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SOME FIXED POINT THEOREMS INVOLVING RATIONAL TYPE CONTRACTIVE OPERATORS IN COMPLETE METRIC SPACES

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Abstract. Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping of X. In 1975 Dass and Gupta introduced the following rational type contractive condition to prove a generalization of Banach's Fixed Point Theorem: For $\alpha,\ \beta \in [0, 1)$, such that $\alpha + \beta < 1$, we have $\forall x, y \in X$,

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)} + \beta d(x, y),$$

where T is continuous.

There are several generalization and extension of Dass and Gupta's result under the hypothesis that T is continuous and $\alpha + \beta < 1$.

In this paper, we prove some fixed point theorems in a complete metric space setting by employing more general rational type contractive conditions than the above one. We show in our results that the continuity of the above operator T is unnecessary and the restrictive condition that $\alpha + \beta < 1$ is also removed. Our results generalize and extend those of Das and Gupta and several known results in the literature.

1 Introduction

As it is mentioned in the abstract, (X, d) is a complete metric space and $T\colon X \to X$ a mapping of X. Suppose that $F_T = \{p \in X \mid Tp = p\}$ is the set of fixed points of T. In a metric space setting, the Picard iterative process $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, n = 0, 1, 2, ... \tag{1.1}$$

has been employed by numerous authors to approximate the fixed point of mappings satisfying the inequality relation

$$d(Tx, Ty) \leq \beta d(x, y),\ \forall\ x,\ \ y \in X \text{ and } \beta \in [0, 1). \tag{1.2}$$

Any operator T satisfying condition (1.2) is necessarily continuous. Condition (1.2) is called *Banach's contraction condition.*

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Banach's Contraction Mapping Principle or Banach's Fixed Point Theorem was established in Banach [1] using the contractive condition (1.2). Banach's Fixed Point Theorem has been generalized and extended in various directions by several authors. Some of the previous nice and interesting works on this great result of Stefan Banach are those of Chatterjea [3], Kannan [16], Rakotch [21], Zamfirescu [22]. Kannan [16] which generalize the Banach's Contraction Mapping Principle by employing the following contractive condition: $\forall\ x,\ y \in X$, there exists $a \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty \leq a[d(x, Tx) + d(y, Ty)]. \tag{1.3}$$

Another generalization of Banach [1] was obtained by Chatterjea [3] by employing a contractive condition independent of that of Kannan [16] but a dual condition as follows: $\forall\ x,\ y \in X$, there exists $b \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Ty) + d(y, Tx)]. \tag{1.4}$$

Zamfirescu obtained a very nice generalization of the results of Banach [1], Chatterjea [3] and Kannan [16] by employing the following contractive condition: $\forall\ x,\ y \in X$, there exist nonnegative real numbers $0 \leq \beta < 1,\ 0 \leq a < \frac{1}{2},\ 0 \leq b < \frac{1}{2}$, such that

$$\begin{array}{lll} (z_1)\ d(Tx, Ty) & \leq & \beta d(x, y); \\ (z_2)\ d(Tx, Ty) & \leq & a[d(x, Tx) + d(y, Ty)]; \\ (z_3)\ d(Tx, Ty) & \leq & b[d(x, Ty) + d(y, Tx). \end{array} \tag{1.5}$$

Dass and Gupta [6] introduced and employed the notion of rational type contractive condition to extend the result of Banach [1]. The authors used the contractive condition

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)} + \beta d(x, y),\ \forall x, y \in X \tag{1.6}$$

(for $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and T continuous) to prove their result. Also, Jaggi [11] used the following rational type contractive condition independent of that of Dass and Gupta [6] to generalize the Banach's Fixed Point Theorem: For continuous mapping $T\colon X \to X$, there exist some $\alpha,\ \beta \in [0, 1)$ with $\alpha + \beta < 1$, such that $\forall\ x,\ y \in X,\ x \neq y$, we have

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, y). \tag{1.7}$$

Jaggi and Dass [12] proved that any continuous mapping $T\colon X \to X$ on a complete metric space (X, d) satisfying the rational type contractive condition

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y) + d(x, Ty) + d(y, Tx)} + \beta d(x, y),\ \forall\ x,\ y \in X \tag{1.8}$$

has a unique fixed point in X, where $\alpha,\ \beta \in [0,1)$ such that $\alpha+\beta<1$.

