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# COMMENTS ON THE DEVELOPMENT OF TOPOS THEORY

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## Presentation

Summarizing several threads in the development of the Elementary Theory of Toposes in its first 30 years 1970-2000, this historical article prepares the reader for later publication such as Johnstone's Elephant (2002) and for the author's own steps toward an improved foundation for algebraic geometry in the Grothendieck spirit, but using the tools of categorical logic and taking up the theme of axiomatic cohesion.

Addendum:
An important fact should be noted. It was inaccessible to me at the time of writing this historical paper. It concerns the origins of the function-space concept that now embodies the basic topological example of cartesian-closed category. I cited seven contributors to that subject at the end of section 4. Later, when I telephoned David Gale to inquire about his 1950 paper, he informed me that indeed it was in lectures at Princeton in the late 1940's that Witold Hurewicz defined and used the notion of k-spaces to present his solution of the problem that he had posed to Fox (and which Fox had solved for the sequential case in the work cited here). It seems that (directly or indirectly) it was Hurewicz himself who by that example inspired the other six works cited here.

There are two corrections to note:
On p. 719, the caption on the photo of the author should read F. William Lawvere.
On p. 734, the author's address should read:
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# Comments on the Development of Topos Theory*

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## 0 The categorical tool applied to algebraic geometry leads to the birth of toposes

Unification and simplification are necessary not only for the dissemination of results, but also for the coherent advance of research in the diverse branches of mathematics. The need for unification and simplification to render coherent some of the many mathematical advances of the 1930's led Eilenberg and Mac Lane [23] to devise the theory of categories, functors and natural transformations in the early 1940's. It is useful to distinguish general categories from the linear ones whose explicit study began with Mac Lane in 1950 [78]. The combination of linearity and exactness known as 'abelian' categories was perfected in the 50's and early 60's. That theory continues to enjoy many applications, for example through the use of derived categories in analysis. A key step, midway in that development, was Grothendieck's Tohoku paper [42], which showed that this conceptual basis for homological algebra over a ring also applies to linear objects varying as sheaves over a space. Then the fact that the exactness concepts also apply in many nonlinear categories became gradually more known and used. The concept of adjoint functors, discovered by Kan (in the mid-1950's) was rapidly incorporated as a key element in Grothendieck's foundation of algebraic geometry [1] and in the new categorical foundation of logic and set theory [70,71]. Grothendieck and his circle at the Institut des Hautes Etudes Scientifiques near Paris developed in the early 1960's the concept of topos for use in geometry; a simplification of that concept with additional uses was proposed by me at the Istituto di Alta Matematica in Rome in 1969. After an initial development 1969-70 in collaboration with the algebraic topologist Tierney (who had independently been lecturing on the need for an axiomatic sheaf theory), this simplified topos theory was presented to the 1970 International Congress of Mathematicians at Nice [72]. Further development of that proposal led to many papers and books, but in spite of these publications, many students find it difficult to discern what topos theory is, where it came from, and where it is going. I hope that the following sketch will help to overcome this difficulty.

The power-set axiom, which defines toposes among all categories, is discussed in detail in section 4 below.

Since I cannot give here a step by step description, I concentrate on the history of the mathematical ideas, both as a guide to those who want to learn and further develop these ideas and also as an aid to those who want to trace the dates and publications. I apologize

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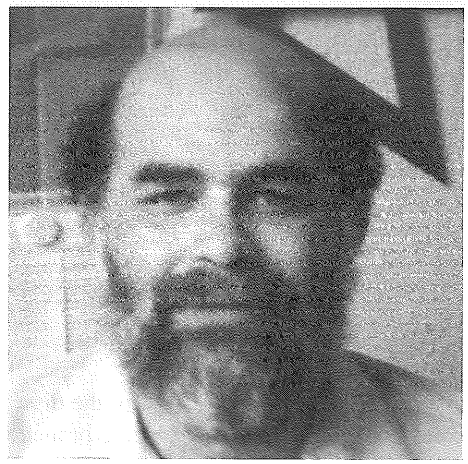


Michael Barr, 1975

for inevitable omissions. Earlier versions of this paper were seen by Barr, Gabriel, Freyd, Johnstone, Kock, Mac Lane, Ramachandran, Schanuel, and Street, whose comments are much appreciated.

