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STABILITY IN NONLINEAR NEUTRAL LEVIN-NOHEL INTEGRO-DYNAMIC EQUATIONS

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Abstract. In this paper we use the Krasnoselskii-Burton's fixed point theorem to obtain asymptotic stability and stability results about the zero solution for the following nonlinear neutral Levin-Nohel integro-dynamic equation

$$x^{\Delta}(t)+\int_{t-\tau(t)}^{t}a(t,s)g\left(x(s)\right)\Delta s+c(t)x^{\tilde{\Delta}}(t-\tau(t))=0.$$

The results obtained here extend the work of Ali Khelil, Ardjouni and Djoudi [5].

1 Introduction

In 1988, Stephan Hilger [24] has initiated the theory of calculus on time scales to unify discrete and continuous analysis for the aim of combining the study of differential and difference equations. Hilger's work has been the foundation of so many investigations in the theory of dynamic equations and has received much attention since its publication.

The study of Levin-Nohel equations brings the traditional research areas of differential and difference equations. It allows researchers to handle these two research areas at the same time, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying more general time scales cases (see [1]-[6], [10], [28]-[30]).

In particular, the fixed point theorem was applied to deduce stability conditions, see also the papers ([7]-[19], [22], [23], [25]-[27]) where different techniques are used to study stability of delay dynamic equations. While, the Lyapunov direct method has been very effective in establishing stability results and the existence of periodic solutions for wide variety of ordinary, functional and partial differential equations. Nevertheless, in the application of Lyapunov's direct method to problems of stability in delay differential equations, serious difficulties occur if the delay is unbounded or if the equation has unbounded terms. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Zhang

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and others began a study in which they noticed that some of this difficulties vanish or might be overcome by means of fixed point theory (see [21], [32]). The fixed point theory does not only solve the problem on stability but has other significant advantage over Lyapunov's direct method. The conditions of the former are often average but those of the latter are usually pointwise (see [20]).

In [5], Ali Khelil, Ardjouni and Djoudi have used the Krasnoselskii-Burton's fixed point theorem to obtain asymptotic stability results about the zero solution for the following nonlinear neutral Levin-Nohel integro-differential equation

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s)g\left(x(s)\right) ds + c(t)x'(t-\tau(t)) = 0.$$

The aim of this paper is to extend the theory established in [5] to neutral Levin-Nohel integro-dynamic equations on time scales. More precisely, we consider the equation

$$x^{\Delta}(t) + \int_{t-\tau(t)}^{t} a(t,s)g\left(x(s)\right) \Delta s + c(t)x^{\widetilde{\Delta}}(t-\tau(t)) = 0,\ t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

with an assumed initial condition

$$x\left(t\right) = \phi\left(t\right),\ \ t \in \left[m\left(t_0\right), t_0\right]_{\mathbb{T}},$$

where $\phi \in C_{rd}\left(\left[m\left(t_0\right), t_0\right]_{\mathbb{T}}, \mathbb{R}\right)$ and

$$m\left(t_0\right) = \inf\left\{t - \tau\left(t\right) : t \in [t_0,\infty)_{\mathbb{T}}\right\}.$$

In order for the functions $x\left(t-\tau(t)\right)$ to be well-defined over $[t_0,\infty)_{\mathbb{T}}$, we assume that $\tau : [t_0,\infty)_{\mathbb{T}} \to \mathbb{T}$ is positive rd-continuous, and that $id-\tau : [t_0,\infty)_{\mathbb{T}} \to \mathbb{T}$ is increasing mapping such that $\left(id-\tau\right)\left([t_0,\infty)_{\mathbb{T}}\right)$ is closed where id is the identity function. Throughout this paper, we assume that $c \in C_{rd}^{1}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}\right)$, $a \in C_{rd}([t_0,\infty)_{\mathbb{T}} \times [m\left(t_0\right),\infty)_{\mathbb{T}}, \mathbb{R}_{+})$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous with respect to its argument. We assume that $g\left(0\right) = 0$ and $\tau \in C_{rd}^{2}\left([t_0,\infty)_{\mathbb{T}}, (0,\infty)_{\mathbb{T}}\right)$ such that

$$\tau^{\Delta}\left(t\right) \neq 1,\ \ t \in [t_0,\infty)_{\mathbb{T}}. \tag{1.2}$$

Our purpose here is to use the Krasnoselskii-Burton's fixed point theorem to show the asymptotic stability and stability of the zero solution for (1.1).

2 Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above and below. Throughout this paper, intervals subscripted with a $\mathbb{T}$ represent real intervals intersected with $\mathbb{T}$. For example, $a, b \in \mathbb{T}$, $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$.

We begin this section by considering some advanced topics in the theory of dynamic equations on time scales. Most of the following definitions, lemmas and theorems can be found in [12, 13].

Definition 1. *The forward and backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ *and the graininess function* $\mu : \mathbb{T} \to [0, \infty)$ *are defined, respectively, by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \ \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \ \mu(t) = \sigma(t) - t.$$

We make the assumption that $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is called right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, and left-scattered if $\rho(t) < t$. If $\mathbb{T}$ has a left-scattered maximum m, define $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$. Finally, if $f : \mathbb{T} \to \mathbb{R}$ we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}.$$

Definition 2. *A function* $f : \mathbb{T} \to \mathbb{R}$ *is called rd-continuous provided it is continuous at every right-dense point* $t \in \mathbb{T}$ *and its left-sided limits exist, and is finite at every left-dense point* $t \in \mathbb{T}$*. The set of rd-continuous functions* $f : \mathbb{T} \to \mathbb{R}$ *will be denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ *that are differentiable and whose derivative is rd-continuous is denoted by*

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 3. *For* $f : \mathbb{T} \to \mathbb{R}$*, we define* $f^\Delta(t)$ *to be the number (if it exists) with the property that for any given* $\varepsilon > 0$*, there exists a neighborhood* U *of* t *such that*

$$\left|(f(\sigma(t)) - f(s)) - f^\Delta(t)\,(\sigma(t) - s)\right| < \varepsilon\,|\sigma(t) - s| \ \textit{for all } s \in U.$$

The function $f^\Delta : \mathbb{T}^k \to \mathbb{R}$ *is called the delta (or Hilger) derivative of* f *on* $\mathbb{T}^k$*.*

Theorem 4. *Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^k$*. Then, we have the following,*

(i) *if* f *is differentiable at* t*, then* f *is continuous at* t*,*

(ii) *if* f *is continuous at* t *and* t *is right-scattered, then* f *is differentiable at* t *with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$

(iii) *if* t *is right-dense, then* f *is differentiable at* t *with*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Theorem 5. *Assume* $f, g : \mathbb{T} \to \mathbb{R}$ *at* $t \in \mathbb{T}^k$. *Then*
(*i*) $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$.
(*ii*) $(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$, *for any constant* α.
(*iii*) *If* $g(t)\, g(\sigma(t)) \neq 0$, *then*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)\, g(t) - f(t)\, g^{\Delta}(t)}{g(t)\, g(\sigma(t))}.$$

The next theorem is the integration by parts.

Theorem 6. *If* $a, b \in \mathbb{T}$ *and* $f, g \in C_{rd}$ *then,*
(*i*) $\int_a^b f(\sigma(t))\, g^{\Delta}(t)\, \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\, g(t)\, \Delta t$,
(*ii*) $\int_a^b f(t)\, g^{\Delta}(t)\, \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\, g(\sigma(t))\, \Delta t$.