Moreover, in [17], Maia introduced a novel technique involving two metrics on the same underlying set to obtain an extension of Banach's Fixed Point Theorem. The result is stated as follows:

Theorem 1 (Maia [17]). *Let X be a nonempty set, d and ρ two metrics on X and $T\colon X \to X$ a mapping. Assume that*

(i) $d(x,y) \le \rho(x,y),\ \forall\ x,\ y \in X$;

(ii) *(X,d) is a complete metric space;*

(iii) *$T\colon (X,d) \to (X,d)$ is continuous;*

(iv) *$T\colon (X,\rho) \to (X,\rho)$ is an $a-$contraction with $a \in [0,1)$.*

Then, T is a Picard mapping.

Maia [17] was the first author to extend the Banach's Fixed Point Theorem in the metric space setting by using two different metrics.

It is noteworthy to say that operators satisfying the contractive conditions (1.3)-(1.5) need not be continuous, while the operators satisfying the rational type contractive conditions (1.6), (1.7) and (1.8) are continuous. That is, the results established by the contractive conditions (1.3)-(1.5) are generalizations of Banach [1], while the results established by the contractive conditions (1.6), (1.7) and (1.8) are extensions of Banach [1]. However, the mapping satisfying contractive conditions (1.3)-(1.5) are continuous at the fixed point.

Motivated by the results of Dass and Gupta [6] and some similar results, in this paper, we prove more general results than those of [6, 11, 12] using more general rational type contractive conditions. Our results are more general because the operator T in the new rational type contractive conditions is not necessarily continuous as in the case of many previous results available in the literature. Our results also extend several others in the literature in the sense of Maia [17] and generalize the result of Maia [17] itself.

Definition 2. *[Berinde [2]] Consider a function $\psi\colon I\!R^+ \to I\!R^+$ satisfying*

(i) *ψ is monotone increasing;*

(ii) *$\psi^n(t) \to\ 0$ as $n \to \infty$;*

(iii) *$\sum_{n=0}^{\infty} \psi^n(t)$ converges for all $t>0$.*

1. *A function ψ satisfying (i) and (ii) above is called a comparison function.*

2. *A function ψ satisfying (i) and (iii) above is called a (c)-comparison function*

Remark 3. **(i)** *Any $(c)-$comparison function is a comparison function.*

(ii) *Every comparison function satisfies $\psi(0) = 0$.*

Example 4. *Let $\psi\colon I\!R^+ \to I\!R^+$ be defined by $\psi(t) = \frac{1}{2}t,\ \forall\ t \in I\!R^+$. Then, ψ is a $(c)-$comparison function.*

In the next section, we shall employ the following contractive conditions: Let (X, d) be a complete metric space and $T\colon X \to X$.

(I) There exist $\alpha,\ p,\ q,\ r,\ \mu,\ \eta \in \mathbb{R}^+$ and $\beta \in [0, 1)$ such that $\forall\ x,\ y \in X$, we have

$$d(Tx, Ty) \leq \alpha \frac{[p + d(x, Tx)][d(y, Ty)]^r[d(y, Tx)]^q}{\mu d(y, Tx) + \eta d(x, Ty) + d(x, y)} + \beta d(x, y), \qquad (\Delta)$$

with $\mu d(y, Tx) + \eta d(x, Ty) + d(x, y) > 0$.

(II) There exist $\alpha,\ p,\ q,\ r,\ \mu,\ \eta \in \mathbb{R}^+$ and a continuous $(c)-$comparison function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\forall\ x,\ y \in X$, we have

$$d(Tx, Ty) \leq \alpha \frac{[p + d(x, Tx)][d(y, Ty)]^r[d(y, Tx)]^q}{\mu d(y, Tx) + \eta d(x, Ty) + d(x, y)} + \psi(d(x, y)), \qquad (\Delta\Delta)$$

with $\mu d(x, Ty) + \eta d(x, Ty) + d(x, y) > 0$.