## 1 Functional analysis and algebraic topology need a common home with a flexible frame

The core of mathematical theories is in the variation of quantity in space and in the emergence of quality within that. The fundamental branches (such as differential geometry and geometric measure theory) gave rise to (and extensively use) the two great auxiliary disciplines of algebraic topology and functional analysis. A great impetus to their crystallization was the electromagnetic theory of Maxwell-Hertz-Heaviside and the materials science of Maxwell-Boltzmann. Both of these disciplines and both of these applications were early made explicit in the work of Volterra. As pointed out by de Rham to Narasimhan [88], it was Volterra who in the 1880's not only proved that the exterior derivative operator satisfies $d^2 = 0$, but proved also the local existence theorem which is usually inexactly referred to as the Poincaré lemma; these results remain the core of algebraic topology as expressed in de Rham's theorem and in the cohomology of sheaves. Volterra's theory of functions of lines, presented in his 1912 Paris lectures and later called 'functional' analysis, was quite effectively developed by his students and by Silva and Zorn (as pointed out by Fichera in [26]), taking open sets and closed sets not as primitive, but as derived from more fundamental structure. In the period 1950-85 that form of functional analysis was largely neglected, but it was revived in the 1980's when some of its key problems were solved and its applications to infinite-dimensional physics were reactivated by the explicitly categorical work of Kriegl and of his collaborators Frölicher, Nel, and Michor [32], [65], [67], and in the explicitly topos-theoretic work of Penon, Dubuc, and Bruno [92], [22], [6].

Indeed, in a sense, the recent work in topos theory finally organically reunites those two strands from Volterra (algebraic topology and the covariant functional analysis explained below), strands which had long been intertwining in

- Grothendieck's work on nuclear spaces [41] and on holomorphic duals [40];
- the Grauert-Cartan-Serre [38], [13] results on coherent analytic sheaves (where, as Houzel and Douady pointed out more explicitly in the 1970's [50], [20], nuclear bornological functional-analysis plays a key role in establishing the finiteness of certain 'algebraic-topological' Betti numbers);
- the Sato-Kashiwara [59] microfunctional approach to the theory of waves.

## 2 A flexible frame for logic and set theory is developed first

In spite of its geometric origin, topos theory has in recent years sometimes been perceived as a branch of logic, partly because of the contributions to the clarification of logic and set theory which it has made possible. However, the orientation of many topos theorists could perhaps be more accurately summarized by the observation that what is usually called mathematical logic can be viewed as a branch of algebraic geometry, and it is useful to make this branch explicit in itself. The central examples studied by the early model-theorists Birkhoff [5], Tarski [100] and Robinson [94] demonstrate algebraic geometry as the historical origin, and the advances made in the past 15 years by their successors van den Dries [21], Macintyre [77], and others strikingly demonstrate the continuing value to geometry. The categorical logic merely systematically shows that there is no need for a separate, special logical terminology and notation, since implication and quantifiers are adjoint functors of kinds that arise much more generally in non-poset categories. (Specifically, implication is the poset case of the function-space transformation which is fundamental to functional analysis, as had been observed by Curry; and quantification is the special application, to truth-valued functors, of the general Kan extension induced by change of domain [60]). Moreover, models themselves are functors [69], since what syntactical 'theories' present is most effectively viewed itself as a certain sort of small category. That observation was worked out by Makkai & Reyes [83], after crucial contributions by Barr concerning regular categories [2] and the existence of Boolean-valued points [3]. Work of Joyal [56] and Freyd [30], revolving around the 1972 discovery that the completeness theorem of first-order logic is a consequence of Deligne's theorem [17] which affirms the existence of points for coherent toposes, also played an important role.

But there is a key refinement to the 'classical' Boolean mathematical logic which is forced by the explicit recognition (which topos theory describes) of the cohesive and variable nature of sets. To workers in algebraic geometry and analysis, it may appear somewhat excessive to detour through an elaborate Mitchell-Bénabou language which in turn requires a Kripke-Joyal semantics in order to get back to the mathematical content of a specific topos. (That sometimes-recommended procedure is strictly analogous to defining a group to be the quotient of the free group generated by itself, which analogously is occasionally useful.) The key clause in that semantics was presupposed in the title 'Quantifiers and Sheaves' [72], but the linear case was a theorem in Godement 1958 [37] and indeed just expresses in terms of 20th century concepts the content of Volterra's local existence theorem. Briefly,

a) the rule of inference for existential quantification is just a symbolic expression of the universal property enjoyed by the geometric image of any map (not only in the category of sets where the axiom of choice holds, but) in any topos, whereas

b) a figure lying in such an image comes in fact only locally from figures in the domain of the map.

