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A SURJECTIVITY PROBLEM FOR MATRICES AND NULL CONTROLLABILITY FOR DIFFERENCE AND DIFFERENTIAL MATRIX EQUATIONS

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Abstract. Let P be a complex polynomial. We prove that the associated polynomial matrix-valued function $\tilde{P}$ is surjective if for each $\lambda \in \mathbb{C}$ the polynomial $P - \lambda$ has at least a simple zero. The null controllability for difference and differential matrix equations is also presented.

1 Introduction

In the general theory of control we meet a state space X and a control space U. Generally speaking these spaces are real or complex Banach spaces and could be different but could be the same. In this note we are referring to the case when $X = U = \mathcal{M}(n, \mathbb{C})$. The inputs (controls) are functions from a real interval $[0, T]$ to $\mathcal{M}(n, \mathbb{C})$.

The problem arising in this note consists in finding conditions on the matrix A and on the polynomial P so that the system

$$X'(t) = AX(t) + \tilde{P}(U)$$

is null controllable.

We provide the following two conditions:

1. The matrix A (that drives the system) is hyperbolic.

2. The polynomial P has the simple zero property.

More sophisticated conditions are also provided in the discrete case.

2 A surjective matrix-valued function

The following result is taken directly from [7].

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Theorem 1. *([[7], Theorem III]) If $\psi(\lambda)$ is a polynomial of degree $m \geq 1$ in λ and the distinct roots of $\psi(\lambda) = 0$ are $\alpha_j, j = 1, 2, \cdots, s$, if $Q(\lambda)$ is a polynomial of degree $p \geq 1$ in λ, whose leading coefficient is unity and $Q(0) = 0$, if the equation $Q(\lambda) - \alpha_j = 0, j = 1, 2, \cdots, s$, has at least one simple root for every α_j which is a multiple root of $\psi(\lambda) = 0$ and if*

$$\psi(A) = 0$$

where A is a square matrix of order n, then there exists at least one matrix X also of order n, such that

$$Q(X) = A$$

and such X is expressible as a polynomial in A with scalar coefficients.

Let $A \in \mathcal{M}(n, \mathbb{C})$. A monic polynomial of least degree (denoted by m_A) having the property that $m_A(A) = 0_n$ is called the minimal polynomial of A. The characteristic polynomial and the minimal polynomial of a matrix A above must have the same zeros but the multiplicity could be different.

For matrices and operators we refer the reader to [3], [4], [5] and [6].

We say that a polynomial $P \in \mathbb{C}[z]$ has the *simple zero property* (**SZP** for short) if for every $m \in \mathbb{C}$ the polynomial $Q := P - m$ has at least a simple zero. For example, the polynomial $P_1(z) = (z-1)(z-2)(z-3)$ has the **SZP** while $P_2(z) = z^3$ does not. Clearly every polynomial of degree 1 has the **SZP** but polynomials of degree 2 might not have the property. The symbol $\mathcal{M}(n, \mathbb{C})$ denotes the Banach algebra of all matrices of order n with complex entries endowed with the usual operator norm.

Whenever

$$P(\lambda) := a_n\lambda^n + \cdots + a_1\lambda + a_0, \quad a_n \neq 0 \tag{2.1}$$

is a polynomial of degree n, with scalar coefficients, the symbol $\tilde{P}$ denotes the matrix-valued function defined by

$$X \mapsto \tilde{P}(X) := a_nX^n + \cdots + a_1X + a_0I_n, \quad X \in \mathcal{M}(n, \mathbb{C}), \tag{2.2}$$

I_n being the unity matrix of order n. Elementary proofs of the next theorem, in the particular cases $n = 2$ and $n = 3$, can be found in [2] and in [1], respectively.

Theorem 2. *Let $P \in \mathbb{C}[z]$ be a polynomial satisfying the* **SZP**. *Then the map*

$$X \mapsto \tilde{P}(X) : \mathcal{M}(n, \mathbb{C}) \to \mathcal{M}(n, \mathbb{C}) \tag{2.3}$$

is surjective.

Proof. Let $Y \in \mathcal{M}(n, \mathbb{C})$ be randomly chosen. Clearly the matrix equation $\tilde{P}(X) = Y$ can be written equivalently as

$$\frac{1}{a_n}[a_nX^n + \cdots + a_1X] = \frac{1}{a_n}Y - \frac{a_0}{a_n}I_n. \tag{2.4}$$

Now we can apply Theorem 1.1 above in the particular case

$$A = \frac{1}{a_n} Y - \frac{a_0}{a_n} I_n$$

and

$$\psi(\lambda) = m_A(\lambda)$$

and

$$Q(\lambda) = \frac{1}{a_n} [a_n \lambda^n + \cdots + a_1 \lambda]$$

in order to get a solution for the equation $\tilde{P}(X) = Y$. □

3 Null controllability for a difference matrix equation

Let $A, B \in \mathcal{X}$ be two given matrices and let $f : \mathcal{X} \to \mathcal{X}$ be a given function, where $\mathcal{X} := \mathcal{M}(n, \mathbb{C})$.

