that the terminal object in our main examples is in the image of $\mathtt{E}$. We will use this in the next section.

3.17. Lemma. *Let $\mathbf{G}$ be an essentially small groupoid. Let ε be the object in $\widehat{!\mathbf{G}}$ such that $\varepsilon U = 1$ if U is in $\mathbf{G}$ and it is empty otherwise. Then $\mathtt{E}\varepsilon = 1$ in $\widehat{!\mathbf{G}}$.*

Proof. Easy. ∎

4. Basic results on E-functors

Let $(\mathbf{D}, \cdot, \mathtt{I})$ satisfy the exponential principle and let C be an object of $\mathbf{D}$. We saw in Lemma 3.10 that the category $\mathbf{D}/(\mathtt{E}C)$ equipped with the monoidal structure induced by the decomposition μ_C satisfies the exponential principle. Any map $f : C \to \mathtt{E}D$ induces a functor $\mathbf{D}_f : \mathbf{D}/(\mathtt{E}C) \to \mathbf{D}/(\mathtt{E}D)$ that assigns to $t : X \to \mathtt{E}C$, the composition $\mu.(\mathtt{E}f).t : X \to \mathtt{E}D$.

4.1. Proposition. *Let $(\mathbf{D}, \cdot, \mathtt{I})$ be symmetric monoidal and satisfy the exponential principle. Then, for any map $f : C \to \mathtt{E}D$, the functor $\mathbf{D}_f : \mathbf{D}/(\mathtt{E}C) \to \mathbf{D}/(\mathtt{E}D)$ is an $\mathtt{E}$-functor.*

Proof. This is a simple calculation using the definitions of $\mathtt{E}/\mu$ and $\mathbf{D}_f$. ∎

When the monoidal category $\mathbf{D}$ is of the form $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$ for some groupoid $\mathbf{G}$, it is possible to give a more explicit description of the functor $\mathbf{D}_f$ along the equivalences $\widehat{!\mathbf{G}}/(\mathtt{E}C) \to \widehat{!\mathrm{el}(C)}$ and $\widehat{!\mathbf{G}}/(\mathtt{E}D) \to \widehat{!\mathrm{el}(D)}$. Indeed, the map $f : C \to \mathtt{E}D$ induces a functor $!\mathrm{el}(f) :!\mathrm{el}(C) \to !\mathrm{el}(\mathtt{E}D) \cong !!\mathrm{el}(D)$. The universal property of ! applied to the identity functor $!\mathrm{el}(D) \to !\mathrm{el}(D)$ provides a functor $!!\mathrm{el}(D) \to !\mathrm{el}(D)$ which can be post-composed with $!\mathrm{el}(f)$ to obtain a functor $\mathbf{f} :!\mathrm{el}(C) \to !\mathrm{el}(D)$. We then have $\mathbf{f}_! : \widehat{!\mathrm{el}(C)} \to \widehat{!\mathrm{el}(D)}$ defined by $(\mathbf{f}_! H)A = \sum HU$ for A in $!\mathrm{el}(D)$ and the sum ranging over the U's such that $\mathbf{f}U = A$. We now describe how some functors that have appeared in [10, 2, 20] arise as in Proposition 4.1.

4.2. Example. The functor $(\_)_x : \mathbf{Joy} \to \mathbf{Joy}_2$ used below Example 3.5 is an $\mathtt{E}$-functor because it is induced by the map $\eta.in_0 : \varepsilon \to \mathtt{E}(\varepsilon + \varepsilon)$.

4.3. Example. There is a unique map $\iota : \varepsilon \to \mathbb{C}$ in $\mathbf{Joy}$. Intuitively, it distinguishes the unique cycle on a one-element set. The map $\eta.\iota : \varepsilon \to \mathtt{E}\mathbb{C} = \mathbb{S}$ induces an $\mathtt{E}$-functor $\mathfrak{K} : \mathbf{Joy} \to \mathbf{Joy}/(\mathtt{E}\mathbb{C})$. At the level of elements the map $\eta.\iota : \varepsilon \to \mathtt{E}\mathbb{C}$ induces the functor $\mathbb{B} \to \mathbb{S}$ from the groupoid of finite sets to that of permutations that assigns to U the permutation (U, id). So that, for each species F, the functor $\mathfrak{K} : \mathbf{Joy} \to \widehat{\mathbb{S}}$ is defined by $(\mathfrak{K}F)(U, \sigma) = FU$ if $\sigma = \mathrm{id}$ and $(\mathfrak{K}F)(U, \sigma) = \emptyset$ otherwise.