For example, the image of the complex exponential map is the whole punctured plane, but complex logarithms exist only locally. This theorem of local existence would be trivial if all objects were projective, as the axiom of choice would require. Long before this logical framework was pointed out (and Bénabou [4] and Joyal [56] formalized it), the mathematical experience of using sheaves in geometry and analysis had produced many correct definitions which extended concepts from the constant to the variable realm. For example, the concept of local ring (Hakim [48]), the concept of multiplicatively-convex bornological algebra (Houzel [49]) and many other concepts were sheafified by inserting the phrase 'there exists a covering on which...' in just the right places in the definition. Similarly, Grothendieck and others unerringly recognized which kinds of mathematical structures are 'preserved by all functors which preserve finite limits and arbitrary colimits'. (A very impressive list was produced by Grothendieck [47] during his 1973 stay in Buffalo; during that same visit he also advocated the abandonment of his earlier complicated definition of 'scheme', but unfortunately the simpler alternative he offered does not seem to have found its way into the textbooks.) However, less experienced mathematicians have found useful an explicit presentation of the positive logic which formalizes those definitions and classes of structures.

The fundamental role of positive logic (also known as coherent logic or geometric logic) suggests a refinement of the standard presentation of predicate logic. Predicates are names for subobjects, and the basic possibility, for two subobjects of the same object (or universe of discourse), that the first be included in the second, is logically the assertion that one predicate entails a second. In a topos, the subobjects of a given domain form a distributive lattice, reflected logically in terms of conjunction and disjunction operations on predicates, satisfying suitable adjointness relations (rules of inference) relative to entailment. Entailment, finite conjunction, and disjunction are preserved by substitution along (the name of) an arbitrary domain-changing map. Substitution signifies inverse image of subobjects, an operation which has 'image' as left adjoint; the latter is known logically as existential quantification along the map. Positive logic does not explicitly include the higher operation of universal quantification (nor its special cases of implica-

William F. Lawvere, 1992

tion and denial) even though right adjoints to internal substitution are also present in any topos, because those right adjoints are typically not preserved by the more general substitution into an arbitrary continuous map between toposes. Indeed, those continuous maps enjoying such additional preservation (of first-order logic with alternating quantifiers) are just the open continuous maps. Thus although in full first-order logic the entailment of two predicates can equivalently be asserted by saying that their universally-quantified implication has the nullary property of being 'true', in positive logic the fundamental relations remain the binary entailments one for each domain (including cartesian products of basic domains). From the positive standpoint, quantifier elimination is related to quantifier definability, in the sense that some favored theories have sufficiently strong axioms permitting the outright definition of universal quantification (e.g. implication and denial) in terms of the positive operations. Moreover, if we restrict to Boolean toposes, positive logic is just as expressive as full first-order logic, by allowing additional predicates; namely, each occurence of a negative formula in an axiom can be viewed as a new primitive predicate, characterized by two lattice axioms as an appropriate complement.

The phrase 'elementary topos' is a confusing relic of the relation with logic, the term 'elementary' having been used by some logicians as synonymous with 'first order'. The origin of the phrase lies in the useful fact that the concept of topos in the Lawvere-Tierney sense is definable in a logical language far weaker than the infinitary higher-order language used originally by Grothendieck and his students. (Internally any topos permits interpretation of higher-order concepts, as explained below, but that is a different matter.) But in fact, this needed external language is actually far weaker than first-order, being essentially equational, even the positive logical operators being needed at that level only to define special classes of toposes (such as the two-valued toposes or the toposes satisfying the axiom of choice).

