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A NEW CONCEPT OF DIFFERENTIABILITY FOR INTERVAL-VALUED FUNCTIONS

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Abstract. In this paper we propose a new concept of differentiability for interval-valued functions. This concept is based on the properties of the Hausdorff-Pompeiu metric and avoids using the generalized Hukuhara difference.

1 Preliminaries

Let us denote by $\mathcal{K}$ the set of all nonempty compact intervals of the real line $\mathbb{R}$. If $A = [a^-, a^+]$, $B = [b^-, b^+] \in \mathcal{K}$, then the usual interval operations, i.e. Minkowski addition and scalar multiplication, are defined by

$$A + B = [a^-, a^+] + [b^-, b^+] = [a^- + b^-, a^+ + b^+]$$

and

$$\lambda A = \lambda[a^-, a^+] = \begin{cases} [\lambda a^-, \lambda a^+] & \text{if } \lambda > 0 \\ \{0\} & \text{if } \lambda = 0 \\ [\lambda a^+, \lambda a^-] & \text{if } \lambda < 0, \end{cases}$$

respectively. If $\lambda = -1$, scalar multiplication gives the opposite

$$-A := (-1)A = (-1)[a^-, a^+] = [-a^+, -a^-].$$

In general, $A + (-A) \neq \{0\}$; that is, the opposite of A is not the inverse of A with respect to the Minkowski addition (unless $A = \{a\}$ is a singleton). The Minkowski difference is $A - B = A + (-1)B = [a^- - b^+, a^+ - b^-]$.
The *generalized Hukuhara difference* (or *gH-difference*) of two intervals $[a^-, a^+]$, $[b^-, b^+] \in \mathcal{K}$ is defined as follows (Markov [6]):

$$[a^-, a^+] \ominus [b^-, b^+] = [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}]. \tag{1.1}$$

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We denote the *width* of an interval $A=[a^-,a^+]$ by $w(A)=a^+-a^-$. Then, for $A=[a^-,a^+]$ and $B=[b^-,b^+]$, we have

$$A\ominus B=\begin{cases}[a^- - b^-, a^+ - b^+], \text{ if } w(A)\geq w(B)\\ [a^+ - b^+, a^- - b^-], \text{ if } w(A)<w(B).\end{cases} \tag{1.2}$$

If $A,B,C\in\mathcal{K}$ then it is easy to see that

$$A\ominus B=C\Longleftrightarrow\begin{cases}A=B+C, \text{ if } w(A)\geq w(B)\\ B=A+(-C), \text{ if } w(A)<w(B).\end{cases} \tag{1.3}$$

If $A,B\in\mathcal{K}$ and $w(A)\geq w(B)$, then the gH-difference $A\ominus B$ will be denoted by $A\ominus B$ and it is called the *Hukuhara difference* (or *H-difference*) of A and B. For other properties involving the operations on $\mathcal{K}$, see Markov [6]. If $A\in\mathcal{K}$, let us define the norm of A by$\|A\|:=\max\{|a^-|,|a^+|\}$. Then it is easy to see that $\|\cdot\|$ is a norm on $\mathcal{K}$, and therefore $(\mathcal{K},+,\cdot,\|\cdot\|)$ is a normed quasilinear space. A metric structure on $\mathcal{K}$ is given by the Hausdorff-Pompeiu distance $\mathcal{H}:\mathcal{K}\times\mathcal{K}\to[0,\infty)$ defined by $\mathcal{H}(A,B)=\max\{|a^- - b^-|,|a^+ - b^+|\}$, where $A=[a^-,a^+]$ and $B=[b^-,b^+]$. Obviously, the metric $\mathcal{H}$ is associated with the norm $\|\cdot\|$ by $\|A\|=\mathcal{H}(A,\{0\})$ and $\mathcal{H}(A,B)=\|A\ominus B\|$. It is well known that $(\mathcal{K},\mathcal{H})$ is a complete.

Proposition 1. *The H-difference $\ominus$ has the following properties (see [3]):*

(a) $A\ominus\theta=A$ *and* $A\ominus A=\theta$ *for all* $A\in\mathcal{K}$.

(b) $(-A)\ominus(-B)=-(A\ominus B)$.

(c) $(A+B)\ominus B=A$.

(d) $A+(B\ominus A)=B$.

(e) $(B+C)\ominus A=B\ominus A+C$.

(f) $A\ominus B+C\ominus D=(A+C)\ominus(B+D)$.

(g) $A\ominus B+B\ominus C=A\ominus C$.

Proposition 2. *The Hausdorff-Pompeiu distance has the following properties (see [3]):*

(i) $\mathcal{H}(A+C,B+C)=\mathcal{H}(A,B)$,

(ii) $\mathcal{H}(\lambda A,\lambda B)=|\lambda|\mathcal{H}(A,B)$ *for* $\lambda\in\mathbb{R}$,

(iii) $\mathcal{H}(A+B,C+D)\leq\mathcal{H}(A,C)+\mathcal{H}(B,D)$,

(iv) $\mathcal{H}(\lambda A,\mu A)=|\lambda-\mu|\mathcal{H}(A,\theta)$ *for* $\lambda\mu\geq 0$,

(v) $\mathcal{H}(A \ominus B, C) = \mathcal{H}(A, B + C)$,

(vi) $\mathcal{H}(A, B) = \mathcal{H}(A \ominus C, B \ominus C)$,

(vii) $\mathcal{H}(A \ominus B, C \ominus D) \le \mathcal{H}(A, C) + \mathcal{H}(B, D)$,

(viii) $\mathcal{H}(A + B, C + D + E \ominus F) \le \mathcal{H}(A, C + E) + \mathcal{H}(D, B + F)$,

(ix) $\mathcal{H}(A \ominus B, C \ominus D + E + F) \le \mathcal{H}(A, C + E) + \mathcal{H}(D, B + F)$,

for all $A, B, C, D, E, F \in \mathcal{K}$.