The next theorem is the chain rule on time scales [13, Theorem 1.93]

Theorem 7 (Chain rule)**.** *Assume that* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. Let* $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. *If* $\nu^{\Delta}(t)$ *and* $\omega^{\widetilde{\Delta}}(\nu(t))$ *exist for* $t \in \mathbb{T}^k$, *then* $(\omega \circ \nu)^{\Delta} = \left(\omega^{\widetilde{\Delta}} \circ \nu\right) \nu^{\Delta}$.

In the sequel we will need to differentiate and integrate functions of the form $f(t - \tau(t)) = f(\nu(t))$, where $\nu(t) := t - \tau(t)$. Our next theorem is the substitution rule [13, Theorem 1.98]

Theorem 8 (Substitution)**.** *Assume that* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. If* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous function and* ν *is differentiable with rd-continuous derivative, then, for* $a, b \in \mathbb{T}$,

$$\int_a^b f(t)\, \nu^{\Delta}(t)\, \Delta t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right) \widetilde{\Delta} s.$$

Definition 9. *A function* $p : \mathbb{T} \to \mathbb{R}$ *is called regressive provided* $1 + \mu(t)p(t) \neq 0$ *for all* $t \in \mathbb{T}$. *The set of all regressive and rd-continuous functions* $p : \mathbb{T} \to \mathbb{R}$ *will be denoted by* $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. *We define the set* $\mathcal{R}^+$ *of all positively regressive elements of* $\mathcal{R}$ *by*

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}.$$

Definition 10. *Let* $p \in \mathcal{R}$, *then the generalized exponential function* e_p *is defined as the unique solution of the initial value problem*

$$x^{\Delta}(t) = p(t)x(t), \;\; x(s) = 1, \;\; \textit{where } s \in \mathbb{T}.$$

An explicit formula for $e_p(t, s)$ *is given by*

$$e_p(t, s) = \exp\left(\int_s^t \zeta_{\mu(\tau)}(p(\tau))\Delta\tau\right), \quad \textit{for all} \;\; s, t \in \mathbb{T},$$

with

$$\zeta_h(\tau) = \begin{cases} \frac{\log(1+h\tau)}{h} & \text{if } h \neq 0, \\ \tau & \text{if } h = 0, \end{cases}$$

where log *is the principal logarithm function.*

Lemma 11. *Let $p \in \mathcal{R}$, then*

(*i*) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$,

(*ii*) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))\, e_p(t,s)$,

(*iii*) $e_p^{\Delta}(t,s) = p(t)e_p(t,s)$,

(*iv*) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ *with* $\ominus p = -\frac{p}{1+\mu p}$,

(*v*) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,

(*vi*) $e_p(t,s)e_p(s,r) = e_p(t,r)$.

Lemma 12. *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(\tau)\Delta\tau\right),$$

for all $t \in [s,\infty)_{\mathbb{T}}$.

Theorem 13 (Variation of constants)**.** *Let $t_0 \in \mathbb{T}$, $p \in \mathcal{R}$ and $x_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$x^{\Delta}(t) = -p(t)\, x^{\sigma}(t) + f(t), \quad x(t_0) = x_0$$

is given by

$$x(t) = e_{\ominus p}(t,t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t,s)f(s)\,\Delta s.$$

3 The inversion and the fixed point theorems

One crucial step in the investigation of an equation using fixed point theory involves the construction of a suitable fixed point mapping. For that end we must invert (1.1) to obtain an equivalent integral equation from which we derive the needed mapping. During the process, an integration by parts has to be performed on the neutral term $x^{\widetilde{\Delta}}(t-\tau(t))$.

Lemma 14. *Suppose that (1.2) holds. Then x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) &= (\phi(t_0) + \gamma(t_0)\phi(t_0 - \tau(t_0)))\, e_{\ominus A}(t,t_0) \\ &\quad + \int_{t_0}^t \left(\int_{s-\tau(s)}^s a(s,u)\,(Gx)(u)du\right) e_{\ominus A}(t,s)\,\Delta s - \gamma(t)x(t-\tau(t)) \\ &\quad - \int_{t_0}^t \left[L_x(s) - \varrho(s)x^{\sigma}(s-\tau(s))\right] e_{\ominus A}(t,s)\,\Delta s, \ t \in [t_0,\infty)_{\mathbb{T}}, \end{aligned} \tag{3.1}$$

where

$$L_x(t) = \int_{t-\tau(t)}^{t} a(t,s)\left(\int_{s}^{\sigma(t)}\left(\int_{u-\tau(u)}^{u} a(u,v)x(v)dv - r(u)x^{\sigma}(u-\tau(u))\right)\Delta u\right.$$
$$\left. +\gamma^{\sigma}(t)\, x(\sigma(t) - \tau^{\sigma}(t)) - \gamma(s)\, x(s-\tau(s))\right)\Delta s \tag{3.2}$$

$$r(t) = \frac{c^{\Delta}(t)(1-\tau^{\Delta}(t)) + \tau^{\Delta\Delta}(t)c(t)}{(1-\tau^{\Delta}(t))\,(1-\tau^{\Delta}(\sigma(t)))},\ \ \gamma(t) = \frac{c(t)}{1-\tau^{\Delta}(t)}, \tag{3.3}$$

$$(Gx)(t) = x(t) - g(x(t)), \tag{3.4}$$

and

$$\varrho(t) = \frac{(c^{\Delta}(t) + c^{\sigma}(t)A(t))(1-\tau^{\Delta}(t)) + \tau^{\Delta\Delta}(t)c(t)}{(1-\tau^{\Delta}(t))\,(1-\tau^{\Delta}(\sigma(t)))},\ \ A(t) = \int_{t-\tau(t)}^{t} a(t,s)\Delta s. \tag{3.5}$$

Proof. Let x be a solution of (1.1). Rewrite (1.1) as

$$x^{\Delta}(t) + \int_{t-\tau(t)}^{t} a(t,s)x(s)\Delta s$$
$$-\int_{t-\tau(t)}^{t} a(t,s)\,(x(s) - g(x(s)))\,\Delta s + c(t)x^{\widetilde{\Delta}}(t-\tau(t)) \ = \ 0,\ \ t \in [t_0,\infty)_{\mathbb{T}}\,.$$

Obviously, we have

$$x(s) = x^{\sigma}(t) - \int_{s}^{\sigma(t)} x^{\Delta}(u)\Delta u.$$

Inserting this relation into (1.1), we get

$$x^{\Delta}(t) + \int_{t-r(t)}^{t} a(t,s)\left(x^{\sigma}(t) - \int_{s}^{\sigma(t)} x^{\Delta}(u)\Delta u\right)\Delta s$$
$$-\int_{t-\tau(t)}^{t} a(t,s)(Gx)\,(s)\,\Delta s + c(t)x^{\widetilde{\Delta}}(t-\tau(t)) \ = \ 0,\ \ t \ \in \ [t_0,\infty)_{\mathbb{T}}\,,$$

or equivalently

$$x^{\Delta}(t) + x^{\sigma}(t)\int_{t-\tau(t)}^{t} a(t,s)\Delta s - \int_{t-\tau(t)}^{t} a(t,s)\left(\int_{s}^{\sigma(t)} x^{\Delta}(u)\Delta u\right)\Delta s$$
$$-\int_{t-\tau(t)}^{t} a(t,s)(Gx)\,(s)\,\Delta s + c(t)x^{\widetilde{\Delta}}(t-\tau(t))) \ = \ 0,\ \ t \ \in \ [t_0,\infty)_{\mathbb{T}}\,.$$