# 3 Parameter spaces and Grothendieck toposes relative to a base topos

The original Grothendieck toposes can be located [18] in the much broader class (of toposes in the 'elementary' sense) using a special case of Grothendieck's relativisation concept, as follows: A continuous map (or geometric morphism) from one topos to another is a functor with a left exact left adjoint. For a fixed topos $S$, an $S$-topos is one equipped with a continuous map to $S$ and boundedly-presented as an $S$-cocomplete category; a map of $S$-toposes is a suitable quasi-commutative triangle of continuous maps with lower vertex $S$ (which implies that the adjointness, in the map between the two $S$-toposes, is itself defined over $S$). Then if $S$ happens to be a universe of abstract sets, the category of $S$-toposes is equivalent to that of $S$-toposes in the sense of Grothendieck. To say that a topos $S$ is a universe of abstract sets is, for mathematical purposes, to say that it satisfies the further properties of two-valuedness and the axiom of choice (which implies [19] that all its subobject lattices are Boolean); axioms of strong infinity can be further imposed on $S$ if needed, since they too are categorical invariants. However, still useful is the continuing program of recasting mathematics relative to an arbitrary base topos $S$ which satisfies much weaker requirements. Theorems in such a recast mathematics (besides having more explicit proofs) often have content to the general effect that classical theorems are stable under suitable continuous variation of parameters, when $S$ is taken to be a topos of sheaves on the parameter space. Moreover, simpler statements can be achieved if the baggage involved in the definition of the particular $S$ can be suppressed. In other words, a reference to classes being 'small' can often be interpreted in more than a merely quantitative sense, since being parametrizable by an object of the base $S$ may be in fact a rich quality.

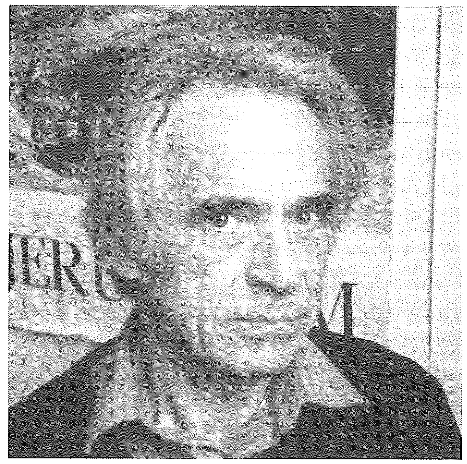
It is a theorem [18] that any $S$-topos can be constructed from $S$ by a three-step zigzag: first, additional parameters are added from a chosen $S$-object, yielding a local homeomorphism $S' \longrightarrow S$, i.e. a continuous map whose inverse functor actually preserves the power-set construction; second, a locally-connected 'surjection' $S' \longrightarrow S''$, i.e. a continuous map whose inverse-image functor has an $S$-adjoint on the left and is faithful, adds the action (among the parameter-levels) of an internal $S$-category; and finally, an 'inclusion' $S''' \longrightarrow S''$, i.e. a continuous map whose forward functor is full and faithful, restricts to those objects of $S''$ which are compatible with a specified notion of covering. The last equivalently amounts to requiring that the objects of $S'''$ satisfy some specified disjunctive and existential axioms with respect to the action they have from $S''$. This elementary theorem includes as a special case the theorem characterizing Grothendieck toposes via sheaves of sets over general small sites. The importance of doing topos theory over a general base topos $S$ (even if $S$ is restricted to be itself a Grothendieck topos, that is, even if one is not concerned with non-standard analysis, independence results in set theory, or higher-order recursion theory), is quite analogous to the importance, often emphasized by Grothendieck, of doing commutative algebra over an arbitrary base ring; the comparison of more-variable sets with less-variable ones arises along with and is similar to the analogous comparison for variable quantities.

Another conceptual characterization of $S$-toposes is that they are the lex-total objects in the realm of large $S$-parameterized categories [98], [99]. Here such a category is called totally cocomplete if its Yoneda embedding has a left adjoint. This notion, due to Street and Walters [99], has applications, described by Kelly [62].