Since $\mathcal{K}$ is a normed quasilinear space, the continuity and the limits of an interval-valued function $F : [a, b] \to \mathcal{K}$ are understood in the sense of the norm $\|\cdot\|$. We recall that if $F : [a, b] \to \mathcal{K}$ is an interval-valued function such that $F(t) = [f^-(t), f^+(t)]$, then $\lim_{t\to t_0} F(t)$ exists, if and only if $\lim_{t\to t_0} f^-(t)$ and $\lim_{t\to t_0} f^+(t)$ exist as finite numbers. In this case, we have

$$\lim_{t\to t_0} F(t) = \left[\lim_{t\to t_0} f^-(t), \lim_{t\to t_0} f^+(t)\right].$$

In particular, F is continuous if and only if f^- and f^+ are continuous. If $F, G : [a, b] \to \mathcal{K}$ are two interval-valued functions, then we define the interval-valued function $F \ominus G : [a, b] \to \mathcal{K}$ by $(F \ominus G)(t) := F(t) \ominus G(t)$, for all $t \in [a, b]$. If there exist $\lim_{t\to t_0} F(t) = A$ and $\lim_{t\to t_0} G(t) = B$, then $\lim_{t\to t_0} (F \ominus G)(t)$ exists, and

$$\lim_{t\to t_0} (F \ominus G)(t) = A \ominus B.$$

In particular, if $F, G : [a, b] \to \mathcal{K}$ are continuous, then the interval-function $F \ominus G$ is a continuous interval-valued function. Let $C([a, b], \mathcal{K})$ denote the set of continuous interval-valued functions from $[a, b]$ into $\mathcal{K}$. Then $C([a, b], \mathcal{K})$ is a complete normed space with respect to the norm $\|F\|_c := \sup_{a\le t\le b} \|F(t)\|$.

Definition 3. *(Markov [6]). Let* $F : [a, b] \to \mathcal{K}$ *be an interval-valued function and let* $t_0 \in [a, b]$. *We define* $D_H F(t_0) \in \mathcal{K}$ *(provided it exists) as*

$$D_H F(t_0) = \lim_{h\to 0} \frac{F(t_0 + h) \ominus F(t_0)}{h}. \tag{1.4}$$

We call $D_H F(t_0)$ *the* generalized Hukuhara derivative *(*gH-derivative *for short) of* F *at* t_0. *Also, we define the* left gH-derivative $D_H^- F(t_0) \in \mathcal{K}$ *(provided it exists) as*

$$D_H^- F(t_0) = \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h},.$$

and the right gH-derivative $D_H^+F(t_0) \in \mathcal{K}$ *(provided it exists) as*

$$D_H^+F(t_0) = \lim_{h\to 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h}.$$

We say that F *is* generalized Hukuhara differentiable *(*gH-differentiable *for short) on* $[a,b]$ *if* $D_HF(t_0) \in \mathcal{K}$ *exists at each point* $t \in [a,b]$*. At the end points of* $[a,b]$ *we consider only the one sided* gH*-derivatives. The interval-valued function* $D_H : [a,b] \to \mathcal{K}$ *is then called the* gH-derivative *of* F *on* $[a,b]$*.*

Proposition 4. *(Markov [6]). Let* $F : [a,b] \to \mathcal{K}$ *be such that* $F(t) = [f^-(t), f^+(t)]$*,* $t \in [a,b]$*. If the real-valued functions* f^- *and* f^+ *are differentiable at* $t \in [a,b]$*, then* F *is* gH*-differentiable at* $t \in [a,b]$ *and*

$$D_HF(t_0) = \left[\min\left\{\frac{d}{dt}f^-(t), \frac{d}{dt}f^+(t)\right\}, \max\left\{\frac{d}{dt}f^-(t), \frac{d}{dt}f^+(t)\right\}\right]. \tag{1.5}$$

The converse of Proposition 4 does not true, that is, the gH-differentiability of F does not imply the differentiability of f^- and f^+ (Markov [6]).

2 A new concept of differentiability for interval-valued functions

Let $F : [a,b] \to \mathcal{K}$ be a given function. We say that F is *left differentiable* at $t_0 \in (a,b]$ if there exists an element $A \in \mathcal{K}$ such that

$$\lim_{h\to 0^+} \frac{1}{h}\mathcal{H}(F(t_0), F(t_0-h) + hA) = 0 \tag{2.1}$$

or

$$\lim_{h\to 0^+} \frac{1}{h}\mathcal{H}(F(t_0-h), F(t_0) - hA) = 0. \tag{2.2}$$

The element $A \in \mathcal{K}$ is called a *left derivative* of F at t_0 and it will be denoted by $F'_-(t_0)$. F is said to be *left differentiable* on $(a,b]$, if F is left differentiable at each $t_0 \in (a,b]$.

Remark 5. *Let* $F : [a,b] \to \mathcal{K}$ *be a given function. If it exists, the left derivative of* F *at a point* $t_0 \in (a,b]$ *is unique. Indeed, suppose that* $A(t_0), B(t_0)$ *are left derivatives of* F *at* $t_0 \in (a,b]$*. Then from the properties of the metric* $\mathcal{H}$ *and (2.1)* *or (2.2) it follows that*

$$\begin{aligned}
\mathcal{H}(A(t_0),B(t_0)) &= \frac{1}{h}\mathcal{H}(hA(t_0),hA(t_0)) \\
&= \frac{1}{h}\mathcal{H}(F(t_0-h)+hA(t_0),F(t_0-h)+hB(t_0)) \\
&\leq \frac{1}{h}\mathcal{H}(F(t_0-h)+hA(t_0),F(t_0)) \\
&\quad +\frac{1}{h}\mathcal{H}(F(t_0-h)+hB(t_0),F(t_0)) \\
&\to 0 \text{ as } h\to 0^+
\end{aligned}$$

or

$$\begin{aligned}
\mathcal{H}(A(t_0),B(t_0)) &= \frac{1}{h}\mathcal{H}(h(-A(t_0)),h(-A(t_0))) \\
&= \frac{1}{h}\mathcal{H}(F(t_0-h)-hA(t_0),F(t_0-h)-hB(t_0)) \\
&\leq \frac{1}{h}\mathcal{H}(F(t_0-h)-hA(t_0),F(t_0)) \\
&\quad +\frac{1}{h}\mathcal{H}(F(t_0-h)-hB(t_0),F(t_0)) \\
&\to 0 \text{ as } h\to 0^+,
\end{aligned}$$

respectively. Therefore, $\mathcal{H}(A(t_0),B(t_0))=0$ *and so* $A(t_0)=B(t_0)$. *Also, we remark that the conditions (2.1) and (2.2) may not be equivalent as we see in the following example.*