After substituting x^{Δ} from (1.1), we obtain

$$\begin{aligned}
&x^{\Delta}(t)+x^{\sigma}(t)\int_{t-\tau(t)}^{t}a(t,s)\Delta s\\
&+\int_{t-\tau(t)}^{t}a(t,s)\left(\int_{s}^{\sigma(t)}\left(\int_{u-\tau(u)}^{u}a(u,v)x(v)\Delta v+c(u)x^{\widetilde{\Delta}}(u-\tau(u))\right)\Delta u\right)\Delta s\\
&-\int_{t-\tau(t)}^{t}a(t,s)(Gx)\left(s\right)\Delta s+c(t)x^{\widetilde{\Delta}}(t-\tau(t))\;=\;0,\;t\;\in\;[t_0,\infty)_{\mathbb{T}}\,. \qquad (3.6)
\end{aligned}$$

By performing the integration by parts, we have

$$\begin{aligned}
&\int_{s}^{\sigma(t)}c(u)x^{\widetilde{\Delta}}(u-\tau(u))\Delta u\\
&=\int_{s}^{\sigma(t)}\frac{c(u)}{1-\tau^{\Delta}(u)}\left(1-\tau^{\Delta}(u)\right)x^{\widetilde{\Delta}}(u-\tau(u))\Delta u\\
&=\gamma^{\sigma}\left(t\right)x(\sigma\left(t\right)-\tau^{\sigma}(t))-\gamma\left(s\right)x(s-\tau(s))-\int_{s}^{\sigma(t)}r(u)x^{\sigma}(u-\tau(u))\Delta u, \qquad (3.7)
\end{aligned}$$

where r and γ are given by (3.3). After substituting (3.7) into (3.6), we have

$$\begin{aligned}
&x^{\Delta}(t)+A(t)x^{\sigma}(t)+L_x(t)\\
&-\int_{t-\tau(t)}^{t}a(t,s)(Gx)\left(s\right)\Delta s+c(t)x^{\widetilde{\Delta}}(t-\tau(t))\;=\;0,\;t\in[t_0,\infty)_{\mathbb{T}}\,,
\end{aligned}$$

where A and L_x are given by (3.5) and (3.2), respectively. By the variation of constants formula, we get

$$\begin{aligned}
&x(t)\\
&=\phi(t_0)e_{\ominus A}\left(t,t_0\right)+\int_{t_0}^{t}\left(\int_{s-\tau(s)}^{s}a(s,u)\left(Gx\right)(u)\Delta u\right)e_{\ominus A}\left(t,s\right)\Delta s\\
&-\int_{t_0}^{t}\left[L_x(s)+c(s)x^{\widetilde{\Delta}}(s-\tau(s))\right]e_{\ominus A}\left(t,s\right)\Delta s,\;t\;\in\;[t_0,\infty)_{\mathbb{T}}\,. \qquad (3.8)
\end{aligned}$$

Letting

$$\begin{aligned}
&\int_{t_0}^{t}c(s)x^{\widetilde{\Delta}}(s-\tau(s))e_{\ominus A}\left(t,s\right)\Delta s\\
&=\int_{t_0}^{t}\frac{c(s)e_{\ominus A}\left(t,s\right)}{1-\tau^{\Delta}(s)}\left(1-\tau^{\Delta}(s)\right)x^{\widetilde{\Delta}}(s-\tau(s))\Delta s.
\end{aligned}$$

By using the integration by parts, we obtain

$$\begin{aligned}
&\int_{t_0}^{t} c(s) x^{\widetilde{\Delta}}(s-\tau(s)) e_{\ominus A}(t,s)\,\Delta s \\
&= \frac{c(t)}{1-\tau^{\Delta}(t)} x(t-\tau(t)) - \frac{c(t_0)}{1-\tau^{\Delta}(t_0)} x(t_0-\tau(t_0)) e_{\ominus A}(t,t_0) \\
&- \int_{t_0}^{t} \varrho(s) x^{\sigma}(s-\tau(s)) e_{\ominus A}(t,s)\,\Delta s, \qquad (3.9)
\end{aligned}$$

where ϱ is given by (3.5). Finally, we obtain (3.1) by substituting (3.9) in (3.8). Since each step is reversible, the converse follows easily. This completes the proof. □

Burton studied the theorem of Krasnoselskii and observed (see [14]) that Krasnoselskii result can be more interesting in applications with certain changes and formulated the Theorem 17 below (see [14] for its proof).

Definition 15. *Let (M,d) be a metric space and $F : M \to M$. F is said to be a large contraction if $\varphi, \psi \in M$ with $\varphi \neq \psi$, then $d(F\varphi, F\psi) < d(\varphi,\psi)$, and if for all $\varepsilon > 0$, there exists $\eta < 1$ such that*

$$[\varphi,\psi \in M,\ d(\varphi,\psi) \geq \varepsilon] \Rightarrow d(F\varphi, F\psi) \leq \eta d(\varphi,\psi).$$

Theorem 16 (Burton)**.** *Let (M,d) be a complete metric space and F be a large contraction. Suppose there is $x \in M$ and $\rho > 0$ such that $d(x, F^n x) \leq \rho$ for all $n \geq 1$. Then F has a unique fixed point in M.*

Below, we state Krasnoselskii-Burton's hybrid fixed point theorem which enables us to establish a stability result of the trivial solution of (1.1). For more details on Krasnoselskii's captivating theorem we refer to Smart [31] or [20].

Theorem 17 (Krasnoselskii-Burton)**.** *Let M be a closed bounded convex nonempty subset of a Banach space $(S, \|.\|)$. Suppose that $\mathcal{A}$, $\mathcal{B}$ map M into M and that*

(i) for all $x, y \in M \Rightarrow \mathcal{A}x + \mathcal{B}y \in M$,

(ii) $\mathcal{A}$ is continuous and $\mathcal{A}M$ is contained in a compact subset of M,

(iii) $\mathcal{B}$ is a large contraction.

Then there is $z \in M$ with $z = \mathcal{A}z + \mathcal{B}z$.

Here we manipulate function spaces defined on infinite t-intervals. So for compactness, we need an extension of Arzela-Ascoli theorem. This extension is taken from [[20], Theorem 1.2.2, p. 20] and is as follows.

Theorem 18. *Let $q : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\varphi_n(t)\}$ is an equicontinuous sequence of $\mathbb{R}^m$-valued functions on $\mathbb{R}_+$ with $|\varphi_n(t)| \leq q(t)$ for $t \in \mathbb{R}_+$, then there is a subsequence that converges uniformly on $\mathbb{R}_+$ to a continuous function $\varphi(t)$ with $|\varphi(t)| \leq q(t)$ for $t \in \mathbb{R}_+$, where $|.|$ denotes the Euclidean norm on $\mathbb{R}^m$.*

4 Stability by Krasnoselskii-Burton's theorem

From the existence theory which can be found in [20], we conclude that for each rd-continuous initial function $\phi : [m_0, t_0]_{\mathbb{T}} \to \mathbb{R}$, there exists a rd-continuous solution $x(t, t_0, \phi)$ which satisfies (1.1) on an interval $[0, \beta)$ for some $\beta > 0$ and $x(t, t_0, \phi) = \phi(t)$ for $t \in [m_0, t_0]_{\mathbb{T}}$.

We need the following stability definitions taken from [20].