The notion of a family of spaces parameterized by a space is most effectively treated by geometers via a single map to the parameter space; the spaces in the family are the fibers of the map; Grothendieck's change-of-base relativization applies within any given topos to yield, for any given object, a new topos of families parameterized by that given base. The operation of sum (or total) of a family can only mean the left adjoint to the inclusion of constant families, i.e. to the change of base; in this case the left adjoint is merely the functor which forgets the map that told how the total was distributed over the base. This tautologous meaning of sums is quite effective in usual mathematics since the families that arise are not arbitrary, but are usually a priori bounded. The same idea applies in set theory, except that the quest for ever-larger ordinals raises the question of the existence of arbitrary families of small sets indexed by a small set. There are two standard sorts of answers to this question in the form of large cardinal axioms: If by families we mean those families definable in the first-order language whose alternating quantifiers range over the set-category that we are describing, then the affirmation of their existence is essentially equivalent to the replacement schema, so that the category is equivalent to a category derived from a model of full Zermelo-Fraenkel set theory. (The inverse of the equivalence is the classical Specker interpretation of ZF 'sets' as tree-like structures, also described in an appendix to some editions of SGA4.) On the other hand, if by families we mean 'arbitrary' families (this can be given a rational interpretation by imagining our topos to be a special category object in another 'larger' topos), then the affirmation that they are each derivable as the fibers of a single map in our topos is equivalent to the 'Grothendieck universe' property: our topos is equivalent to a category object which has strongly inaccessible cardinality in the sense of the larger topos. Strong inaccessibility is usually defined in terms of the existence of products of small families of small sets, but that follows from the fact that our topos has map-spaces because the 'product' of the family of fibers of a map is just the set of sections of the map. Of course, the product functor is defined as the right adjoint to the inclusion of constant families, i.e. to the change-of-base. Since that right adjoint exists for any topos (where, however, it consists of the sections that are smooth in the sense of being themselves maps in the topos) one continues to call it the infinite product and even to denote it by $\Pi$; it is doubtless not coincidental that Weil much earlier used $\Pi$ to denote a special case of this construction arising in algebraic geometry.

## 4 Function spaces and cohesive power sets

The key advance in the Lawvere-Tierney formulation after the Grothendieck-Giraud formulation of the topos concept is the explicit recognition of internal representability of power sets (even in categories of non-abstract sets such as cohesive or variable sets). Thus the power exploited by Cantor, Dedekind, and Hausdorff and other pioneers in the case




Ernst Specker, 1985

of abstract sets became available also for direct use in geometry and analysis. The continuous maps (or geometric morphisms) in the 2-category of toposes do not necessarily preserve this central structure up to isomorphism, but only up to a natural map. (This 'laxity' phenomenon would be strange for ordinary categories, but is common for other 2-categories, such as that of closed categories). Because power sets are injective objects, their algebra can faithfully reflect the geometry (as shown in detail in Mikkelsen's thesis [84]); by contrast, most toposes do not have enough projective objects, which implies that the commutation rule for internal existential quantification, as emphasized in (a) and (b) in section 2, is really needed to move calculations ahead. The way in which the mere existence of the power set functor implies the needed properties of toposes was elegantly shown by Paré [91]. Thereafter, the simplest definition of 'topos' has been 'category with power set'.

The power set construction can usefully be seen as divided into two parts: First, there is the map-space construction (discussed below) which is essential for calculus of variations and for functional analysis generally (and for continuum physics), but which when applied to a special codomain space yields the power set space of the domain space. (In Synthetic Differential Geometry [64], with its projected application to Continuum Dynamics [73,74], the application of the map-space construction to special, infinitesimal domains yields the tangent bundle functor and higher jet-bundle functors in a form very amenable to manipulation.)

Second, that special codomain, specifically a truth-value space or subobject classifier, is assumed. This 'objectifies the subjective' in the sense that it postulates an object which perfectly parameterizes the truth-values of judgements of the form 'such and such a figure belongs to such and such a subobject of its codomain'. By a method well understood by the pioneers of set theory and formalized for general toposes by Freyd in

1972 [31], this in turn implies (provided the topos contains at least one object which is not Dedekind-finite) that the subjective process of iteration can also be perfectly parameterized (by an absolutely-free Peano algebra; the method uses the fact that the class of all subalgebras containing a given point is parameterizable and that any parameterizable class of subobjects of a given object has an intersection, both of those facts following easily from the universal property of power sets). But in turn the parameterizability of completed iteration implies some physically counter-intuitive consequences such as Peano's space-filling curve and some methodologically awkward consequences such as Gödel's incompleteness theorem. Thanks to recent detailed work by van den Dries [21] and others (a kind of work that had been partly foreseen by Grothendieck in a discussion with me in January 1982), it is now known that such consequences can be bypassed in the following way: A suitable topos can be generated by a subcategory which contains sufficiently many geometrically-reasonable spaces and maps, but which does not contain the infinite discrete Peano algebras; although the latter do appear as subobjects (of the geometrical spaces) defined by truth equations, they cannot be defined by equations valued in spaces in the geometrical category itself.