Example 6. *Consider the function* $F:\mathbb{R}\to\mathcal{K}$ *given by* $F(t)=[0,|t|]$. *For* $A=[a^-,a^+]\in\mathcal{K}$, *we have that*

$$\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}(F(0-h),F(0)-hA)=\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}([0,h],[-ha^+,-ha^-]$$

$$=\lim_{h\to 0^+}\tfrac{1}{h}\max\{|ha^+|,|h+ha^-|\}=\max\{|a^+|,|1+a^-|\}=0$$

only if $a^-=-1<a^+=0$. *It follows that (2.2) holds. Therefore,* F *is left differentiable at* $t_0=0$ *and* $F'_-(0)=[-1,0]$. *Since* $\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(0),F(0-h)+hA)=0$ *only if* $a^-=0>a^+=-1$, *then there exists a contradiction with the assumption that* $A=[a^-,a^+]\in\mathcal{K}$, *and so (2.1) does not hold.*

We say that F is *right differentiable* at $t_0\in[a,b)$ if there exists an element $A\in\mathcal{K}$ such that

$$\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0+h),F(t_0)+hA)=0 \tag{2.3}$$

or

$$\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0),F(t_0+h)-hA)=0. \tag{2.4}$$

The element $A\in\mathcal{K}$ is called a *right derivative* of F at t_0 and it will be denoted by $F'_+(t_0)$. F is said to be *right differentiable* on $[a,b)$, if F is right differentiable at each $t_0\in[a,b)$.

Remark 7. *Using the same reasoning as in Remark 5, we can show that if it exists, the right derivative of F at a point $t_0\in[a,b)$ is unique. Also, we remark that the conditions (2.3) and (2.4) may not be equivalent as we see in the following example.*

Example 8. *Consider the function $F:\mathbb{R}\to\mathcal{K}$ given by $F(t)=[0,|t|]$. For $A=[a^-,a^+]\in\mathcal{K}$, we have that*

$$\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}(F(0+h),F(0)+hA)=\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}([0,h],[ha^-,ha^+]$$

$$=\lim_{h\to 0^+}\tfrac{1}{h}\max\{|ha^+|,|h-ha^-|\}=\max\{|a^+|,|1-a^-|\}=0$$

only if $a^-=0<a^+=1$. It follows that (2.3) holds. Therefore, F is right differentiable at $t_0=0$ and $F'_+(0)=[0,1]$. Since $\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(0),F(0+h)-hA)=0$ only if $a^-=1>a^+=0$, then there exists a contradiction with the assumption that $A=[a^-,a^+]\in\mathcal{K}$, and so (2.4) does not hold.

We say that F is *differentiable* at $t_0\in[a,b]$ if F is left and right differentiable at t_0, and $F'_-(t_0)=F'_+(t_0)$. The element $F'_-(t_0)$ or $F'_+(t_0)$ will be denoted by $F'(t_0)$ and it is called a *derivative* of F at t_0. F is said to be *differentiable* on $[a,b]$, if F is differentiable at each $t_0\in[a,b]$. At the end points of $[a,b]$, we consider only the one-side derivatives.

Remark 9. *From Remarks 5 and 7, it is clear if it exists, the derivative of F at a point $t_0\in[a,b]$ is unique.*

Example 10. *Let $F:\mathbb{R}\to\mathcal{K}$ be the function given by $F(t)=[0,|t|]$. From Examples 8 and 10 we have that $F'_-(0)\neq F'_+(0)$, and so F is not differentiable at $t_0=0$.*

Theorem 11. *If $F:[a,b]\to\mathcal{K}$ is left (right) differentiable at $t_0\in(a,b]$ ($t_0\in[a,b)$), then F is left (right) continuous at t_0. In particular, if F is differentiable at t_0, then F is continuous at t_0.*

Proof. Suppose that F is left differentiable at t_0 and $F'_-(t_0)=A$ and let $\varepsilon>0$. Then from (2.1) or (2.2) it follows that there exists a $\delta>0$ such that for all $h\in(0,\delta)$ we have

$$\begin{aligned}\mathcal{H}(F(t_0-h),F(t_0)) &\leq \mathcal{H}(F(t_0-h),F(t_0)+hA)+\mathcal{H}(F(t_0)+hA,F(t_0))\\ &= \mathcal{H}(F(t_0-h),F(t_0)+hA)+\mathcal{H}(hA,\theta)\\ &\leq \varepsilon h+h\mathcal{H}(A,\theta)\end{aligned}$$

or

$$\begin{aligned}\mathcal{H}(F(t_0-h),F(t_0)) &\leq \mathcal{H}(F(t_0-h),F(t_0)-hA)+\mathcal{H}(F(t_0)-hA,F(t_0))\\ &= \mathcal{H}(F(t_0-h),F(t_0)-hA)+\mathcal{H}(h(-A),\theta)\\ &\leq \varepsilon h+h\mathcal{H}(-A,\theta),\end{aligned}$$

respectively. Therefore, $\lim_{h\to 0^+}\mathcal{H}(F(t_0-h),F(t_0))=0$, and so F is left continuous at t_0. The proof is similar when it is assumed that F is right differentiable at t_0. □

Remark 12. *From the definition it follows that a function $F:[a,b]\to\mathcal{K}$ is differentiable at $t_0\in[a,b]$ if there exists an $A\in\mathcal{K}$ such that one of the following conditions is true*

$$\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0+h),F(t_0)+hA)=\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0),F(t_0-h)+hA)=0, \quad (2.5)$$

$$\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0),F(t_0+h)-hA)=\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0-h),F(t_0)-hA)=0, \quad (2.6)$$

$$\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0),F(t_0+h)-hA)=\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0),F(t_0-h)+hA)=0, \quad (2.7)$$

$$\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0+h),F(t_0)+hA)=\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0-h),F(t_0)-hA)=0. \quad (2.8)$$

The element $A\in\mathcal{K}$ is the derivative of F at t_0; that is, $A=F'(t_0)$.