4.4. Example. Consider now the map $\mathbb{C} \to 1 = \mathtt{E}\varepsilon$ in $\mathbf{Joy}$ which collapses cycles. This map induces an $\mathtt{E}$-functor $\mathfrak{U} : \mathbf{Joy}/(\mathtt{E}\mathbb{C}) \to \mathbf{Joy}$. At the level of elements the map $\mathbb{C} \to 1$ induces the functor $\mathbb{S} \to \mathbb{B}$ which assigns to each permutation (U, σ) its underlying set. So that $\mathfrak{U} : \widehat{\mathbb{S}} \to \mathbf{Joy}$ is explicitely defined by

$$(\mathfrak{U}T)A = \sum_{\sigma \in \mathbb{S}A} T\sigma$$

and it is explained in [2] that when one thinks of T as being the combinatorial interpretation of a power-series $t(x_1, x_2, x_3, \ldots)$ on an infinite set of variables $x_1, x_2, \ldots$ as in the introduction then $\mathfrak{U}$ corresponds to the assignment $t \mapsto t(x, x^2, x^3, \ldots)$.

4.5. Example. Recall the species $\mathbb{U}$ of non-empty sets and consider the map $\mathbb{C} \to \mathbb{U}$ that forgets cycles. At the level of elements the induced map $\mathtt{E}\mathbb{C} \to \mathtt{E}\mathbb{U}$ corresponds to the functor $\mathbb{S} \to \mathbb{P}$ that assigns to each permutation (U, σ) the partition that σ induces on U. In [2], Bergeron proposes to compare the categories of permutationals and of partitionals via the resulting functor $\mathfrak{S} : \widehat{\mathbb{S}} \to \widehat{\mathbb{P}}$ whose explicit definition is, for each permutational T and each partition π of a finite set U, $(\mathfrak{S}T)\pi = \sum T\sigma$ where the sum is taken over all the permutations σ whose induced partition is π.

In Section 6 of [10], Joyal introduces the *espèces pondérées* and uses them to give combinatorial interpretations of four well known formulas. We now show that Examples 36 and 38 in [10] can be seen as the application of an $\mathtt{E}$-functor to a terminal object. For any C in **Joy**, Lemma 3.17 implies that the terminal object in $\mathbf{Joy}/(\mathtt{E}C)$ is such that $1 = \mathtt{E}\varepsilon$ for some ε. So for any $\phi : C \to \mathtt{E}D$ in **Joy**, the induced $\mathtt{E}$-functor $\phi : \mathbf{Joy}/(\mathtt{E}C) \to \mathbf{Joy}/(\mathtt{E}D)$ satisfies $\phi 1 = \phi \mathtt{E}\varepsilon = \mathtt{E}\phi\varepsilon$.

4.6. Example. (Example 36 in [10].) Let $x_1, x_2, \ldots$ be a denumerable set of formal variables. For any permutation σ on a finite set U we define $p\sigma = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ where n is the cardinality of U and d_i is the number of σ-cycles of length i. The *cycle indicator polynomial* associated with U is the formal expression

$$Z_n = \frac{1}{n!} \sum_{\sigma \in \mathbb{S}U} p\sigma$$

where n is the cardinality of U and $\mathbb{S}U$ denotes the set of permutations on the set U. Consider the terminal object 1 in the category of permutationals. We can assume that $1(U, \sigma)$ is the singleton set with $p\sigma$ as its unique element and apply the $\mathtt{E}$-functor $\mathfrak{U} : \widehat{\mathbb{S}} \to \mathbf{Joy}$ (Example 4.4) so that $(\mathfrak{U}1)U = \sum_{\sigma \in \mathbb{S}U} p\sigma$ and

$$(\mathfrak{U}\varepsilon)U = \sum_{\sigma \in \mathbb{S}U} \varepsilon\sigma = \sum_{\sigma \in \mathbb{C}U} \varepsilon\sigma = (n-1)! x_n$$

where n is the cardinality of U. Switching to formal power series we obtain the identity

$$\sum_{n \geq 0} Z_n x^n = \mathfrak{U}1 = \mathtt{E}\mathfrak{U}\varepsilon = \mathtt{E}\left(\sum_{n \geq 0} x_n \frac{x^n}{n} \right)$$

and the reader should also compare with Proposition 7.3 in [4].