Definition 19. *The zero solution of (1.1) is said to be stable at $t = t_0$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\phi : [m_0, t_0]_{\mathbb{T}} \to (-\delta, \delta)$ implies that $|x(t, t_0, \phi)| < \varepsilon$ for all $t \geq m_0$.*

Definition 20. *The zero solution of (1.1) is said to be asymptotically stable if it is stable at $t = t_0$ and $\delta > 0$ exists such that for any continuous function $\phi : [m_0, t_0]_{\mathbb{T}} \to (-\delta, \delta)$ the solution $x(t, t_0, \phi)$ with $x(t, t_0, \phi) = \phi(t)$ on $[m_0, t_0]_{\mathbb{T}}$ tends to zero as $t \to \infty$.*

To apply Theorem 17, we have to choose carefully a Banach space depending on the initial function ϕ and construct two mappings, a large contraction and a compact operator which obey the conditions of the theorem. So let S be the Banach space of rd-continuous bounded functions $\varphi : [m_0, \infty]_{\mathbb{T}} \to \mathbb{R}$ with the supremum norm $\|.\|$. Let $L > 0$ and define the set

$$\begin{aligned} S_\phi \quad = \quad & \{\varphi \in S : \varphi \text{ is } k\text{-Lipschitzian}, \ |\varphi(t)| \leq L, \ t \in [m_0, \infty)_{\mathbb{T}}, \\ & \varphi(t) = \phi(t) \text{ if } t \in [m_0, t_0]_{\mathbb{T}} \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}. \end{aligned}$$

Clearly, if $\{\varphi_n\}$ is a sequence of k-Lipschitzian functions converging to a function φ then

$$\begin{aligned} |\varphi(u) - \varphi(v)| &\leq |\varphi(u) - \varphi_n(u)| + |\varphi_n(u) - \varphi_n(v)| + |\varphi_n(v) - \varphi(v)| \\ &\leq \|\varphi - \varphi_n\| + k\,|u - v| + \|\varphi - \varphi_n\|. \end{aligned}$$

Consequently, as $n \to \infty$, we see that φ is k-Lipschitzian. It is clear that S_ϕ is convex, bounded and complete endowed with $\|.\|$.

For $\varphi \in S_\phi$ and $t \geq t_0$, define the maps $\mathcal{A}$, $\mathcal{B}$ and H on S_ϕ as follows

$$\begin{aligned} (\mathcal{A}\varphi)(t) = -\gamma(t)\varphi(t - \tau(t)) - \int_{t_0}^{t} L_x(s) e_{\ominus A}(t, s)\, \Delta s \\ + \int_{t_0}^{t} \varrho(s)\varphi^{\sigma}(s - \tau(s)) e_{\ominus A}(t, s)\, \Delta s, \end{aligned} \tag{4.1}$$

$$\begin{aligned} (\mathcal{B}\varphi)(t) = (\phi(t_0) + \gamma(t_0)\phi(t_0 - \tau(t_0)))\, e_{\ominus A}(t, t_0)\, \Delta s \\ + \int_{t_0}^{t} \left(\int_{s-\tau(s)}^{s} a(s, u)\, (G\varphi)\,(u) \Delta u \right) e_{\ominus A}(t, s)\, \Delta s, \end{aligned} \tag{4.2}$$

and

$$(H\varphi)(t) = (\mathcal{A}\varphi)(t) + (\mathcal{B}\varphi)(t). \tag{4.3}$$

If we are able to prove that H possesses a fixed point φ on the set S_ϕ, then $x(t,t_0,\phi) = \varphi(t)$ for $t \geq t_0$, $x(t,t_0,\phi) = \phi(t)$ on $[m_0,t_0]_{\mathbb{T}}$, $x(t,t_0,\phi)$ satisfies (1.1) when its derivative exists and $x(t,t_0,\phi) \to 0$ as $t \to \infty$.

Let

$$\begin{aligned}\omega(t) = \int_{t-\tau(t)}^{t} |a(t,s)| \left(\int_{s}^{\sigma(t)} \left(\int_{u-\tau(u)}^{u} |a(u,v)|\, \Delta v + |r(u)| \right) \Delta u \right.\\ \left. + |\gamma^\sigma(t)| + |\gamma(s)| \right) \Delta s,\end{aligned}$$

and assume that there are constants $k_1, k_2, k_3 > 0$ such that for $t_0 \leq t_1 \leq t_2$,

$$\left| \int_{t_1}^{t_2} A(z)\Delta z \right| \leq k_1 \left| t_2 - t_1 \right|, \tag{4.4}$$

$$\left| \tau(t_2) - \tau(t_1) \right| \leq k_2 \left| t_2 - t_1 \right|, \tag{4.5}$$

and

$$\left| \gamma(t_2) - \gamma(t_1) \right| \leq k_3 \left| t_2 - t_1 \right|. \tag{4.6}$$

Suppose for $t \geq t_0$,

$$|\varrho(t)| \leq \delta A(t), \tag{4.7}$$

$$\omega(t) \leq \lambda A(t), \tag{4.8}$$

$$\sup_{t \geq t_0} |\gamma(t)| = \alpha_0, \tag{4.9}$$

and that

$$J(\alpha_0 + \lambda + \delta) < 1, \tag{4.10}$$

$$\max\left(|G(-L)|, |G(L)| \right) \leq \frac{2L}{J}, \tag{4.11}$$

$$\left(\alpha_0 + \alpha_0 k_2 \right) k + L k_3 + 3L \left(\delta + \lambda + \frac{2}{J} \right) k_1 < k, \tag{4.12}$$

where α_0, δ, λ, J are positive constants with $J > 3$.

Choose $\theta > 0$ small enough and such that

$$(1 + \gamma(t_0))\theta + \left(\alpha_0 + \alpha_0 k_2 \right) k + L k_3 + 3L \left(\delta + \lambda + \frac{2}{J} \right) k_1 \leq k, \tag{4.13}$$

and

$$(1 + \gamma(t_0))\theta + \frac{3L}{J} \leq L. \tag{4.14}$$

The chosen θ in the relation (4.14) is used below in Lemma 23 to show that if $\varepsilon = L$ and if $\|\phi\| < \theta$, then the solutions satisfy $x(t, t_0, \phi) < \varepsilon$.

Assume further that

$$t - \tau(t) \to \infty \text{ as } t \to \infty \text{ and } \int_0^t A(z)\Delta z \to \infty \text{ as } t \to \infty, \tag{4.15}$$

$$\gamma(t) \to 0 \text{ as } t \to \infty, \tag{4.16}$$

$$\frac{\varrho(t)}{A(t)} \to 0 \text{ as } t \to \infty, \tag{4.17}$$

and

$$\frac{\omega(t)}{A(t)} \to 0 \text{ as } t \to \infty. \tag{4.18}$$

We begin by showing that G given by (3.4) is a large contraction on the set S_ϕ. So, we suppose that $g : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions.

$(H1)$ $g : \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,

$(H2)$ the function g is strictly increasing on $[-L, L]$,

$(H3)$ $\sup_{t\in(-L,L)} g'(t) \le 1$.

Theorem 21 ([2])**.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H1) - (H3)$. Then the mapping G in (3.4) is a large contraction on the set S_ϕ.*

By step we will prove the fulfillment of (i), (ii) and (iii) in Theorem 17.