Example 13. *Consider the function $F:\mathbb{R}\to\mathcal{K}$ given by $F(t)=[-t^2,t^2]$. For $A=[a^-,a^+]\in\mathcal{K}$, we have that*

$$\begin{aligned}&\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}(F(t),F(t-h)+hA)=\\ &=\lim_{h\to 0^+}\tfrac{1}{h}\max\{|2th-h^2+ha^-|,|-2th+h^2+ha^+|\}\\ &=\max\{|2t+a^-|,|-2t+a^+|\}=0\end{aligned}$$

only if $a^-=-2t\leq a^+=2t$ and $t\geq 0$. Also, we have that

$$\begin{aligned}&\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}(F(t-h),F(t)-hA)=\\ &=\lim_{h\to 0^+}\tfrac{1}{h}\max\{|2th-h^2+ha^+|,|-2th+h^2+ha^-|\}\\ &=\max\{|2t+a^+|,|-2t+a^-|\}=0\end{aligned}$$

only if $a^- = 2t < a^+ = -2t$ *and* $t < 0$*. It follows that* F *is left differentiable at* $t \in \mathbb{R}$ *and*

$$F'_-(t) = \begin{cases} [-2t, 2t] & \text{if } t \geq 0 \\ [2t, -2t] & \text{if } t < 0. \end{cases}$$

Further, we have that

$$\lim_{h \to 0^+} \tfrac{1}{h} \mathcal{H}(F(t+h), F(t) + hA) =$$

$$= \lim_{h \to 0^+} \tfrac{1}{h} \max\{| - 2th - h^2 - ha^-|, |2th + h^2 - ha^+|\}$$

$$= \max\{| - 2t - a^-|, |2t - a^+|\} = 0$$

only if $a^- = -2t \leq a^+ = 2t$ *and* $t \geq 0$*. Also, we have that*

$$\lim_{h \to 0^+} \tfrac{1}{h} \mathcal{H}(F(t), F(t+h) - hA) =$$

$$= \lim_{h \to 0^+} \tfrac{1}{h} \max\{| - 2th - h^2 - ha^+|, |2th + h^2 - ha^-|\}$$

$$= \max\{| - 2t - a^+|, |2t - a^-|\} = 0$$

only if $a^- = 2t < a^+ = -2t$ *and* $t < 0$*. Therefore,* F *is right differentiable at each* $t \in \mathbb{R}$ *and*

$$F'_+(t) = \begin{cases} [-2t, 2t] & \text{if } t \geq 0 \\ [2t, -2t] & \text{if } t < 0. \end{cases}$$

Since $F'_+(t) = F'_-(t)$ *for all* $t \in \mathbb{R}$*, it follows that* F *is differentiable at each* $t \in \mathbb{R}$*. We remark that*

$$\lim_{h \to 0^+} \frac{1}{h} \mathcal{H}(F(t+h), F(t) + hA) = \lim_{h \to 0^+} \frac{1}{h} \mathcal{H}(F(t), F(t-h) + hA) = 0$$

for $A = [-2t, 2t]$ *and* $t > 0$*; that is, (2.5) holds for each* $t > 0$*, but (2.6)-(2.8) do not hold for* $t > 0$*. Also,*

$$\lim_{h \to 0^+} \frac{1}{h} \mathcal{H}_{\mathcal{I}}(F(t), F(t+h) - hA) = \lim_{h \to 0^+} \frac{1}{h} \mathcal{H}(F(t-h), F(t) - hA) = 0$$

for $A = [2t, -2t]$ *and* $t < 0$*; that is, (2.8) holds for each* $t > 0$*, but (2.5)-(2.7) do not hold for* $t > 0$*. If* $t = 0$*, then it is easy to check that* F *is differentiable at* $t = 0$*,* $F'(0) = \{0\}$*, and (2.5)-(2.8) are equivalent for* $A = \{0\}$*.*

Remark 14. *In [3], a function* $F : [a, b] \to \mathcal{K}$ *is called differentiable at* $t_0 \in [a, b]$ *if there exists an element* $A \in \mathcal{F}$ *such that*

$$\lim_{h \to 0^+} \frac{1}{h} \mathcal{H}(F(t_0 + h), F(t_0) + hA) = 0.$$

In this case, the element $A \in \mathcal{K}$ is called the derivative of F at t_0. In [7] it is shown that the function $F : \mathbb{R} \to \mathcal{K}$ given by $F(t) = [e^{-t}, 2e^{-t}]$ is not differentiable in this sense since, for a $t_0 \in \mathbb{R}$ and $A = [a^-, a^+] \in \mathcal{K}$, we have that $\lim_{h\to 0^+} \frac{1}{h}\mathcal{H}(F(t_0 + h), F(t_0) + hA) = 0$ only if $a^- = -e^{-t_0} > a^+ = -2e^{-t_0}$ which is a contradiction with the assumption that $A = [a^-, a^+] \in \mathcal{K}$. However, for $t_0 \in \mathbb{R}$ and $A = [-2e^{-t_0}, -e^{-t_0}] \in \mathcal{K}$ we have that

$$
\begin{aligned}
&\lim_{h\to 0^+} \tfrac{1}{h}\mathcal{H}(F(t_0), F(t_0 + h) - hA) = \\
&= \lim_{h\to 0^+} \tfrac{1}{h}\mathcal{H}([e^{-t_0}, 2e^{-t_0}], [e^{-t_0-h}, 2e^{-t_0-h}] - h[-2e^{-t_0}, -e^{-t_0}]) \\
&= \lim_{h\to 0^+} \tfrac{1}{h}\mathcal{H}([e^{-t_0}, 2e^{-t_0}], [e^{-t_0-h} + he^{-t_0}, 2e^{-t_0-h} + 2he^{-t_0}]) \\
&= \lim_{h\to 0^+} \tfrac{1}{h} \max\left\{|e^{-t_0-h} - e^{-t_0} + he^{-t_0}|, 2|e^{-t_0-h} - e^{-t_0} + he^{-t_0}|\right\} \\
&= \lim_{h\to 0^+} 2|e^{-t_0}\tfrac{e^{-h}-1}{h} + e^{-t_0}| = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{h\to 0^+} \tfrac{1}{h}\mathcal{H}(F(t_0 - h), F(t_0) - hA) = \\
&= \lim_{h\to 0^+} \tfrac{1}{h}\mathcal{H}([e^{-t_0+h}, 2e^{-t_0+h}], [e^{-t_0}, 2e^{-t_0}] - h[-2e^{-t_0}, -e^{-t_0}]) \\
&= \lim_{h\to 0^+} \tfrac{1}{h}\mathcal{H}([e^{-t_0+h}, 2e^{-t_0+h}], [e^{-t_0} + he^{-t_0}, 2e^{-t_0} + 2he^{-t_0}]) \\
&= \lim_{h\to 0^+} \tfrac{1}{h} \max\left\{|e^{-t_0+h} - e^{-t_0} - he^{-t_0}|, 2|e^{-t_0+h} - e^{-t_0} - he^{-t_0}|\right\} \\
&= \lim_{h\to 0^+} 2|e^{-t_0}\tfrac{e^{h}-1}{h} - e^{-t_0}| = 0.
\end{aligned}
$$

It follows that

$$
\lim_{h\to 0^+} \frac{1}{h}\mathcal{H}(F(t_0), F(t_0 + h) - hA) = \lim_{h\to 0^+} \frac{1}{h}\mathcal{H}(F(t_0 - h), F(t_0) - hA) = 0.
$$

Therefore, (2.6) holds and so F is differentiable at t_0 with $F'(t_0) = \left[-2e^{-t_0}, -e^{-t_0}\right]$.