The first part of the power set construction, the notion of map space, has seemed objectively inevitable since Bernoulli and others pioneered the calculus of variations. Namely, if a time interval, a body, and an ordinary space can be modelled as objects of a category, then the space of all paths in the space, parameterized by the time interval in ways allowed by the category, should also be an object, as should be the space of all allowable placements of the body in the space. Then a quantity which depends on placement, such as potential energy, or a quantity which depends on path, such as squared velocity, can be treated as another map in the category. Yet the three descriptions of a motion as a path in placement space, a placement in path space, or simply as a map into ordinary space from a space of pairs <particle, instant>, are all equivalent and indeed that equivalence (for any three objects in the topos) is the axiom defining the map-space as an adjoint functor. Typically there is an adequate notion of generalized 'path', so that an admissible functional is just one which covariantly takes paths to paths; this was a key concept of the Volterra school of functional analysis [24]. The fact that map-spaces enjoying that simple adjointness axiom are not generally present in the usual category of topological spaces was already noted by Fox [27], Kelley [61], Brown [8], Spanier [96], Steenrod [97], and Day [16] and later by Frölicher [32], all of whom proposed more path-oriented categories in order to achieve this fundamental construction.

## 5 Some classes of examples

How are examples of toposes constructed? Of course, sheaves of sets were needed as a base for the categories of abelian sheaves on particular analytic or algebraic spaces treated as in Leray, Oka, Cartan and Serre. Indeed, that is a very important line of development continuing intensely today in partial differential equations, but for the history of that line I refer to Gray [39] and Houzel [51]. Classically a sheaf on a space is a contravariant functor (with pasting condition) on a poset of open regions in the space. But in some ways more typical are the toposes of set-valued functors on non-posets. Consider for example the topos of simplicial sets which has enjoyed widespread use in algebraic topology since 1950, and whose special classifying role is explained in detail in Mac Lane and Moerdijk [81], or the topos of functors from rings to sets which was the base for Cartier's simplified definition [14] of algebraic groups and which was the precursor of almost all constructions of particular models of synthetic differential geometry [73,87]. A detailed treatment of a topos of this kind which contains all analytic spaces as a full subcategory was given by Grothendieck in the 1960 Cartan seminar [43] on families of analytic spaces, even before Grothendieck's general definition of topos was crystallized by Giraud in 1963 [36].

Intermediate in generality between a topos of general spaces and a topos of sheaves on a particular classical space is the idea of a topos of sheaves on a generalized space; most famous is the fact that sheaves on a particular kind of space, over which the implicit function theorem does not hold, are not entirely determined by their restriction to subregions. The fact that the implicit function theorem does not hold in algebraic geometry was turned into a virtue by Grothendieck's brilliant construction of the étale topos of an algebraic space, based on a particular site which is not a poset though still quite special. Even older is the idea of a generalized space equipped not only with a poset of open regions, but also with an action by homeomorphisms of a group. This category leads to a sub-2-category of toposes which very effectively unites the whole development which started with the Hurewicz-Hopf discovery of the effect of the fundamental group of a space on its homology [79], by providing a universe in which the space, its covering spaces, and the fundamental group itself are on the same footing and are connected by maps, and in which the cohomology of the space and of the group are strictly instances of the same construction, namely that of the derived category of the abelianization of a topos. (The actions of a group on sets constitute the objects of the simplest kind of Boolean topos.) Results of Joyal and Tierney [58] and of Joyal and Moerdijk [57] extend these ideas to give a localic groupoid presentation of the general topos, somewhat as the general space could classically be presented as a quotient of a zero-dimensional space; as Mac Lane and Moerdijk point out in their useful summary of the field [82], this extended presentation has not yet been analyzed in detail nor much applied. Johnstone [54] in proving some powerful representation theorems concerning these extensions of the notion of generalized space, in effect showed that the philosophical account common in the 1970's of toposes (based on parameterized variation and internal logic) is too restrictive, which led me in this paper to base my account on the dialectical relations between cohesion and variation and between the logic internal to a topos and internal to the category of toposes.

## 6 Some recent developments

Although the 25th anniversary of 'elementary' topos theory (along with the 50th anniversary of category theory itself) was celebrated in Halifax in 1995, and although the fundamentals would seem to be well-established in the many papers and books previously mentioned, new qualitative advances are ongoing. An example is the work of Funk, and of Bunge-Funk and Bunge-Carboni [33], [12], [11].