Lemma 22. *Suppose that (4.7)–(4.10) and (4.15) hold. For $\mathcal{A}$ defined in (4.1), if $\varphi \in S_\phi$, then $|(\mathcal{A}\varphi)(t)| \le L/J \le L$. Moreover, $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$.*

Proof. Using the conditions (4.7)–(4.10) and the expression (4.1) of the map $\mathcal{A}$, we get

$$\begin{aligned}
|(\mathcal{A}\varphi)(t)| &\le |\gamma(t)|\,|\varphi(t-\tau(t))| + \int_{t_0}^t |L_\varphi(s)|\, e_{\ominus A}(t,s)\,\Delta s \\
&\quad + \int_{t_0}^t |\varrho(s)|\,|\varphi(s-\tau(s))|\, e_{\ominus A}(t,s)\,\Delta s \\
&\le \alpha_0 L + L\int_{t_0}^t \omega(s) e_{\ominus A}(t,s)\,\Delta s + L\int_{t_0}^t |\varrho(s)|\, e_{\ominus A}(t,s)\,\Delta s \\
&\le \alpha_0 L + \lambda L\int_{t_0}^t A(s) e_{\ominus A}(t,s)\,\Delta s + \delta L\int_{t_0}^t A(s) e_{\ominus A}(t,s)\,\Delta s \\
&\le (\alpha_0 + \lambda + \delta)L \le \frac{L}{J} < L.
\end{aligned}$$

So $\mathcal{A}S_\phi$ is bounded by L as required.

Let $\varphi \in S_\phi$ be fixed. We will prove that $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$. Due to the conditions $t - \tau(t) \to \infty$ as $t \to \infty$ in (4.15) and (4.9), it is obvious that the first term on the right hand side of $\mathcal{A}$ tends to 0 as $t \to \infty$. That is

$$|\gamma(t)\varphi(t-\tau(t))| \leq \alpha_0 |\varphi(t-\tau(t))| \to 0 \text{ as } t \to \infty.$$

It is left to show that the two remaining integral terms of $\mathcal{A}$ go to zero as $t \to \infty$. Let $\varepsilon > 0$ be given. Find T such that $|\varphi(t-\tau(t))| < \varepsilon$ for $t \geq T$. Then we have

$$\begin{aligned}
&\left|\int_{t_0}^{t} L_\varphi(s) e_{\ominus A}(t,s)\,\Delta s\right| \\
&\leq \int_{t_0}^{T} |L_\varphi(s)|\, e_{\ominus A}(t,s)\,\Delta s + \int_{T}^{t} |L_\varphi(s)|\, e_{\ominus A}(t,s)\,\Delta s \\
&\leq L e_{\ominus A}(t,T) \int_{t_0}^{T} \omega(s) e_{\ominus A}(T,s)\,\Delta s + \varepsilon \int_{T}^{t} \omega(s) e_{\ominus A}(t,s)\,\Delta s \\
&\leq L\lambda e_{\ominus A}(t,T) + \varepsilon\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{t_0}^{t} \varrho(s)\varphi^\sigma(s-\tau(s)) e_{\ominus A}(t,s)\,\Delta s\right| \\
&\leq \int_{t_0}^{T} |\varrho(s)|\, |\varphi^\sigma(s-\tau(s))|\, e_{\ominus A}(t,s)\,\Delta s \\
&+ \int_{T}^{t} |\varrho(s)|\, |\varphi^\sigma(s-\tau(s))|\, e_{\ominus A}(t,s)\,\Delta s \\
&\leq L e_{\ominus A}(t,T) \int_{t_0}^{T} |\varrho(s)|\, e_{\ominus A}(T,s)\,\Delta s + \varepsilon \int_{T}^{t} |\varrho(s)|\, e_{\ominus A}(t,s)\,\Delta s \\
&\leq L\delta e_{\ominus A}(t,T) + \varepsilon\delta.
\end{aligned}$$

The terms $L\lambda e_{\ominus A}(t,T)$ and $L\delta e_{\ominus A}(t,T)$ are arbitrarily smalls as $t \to \infty$, because of (4.15). This ends the proof. □

Lemma 23. *Let (4.7)–(4.12) and (4.15) hold. For $\mathcal{A}$ and $\mathcal{B}$ defined in (4.1) and (4.2), if $\varphi, \psi \in S_\phi$ are arbitrary, then*

$$\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq L.$$

Moreover, $\mathcal{B}$ is a large contraction on S_ϕ with a unique fixed point in S_ϕ and $(\mathcal{B}\psi)(t) \to 0$ as $t \to \infty$.

Proof. Using the definitions (4.1), (4.2) of $\mathcal{A}$ and $\mathcal{B}$ and applying (4.7)–(4.11), we obtain

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t)+(\mathcal{B}\psi)(t)\right| \\
&\leq \left|(\mathcal{A}\varphi)(t)\right|+\left|(\mathcal{B}\psi)(t)\right| \\
&\leq \alpha_0 L+\lambda L\int_{t_0}^{t} A(s)e_{\ominus A}(t,s)\,\Delta s+L\int_{t_0}^{t}\left|\varrho(s)\right| e_{\ominus A}(t,s)\,\Delta s \\
&+(1+\gamma(t_0))\left\|\phi\right\| e_{\ominus A}(t,t_0)+\frac{2L}{J}\int_{t_0}^{t} A(s)e_{\ominus A}(t,s)\,\Delta s \\
&\leq (1+\gamma(t_0))\left\|\phi\right\|+(\alpha_0+\lambda+\delta)L+\frac{2L}{J} \\
&\leq (1+\gamma(t_0))\left\|\phi\right\|+\frac{L}{J}+\frac{2L}{J},
\end{aligned}$$

by the monotonicity of the mapping G. So from the above inequality, by choosing the initial function ϕ having small norm, say $\|\phi\|\leq\theta$, then, and referring to (4.14), we obtain

$$\|\mathcal{A}\varphi+\mathcal{B}\psi\|\leq(1+\gamma(t_0))\theta+\frac{3L}{J}\leq L.$$

Since $0\in S_\phi$, we have also proved that $|(\mathcal{B}\psi)(t)|\leq L$. The proof that $\mathcal{B}\psi$ is k-Lipschitzian is similar to that of the map $\mathcal{A}\varphi$ below. To see that $\mathcal{B}$ is a large contraction on S_ϕ with a unique fixed point, we know from Theorem 21 that $G(\varphi)=\varphi-g(\varphi)$ is a large contraction within the integrand. Thus, for any ε, from the proof of that Theorem 21, we have found $\eta<1$ such that

$$\begin{aligned}
&\left|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)\right| \\
&\leq \int_{t_0}^{t}\left(\int_{s-\tau(s)}^{s}\left|a(s,u)\right|\left|(G\varphi)(u)-(G\psi)(u)\right|du\right)e_{\ominus A}(t,s)\,\Delta s \\
&\leq \eta\int_{t_0}^{t}\left(\int_{s-\tau(s)}^{s}a(s,u)\left\|\varphi-\psi\right\|\Delta u\right)e_{\ominus A}(t,s)\,\Delta s \\
&\leq \eta\int_{t_0}^{t}A(s)\left\|\varphi-\psi\right\|e_{\ominus A}(t,s)\,\Delta s \\
&\leq \eta\left\|\varphi-\psi\right\|.
\end{aligned}$$

To prove that $(\mathcal{B}\psi)(t)\to 0$ as $t\to\infty$, we use (4.15) for the first term, and for the second term, we argue as above for the map $\mathcal{A}$. □

Lemma 24. *Suppose (4.7)–(4.10) hold. Then the mapping $\mathcal{A}$ is continuous on S_ϕ.*