The solution $j \mapsto X(j, 0, U, B)$ of the initial value matrix difference equation

$$X_{j+1} = AX_j + f(U), \quad j \in \mathbb{Z}_+, X_j \in \mathcal{X}, U \in \mathcal{X}, \quad X_0 = B \tag{3.1}$$

is given by

$$X(j, 0, U, B) = A^j B + \sum_{k=0}^{j-1} A^{j-k-1} f(U), \quad j \in \{1, 2, \cdots\}. \tag{3.2}$$

Definition 3. *The system*

$$X_{j+1} = AX_j + f(U), \quad j \in \mathbb{Z}_+ \tag{3.3}$$

is called null controllable if there exists a positive integer N such that for every matrix $B \in \mathcal{X}$ there exists a $U = U_B \in \mathcal{X}$ satisfying

$$X(N, 0, U, B) = 0. \tag{3.4}$$

If, in addition, assume that the matrix A is invertible then (3.4) is equivalent to

$$B = -\left(\sum_{k=0}^{N-1} A^{-k-1} \right) f(U). \tag{3.5}$$

Recall that for any $A \in \mathcal{X}$, $\sigma(A)$ denotes the spectrum of A i.e. the set consisting of all its eigenvalues.

Theorem 4. *Let $\mathcal{A}$ be the set of all complex numbers $\omega_j := \cos \frac{2\pi}{j} + i \sin \frac{2\pi}{j}$ with j being any positive integer. Assume that the matrix A is invertible and that its spectrum does not intersect the set $\mathcal{A}$. The system (3.3) is null controllable if and only if the map $U \mapsto f(U)$, acting on $\mathcal{X}$, is surjective.*

Proof. Based on the assumptions is easy to see that for every positive integer N the matrix

$$C_N := -\left(\sum_{k=0}^{N-1} A^{-k-1}\right) = -A^{-1}(I_n - A^{-N})(I_n - A^{-1})^{-1} \tag{3.6}$$

is well defined and invertible and

$$C_N^{-1} = -(I_n - A^{-1})(I_n - A^{-N})^{-1}A.$$

From this the proof consists of two steps.

1. Assume that the system (3.3) is null controllable and let Y be randomly chosen in $\mathcal{X}$. Thus for some positive integer N and $B := -C_N Y \in \mathcal{X}$ there exists a $U = U_B \in \mathcal{X}$ verifying (3.5), that is $Y = f(U_B)$.

2. Assume that f is surjective and let N be randomly chosen. For every $B \in \mathcal{X}$ let $U = U_B$ such that $f(U_B) = -C_N^{-1}B$. Thus the system (3.3) is null controllable. □

Corollary 5. *Let $P(z)$ be a scalar polynomial and $A \in \mathcal{X}$ be an invertible matrix such that $\sigma(A) \cap \mathcal{A}$ is empty. The system*

$$X_{j+1} = AX_j + \tilde{P}(U), \quad j \in \mathbb{Z}_+, X_j \in \mathcal{X}, U \in \mathcal{X}$$

*is null controllable if P has the **SZP**.*

Proof. Follows from Theorem 2 and Theorem 4. □

4 Null controllability for a differential matrix equation

We use the same notation as in the previous section.

The solution $t \mapsto X(t, 0, U, B)$ of the initial value differential matrix equation

$$\dot{X}(t) = AX(t) + f(U), \quad t \in \mathbb{R}_+, X(t) \in \mathcal{X}, U \in \mathcal{X}, \quad X(0) = B \tag{4.1}$$

is given by

$$X(t, 0, U, B) = e^{tA}B + \int_0^t e^{(t-s)A} f(U)ds, \quad t \in \mathbb{R}_+. \tag{4.2}$$

Definition 6. *The system*

$$\dot{X}(t) = AX(t) + f(U), \quad t \in \mathbb{R}_+ \tag{4.3}$$

is called null controllable if there exists a positive real number T such that for every matrix $B \in \mathcal{X}$ there exists a matrix $U = U_B \in \mathcal{X}$ satisfying

$$X(T, 0, U, B) = 0. \tag{4.4}$$

Clearly (4.4) is equivalent to

$$B = -\left(\int_0^T e^{-sA} ds\right) f(U). \tag{4.5}$$

Theorem 7. *Let* $i\mathbb{R}$ *be the set of all complex numbers* $z = im$ *with* m *a real number. Assume that the matrix* $A \in \mathcal{X}$ *is hyperbolic, that is its spectrum does not intersect the set* $i\mathbb{R}$*. The system (4.3) is null controllable if and only if the map* $U \mapsto f(U)$*, acting on* $\mathcal{X}$*, is surjective.*

Proof. Taking into account that A is hyperbolic is easy to see that the matrix

$$D_T := -\left(\int_0^T e^{-sA} ds\right) = -A^{-1}(e^{-TA} - I_3) \tag{4.6}$$

is well defined and invertible.

1. Assume that the system (3.3) is null controllable and let Y be randomly chosen in $\mathcal{X}$. Thus for some positive real number T and $B := -D_T Y \in \mathcal{X}$ there exists a $U = U_B \in \mathcal{X}$ verifying (4.5), that is, $Y = f(U_B)$.

2. Assume that f is surjective and let $T > 0$ be randomly chosen. For every $B \in \mathcal{X}$ let $U = U_B$ such that $f(U_B) = -D_T^{-1}B$. Thus $D_T f(U_B) = -B$, i.e. the system (4.3) is null controllable. □

Corollary 8. *Let* $P(z)$ *be a scalar polynomial and* $A \in \mathcal{X}$ *be a hyperbolic matrix. The system*

$$\dot{X}(t) = AX(t) + \tilde{P}(U), \quad t \in \mathbb{R}_+, X(t) \in \mathcal{X}, U \in \mathcal{X}$$

is null controllable if P *has the* ***SZP****.*

Proof. Follows from Theorem 2 and Theorem 7. □

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