Theorem 15. *Let $F : [a, b] \to \mathcal{K}$ be a given function. If there exists an $A \in \mathcal{K}$ such that (2.1) and (2.2) or (2.3) and (2.4) occur simultaneously, then A is a singleton.*

Proof. Suppose that (2.3) and (2.4) simultaneously hold. Since

$$\mathcal{H}(A-A,\theta)=\tfrac{1}{h}\mathcal{H}(hA-hA,\theta)=\tfrac{1}{h}\mathcal{H}(F(t_0)+hA-hA,F(t_0))$$

$$\leq \tfrac{1}{h}\mathcal{H}(F(t_0)+hA-hA,F(t_0+h)-hA)+\tfrac{1}{h}\mathcal{H}(F(t_0+h)-hA,F(t_0))$$

$$=\tfrac{1}{h}\mathcal{H}(F(t_0)+hA,F(t_0+h))+\tfrac{1}{h}\mathcal{H}(F(t_0),F(t_0+h)-hA)$$

$$\to 0 \text{ as } h\to 0^+,$$

it follows that $\mathcal{H}(A-A,\theta)=0$. Therefore, $A-A=\theta$ and so A is a singleton. A similar proof establishes the result if (2.1) and (2.2) simultaneously hold. □

Corollary 16. *If for a given function $F:[a,b]\to\mathcal{K}$ and $t_0\in[a,b]$ there exists an $A\in\mathcal{K}$ such that at least two from the conditions (2.5)-(2.8) occur simultaneously, then A is a singleton.*

Proposition 17. *If $F:[a,b]\to\mathcal{K}$ is a given function and $t_0\in(a,b)$. Then the following statements are true.*

(a) If there exists $A\in\mathcal{K}$ such that (2.5) holds, then

$$\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0+h),F(t_0-h)+2hA)=0. \tag{2.9}$$

(b) If there exists $A\in\mathcal{K}$ such that (2.6) holds, then

$$\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0-h),F(t_0+h)-2hA)=0. \tag{2.10}$$

(c) If there exists $A\in\mathcal{K}$ such that (2.7) or (2.8) holds, then

$$\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t_0+h)-hA,F(t_0-h)+hA)=0. \tag{2.11}$$

Proof. Suppose there exists $A\in\mathcal{K}$ such that (2.5) holds. Then we have

$$\begin{aligned}&\frac{1}{h}\mathcal{H}(F(t_0+h),F(t_0-h)+2hA)\\ =\;&\frac{1}{h}\mathcal{H}(F(t_0)+F(t_0+h),F(t_0)+F(t_0-h)+2hA)\\ \leq\;&\frac{1}{h}\mathcal{H}(F(t_0),F(t_0-h)+hA)+\frac{1}{h}\mathcal{H}(F(t_0+h),F(t_0)+hA)\to 0\end{aligned}$$

as $h\to 0^+$, and so (2.9) is true. With a similar reasoning we can prove statement (b). Now, suppose that there exists $A\in\mathcal{K}$ such that (2.7) holds. Then we have

$$\begin{aligned}&\frac{1}{h}\mathcal{H}(F(t_0+h)-hA,F(t_0-h)+hA)\\ =\;&\frac{1}{h}\mathcal{H}(F(t_0)+F(t_0+h)-hA,F(t_0)+F(t_0-h)+hA)\\ \leq\;&\frac{1}{h}\mathcal{H}(F(t_0),F(t_0-h)+hA)+\frac{1}{h}\mathcal{H}(F(t_0),F(t_0+h)-hA)\to 0\end{aligned}$$

as $h \to 0^+$, and so (2.11) is true. If there exists an $A \in \mathcal{K}$ such that (2.8) holds, then we have

$$\begin{aligned} & \frac{1}{h}\mathcal{H}(F(t_0+h)-hA, F(t_0-h)+hA) \\ = & \frac{1}{h}\mathcal{H}(F(t_0)+F(t_0+h)-hA, F(t_0)+F(t_0-h)+hA) \\ \leq & \frac{1}{h}\mathcal{H}(F(t_0)-hA, F(t_0-h))+\frac{1}{h}\mathcal{H}(F(t_0+h), F(t_0)+hA) \to 0 \end{aligned}$$

as $h \to 0^+$, and so (2.11) is again true. □

Theorem 18. *Let $F : [a, b] \to \mathcal{K}$ be a given function. If there exists an $A \in \mathcal{K}$ such that (2.7) or (2.8) holds, then A is a singleton.*

Proof. Suppose that (2.7) holds. Then (2.11) is also true and it follows that

$$\begin{aligned} & \left|\tfrac{1}{h}d(F(t_0), F(t_0+h)-hA) - \tfrac{1}{h}d(F(t_0), F(t_0-h)+hA)\right| \\ & \leq \tfrac{1}{h}d(F(t_0+h)-hA, F(t_0-h)+hA) \to 0 \text{ as } h \to 0^+; \end{aligned}$$

that is,

$$\lim_{h\to 0^+}\frac{1}{h}d(F(t_0), F(t_0+h)-hA) = \lim_{h\to 0^+}\frac{1}{h}d(F(t_0), F(t_0-h)+hA). \tag{2.12}$$

On the other hand, if we take in (2.1) $t = t+h$ we obtain $\lim\limits_{h\to 0^+}\frac{1}{h}d(F(t_0), F(t_0+h)-hA) = 0$. Then from (2.12) it follows that (2.8) also holds. Therefore (2.7) and (2.8) occur simultaneously and thus by Corollary 16 we infer that A is a singleton. A similar proof works if (2.8) holds. □

Remark 19. *From Theorem 15 and Theorem 18 it follows that a function $F : [a, b] \to \mathcal{K}$ can be differentiable in the sense of (2.5) or in the sense of (2.6). We will say that F is $\mathcal{H}^1$-*differentiable *if it is differentiable in the sense of (2.5). Also, we will say that F is $\mathcal{H}^2$-*differentiable *if it is differentiable in the sense of (2.6).*

Theorem 20. *Let $F, G : [a, b] \to \mathcal{K}$ be two given function.*

(a) If F is differentiable and $\lambda \in \mathbb{R}$, then the function λF is differentiable and $(\lambda F)' = \lambda F'$.