We can modify Examples 4.4 and 4.6 to apply to partitionals.

4.7. Example. (Example 38 in [10]) Let $x_1, x_2, \ldots$ a set of formal variables. For any partition π of a set U we define $p\pi = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ where n is the cardinality of U and d_i is the number of π-classes of cardinality i. The *exponential polynomial* associated with U is the formal sum

$$Y_n = \sum_{\pi \in \operatorname{Part} U} p\pi.$$

Consider the terminal object 1 in the category of partitionals. The map $\mathbb{U} \to 1 = \mathtt{E}\varepsilon$ corresponds, at the level of elements, to the forgetful functor from $\mathbb{P} \to \mathbb{B}$ and it induces an

$\mathtt{E}$-functor $\phi : \widehat{\mathbb{P}} \to \mathbf{Joy}$ such that for each partitional F and finite set U, $(\phi F)U = \sum F\pi$ where the sum is taken over the partitions of U. As before, we can assume that the terminal object is defined by $1(U, \pi) = \{p\pi\}$ so that $(\phi 1)U = Y_n$ where n is the cardinality of U and $(\phi\varepsilon)U = x_n$. Switching to formal power series we obtain the identity

$$\sum_{n\geq 0} Y_n \frac{x^n}{n!} = \phi 1 = \mathtt{E}\phi\varepsilon = \mathtt{E}\left(\sum_{n\geq 0} x_n \frac{x^n}{n!}\right).$$

4.8. Example. Fix a set I thought of as a set of nodes and let J be $I \times I$ and let K be the set of unordered pairs of elements of I. There is an obvious epi $d : J \to K$. It induces an $\mathtt{E}$-functor from digraphical to graphical species (recall Examples 3.6 and 3.14) with a right adjoint which we know, by Corollary 1.7, that it is a weak $\mathtt{E}$-functor. Compare with Proposition 6.2 in [20]. We discuss more examples of weak $\mathtt{E}$-functors below.

4.9. Weak $\mathtt{E}$-functors. Recall from the discussion below Corollary 1.7 that the right adjoints $\mathbf{D}/(\mathtt{E}D) \to \mathbf{D}/(\mathtt{E}C)$ induced by the maps $C \to \mathtt{E}D$ in $\mathbf{D}$ are weak $\mathtt{E}$-functors (Definition 1.6). As in the case of the left adjoints (see below Proposition 4.1) when $\mathbf{D}$ is of the form $\widehat{!\mathbf{G}}$ the right adjoints have a more explicit description. If the map $C \to \mathtt{E}D$ induces a functor $K : !\mathrm{el}(C) \to !\mathrm{el}(D)$ then the right adjoint $K^* : \widehat{!\mathrm{el}(D)} \to \widehat{!\mathrm{el}(C)}$ is defined simply as $K^*F = FK$, that is, pre-composition with K. For example, the right adjoint $R : \mathbf{Joy}_2 \to \mathbf{Joy}$ to the functor $(\_)_x : \mathbf{Joy} \to \mathbf{Joy}_2$ of Example 4.2 is defined by $(RF)U = F(U, \varnothing)$. It should be noticed that R is an $\mathtt{E}$-functor.