Remark 5. **(i)** *If $\alpha = 0$ in (Δ) or, if in $(\Delta\Delta)$, $\alpha = 0,\ \psi(u) = \beta u,\ \forall\ u \in I\!R^+,\ \beta \in [0, 1)$, we obtain the contractive condition employed in Banach [1].*

(ii) *The contractive condition (Δ) reduces to the contractive condition (1.6) when $p = r = 1,\ q = 0,\ \mu d(y, Tx) + \eta d(x, Ty) = 1,\ \alpha,\ \beta,\ \in [0, 1),\ \alpha + \beta < 1$.*

The contractive condition (1.6) was used in Dass and Gupta [6].

(iii) *The contractive condition (Δ) reduces to the contractive condition (1.7) when $p = q = \mu = \eta = 0,\ r = 1,\ \alpha,\ \beta,\ \in [0, 1),\ \alpha + \beta < 1$ and $x \neq y$.*

Again, contractive condition (1.7) was employed in Jaggi [11].

(iv) *The contractive condition (Δ) reduces to the contractive condition (1.8) when $p = q = 0,\ \mu = \eta = r = 1,\ \alpha,\ \beta,\ \in [0, 1),\ \alpha + \beta < 1$.*

Again, the contractive condition (1.8) was used in Jaggi and Dass [12].

(v) *The contractive condition (Δ) reduces to several others in the literature. Similarly, we can reduce the contractive condition $(\Delta\Delta)$ to the contractive conditions stated in (1.6), (1.7), (1.8) and many others in the literature.*

2 Main Results

Theorem 6. *Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping satisfying (Δ). For $x_0 \in X$, let $\{x_n\}_{n=0}^{\infty} \subset X$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$, be the Picard iteration associated to T. Then T has a unique fixed point in X.*

Proof. For $x_0 \in X$, $\{x_n\} \subset X$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$, is the Picard iteration associated to T. Thus, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha \tfrac{[p+d(x_{n-1},Tx_{n-1})][d(x_n,Tx_n)]^r[d(x_n,Tx_{n-1})]^q}{\mu d(x_n,Tx_{n-1})+d(x_{n-1},x_n)} + \beta d(x_{n-1}, x_n) \\ &= \beta d(x_{n-1}, x_n), \end{aligned}$$

that is

$$d(x_n, x_{n+1}) \leq \beta d(x_{n-1}, x_n). \tag{2.1}$$

Using (2.1) inductively in the repeated application of the triangle inequality, we obtain for $p \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq \tfrac{\beta^n(1-\beta^p)}{1-\beta} d(x_0, x_1) \to 0 \end{aligned}$$

as $n \to \infty$, which shows that the sequence $\{x_n\}$ is Cauchy. Since (X, d) is a complete metric space, then $\{x_n\}$ converges to some point $u \in X$.
We claim that $Tu = u$. Indeed, by triangle inequality and noting that $x_n = Tx_{n-1}$, we have that

$$\begin{aligned} d(Tu, u) &\leq d(Tu, Tx_{n-1}) + d(x_n, u) \\ &\leq \alpha \tfrac{[p+d(u,Tu)][d(x_{n-1},x_n)]^r[d(x_{n-1},Tu)]^q}{\mu d(x_{n-1},Tu)+\eta d(u,x_n)+d(u,x_{n-1})} + \beta d(u, x_{n-1}) + d(x_n, u) \to 0 \end{aligned}$$

as $n \to \infty$, which yields $d(u, Tu) \leq 0$.

Therefore, $d(u, Tu) = 0 \iff Tu = u$.

Next, we show that $u \in F_T$ is unique. To this end, suppose there exists another $v \in F_T$ such that $Tu = u$, $Tv = v$, $u \neq v$, $d(u, v) > 0$. Then,

$$0 < d(u, v) = d(Tu, Tv) \leq \alpha . \frac{[p + d(u, Tu)][d(v, Tv)]^r[d(v, Tu)]^q}{\mu d(v, Tu) + \eta d(u, Tv) + d(u, v)} + \beta d(u, v) = \beta d(u, v),$$

from which we have that $(1 - \beta)d(u, v) \leq 0$.

Since $1 - \beta > 0$, $\beta \in [0, 1)$, we have $d(u, v) \leq 0$ (which is a contradiction).