(b) If $F, G \in \mathcal{H}^i$ $(i = 1, 2)$ and $F \ominus G$, $F' \ominus G'$ exist, then $F + G$, $F \ominus G \in \mathcal{H}^i$ $(i = 1, 2)$ and

$$(F+G)' = F' + G',$$

$$(F \ominus G)' = F' \ominus G'.$$

(c) If $F \in \mathcal{H}^i$, $G \in \mathcal{H}^j$ $(i, j = 1, 2)$ for $i \neq j$ and $F \ominus G$, $F' \ominus (-G')$ exist, then $F + G$, $F \ominus G \in \mathcal{H}^i$ and

$$(F + G)' = F' \ominus (-G'),$$

$$(F \ominus G)' = F' + (-G').$$

Proof. (a) is obvious. (b) Suppose that $F, G \in \mathcal{H}^1$. Using Proposition 1 and Propostion 2 we have

$$\begin{aligned}
& \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}((F \ominus G)(t+h), (F \ominus G)(t) + h(F' \ominus G')(t)) \\
= & \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}(F(t+h) \ominus G(t+h), (F(t) + hF'(t)) \ominus (G(t) + hG'(t))) \\
\leq & \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}(F(t+h), F(t) + hF'(t)) \\
& + \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}(G(t+h), G(t) + hG'(t)) = 0
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}((F \ominus G)(t), (F + G)(t-h) \ominus h(F' + G')(t)) \\
= & \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}(F(t) + G(t), (F(t-h) + hF'(t)) \ominus (G(t-h) + hG'(t))) \\
\leq & \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}(F(t), F(t-h) + hF'(t)) \\
& + \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}(G(t), G(t-h) + hG'(t)) = 0.
\end{aligned}$$

It follows that $F \ominus G \in \mathcal{H}^1$ and $(F \ominus G)' = F' \ominus G'$. Also, it is easy to check that $F + G \in \mathcal{H}^1$ and $(F + G)' = F' + G'$. A similar proof establishes the result if $F, G \in \mathcal{H}^2$. (c) Suppose that $F \in \mathcal{H}^1$ and $G \in \mathcal{H}^2$. Then using Propostion 2 we have

$$\begin{aligned}
& \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}((F + G)(t+h), (F + G)(t) + h(F' \ominus (-G'))(t)) \\
= & \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}(F(t+h) + G(t+h), F(t) + G(t) + hF'(t)) \ominus (-hG'(t))) \\
\leq & \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}(F(t+h), F(t) + hF'(t)) \\
& + \lim_{h\to 0^+} \frac{1}{h} \mathcal{H}(G(t), G(t+h) - hG'(t)) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}((F+G)(t),(F+G)(t-h)+h(F'\ominus(-G'))(t))\\
=&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t)+G(t),(F(t-h)+G(t-h)+hF'(t))\ominus(-hG'(t)))\\
\leq&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t),F(t-h)+hF'(t))\\
&+\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(G(t),G(t-h)-hG'(t))=0.
\end{aligned}$$

It follows that $F+G\in\mathcal{H}^1$ and $(F+G)'=F'\ominus(-G')$. Also we have

$$\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}((F\ominus G)(t+h),(F\ominus G)(t)+h(F'+(-G'))(t))\\
=&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t+h)\ominus G(t+h),F(t)\ominus G(t)+hF'(t)-hG'(t))\\
\leq&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t+h),F(t)+hF'(t))\\
&+\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(G(t),G(t+h)-hG'(t))=0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}((F\ominus G)(t),(F\ominus G)(t-h)+h(F'+(-G'))(t))\\
=&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t)\ominus G(t),F(t-h)\ominus G(t-h)+hF'(t)-hG'(t))\\
\leq&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t-h),F(t)+hF'(t))\\
&+\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(G(t-h),G(t)-hG'(t))=0.
\end{aligned}$$

It follows that $F\ominus G\in\mathcal{H}^1$ and $(F\ominus G)'=F'+(-G')$. Now, we suppose that $F\in\mathcal{H}^2$ and $G\in\mathcal{H}^1$. Then using Propostion 2, we obtain that

$$\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}((F+G)(t),(F+G)(t+h)-h(F'\ominus(-G'))(t))\\
=&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t)+G(t),F(t+h)+G(t+h)+(-hF'(t))\ominus(hG'(t)))\\
\leq&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t),F(t+h)-hF'(t))\\
&+\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(G(t+h),G(t)+hG'(t))=0
\end{aligned}$$

and

$$
\begin{aligned}
& \lim_{h\to 0^+}\frac{1}{h}\mathcal{H}((F+G)(t-h),(F+G)(t)-h(F'\ominus(-G'))(t)) \\
= & \lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t-h)+G(t-h),F(t)+G(t)+(-hF'(t))\ominus(hG'(t))) \\
\leq & \lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t-h),F(t)-hF'(t)) \\
& +\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(G(t),G(t-h)+hG'(t))=0
\end{aligned}
$$

It follows that $F+G\in\mathcal{H}^2$ and $(F+G)'=F'\ominus(-G')$. Finally using Propostion 2, we have that

$$
\begin{aligned}
& \lim_{h\to 0^+}\frac{1}{h}\mathcal{H}((F\ominus G)(t),(F\ominus G)(t+h)-h(F'+(-G'))(t)) \\
= & \lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t)\ominus G(t),F(t+h)\ominus G(t+h)+(-hF'(t))+hG'(t)) \\
\leq & \lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t),F(t+h)-hF'(t)) \\
& +\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(G(t+h),G(t)+hG'(t))=0
\end{aligned}
$$

and

$$
\begin{aligned}
& \lim_{h\to 0^+}\frac{1}{h}\mathcal{H}((F\ominus G)(t-h),(F\ominus G)(t)-h(F'+(-G'))(t)) \\
= & \lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t-h)\ominus G(t-h),F(t)\ominus G(t)+(-hF'(t))+hG'(t)) \\
\leq & \lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t-h),F(t)-hF'(t)) \\
& +\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(G(t),G(t-h)+hG'(t))=0.
\end{aligned}
$$