Following up on a 1966 Oberwolfach talk where I had proposed a theory of distributions (not only in but) on presheaf toposes, in 1983 at Aarhus I posed several questions concerning distributions on $S$-toposes. The base for the definition and questions is a pair of analogies with known theories (commutative algebra and measure theory) for variable quantities, coupled with the fact that there are many important examples of variable $S$-valued 'quantities' where the domains of variation are $S$-toposes. The intensively variable quantities are taken to be the sheaves on the topos, i.e. simply the objects in the category. (Of course, the term topos means 'place' or 'situation', but Grothendieck treats the general situation by dealing instead with the category of $S$-valued quantities which vary continuously over it, as an affine $K$-scheme is described by dealing with the $K$-algebra of functions on it. Perhaps to avoid confusion one should have used a distinct notion, $E$ for the situation and $C(E)$ for the category of sheaves on $E$, but notational practice has used the same symbol for both, even though functorially they are opposite in the same way that $X$ and $C(X)$ are opposed in classical topology. In classical topology there is no strict analogue, for real or complex continuous functions $C(X)$, of the spatially-forward operation right adjoint to the homomorphism $f^*$ which pulls back continuous functions along a general continuous map $f$.) These set-valued quantities can be added via $S$-parameterized colimits in the topos, and they can be multiplied via finite limits of sheaves. A point is just a continuous map to the given $S$-topos from $S$ itself (which is of course the terminal object in the 2-category of $S$-toposes); the inverse image part of (or evaluation at) the point is a functor which preserves both the 'addition' and the 'multiplication'.

Thus we follow the lead of analysis and define a distribution or extensively variable quantity on an $S$-topos to be a continuous linear functional, or generalized point, i.e. a functor to $S$ which preserves $S$-colimits, but not necessarily the finite limits. For example, given a small category $C$ in $S$, the left actions of $C$ on objects of $S$ is an $S$-topos of functions whose corresponding category of distributions or integration processes is just the $S$-cocomplete category of right actions of $C$ on objects of $S$ (by a special case of a theorem of Kan [60], [35].) It is easy to see that in that special example, where no Grothendieck coverings intervene, the answer to the following question is affirmative. Is there, for any given $S$-topos $E$ another one $M(E)$ whose points are just the distributions on $E$ ? This geometrical idea of the space of all measures on a given space can be described in terms of the commutative algebra analogy as the symmetric algebra, that is, for a suitable $S$-cocomplete category is it always possible to adjoin finite products (and more generally fibered products) in a free way compatible with distributivity to obtain a category which is the (category of sheaves on) an $S$-topos? That formulation relativizes and strengthens the idea of $M(E)$, so that like all 2-adjoints $M$ is unique up to equivalence if it exists.

The question of existence seemed serious, since it is known that there is in general no corresponding space $F(E)$ of intensive quantities: while there is an easily-described [104] topos $F(1) = S(X)$ over $S$ which is the 'polynomial algebra' or 'affine line' in that the $S$-morphisms to it from any $S$-topos $E$ precisely classify all the sheaves on (objects in) $E$, function-spaces exist in Top/$S$ only for 'locally compact' exponents $E$ (i.e. only [55] for retracts of coherent toposes). In 1995 Bunge showed that $M(E)$ exists as an $S$-Topos for every $S$-Topos $E$, a more elegant proof being given by Bunge and Carboni in [11].

What examples of distributions can we expect to find? For any locally connected $S$-topos $A$ there is by definition a left adjoint to the left adjoint to the structural map to $S$; this further left adjoint or 'set of connected components' functor is then automatically a distribution on $A$ which perhaps should be thought of as the counting measure since it is invariant under all automorphisms of $A$ (and even under all essential endomaps of $A$). Now, as with any doctrine of extensive quantity, distributions can be pushed forward along any continuous map between toposes, here just by composing the integration process with the inverse image part of the map. Funk's remarkable results include the fact that any distribution on any $E$ is the push forward along some map, from some locally-connected $A$, of the components counting measure on $A$. Among all such locally-connected $A$ over $E$ which so represent a given distribution on $E$, there is a unique one closest to $E$; that one's relation to $E$ turns out, according to work of Bunge and Funk, to be topos-theoretically the same as that of a complete spread over $E$, the notion discovered by Fox [28] in his topological investigation of ramified coverings. Still other relations with classical topology continue to be discovered, for example Plewe's description of a large class of descent maps as triquotients in the sense of Michael [93], [85], also reported at the Halifax celebration.