The right adjoint to the functor $\mathfrak{K} : \mathbf{Joy} \to \widehat{\mathbb{S}}$ in Example 4.3 is the (weak $\mathtt{E}$-functor) $\mathfrak{T} : \widehat{\mathbb{S}} \to \mathbf{Joy}$ explicitely defined in [2] by $(\mathfrak{T}T)A = T(A, id)$. Bergeron explains that when one thinks of T as being the combinatorial interpretation of a power-series $t(x_1, x_2, x_3, \ldots)$ then $\mathfrak{T}$ corresponds to the assignment $t \mapsto t(x, 0, 0, \ldots)$. As in the previous case, this functor is not just a weak $\mathtt{E}$-functor but also an $\mathtt{E}$-functor.

Consider now the right adjoint induced by the map $\mathbb{C} \to 1 = \mathtt{E}\varepsilon$ of Example 4.4. This is the weak $\mathtt{E}$-functor $R : \mathbf{Joy} \to \widehat{\mathbb{S}}$ defined by $(RF)(U, \sigma) = FU$. In this case, R is not an $\mathtt{E}$-functor (it is easy to see this by applying $\mathtt{E}(RF)$ and $R(\mathtt{E}F)$ to a non-trivial cycle).

The functor $\mathbf{Fix} : \mathbf{Joy} \to \widehat{\mathbb{S}}$ defined by $(\mathbf{Fix}F)(U, \sigma) = \{x \in FU \mid (F\sigma)x = x\}$ is discussed from slightly different perspectives in [10] and [2]. The functor $\mathbf{Fix}$ is monoidal and preserves coproducts (see Proposition 3 in [2]) but it does not preserve epis (for example, it does not preserve the unique epi from the species $\mathbb{L}$ of totally ordered sets to the terminal species 1). So it does not have a right adjoint and hence it is not induced by a map $\varepsilon \to \mathtt{E}\mathbb{C}$ in $\mathbf{Joy}$. But it is easy to prove that $\mathbf{Fix}$ preserves pullbacks so it is a weak $\mathtt{E}$-functor by Corollary 1.7. That is, for any N in $\mathbf{Joy}$, $\mathbf{Fix}(\mathtt{E}N)$ is a free $\mathtt{E}$-algebra in the category $\widehat{\mathbb{S}}$ of permutationals. Indeed, there is an interesting way of expressing $\mathbf{Fix}(\mathtt{E}N)$ as an exponential (see Section 3 in [10]). A *crown of N-structures* for some species N is an element of $\{x_p\}_{p\in\pi}$ of $(\mathtt{E}N)U$ (for some finite set U and partition π of U) together with an automorphism σ (given by an iso $\sigma : U \to U$) whose action permutes the x_p's in a cycle. We denote the set of N-crowns with underlying iso σ by $(\mathbf{Cr}N)\sigma$. Because $(\mathbf{Fix}(\mathtt{E}N))(U,\sigma)$ is the set of elements of $(\mathtt{E}N)U$ that are fixed by σ it is easy to see that the connected elements of the free simple monoid $\mathbf{Fix}(\mathtt{E}N)$ are the crowns of N-structures. In other words, $\mathbf{Fix}(\mathtt{E}N) = \mathtt{E}(\mathbf{Cr}N)$. It is then clear that $\mathbf{Fix}$ is not an $\mathtt{E}$-functor. Is $\mathbf{Fix}$ the right adjoint induced by a map $\mathbb{C} \to 1$ in $\mathbf{Joy}$? The answer is again no. For suppose it is, then there exists a functor $\Gamma : \mathbb{S} \to \mathbb{B}$ such that $(\mathbf{Fix}F)(U,\sigma) = F(\Gamma(U,\sigma))$. Consider now the species $\mathbb{L}$ of total orders. For every non trivial cycle σ, $(\mathbf{Fix}\mathbb{L})\sigma = \emptyset$. On the other hand for every set U, $\mathbb{L}U$ is not empty. So no such Γ exists.

The reader is invited to check if the weak $\mathtt{E}$-functors that are right adjoints to the $\mathtt{E}$-functors described in Examples 4.5 and 4.8 are also $\mathtt{E}$-functors.

Finally, it is well known that every functor $K : \mathbf{C} \to \mathbf{C}'$ between small categories induces a right adjoint $K_* : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}'}$ to K^*. Using the same argument that we used for the case of the functors $K : !\mathrm{el}(C) \to !\mathrm{el}(D)$ one shows that K_* is a weak $\mathtt{E}$-functor. I have not found any use of these functors in the literature on species, though.