It follows that $F\ominus G\in\mathcal{H}^2$ and $(F\ominus G)'=F'+(-G')$. □

Theorem 21. *If $F:[a,b]\to\mathcal{K}$ is left (right) gH-differentiable at $t_0\in(a,b]$ ($t_0\in[a,b)$), then F is left (right) differentiable at $t_0\in(a,b]$ ($t_0\in[a,b)$) and $D_H^-F(t_0)=F'_-(t_0)$ ($D_H^+F(t_0)=F'_+(t_0)$).*

Proof. If F is left gH-differentiable at $t_0\in(a,b]$, then there exist an element $A=D_H^-F(t_0)\in\mathcal{K}$ such that

$$
\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}(F(t_0),F(t_0-h)+hA)=\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}(F(t_0)\ominus F(t_0-h),hA)
$$

$$
=\lim_{h\to 0^+}\mathcal{H}\left(\tfrac{1}{h}(F(t_0)\ominus F(t_0-h)),A\right)=0.
$$

or

$$\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}(F(t_0-h),F(t_0)-hA)=\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}(F(t_0-h)\ominus F(t_0),(F(t_0)-hA)\ominus F(t_0))$$

$$\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}(F(t_0-h)\ominus F(t_0),-hA)=\lim_{h\to 0^+}\tfrac{1}{h}\mathcal{H}(-(F(t_0)\ominus F(t_0-h)),-hA)$$

$$=\lim_{h\to 0^+}\mathcal{H}\left(\tfrac{1}{h}(F(t_0)\ominus F(t_0-h)),A\right)=0.$$

It follows that F is left differentiable at $t_0\in[a,b)$ and $F'_-(t_0)=D_H^-F(t_0)$. A similar proof establishes the result if F is right gH-differentiable at $t_0\in[a,b)$.

□

Corollary 22. *If $F:[a,b]\to\mathcal{Q}$ is H-differentiable at $t_0\in[a,b]$, then F is differentiable at $t_0\in[a,b]$ and $D_HF(t_0)=F'(t_0)$.*

Remark 23. *The converse of the theorem is not true in general as we will show in next example.*

Example 24. *Consider the function $F:[a,b]\to\mathcal{K}$ defined by $F(t)=(2+\sin t)[-1,1]$, $t\in(0,2\pi)$. Then for any $t\in(0,2\pi)$ and $U=[-1,1]$, we have*

$$\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t+h),F(t)+h\cos t\cdot U)\\
=&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}((2+\sin(t+h))U,(2+\sin t)U+(h\cos t)U)\\
=&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(\sin(t+h)U,(\sin t+h\cos t)U)\\
=&\lim_{h\to 0^+}\frac{1}{h}\left|\sin(t+h)-\sin t-h\cos t\right|\mathcal{H}_{\mathcal{K}}(U,\theta))\\
=&\lim_{h\to 0^+}\left|\frac{\sin(t+h)-\sin t}{h}-\cos t\right|=0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}(F(t),F(t-h)+h\cos t\cdot U)\\
=&\lim_{h\to 0^+}\frac{1}{h}\mathcal{H}((2+\sin t)U,(2+\sin(t-h))U+(h\cos t)U)\\
=&\lim_{h\to 0^+}\left|\frac{\sin t-\sin(t-h)}{h}-\cos t\right|=0.
\end{aligned}$$

It follows that F is differentiable (in fact, $\mathcal{H}^2$-differentiable) on $(0,2\pi)$ and $F'(t)=(\cos t)U$, $t\in(0,2\pi)$. On the other hand, F is not right gH-differentiable nor left gH-differentiable on $(0,2\pi)$ since diam$(F(t))=2(2+\sin t)$ changes its monotonicity on $(0,2\pi)$ (see [1, Theorem 4.1]).

3 Riemann integral for interval-valued functions

Let $F : [a, b] \to \mathcal{K}$ be a given function. For each finite partition $\Delta_n = \{t_0, t_1, ..., t_n\}$, $a = t_0 < t_1 < ... < t_n = b$, of interval $[a, b]$ and for arbitrary system $\xi = (\xi_1, \xi_2, ..., \xi_n)$ of intermediate points $\xi_i \in [t_{i-1}, t_i]$, $i = 1, 2, ..., n$, we consider Riemann sum

$$R_F(\Delta_n, \xi) = \sum_{i=1}^{n}(t_i - t_{i-1})F(\xi_i) \text{ and } |\Delta_n| := \max_{1\le i\le n}(t_i - t_{i-1}).$$

We say that the function $F : [a, b] \to \mathcal{K}$ is *Riemann integrable* on $[a, b]$ if there exists an $A \in \mathcal{K}$ such that for each $\varepsilon > 0$, there exists a $\delta > 0$ so that if Δ_n is any partition of $[a, b]$ and ξ an arbitrary system of intermediate points, then

$$\mathcal{H}(R_F(\Delta_n, \xi), A) < \varepsilon.$$

We write $A = \int_a^b F(t)dt$. It easy to see that, if $F : [a, b] \to \mathcal{K}$ is Riemann integrable on $[a, b]$, then the value of the integral is unique. We observe that, for each partition Δ_n of $[a, b]$ with $|\Delta_n| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} R_F(\Delta_n, \xi) = \int_a^b F(t)dt.$$

Theorem 25. *If $F : [a, b] \to \mathcal{K}$ is a continuous function, then F is Riemann integrable on $[a, b]$.*

Proof. As in the classical proof, the uniform continuity of F implies that $R_F(\Delta_n, \xi)$ is a Cauchy sequence for all sequences of partitions which have $|\Delta_n| \to 0$ as $n \to \infty$. Consideration of the interleaved sequence $R_F(\Delta_1, \xi)$, $R_F(\Delta_1', \xi)$, $R_F(\Delta_2, \xi)$, $R_F(\Delta_2', \xi)$,... shows that all sequences $R_F(\Delta_n, \xi)$ and $R_F(\Delta_n', \xi)$ for which $|\Delta_n| \to 0$ and $|\Delta_n'| \to 0$ as $n \to \infty$, have the same limit. $\square$

Let $F, G : [a, b] \to \mathcal{K}$ be Riemann integrable on $[a, b]$. Then the following properties are obvious by passing to the limit form corresponding relations for Riemann sums.