Proof. Let $\varphi, \psi \in S_\phi$, then

$$
\begin{aligned}
&|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \\
&\le \alpha_0 \left|\varphi(t-\tau(t)) - \psi(t-\tau(t))\right| + \int_{t_0}^{t} \left|L_\varphi(s) - L_\psi(s)\right| e_{\ominus A}(t,s)\,\Delta s \\
&+ \int_{t_0}^{t} |\varrho(s)| \left|\varphi(s-\tau(s)) - \psi(s-\tau(s))\right| e_{\ominus A}(t,s)\,\Delta s \\
&\le \alpha_0 \|\varphi - \psi\| + \|\varphi - \psi\| \int_{t_0}^{t} \omega(s) e_{\ominus A}(t,s)\,\Delta s \\
&+ \|\varphi - \psi\| \int_{t_0}^{t} |\varrho(s)|\, e_{\ominus A}(t,s)\,\Delta s \\
&\le \alpha_0 \|\varphi - \psi\| + \lambda \|\varphi - \psi\| \int_{t_0}^{t} A(s) e_{\ominus A}(t,s)\,\Delta s \\
&+ \delta \|\varphi - \psi\| \int_{t_0}^{t} A(s) e_{\ominus A}(t,s)\,\Delta s \\
&\le (\alpha_0 + \lambda + \delta) \|\varphi - \psi\| \le \frac{1}{J} \|\varphi - \psi\|.
\end{aligned}
$$

Let $\varepsilon > 0$ be arbitrary. Define $\eta = \varepsilon J$. Then for $\|\varphi - \psi\| \le \eta$, we obtain

$$\|\mathcal{A}\varphi - \mathcal{A}\psi\| \le \frac{1}{J} \|\varphi - \psi\| \le \varepsilon.$$

Therefore, $\mathcal{A}$ is continuous. □

Lemma 25. *Let (4.4)–(4.12) and (4.16)–(4.18) hold. The function $\mathcal{A}\varphi$ is k-Lipschitzian and the operator $\mathcal{A}$ maps S_ϕ into a compact subset of S_ϕ.*

Proof. Let $\varphi \in S_\phi$ and let $0 \le t_1 < t_2$. Then

$$
\begin{aligned}
&|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \\
&\le |\gamma(t_2)\varphi(t_2 - \tau(t_2)) - \gamma(t_1)\varphi(t_1 - \tau(t_1))| \\
&+ \left| \int_{t_0}^{t_2} L_\varphi(s) e_{\ominus A}(t_2,s)\,\Delta s - \int_{t_0}^{t_1} L_\varphi(s) e_{\ominus A}(t_1,s)\,\Delta s \right| \\
&+ \left| \int_{t_0}^{t_2} \varrho(s)\varphi^\sigma(s-\tau(s)) e_{\ominus A}(t_2,s)\,\Delta s - \int_{t_0}^{t_1} \varrho(s)\varphi^\sigma(s-\tau(s)) e_{\ominus A}(t_1,s)\,\Delta s \right|.
\end{aligned}
\tag{4.19}
$$

By hypotheses (4.5)–(4.6), we have

$$
\begin{aligned}
&|\gamma(t_2)\varphi(t_2 - \tau(t_2)) - \gamma(t_1)\varphi(t_1 - \tau(t_1))| \\
&\le |\gamma(t_2)|\, |\varphi(t_2 - \tau(t_2)) - \varphi(t_1 - \tau(t_1))| + |\varphi(t_1 - \tau(t_1))|\, |\gamma(t_2) - \gamma(t_1)| \\
&\le \alpha_0 k \left|(t_2 - t_1) - (\tau(t_2) - \tau(t_1))\right| + L k_3 |t_2 - t_1| \\
&\le (\alpha_0 k + \alpha_0 k k_2 + L k_3) |t_2 - t_1|,
\end{aligned}
\tag{4.20}
$$

where k is the Lipschitz constant of φ. By hypotheses (4.4) and (4.7), we have

$$
\begin{aligned}
&\left|\int_{t_0}^{t_2} \varrho(s)\varphi^{\sigma}(s-\tau(s))e_{\ominus A}(t_2,s)\,\Delta s - \int_{t_0}^{t_1} \varrho(s)\varphi^{\sigma}(s-\tau(s))e_{\ominus A}(t_1,s)\,\Delta s\right| \\
&\leq L\left|e_{\ominus A}(t_2,t_1)-1\right| \int_{t_0}^{t_1} \delta A(s)e_{\ominus A}(t_1,s)\,\Delta s + L\int_{t_1}^{t_2} |\varrho(s)|\, e_{\ominus A}(t_2,s)\,\Delta s \\
&\leq L\delta \int_{t_1}^{t_2} A(s)\Delta s + L\int_{t_1}^{t_2} e_{\ominus A}(t_2,s)\left(\int_{t_1}^{s} |\varrho(v)|\,\Delta v\right)^{\Delta} \Delta s \\
&\leq L\delta \int_{t_1}^{t_2} A(s)\Delta s + L\int_{t_1}^{t_2} |\varrho(v)|\,\Delta v \left(1+\int_{t_1}^{t_2} A(s)e_{\ominus A}(t_2,s)\,\Delta s\right) \\
&\leq L\delta \int_{t_1}^{t_2} A(s)\Delta s + 2L\int_{t_1}^{t_2} |\varrho(v)|\,\Delta v \\
&\leq L\delta \int_{t_1}^{t_2} A(s)\Delta s + 2L\delta\int_{t_1}^{t_2} A(v)\Delta v \\
&\leq 3L\delta k_1 \left|t_2-t_1\right|.
\end{aligned}
\tag{4.21}
$$

Similarly, by (4.4) and (4.8), we deduce

$$
\begin{aligned}
&\left|\int_{t_0}^{t_2} L_{\varphi}(s)e_{\ominus A}(t_2,s)\,\Delta s - \int_{t_0}^{t_1} L_{\varphi}(s)e_{\ominus A}(t_1,s)\,\Delta s\right| \\
&\leq L\left|e_{\ominus A}(t_2,t_1)-1\right| \int_{t_0}^{t_1} \omega(s)e_{\ominus A}(t_1,s)\,\Delta s + L\int_{t_1}^{t_2} \omega(s)e_{\ominus A}(t_2,s)\,\Delta s \\
&\leq L\left|e_{\ominus A}(t_2,t_1)-1\right| \int_{t_0}^{t_1} \lambda A(s)e_{\ominus A}(t_1,s)\,\Delta s + L\int_{t_1}^{t_2} \omega(s)e_{\ominus A}(t_2,s)\,\Delta s \\
&\leq \lambda L \int_{t_1}^{t_2} A(z)dz + L\int_{t_1}^{t_2} e_{\ominus A}(t_2,s)\left(\int_{t_1}^{s} \omega(v)\Delta v\right)^{\Delta} \Delta s \\
&\leq \lambda L \int_{t_1}^{t_2} A(z)dz + L\int_{t_1}^{t_2} \omega(v)\Delta v \left(1+\int_{t_1}^{t_2} A(s)e_{\ominus A}(t_2,s)\,\Delta s\right) \\
&\leq \lambda L \int_{t_1}^{t_2} A(z)dz + 2L\int_{t_1}^{t_2} \omega(v)\Delta v \\
&\leq \lambda L \int_{t_1}^{t_2} A(z)dz + 2L\lambda\int_{t_1}^{t_2} A(v)\Delta v \\
&\leq 3\lambda L k_1 \left|t_2-t_1\right|.
\end{aligned}
\tag{4.22}
$$