5. Infinitesimals

It is easy to see that the monoidal structure $(\mathbf{D}, +, 0)$ given by coproducts does not have infinitesimals. The same happens with products. In order to characterize the infinitesimals in the monoidal structures of the form $(\widehat{!\mathbf{G}}, \cdot, \mathtt{I})$ it is useful to have the following simple result.

5.1. Lemma. *In any extensive symmetric monoidal category* $(\mathbf{D}, \cdot, \mathtt{I})$ *the following hold:*

1. *if* $\mathtt{I}$ *is connected then so is every object* D *such that* $D \cdot (\_)$ *has a right adjoint that preserves finite coproducts*
2. *if* $\mathtt{I}$ *is projective then so is every object* D *such that* $D \cdot (\_)$ *has a right adjoint that preserves regular epis*
3. *if* $\mathtt{I}$ *is infinitesimal then* $\mathtt{I}$ *is initial.*

Proof. For the first item let ∂ denote the right adjoint to $D \cdot (\_)$ and assume that $\mathtt{I}$ is connected and that ∂ preserves coproducts. We first show that D is not initial. For assume that it is, then there exists a map $D \to 0$ and by transposition a map $\mathtt{I} \to \partial 0 \cong 0$ which is absurd. It is also easy to show that $\mathbf{C}(D, \_)$ preserves finite coproducts and so D is connected. The case of projectivity is also easy. For the last item let ∂ be the right adjoint to $\mathtt{I} \cdot (\_)$. Then the identity $\mathtt{I} \to \mathtt{I}$ induces a map $\mathtt{I} \to \partial\mathtt{I} \cong 0$ and as 0 is strict (the underlying category is extensive), $\mathtt{I}$ is initial. ∎

We can now characterize infinitesimals in our main examples of monoidal categories.

Proof of Proposition 1.10. First we show that every object C in $\mathbf{G}$ induces an amazing infinitesimal in $\widehat{!\mathbf{G}}$. Let $\mathbf{x}$ denote the induced object in $\widehat{!\mathbf{G}}$ and let ∂ be the right adjoint to $\mathbf{x}\cdot(\_)$. Using that Yoneda preserves the tensor it is easy to show that for any F in $\widehat{!\mathbf{G}}$ and U in $!\mathbf{G}$, $(\partial F)U = F(C \sqcup U)$. The strength $\mathrm{str}_l : \partial F \cdot G \to \partial(F\cdot G)$ is given as follows. An element of $\partial F\cdot G$ at stage U is given by a binary partition (U_0, U_1) of U together with an element $x \in F(C\sqcup U_0)$ and an element $y \in GU_1$. The strength then assigns to this information the pair $(C \sqcup U_0, U_1)$ (seen as a partition of $C \sqcup U$) together with x and y. To construct an inverse to the map $(\partial F\cdot G) + (F\cdot \partial G) \to \partial(F\cdot G)$ we must start with a binary partition (V_0, V_1) of $C\sqcup U$ and a pair (x, y) in $FV_0 \times GV_1$. As C is in $\mathbf{G}$, it must be the case that C is in V_0 or C is in V_1. In the first case, it must be the case that $V_0 = C \sqcup W$ with $W \sqcup V_1 = U$. So we have a binary partition (W, V_1) of U and a pair (x, y) in $F(C \sqcup W)\times GV_1$ which altogether give an element of $\partial F\cdot G$. Similarly in the case when C is in V_1. So we obtain a map $\partial(F\cdot G) \to \partial F\cdot G + F\cdot\partial G$. It is easy to see that it is the inverse of the map in the opposite direction given by the strengths. It is also clear that the map $0 \to \partial \mathtt{I}$ is an iso so the Leibniz rule holds. To prove that $\mathbf{x}$ is an infinitesimal we need to show that ∂ preserves finite colimits. For this it is enough to show that it is a left adjoint (making $\mathbf{x}$ amazing). But given the explicit description of ∂ it is easy to show that its right adjoint $\Diamond$ is given by the following formula

$$(\Diamond F)U = \prod_V FV$$

where V ranges over the objects of $!\mathbf{G}$ obtained from $U = \{C_i\}_{i\in I}$ by removing some C_k such that $C_k \cong C$.