## 7 From and to continuum physics

What was the impetus which demanded the simplification and generalization of Grothendieck's concept of topos, if indeed the applications to logic and set theory were not decisive? Tierney had wanted sheaf theory to be axiomatized for efficient use in algebraic topology. My own motivation came from my earlier study of physics. The foundation of the continuum physics of general materials, in the spirit of Truesdell, Noll, and others, involves powerful and clear physical ideas which unfortunately have been submerged under a mathematical apparatus including not only Cauchy sequences and countably additive measures, but also ad hoc choices of charts for manifolds and of inverse limits of Sobolev Hilbert spaces, to get at the simple nuclear spaces of intensively and extensively variable quantities. But, as Fichera [25] lamented, all this apparatus gives often a very uncertain fit to the phenomena. This apparatus may well be helpful in the solution of certain problems, but can the problems themselves and the needed axioms be stated in a direct and clear manner? And might this not lead to a simpler, equally rigorous account? These were the questions to which I began to apply the topos method in my 1967 Chicago lectures [73], [74]. It was clear that work on the notion of topos itself would be needed to achieve the goal. I had spent 1961-62 with the Berkeley logicians, believing that listening to experts on foundations might be a road to clarifying foundational questions. (Perhaps my first teacher Truesdell had a similar conviction 20 years earlier when he spent a year [15] with the Princeton logicians.) Though my belief became tempered, I learned about constructions such as Cohen forcing which also seemed in need of simplification if large numbers of people were to understand them well enough to advance further. Thus the theory (as I had proposed in 1969 in Rome, and reported at the 1970 ICM) was first applied by Tierney [101] and Bunge [9] to questions such as the independence of the continuum hypothesis and Souslin's conjecture.

Indeed the key role of the power set (and the relative unimportance for the mathematics of space and quantity of the replacement schema) had emerged clearly from a study of Scott's many reformulations of mid-century set theory. But the key discovery reported in the Rome lecture (after having been worked out at Eckmann's Forschungsinstitut in Zurich) was that, not only does a power set functor $P$ unambiguously exist in a vast number of mathematically-arising categories, but that any Grothendieck topology, i.e. any reasonable notion of preferred 'covering' permitting a restriction to 'sheaf' objects satisfying additional disjunctive or 'existential' conditions, is expressible as a single map in such a category, indeed as an endomap of the truth-object $P1$ which subjectively can be considered as a modal operator of the kind 'it is locally the case that…'.

That observation, together with the previous adjoint axiomatization of function spaces, made it clear that by working at the topos level nearly all constructions and assertions require only a finitary essentially algebraic equational language for formalization, the infinitary higher-order languages with alternations of external Fregean quantifiers being unnecessary, not only for the axioms of the general theory of a topos, but also for particularizing many very special kinds of topos, such as those arising in combinatorics and differential geometry.

The negation-of-negation operator in Heyting logic was easily seen to satisfy the Lawvere-Tierney axiom and is thus a particular example of a Grothendieck topology available in any topos; it is not only central to the constructions which 'force' independence results in set theory and which extract the minimal dense part [52] of a space in topology that consists of the substantial subbodies [89], [74] in continuum physics (all those just being for the case of a topos with a poset site), but it permits a very succinct formulation of the condition that a topos satisfy the Hilbert Nullstellensatz, expressing an important role of zero-dimensional Galois theory within algebraic geometry as a whole. Namely, the sheaves for the double negation, which in a sense form the Boolean part of an arbitrary topos, constitute, in the case of a topos generated by all the algebraic spaces defined over a given ground field, the classifying topos for algebraic extension fields of the base (this subtopos is perhaps a more tractable entity than the algebraic closure which only exists as a non-canonically rigidified condensation of it). The Nullstellensatz concerns those few toposes in which all non-empty objects have zero-dimensional figures, i.e. points whose domains are dialectical companions of such Boolean sheaves [75].

Some of the geometric aspects of the 1967 program, such as the role of map-spaces of infinitesimal objects, were worked out under the name of Synthetic Differential Geometry by Wraith [104], Kock [64], Reyes [87], Bunge & Gago [10], Dubuc [22], Yetter [105], Penon [92], Bruno [7], Moerdijk [87] and others. Several books treating the simplified topos theory (Mac Lane & Moerdijk being the most recent and readable text), together with the three excellent books on Synthetic Differential Geometry [64], [87], [68] provide a solid basis on which further treatment of functional analysis and the needed development of continuum physics can be based.

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