Hence, we obtain $d(u, v) = 0 \iff u = v$. □

Example 7. *The following example shows that T satisfies contractive condition (Δ) of Theorem 6, but fails to satisfy contractive condition (1.7) (which is that of Jaggi [11]):*

Consider $X = [0,1] \subset I\!R$ *and* $T\colon X \to X$ *defined by*

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, \frac{1}{2}) \\ 1 - \frac{1}{2}x, & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

with the usual metric(i.e. $d(x,y) = |x-y|,\ x,\ y \in X$*).*

The operator T *is discontinuous at* $x = \frac{1}{2}$.

Solution <u>A:</u> We shall first show that T does not satisfy the contractive condition (1.7)(which is that of Jaggi [11]) when $x = \frac{1}{2}$ and $y = \frac{1}{4}$ as follows:

We obtain $Tx = \frac{3}{4},\ Ty = \frac{1}{2}$ and also, $d(x,Tx) = d(y,Ty) = d(x,y) = d(Tx,Ty) = \frac{1}{4}$.

But,

$$\begin{aligned} \tfrac{1}{4} = d(Tx,Ty) &\leq \alpha \tfrac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) \\ &= \alpha \tfrac{(\frac{1}{4})^2}{\frac{1}{4}} + \beta \tfrac{1}{4} = \tfrac{\alpha+\beta}{4}, \end{aligned}$$

from which it follows that $\alpha + \beta \geq 1$ (contradicting the fact that $\alpha + \beta < 1$). Thus, there do not exist $\alpha,\ \beta \in [0,1)$ such that $\alpha + \beta < 1$.

So, T does not satisfy the contractive condition (1.7) (which is that of Jaggi [11]).

<u>**B:**</u> We now show that T satisfies the contractive condition (Δ) of Theorem 6 in the present paper for $x = \frac{1}{2},\ y = \frac{1}{4},\ p = 0,\ r = q = \eta = \mu = 1$ and $\alpha = 1$ as follows:
We have $Tx = \frac{3}{4},\ Ty = \frac{1}{2},\ d(y,Tx) = \frac{1}{2},\ d(x,Ty) = 0,$ and
$d(x,Tx) = d(y,Ty) = d(x,y) = \frac{1}{4},\ d(Tx,Ty) = \frac{1}{4}$. Indeed,

$$\begin{aligned} \tfrac{1}{4} = d(Tx,Ty) &\leq \tfrac{d(x,Tx)d(y,Ty)d(y,Tx)}{d(y,Tx)+d(x,Ty)+d(x,y)} + \beta d(x,y) \\ &= \tfrac{(\frac{1}{4})^2(\frac{1}{2})}{\frac{1}{2}+\frac{1}{4}} + \beta(\tfrac{1}{4}) \\ &= \tfrac{1}{24} + \beta(\tfrac{1}{4}), \end{aligned}$$

from which it follows that $\beta \geq \frac{5}{6}$, that is, $\beta \in [0,1)$. Hence, T satisfies the contractive condition (Δ) of Theorem 6.

Theorem 8. *Let* X *be a nonempty set,* d *and* ρ *two metrics on* X *and* $T\colon X \to X$ *a mapping. For* $x_0 \in X$, $\{x_n\}_{n=0}^{\infty} \subset X$ *defined by* $x_{n+1} = Tx_n,\ n = 0,1,2,\cdots,$ *is the Picard iteration associated to* T*. Suppose that*

(i) *there exists* $c > 0$ *such that* $\rho(Tx,Ty) \leq cd(x,y),\ \forall\ x,\ y \in X$;

(ii) (X,ρ) *is a complete metric space;*

(iii) $T\colon (X,\rho) \to (X,\rho)$ *is continuous;*

(iv) $T\colon (X,d) \to (X,d)$ *satisfies the contractive condition* (Δ).

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and $x_n = T^n x_0,\ n = 1, 2, \cdots$, be Picard iteration associated to T.

From (iv), we have as in Theorem 6 that $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$.

That is, $\{x_n\}$ is a Cauchy sequence in (X, d).