Now assume that $\mathbf{x}$ is an infinitesimal. As $\mathtt{I}$ is representable we have that it is connected and projective. It follows by Lemma 5.1 that so is $\mathbf{x}$ and then, as $\widehat{!\mathbf{G}}$ is a presheaf topos, we have that $\mathbf{x}$ is an object D from $!\mathbf{G}$. So we need to show that D is actually in $\mathbf{G}$. Assume that D is $\varnothing$. Then $\mathtt{I} = \mathbf{y}\varnothing = \mathbf{y}D = \mathbf{x}$ is infinitesimal and by Lemma 5.1, $\mathtt{I}$ is initial. Absurd, so D is not $\varnothing$. Now assume that there is an iso $D \to D_0 \sqcup D_1$ in $!\mathbf{G}$ and let $\mathbf{y}_0$ and $\mathbf{y}_1$ be the representables induced by D_0 and D_1. Via Yoneda we obtain a map $\mathbf{x} \to \mathbf{y}_0\cdot\mathbf{y}_1$ in $\widehat{!\mathbf{G}}$ and by transposition we get a map $\mathtt{I} \to \partial(\mathbf{y}_0\cdot\mathbf{y}_1)$. Using the Leibniz rule and without loss of generality we can assume that we obtain a map $\mathtt{I} \to \partial\mathbf{y}_0\cdot\mathbf{y}_1$. So that there is an element in $(\partial\mathbf{y}_0\cdot\mathbf{y}_1)\varnothing$. But this can only happen if D_0 is iso to D and D_1 is iso to $\varnothing$ in $!\mathbf{G}$. So D is indeed an object of $\mathbf{G}$. ∎

In the context of categories of combinatorial structures, the only explicit appearance, I am aware of, of a right adjoint to a derivative functor is in [22] where the (essentially unique) amazing infinitesimal in the topos **Joy** is considered in detail. Some properties discussed loc. cit. can be lifted to infinitesimals in the monoidal categories discussed in Proposition 1.10 above.

Finally, let us consider the example of an object D which fails to be an infinitesimal but for the preservation of epis. Consider the topos of "irreflexive" graphs. This is the topos of presheaves on the category with two objects E and N and two non-trivial maps $s, t : N \to E$. So an object F of this topos is a graph with set of nodes FN and set of directed edges FE. The source and target of an edge $e \in FE$ given by $s \centerdot e$ and $t \centerdot e$ respectively. Define the tensor $\cdot$ as follows.

$$(F \cdot G)N = FN \times GN \qquad (F \cdot G)E = (FE \times GN) + (FN \times GE)$$

The source of an edge (e, n) in $(FE \times GN)$ is given by $(s \centerdot e, n)$ and its target by $(t \centerdot e, n)$. The unit is the graph $\mathtt{I}$ with one node and no edges. It is easy to show that this is a symmetric monoidal structure on the topos of graphs. This monoidal structure is obviously different from that of products or coproducts. Consider for example the graph F given by a discrete set of points (no edges) and think of it as a line. Then picture the graph G induced by a cyclic permutation on a finite set as a circle. It is fair to think of $F \cdot G$ as a "tube" while $F \times G$ and $F + G$ give completely different things. Let us look for infinitesimals. Since $\mathtt{I}$ is representable by N and the underlying category is a presheaf topos, Lemma 5.1 implies that any infinitesimal must be representable. Indeed, item 3 of Lemma 5.1 leaves only one option: the object representable by E. This is the graph (also denoted by E) with two nodes and one edge joining them. It is not difficult to show that $E \cdot (\_)$ has a right adjoint $\eth$. The graph $\eth G$ has the edges of G as nodes and an edge between two such nodes is a pair of edges in G forming a square in the obvious way. It is easy to see that $\eth$ satisfies Leibniz rule. It also preserves coproducts but it does not preserve epis as it can be seen by applying $\eth$ to the epi that quotients the graph with two nodes each with a loop by the equivalence relation that relates the two nodes but not the loops.