Thus, by substituting (4.20)–(4.22) in (4.19), we obtain

$$
\begin{aligned}
&\left|(\mathcal{A}\varphi)(t_2)-(\mathcal{A}\varphi)(t_1)\right| \\
&\leq (\alpha_0 k + \alpha_0 k k_2 + L k_3)\left|t_2-t_1\right| + 3L\delta k_1\left|t_2-t_1\right| + 3L\lambda k_1\left|t_2-t_1\right| \\
&\leq k\left|t_2-t_1\right|.
\end{aligned}
\tag{4.23}
$$

This shows $\mathcal{A}\varphi$ that is k-Lipschitzian if φ is and that $\mathcal{A}S_\phi$ is equicontinuous. Next, we notice that for arbitrary $\varphi \in S_\phi$, we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)| \\
&\le |\gamma(t)\varphi(t-\tau(t))| + \int_{t_0}^{t} |L_\varphi(s)|\, e_{\ominus A}(t,s)\,\Delta s \\
&+ \int_{t_0}^{t} |\varrho(s)|\,|\varphi(s-\tau(s))|\, e_{\ominus A}(t,s)\,\Delta s \\
&\le L\,|\gamma(t)| + L\int_{t_0}^{t} \omega(s) e_{\ominus A}(t,s)\,\Delta s + L\int_{t_0}^{t} |\varrho(s)|\, e_{\ominus A}(t,s)\,\Delta s \\
&\le L\,|\gamma(t)| + L\int_{t_0}^{t} A(s)\frac{\omega(s)}{A(s)} e_{\ominus A}(t,s)\,\Delta s + L\int_{t_0}^{t} A(s)\frac{|\varrho(s)|}{A(s)} e_{\ominus A}(t,s)\,\Delta s \\
&:= q(t),
\end{aligned}$$

because of (4.16)–(4.18). Using a method like the one used for the map $\mathcal{A}$, we see that $q(t) \to 0$ as $t \to \infty$. By Theorem 18, we conclude that the set $\mathcal{A}S_\phi$ resides in a compact set. □

Theorem 26. *Let $L > 0$. Suppose that the conditions $(H1) - (H3)$, (1.2), (4.4)–(4.12) and (4.16)–(4.18) hold. If ϕ is a given initial function which is sufficiently small, then there is a solution $x(t, t_0, \phi)$ of (1.1) with $|x(t, t_0, \phi)| \le L$ and $x(t, t_0, \phi) \to 0$ as $t \to \infty$.*

Proof. From Lemmas 22 and 25 we have $\mathcal{A}$ is bounded by L, k-Lipschitzian and $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$. So $\mathcal{A}$ maps S_ϕ into S_ϕ. From Lemmas 23 and 25 for arbitrary, we have $\varphi, \psi \in S_\phi$, $\mathcal{A}\varphi + \mathcal{B}\psi \in S_\phi$ since $\mathcal{A}\varphi + \mathcal{B}\psi$ is k-Lipschitzian bounded by L and $(\mathcal{B}\psi)(t) \to 0$ as $t \to \infty$. From Lemmas 23–25, we have proved that $\mathcal{B}$ is large contraction, $\mathcal{A}$ is continuous and $\mathcal{A}S_\phi$ resides in a compact set. Thus, all the conditions of Theorem 17 are satisfied. Therefore, there exists a solution of (1.1) with $|x(t, t_0, \phi)| \le L$ and $x(t, t_0, \phi) \to 0$ as $t \to \infty$. □

5 Stability in weighted Banach spaces

Referring to Burton [20], except for the fixed point method, we know of no other way proving that solutions of (1.1) converge to zero. Nevertheless, if all we need is stability and not asymptotic stability, then we can avoid conditions (4.16)–(4.18) and still use Krasnoselskii-Burton's theorem on a Banach space endowed with a weighted norm.

Let $h : [m_0, \infty)_{\mathbb{T}} \to [1, \infty)$ be any strictly increasing and continuous function with $h(m_0) = 1$, $h(s) \to \infty$ as $s \to \infty$. Let $(S, |.|_h)$ be the Banach space of continuous

$\varphi : [m_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ for which

$$|\varphi|_h = \sup_{t \geq m_0} \left| \frac{\varphi(t)}{h(t)} \right| < \infty,$$

exists. We continue to use $\|.\|$ as the supremum norm of any $\varphi \in S$ provided φ bounded. Also, we use $\|\phi\|$ as the bound of the initial function. Further, in a similar way as Theorem 21, we can prove that the function $G(\varphi) = \varphi - g(\varphi)$ is still a large contraction with the norm $|.|_h$.

Theorem 27. *If the conditions of Theorem 26 hold, except for (4.16)–(4.18), then the zero solution of (1.1) is stable.*

Proof. We prove the stability starting at t_0. Let $\varepsilon > 0$ be given such that $0 < \varepsilon < L$, then for $|x| \leq \varepsilon$, find α^* with $|x - g(x)| \leq \alpha^*$ and choose a number α such that

$$\alpha + \alpha^* + \frac{\varepsilon}{J} \leq \varepsilon. \tag{5.1}$$

In fact, since $x - g(x)$ is increasing on $(-L, L)$, we may take $\alpha^* = \frac{2\varepsilon}{J}$. Thus, inequality (5.1) allows $\alpha > 0$. Now, remove the condition $\varphi(t) \to 0$ as $t \to \infty$ from S_ϕ defined previously and consider the set

$$\begin{aligned} E_\phi \quad = \quad & \{\varphi \in S : \varphi \ k\text{-Lipshitzian}, \ |\varphi(t)| \leq \varepsilon, \ t \in [m_0, \infty)_{\mathbb{T}} \\ & \text{and } \varphi(t) = \phi(t) \text{ for } t \in [m_0, t_0]_{\mathbb{T}}\} \,. \end{aligned}$$

Define $\mathcal{A}$ and $\mathcal{B}$ on E_ϕ as before by (4.1), (4.2). We easily check that if $\varphi \in E_\phi$, then $|(\mathcal{A}\varphi)(t)| \leq \varepsilon$, and $\mathcal{B}$ is a large contraction on E_ϕ. Also, by choosing $\|\phi\| \leq \alpha$ and referring to (5.1), we verify that for $\varphi, \psi \in E_\phi$, $|(\mathcal{A}\varphi)(t) + (\mathcal{B}\psi)(t)| \leq \varepsilon$ and $|(\mathcal{B}\psi)(t)| \leq \varepsilon$. $\mathcal{A}E_\phi$ is an equicontinuous set. According to [[20], Theorem 4.0.1], in the space $(S, |.|_h)$ the set $\mathcal{A}E_\phi$ resides in a compact subset of E_ϕ. Moreover, the operator $\mathcal{A}: E_\phi \to E_\phi$ is continuous. Indeed, for $\varphi, \psi \in S_\phi$,

$$\begin{aligned}
&\frac{|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)|}{h(t)} \\
&\leq \frac{1}{h(t)} \{|\gamma(t)| \, |\varphi(t-\tau(t)) - \psi(t-\tau(t))| \\
&+ \left| \int_{t_0}^{t} (L_\varphi(s) - L_\psi(s)) \, e_{\ominus A}(t,s) \, \Delta s \right| \\
&+ \left| \int_{t_0}^{t} \varrho(s) \left(\varphi^\sigma(s-\tau(s)) - \psi^\sigma(s-\tau(s))\right) e_{\ominus A}(t,s) \, \Delta s \right| \Big\} \\
&\leq \alpha_0 \, |\varphi - \psi|_h + |\varphi - \psi|_h \int_{t_0}^{t} \omega(s) \frac{h(s)}{h(t)} e_{\ominus A}(t,s) \, \Delta s \\
&+ |\varphi - \psi|_h \int_{t_0}^{t} |\varrho(s)| \, \frac{h(s-\tau(s))}{h(t)} e_{\ominus A}(t,s) \, \Delta s \\
&\leq \alpha_0 \, |\varphi - \psi|_h + \lambda \, |\varphi - \psi|_h \int_{t_0}^{t} A(s) e_{\ominus A}(t,s) \, \Delta s \\
&+ \delta \, |\varphi - \psi|_h \int_{t_0}^{t} A(s) e_{\ominus A}(t,s) \, \Delta s \\
&\leq (\alpha_0 + \lambda + \delta) \, |\varphi - \psi|_h \leq \frac{1}{J} \, |\varphi - \psi|_h \, .
\end{aligned}$$

The conditions of Theorem 17 are satisfied on E_ϕ, and so there exists a fixed point lying in E_ϕ and solving (1.1). □

References

[1] M. Adivar and Y. N. Raffoul, *Stability and periodicity in dynamic delay equations*, Computers and Mathematics with Applications **58** (2009), 264-272. MR2535793. Zbl 1189.34143.