We now show that $\{x_n\}$ is a Cauchy sequence in (X, ρ) too as follows:

So, by condition (i), we obtain

$$\rho(x_n, x_{n+p}) = \rho(Tx_{n-1}, Tx_{n+p-1}) \leq cd(x_{n-1}, x_{n+p-1}) \leq \frac{c\beta^{n-1}(1-\beta^p)}{1-\beta} d(x_0, x_1) \to 0$$

as $n \to \infty$. Therefore, $\{x_n\}$ is also a Cauchy sequence in (X, ρ).

By condition (ii), since (X, ρ) is complete, then there exists $u \in X$ such that $\lim_{n\to\infty} \rho(x_n, u) = 0$, that is, $\lim_{n\to\infty} x_n = u$.

Using condition (iii) and the continuity of the metric, we have

$$0 = \lim_{n\to\infty} \rho(x_{n+1}, u) = \lim_{n\to\infty} \rho(Tx_n, u) = \rho(T(\lim_{n\to\infty} x_n), u) = \rho(Tu, u).$$

Hence, $\rho(u, Tu) = 0 \iff Tu = u$. So, u is a fixed point of T.

By condition (iv) again, we have as in Theorem 6 that T has a unique fixed point. □

Theorem 9. *Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping satisfying ($\Delta\Delta$). For $x_0 \in X$, let $\{x_n\}_{n=0}^{\infty} \subset X$ defined by $x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$ be the Picard iteration associated to T. Suppose that $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a $(c)-$comparison function. Then T has a unique fixed point in X.*

Proof. For $x_0 \in X,\ \{x_n\} \subset X$ defined by $x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$, is the Picard iteration associated to T. Then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha \tfrac{[p+d(x_{n-1},Tx_{n-1})][d(x_n,Tx_n)]^r[d(x_n,Tx_{n-1}]^q}{\mu d(x_n,Tx_{n-1})+\eta d(x_{n-1},Tx_n)+d(x_{n-1},x_n)} + \psi(d(x_n, x_{n-1})) \\ &= \alpha \tfrac{d(x_{n-1},x_n)[d(x_n,x_{n+1})]^r[d(x_n,x_n)]^q}{\mu d(x_n,x_n)+\eta d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)} + \psi(d(x_n, x_{n-1})) \\ &= \psi(d(x_{n-1}, x_n)) \end{aligned}$$

By repeated application of triangle inequality in (2.1), we have for $p \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, X_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq \psi^n(d(x_0, x_1) + \psi^{n+1}(d(x_0, x_1)) + \cdots + \psi^{n+p-1}(d(x_0, x_1)) \\ &= \textstyle\sum_{k=n}^{n+p-1} \psi^k(d(x_0, x_1)) \\ &= \textstyle\sum_{k=0}^{n+p-1} \psi^k(d(x_0, x_1)) - \sum_{k=0}^{n-1} \psi^k(d(x_0, x_1)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, showing that $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$. Hence, the sequence $\{x_n\}$ is a Cauchy sequence. Since X is complete, $x_n \to x^\star \in X$ as $n \to \infty$. That is,

$$\lim_{n\to\infty} x_n = x^\star$$

We next show that $x^\star$ is the fixed point of T. That is, $Tx^\star = x^\star$. Indeed, using the triangle inequality, we have

$$\begin{aligned} d(x^\star, Tx^\star) &= d(x^\star, x_n) + d(x_n, Tx^\star) = d(x^\star, x_n) + d(Tx_{n-1}, Tx^\star) \\ &\leq d(x^\star, x_n) + d(Tx^\star, Tx_{n-1}) \\ &\leq d(x^\star, x_n) + \alpha\frac{[p + d(x^\star, Tx^\star)][d(x_{n-1}, x_n)]^r[d(x_{n-1}, Tx^\star)]^q}{\mu d(x_{n-1}, Tx^\star) + \eta d(x^\star, Tx_{n-1}) + d(x^\star, x_{n-1})} + \psi d(x^\star, x_{n-1}) \\ &\to \psi(0), \text{ as } n \to \infty. \end{aligned} \tag{2.2}$$

That is, as $n \to \infty$ in (2.2), we have $d(x^\star, Tx^\star) \leq 0$ (since $\psi(0) = 0$).

Therefore, $d(Tx^\star, x^\star) = 0 \iff Tx^\star = x^\star$.