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References

[1] E. Bender and J. Goldman. Enumerative uses of generating functions. *Indiana university mathematics journal*, 20:753–764, 1971.

[2] F. Bergeron. Une combinatoire du pléthysme. *Journal of combinatorial theory (Series A)*, 46:291–305, 1987.

[3] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial species and tree-like structures*, volume 67. Cambridge University Press, 1998.

[4] Flavio Bonetti, Gian-Carlo Rota, Domenico Senato, and Antonietta M. Venezia. On the foundation of combinatorial theory. X. A categorical setting for symmetric functions. *Stud. Appl. Math.*, 86(1):1–29, 1992.

[5] A. Dress and T. Müller. Decomposable functors and the exponential principle. *Advances in mathematics*, 129:188–221, 1997.

[6] S. Eilenberg and G. M. Kelly. Closed categories. In *Proceedings of the Conference on Categorical Algebra, La Jolla 1965*, pages 421–562. Springer, 1966.

[7] G. B. Im and G. M. Kelly. A universal property of the convolution monoidal structure. *Journal of pure and applied algebra*, 43:75–88, 1986.

[8] P. Johnstone. Cartesian monads on toposes. *Journal of pure and applied algebra*, 116:199–220, 1997.

[9] P. T. Johnstone. Adjoint lifting theorems for categories of algebras. *Bulletin of the London Mathematical Society*, 7:294–297, 1975.

[10] A. Joyal. Une theorie combinatoire des séries formelles. *Advances in mathematics*, 42:1–82, 1981.

[11] A. Joyal. Foncteurs analytiques et especès de structures. In G. Labelle and P. Leroux, editors, *Combinatoire énumérative*, volume 1234 of *Lecture notes in mathematics*, pages 126–159. Springer-Verlag, 1986. Proceedings, Montréal, Quebec 1985.

[12] G. M. Kelly. Doctrinal adjunction. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 257–280. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[13] Gilbert Labelle and Pierre Leroux. An extension of the exponential formula in enumerative combinatorics. *Electron. J. Combin.*, 3(2):Research Paper 12, approx. 14 pp. (electronic), 1996. The Foata Festschrift.

[14] Stephen Lack. On the monadicity of finitary monads. *J. Pure Appl. Algebra*, 140(1):65–73, 1999.

[15] P. Leroux and G. X. Viennot. Combinatorial resolution of systems of differential equations, I. Ordinary differential equations. In G. Labelle and P. Leroux, editors, *Combinatoire énumérative*, volume 1234 of *Lecture notes in mathematics*, pages 210–245. Springer-Verlag, 1986. Proceedings, Montréal, Quebec 1985.

[16] S. Mac Lane. *Categories for the Working Mathematician.* Graduate Texts in Mathematics. Springer Verlag, 1971.

[17] Angus J. Macintyre. Exponential algebra. In *Logic and algebra (Pontignano, 1994)*, volume 180 of *Lecture Notes in Pure and Appl. Math.*, pages 191–210. Dekker, New York, 1996.

[18] M. Méndez and O. Nava. Colored species, c-monoids and plethysm, I. *Journal of combinatorial theory, Series A*, 64:102–129, 1993.

[19] M. Méndez and J. Yang. Möbius species. *Advances in mathematics*, 85:83–128, 1991.

[20] Miguel A. Méndez. Species on digraphs. *Adv. Math.*, 123(2):243–275, 1996.

[21] O. Nava and G-C. Rota. Plethysm, categories and combinatorics. *Advances in mathematics*, 58:61–88, 1985.

[22] D. S. Rajan. The adjoints to the derivative functor on species. *Journal of combinatorial theory (Series A)*, 62:93–106, 1993.

[23] R. P. Stanley. *Enumerative combinatorics 2*, volume 62 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1999.

[24] Herbert S. Wilf. *Generatingfunctionology*. Academic Press Inc., Boston, MA, second edition, 1994.

[25] Tomoyuki Yoshida. Categorical aspects of generating functions. I. Exponential formulas and Krull-Schmidt categories. *J. Algebra*, 240(1):40–82, 2001.

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