(a) For each $\alpha, \beta \in \mathbb{R}$, $\alpha F + \beta G$ is Riemann integrable on $[a, b]$ and

$$\int_a^b (\alpha F(t) + \beta G(t))dt = \alpha \int_a^b F(t)dt + \beta \int_a^b G(t)dt. \tag{3.1}$$

(b) F is Riemann integrable on each subinterval of $[a, b]$ and

$$\int_a^b F(t)dt = \int_a^c F(t)dt + \int_c^b F(t)dt,\ a \le c \le b. \tag{3.2}$$

(c) $t \mapsto \mathcal{H}(F(t), G(t))$ is Riemann integrable on $[a, b]$, and

$$\mathcal{H}\left(\int_a^b F(t)dt, \int_a^b G(t)dt\right) \leq \int_a^b \mathcal{H}(F(t), G(t))dt. \tag{3.3}$$

(d)

$$\frac{1}{b-a}\int_a^b F(t)dt \in \overline{co}\{F(t); t \in [a, b]\}, \tag{3.4}$$

where $\overline{co}\mathcal{M}$ means the closed convex hull of subset $\mathcal{M} \subset \mathcal{K}$.

Theorem 26. *If $F : [a, b] \to \mathcal{K}$ is a continuous function on $[a, b]$, then the function $G : [a, b] \to \mathcal{Q}$, defined by*

$$G(t) = \int_a^t F(s)ds, \ t \in [a, b], \tag{3.5}$$

is $\mathcal{H}^1$-differentiable on $[a, b]$ and $G'(t) = F(t)$ for each $t \in [a, b]$.

Proof. Let $t \in [a, b]$ and $h > 0$ such that $t + h, t - h \in [a, b]$. Since

$$\frac{1}{h}\mathcal{H}(G(t), G(t-h) + hF(t)) \leq \frac{1}{h}\int_{t-h}^t \mathcal{H}\left(F(s), F(t)\right) ds,$$

$$\frac{1}{h}\mathcal{H}(G(t-h), G(t) - hF(t)) \leq \frac{1}{h}\int_{t-h}^t \mathcal{H}\left(\theta, F(s) - F(t)\right) ds,$$

$$\frac{1}{h}\mathcal{H}(G(t+h), G(t) + hF(t)) \leq \frac{1}{h}\int_t^{t+h} \mathcal{H}\left(F(s), F(t)\right) ds,$$

$$\frac{1}{h}\mathcal{H}(G(t), G(t+h) - hF(t)) \leq \frac{1}{h}\int_t^{t+h} \mathcal{H}\left(\theta, F(s) - F(t)\right) ds,$$

and

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h} \mathcal{H}\left(F(s), F(t)\right) ds = \lim_{h\to 0^+}\frac{1}{h}\int_{t-h}^t \mathcal{H}\left(F(s), F(t)\right) ds = 0,$$

we infer that G is $\mathcal{H}^1$-differentiable on $[a, b]$ and $G'(t) = F(t)$ for each $t \in [a, b]$. □

Theorem 27. *Let $F : [a, b] \to \mathcal{K}$ be a differentiable fuction on $[a, b]$ such that F' is continuous on $[a, b]$.*

(a) *If F is $\mathcal{H}^1$-differentiable, then*

$$F(t) = F(a) + \int_a^t F'(s)ds \tag{3.6}$$

for any $t \in [a, b]$.

(b) *If F is $\mathcal{H}^2$-differentiable, then*

$$F(t) = F(a) \ominus \left(-\int_a^t F'(s)ds\right) \tag{3.7}$$

for any $t \in [a,b]$.

Proof. (a) Suppose that F is $\mathcal{H}^1$-differentiable on $[a,b]$ and F' is continuous on $[a,b]$. If we put $G(t) := F(a) + \int_a^t F'(s)ds$, $t \in [a,b]$, then $G'(t) = F'(t)$ for all $t \in [a,b]$. Define $m(t) := \mathcal{H}(F(t), G(t))$, $t \in [a,b]$. Then, we have

$$\begin{aligned} m(t+h) - m(t) &= \mathcal{H}(F(t+h), G(t+h)) - \mathcal{H}(F(t), G(t)) \\ &\leq \mathcal{H}(F(t+h), F(t) + hF'(t)) + \\ &\quad + \mathcal{H}(F(t) + hF'(t), G(t) + hF'(t)) \\ &\quad + \mathcal{H}(G(t) + hF'(t), G(t+h)) - \mathcal{H}(F(t), G(t)) \\ &= \mathcal{H}(F(t+h), F(t) + hF'(t)) + \\ &\quad + \mathcal{H}(G(t+h), G(t) + hF'(t)), \end{aligned}$$

and thus

$$\begin{aligned} D^+m(t) &= \limsup_{h \to 0^+} \frac{m(t+h) - m(t)}{h} \\ &\leq \lim_{h \to 0^+} \frac{1}{h} \mathcal{H}(F(t+h), F(t) + hF'(t)) \\ &\quad + \lim_{h \to 0^+} \frac{1}{h} \mathcal{H}(G(t+h), G(t) + hF'(t)) = 0. \end{aligned}$$

Therefore, $D^+m(t) \leq 0$ for all $t \in [a,b)$ and so m is a decreasing function on $[a,b]$. Since $m(a) = 0$, it follows that $m(t) \leq m(a) = 0$ for all $t \in [a,b]$. On the other hand, we have that $m(t) \geq 0$ for all $t \in [a,b]$ and so $m(t) = 0$ for all $t \in [a,b]$; that is, (3.6). (b) Suppose that F is $\mathcal{H}^2$-differentiable on $[a,b]$. If we put $G(t) := F(a) \ominus \left(-\int_a^t F'(s)ds\right)$, $t \in [a,b]$, then by Corollary 3 in [4] it follows that $G'(t) = F'(t)$ for all $t \in [a,b]$. As above we obtain that $m(t) = 0$ for all $t \in [a,b]$; that is, (3.7). □

Conclusion. *This new concept of differentiability for interval-valued functions avoids the use of generalized Hukuhara difference. It is well known that the generalized difference Hukuhara $A \ominus B$ does not generally exist if A and B are compact sets in R^n with $n \geq 2$ or if A and B are fuzzy sets. Therefore, this new concept of differentiability can be much more efficient in these situations than the concepts previously known. The extension of these results to fuzzy functions, as well as their applications to differential equations, will be developed in a few future works. We will also extend this concept to functions with valuesin much more general spaces, namely to functions with values in quasilinear spaces [5].*

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