We claim that $x^\star \in F_T$ is a unique fixed point of T. Suppose that there exists another $y^\star \in F_T$, $x^\star \neq y^\star$ such that $d(x^\star, y^\star) > 0$. Then

$$\begin{aligned} d(x^\star, y^\star) &= d(Tx^\star, Ty^\star) \\ &\leq \alpha\frac{[p+d(x^\star,Tx^\star)][d(y^\star,y^\star)]^r[d(y^\star,Tx^\star)]^q}{\mu d(y^\star,Tx^\star)+d(x^\star,y^\star)} + \psi(d(x^\star, y^\star)) \\ &= \psi(d(x^\star, y^\star)) < d(x^\star, y^\star) \text{ (which is a contradiction).} \end{aligned}$$

Hence, $x^\star = y^\star$. □

Theorem 10. *Let X be a nonempty set, d and ρ two metrics on X and $T\colon X \to X$ a mapping. For $x_0 \in X$, $\{x_n\} \subset X$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$, is the Picard iteration associated to T. Suppose that*

(i) *there exists $c > 0$ such that $\rho(Tx, Ty) \leq cd(x, y)$, $\forall\ x,\ y \in X$;*

(ii) *(X, ρ) is a complete metric space;*

(iii) *$T\colon (X, \rho) \to (X, \rho)$ is continuous;*

(iv) *$T\colon (X, d) \to (X, d)$ satisfies the contractive condition ($\Delta\Delta$), where $\psi\colon I\!R^+ \to I\!R^+$ is a continuous comparison function.*

Then T has a unique fixed point.

Proof. For arbitrary $x_0 \in X$, $x_n = T^n x_0$, $n = 1, 2, \cdots$ is the Picard iteration associated to T.

Using condition(iv), we have as in Theorem 9 that $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$. That is, $\{x_n\}$ is a Cauchy sequence in (X, d).

We now show that $\{x_n\}$ is a Cauchy sequence in (X, ρ) too:

So by condition (i), we have

$$\begin{aligned} \rho(x_n, x_{n+p}) &= \rho(Tx_{n-1}, Tx_{n+p-1}) \\ &\leq cd(x_{n-1}, x_{n+p-1}) \\ &\leq c[\textstyle\sum_{k=0}^{n+p-2} \psi^k(d(x_0, x_1)) - \sum_{k=0}^{n-2} \psi^k(d(x_0, x_1))] \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\{x_n\}$ is also a Cauchy sequence in (X, ρ).

By condition (ii), since (X, ρ) is complete, then there exists $u \in X$ such that $\lim_{n\to\infty} \rho(x_n, u) = 0$, that is, $\lim_{n\to\infty} x_n = u$.

By condition (iii), we have

$$0 = \lim_{n\to\infty} \rho(x_{n+1}, u) = \lim_{n\to\infty} \rho(Tx_n, u) = \rho(T(\lim_{n\to\infty} x_n), u) = \rho(Tu, u).$$

Hence, $\rho(u, Tu) = 0 \iff Tu = u$. So, u is a fixed point of T.

By condition (iv) again, we have as in Theorem 6 that T has a unique fixed point. □

Remark 11. *Our results show that continuity condition needs not necessarily be imposed on the operator T involved in our rational type contractive conditions just as in the cases of Chatterjea [3], Kannan [16], Zamfirescu [22] and many others in the literature where rational type contractive conditions are not involved. Although, many authors that have used rational type contractive conditions were unable to remove the continuity condition on T, especially, Dass and Gupta [6], Jaggi [11], Jaggi and Dass [12]. Furthermore, through the aid of our rational type contractive conditions, the restrictive condition $\alpha + \beta < 1$ has been removed.*

Remark 12. *Both Theorem 6 and Theorem 9 are generalizations of the results of Banach [1], Dass and Gupta [6], Jaggi [11], Jaggi and Dass [12] and many others in the literature. Similarly, both Theorem 8 and Theorem 10 are extensions of Theorem 6 and Theorem 9, respectively. Indeed, both Theorem 8 and Theorem 10 generalize the result of Maia [17] and they also generalize and extend several classical results in the literature.*

Remark 13. *In Theorem 6, Theorem 8, Theorem 9 and Theorem 10, the defined contractivity conditions are not symmetric in the sense that, if we change x by y, the contractivity conditions change. From a practical point of view, it is convenient that these contractivity conditions are symmetric.*

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References

[1] S. Banach,*Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundamenta Mathematicae **3** (1922), 133-181. JFM 48.0201.01.