[2] M. Adivar and Y. N. Raffoul, *Existence of periodic solutions in totally nonlinear delay dynamic equations*, Electronic Journal of Qualitative Theory of Differential Equations **2009(1)** (2009), 1-20. MR2558826. Zbl 1195.34138.

[3] E. Akin-Bohner, Y. N. Raffoul and C. C. Tisdell, *Exponential stability in functional dynamic equations on time scales*, Commun. Math. Anal. **9** (2010), 93-108. MR2640308. Zbl 1194.34173.

[4] K. Ali Khelil, F. Bouchelaghem and L. Bouzettouta, *Exponential stability of linear Levin-Nohel integro-dynamic equations on time scales*, Int. J. Appl. Math. Stat. **56(6)** (2017), 138-149. MR3685491.

[5] K. Ali Khelil, A. Ardjouni and A. djoudi, *Stability in nonlinear neutral Levin-Nohel integro-differential equations*, Korean J. Math. **25(3)** (2017), 303-321. [MR3713747.

[6] A. Ardjouni and A. Djoudi, *Stability in neutral nonlinear dynamic equations on time scale with unbounded delay*, Stud. Univ. Babes-Bolyai Math. **57(4)** (2012), 481-496. MR3034097. Zbl 1289.34257.

[7] A. Ardjouni and A. Djoudi, *Stability in nonlinear neutral integro-differential equations with variable delay using fixed point theory*, J. Appl. Math. Comput. **44** (2014), 317-336. MR3147744. Zbl 1298.34134.

[8] A. Ardjouni and A. Djoudi, *Fixed point and stability in neutral nonlinear differential equations with variable delays*, Opuscula Mathematica **32(1)** (2012), 5-19. MR2852465. Zbl 1254.34110.

[9] A. Ardjouni, A. Djoudi and I. Soualhia, *Stability for linear neutral integro-differential equations with variable delays*, Electronic journal of Differential Equations **2012(172)** (2012), 1-14. MR2991406. Zbl 1255.34074.

[10] F. M. Atici, G. Sh. Guseinov and B. Kaymakcalan, *Stability criteria for dynamic equations on time scales with periodic coefficients*, Proceedings of the International Confernce on Dynamic Systems and Applications **3(3)** (2001), 43-48. MR1864659. Zbl 0998.34040.

[11] L. C. Becker and T. A. Burton, *Stability, fixed points and inverse of delays*, Proc. Roy. Soc. Edinburgh **136A** (2006), 245-275. MR2218152.

[12] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003. MR1962542. Zbl 1025.34001.

[13] M. Bohner and A. Peterson, *Dynamic equations on time scales, An introduction with applications*, Birkhäuser, Boston, 2001. MR1843232.

[14] T. A. Burton, *Integral equations, implicit functions and fixed points*, Proc. Amer. Math. Soc. **124** (1996), 2383-2390. MR1346965. Zbl 0873.45003.

[15] T. A. Burton, *Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem*, Nonlinear Studies **9** (2001), 181-190. MR1898587. Zbl 1084.47522.

[16] T. A. Burton, *Stability by fixed point theory or Liapunov's theory, A comparison*, Fixed Point Theory **4** (2003), 15-32. MR2031819. Zbl 1061.47065.

[17] T. A. Burton and T. Furumochi, *Asymptotic behavior of solutions of functional differential equations by fixed point theorems*, Dynamic Systems and Applications **11** (2002), 499-519. MR1946140. Zbl 1044.34033.

[18] T. A. Burton and T. Furumochi, *Krasnoselskii's fixed point theorem and stability*, Nonlinear Analysis **49** (2002), 445-454. MR1886230. Zbl 1015.34046.

[19] T. A. Burton, *Fixed points and stability of a nonconvolution equation*, Proceedings of the American Mathematical Society **132** (2004), 3679-3687. MR2084091. Zbl 1050.34110.

[20] T. A. Burton, *Stability by fixed point theory for functional differential equations*, Dover Publications, New York, 2006. MR2281958. Zbl 1160.34001.

[21] T.A. Burton and T. Furumochi, *Fixed points and problems in stability theory for ordinary and functional differential equations*, Dynamic Systems and Appl. **10** (2001), 89-116. MR1844329. Zbl 1021.34042.

[22] N. T. Dung, *Asymptotic behavior of linear advanved differential equations*, Acta Mathematica Scientia **35B(3)** (2015), 610-618.

[23] N. T. Dung, *New stability conditions for mixed linear Levin-Nohel integro-differential equations*, Journal of Mathematical Physics **54**, (2013), 1-11. MR3135476. Zbl 1286.45006.

[24] S. Hilger, *Ein Maßkettenkalkül mit anwendung auf zentrumsmannigfaltigkeiten*, Ph. D. thesis, Universität Würzburg, 1988.

[25] C. H. Jin and J. W. Luo, *Stability of an integro-differential equation*, Computers and Mathematics with Applications **57** (2009), 1080-1088. MR2508538. Zbl 1186.45011.

[26] C. H. Jin and J. W. Luo, *Stability in functional differential equations established using fixed point theory*, Nonlinear Anal. **68** (2008), 3307-3315. MR2401344. Zbl 1165.34042.

[27] C. H. Jin and J. W. Luo, *Fixed points and stability in neutral differential equations with variable delays*, Proceedings of the American Mathematical Society **136(3)** (2008), 909-918. MR2361863. Zbl 1136.34059.

[28] E. R. Kaufmann and Y.N. Raffoul, *Periodicity and stability in neutral nonlinear dynamic equations with functional delay on a time scale*, Electron. J. Differential Equations, **2007(27)** (2007), 1-12. MR2299581. Zbl 1118.34058.

[29] E. R. Kaufmann and Y. N. Raffoul, *Stability in neutral nonlinear dynamic equations on a time scale with functional delay*, Dynamic Systems and Applications **16** (2007), 561-570. MR2356339. Zbl 1140.34430.

[30] A. A. Martynyuk, *On the exponential stability of a dynamical system on a time scale*, Dokl. Math. **78** (2008), 535-540. MR2464521. Zbl 1234.34056.

[31] D. R. Smart, *Fixed point theorems*, Cambridge Tracts in Mathematics, No. **66**. Cambridge University Press, London-New York, 1974. MR0467717.

[32] B. Zhang, *Fixed points and stability in differential equations with variable delays*, Nonlinear Anal. **63** (2005), e233-e242. Zbl 1159.34348.

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