[2] V. Berinde, *Iterative approximation of fixed points. 2nd revised and enlarged ed.*, Lecture Notes in Mathematics **1912** Springer-Verlag Berlin Heidelberg, 2007. MR2323613. Zbl 1165.47047.

[3] S. K. Chatterjea,*Fixed-point theorems*, C. R. Acad. Bulgare, Sci. **25** (1972), 727-730. MR0324493. Zbl 0274.54033.

[4] Lj. B. Ćirić, *On contraction type mappings*, Math. Balk **1** (1971), 52-57. MR0324494. Zbl 0223.54018.

[5] Lj. B. Ćirić, *Some recent results in metrical fixed point theory*, University of Belgrade, 2003.

[6] B. K. Dass and S. Gupta, *An extension of Bannach contraction principle through rational expression*, Indian J. Pure and Appl. Math **6** (1975), 1455-1458. MR0467708. Zbl 0371.54074.

[7] P. Dass, *A fixed point theorem on a class of generalized metric spaces*, Korean J. Math. Sci. 1 (2002), 29-55.

[8] P. Das and L. K. Dey, *A fixed point theorem in a generalized metric space*, Soochow Journal of Mathematics **33** (2007), 33-39. MR2294745. Zbl 1137.54024.

[9] P. Das, L. K. Dey, *Fixed point of contractive mappings in generalized metric spaces*, Math. Slovaca **59**(4) (2009), 499-504. MR2529258. Zbl 1240.54119.

[10] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. Lond. Math. Soc. **37** (1962), 74 - 79. MR0133102. Zbl 0113.16503.

[11] D. S. Jaggi, *Some unique fixed point theorems*, Indian Journal of Pure and Applied Mathematics **8**(2) (1977), 223-230. MR0669594. Zbl 0379.54015.

[12] D. S. Jaggi and B. K. Dass, *An extension of Banach's fixed point theorem through rational expression*, Bull Cal. Math. **72** (1980), 261-266. MR0500887. Zbl 0476.54044.

[13] J. Harjani, B. Lopez and K. Sadaragani, *A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space*, Abstract and Applied Analysis (2010). MR2726609. Zbl 1203.54041.

[14] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Soc. **16** (1973), 201-206. MR0324495. Zbl 0266.54015.

[15] M. A. Khamsi, *Introduction to metric fixed theory*, International workshop on non-linear functional anal. and its App., Shahid Behesht Uni. (2002), 20-24.

[16] R. Kannan, *Some results on fixed points*, Bull Calcutta Math. Soc. **60** (1968), 71-76. MR0257837. Zbl 0209.27104.

[17] M. G. Maia, *Un'osservazione sulle contrazioni metriche*, Rend. Sem. Mat. Univ. Padova **40** (1968), 139–143. MR0229103. Zbl 0188.45603.

[18] J. A. Meszáros, *A comparison of various definitions of contractive type mappings*, Bull Calcutta Math. Soc. **84** (2) (1992), 167-194. MR1210588. Zbl 0782.54040.

[19] M. R. Takovic, *A generalization of Banach's contraction principle*, Publications de l'Institut Mathématique. Nouvelle Série (Beograd) **23**(37) (1978), 179-191 MR0508142. Zbl 0403.54042.

[20] M. O. Olatinwo, *Some new fixed point theorems in complete metric spaces*, Creative Math. Inform. **21**(2) (2012), 189-196. MR3027361. Zbl 1289.54141.

[21] E. Rakotch, *A note on contractive mappings*, Proc. Amer Math. Soc. **13** (1962), 459-465. MR0148046. Zbl 0105.35202.

[22] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math. (Basel) **23** (1972), 292-298. MR0310859. Zbl 0239.54030.

[23] E. Zeidler, *Nonlinear functional analysis and its applications. I. Fixed-point theorems.* Translated from the German by Peter R. Wadsack, Springer-Verlag, New York, 1986. MR0816732. Zbl 0583.47050.

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