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INITIAL VALUE PROBLEMS FOR FRACTIONAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH HADAMARD TYPE DERIVATIVES IN BANACH SPACES

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Abstract. The authors establish sufficient conditions for the existence of solutions to boundary value problems for fractional differential inclusions involving the Hadamard type derivatives of order $\alpha \in (0,1]$ in Banach spaces.

1 Introduction

This paper is concerned with the existence of solutions to initial value problems (IVP for short) for fractional order functional differential inclusions. We consider the initial value problem

$$^{H}D^{\alpha}y(t) \in F(t, y_t), \quad \text{for a.e. } t \in J = [1, T],\ 0 < \alpha \leq 1, \tag{1.1}$$

$$y(t) = \varphi(t), \quad t \in [1-r, 1], \tag{1.2}$$

where $^{H}D^{\alpha}$ is the Hadamard fractional derivative, $\mathbb{E}$ is a Banach space, $\mathcal{P}(\mathbb{E})$ is the family of all nonempty subsets of $\mathbb{E}$, $F : [1-r, T] \times \mathbb{E} \to \mathcal{P}(\mathbb{E})$ is a multivalued map, and $\varphi \in C([1-r, 1], \mathbb{E})$ with $\varphi(1) = 0$. For any function y defined on $[1-r, T]$ and any $t \in J$, we denote by y_t the element of $C([1-r, 1], \mathbb{E})$ defined by

$$y_t = y(t+\theta),\ \theta \in [1-r, 1].$$

Here, $y_t(\cdot)$ represents the history of the state of the system from the time $t - r$ up to the present time t.

Differential equations of fractional order have recently proved to be valuable tools in modeling many phenomena in various fields of science and engineering. There are

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numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. documented in the literature (see [29, 32, 38]). There have been significant developments in the theory of fractional differential equations in recent years; see, for example, the monographs of Hilfer [30], Kilbas *et al.* [32], Momani *et al.* [35], and Podlubny [38], as well as the papers [1, 2, 11, 12, 13, 22, 23, 27, 29, 35]. However, the literature on Hadamard-type fractional differential equations has not undergone as much development; see, for example, [4, 10, 24, 25, 40]. The fractional derivative that Hadamard [26] introduced in 1892 differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of the Hadamard derivative contains a logarithmic function with an arbitrary exponent (see Definition 6 below). A detailed description of the Hadamard fractional derivative and integral can be found in [15, 16, 17].

In this paper, we present existence results for the problem (1.1)–(1.2) in the case where the right hand side is convex valued. This result relies on the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. Recently, this has proved to be a valuable tool in studying fractional differential equations and inclusions in Banach spaces; for details, see the papers of Agarwal *et al.* [2], Benchohra *et al.* [12, 13, 14], Graef *et al.* [25], and Laosta *et al.* [34]. The results here extend to the multivalued case some previous results in the literature, and we believe constitutes an interesting contribution to this emerging field of study.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $C(J,\mathbb{E})$ be the Banach space of all continuous functions from J into $\mathbb{E}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\},$$

and let $L^1(J,\mathbb{E})$ denote the Banach space of functions $y : J \to \mathbb{E}$ that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

We take $AC(J,\mathbb{E})$ to be the space of functions $y : J \to \mathbb{E}$ that are absolutely continuous. We endow the space $C([1-r,1],\mathbb{E})$ with the norm

$$\|\varphi\|_C = \sup\{|\varphi(\theta)| : 1-r \le \theta \le 1\}.$$

For any Banach space $(X, \|\cdot\|)$, we let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multivalued map $G : X \to \mathcal{P}(X)$ is *convex* (*closed*) valued if $G(X)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B) = \cup_{x\in B}G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\}$).

The mapping G is called *upper semi-continuous* ($u.s.c.$) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. Also, G is said to be *completely continuous* if $G(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). The mapping G has a *fixed point* if there is $x \in X$ such that $x \in G(x)$. The set of fixed point of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to P_{cl}(X)$ is said to be *measurable* if for every $y \in X$, the function

$$t \to d(y, G(t)) = inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 1. *A multivalued map $F : J \times \mathbb{E} \to \mathcal{P}(\mathbb{E})$ is said to be Carathéodory if:*

(1) $t \to F(t, u)$ is measurable for each $u \in \mathbb{E}$;

(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in AC(J, \mathbb{E})$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J, \mathbb{E}) : v(t) \in F(t, y_t) \text{ a.e. } t \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. The function $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$$

is known as the Hausdorff-Pompeiu metric.

For more details on multivalued maps see the books of Aubin and Cellina [6], Aubin and Frankowska [7], Castaing and Valadier [19], and Deimling [21].

Next, we define the Kuratowski measure of noncompactness and give some of its important properties.

Definition 2. ([5, 8]) *Let $\mathbb{E}$ be a Banach space and let $\Omega_{\mathbb{E}}$ be the set of all bounded subsets of $\mathbb{E}$. The Kuratowski measure of noncompactness is the map $\beta : \Omega_{\mathbb{E}} \to [0, \infty)$ defined by*

$$\beta(B) = \inf\{\epsilon > 0 : B \subset \bigcup_{j=1}^{m} B_j,\ B \in \Omega_{\mathbb{E}},\ and\ diam\,(B_j) \leq \epsilon\}.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [5, 8]).

(1) $\beta(B) = 0$ if and only if $\overline{B}$ is compact (B is relatively compact).

(2) $\beta(B) = \beta(\overline{B})$.

(3) $A \subset B$ implies $\beta(A) \leq \beta(B)$.

(4) $\beta(A + B) \leq \beta(A) + \beta(B)$.

(5) $\beta(cB) = |c|\beta(B),\ c \in \mathbb{R}$.

(6) $\beta(conB) = \beta(B)$.

Here $\overline{B}$ and $conB$ denote the closure and the convex hull of the bounded set B, respectively.

Theorem 3. ([28], [37, Theorem 1.3]) *Let $\mathbb{E}$ be a Banach space and $C \subset L^1(J, \mathbb{E})$ be a countable set with $|u(t)| \leq h(t)$ for a.e. $t \in J$ and every $u \in C$, where $h \in L^1(J, \mathbb{R}_+)$. Then the function $\varphi(t) = \beta(C(t))$ belongs to $L^1(J, \mathbb{R}_+)$ and satisfies*

$$\beta\left(\int_0^T u(s)ds : u \in C\right) \leq 2\int_0^T \beta(C(s))ds.$$

Lemma 4. ([34, Lemma 2.6]) *Let J be a compact real interval, let F be a Carathéodory multivalued map, and let θ be a linear continuous map from $L^1(J, \mathbb{E}) \mapsto C(J, \mathbb{E})$. Then the operator*

$$\theta \circ S_{F,y} : C(J, \mathbb{E}) \mapsto P_{cp,c}(C(J, \mathbb{E})), \quad y \mapsto (\theta \circ S_{F,y})(y) = \theta(S_{F,y})$$

is a closed graph operator in $C(J, \mathbb{E}) \times C(J, \mathbb{E})$.

In the remainder of this paper we use the notation that $\log(\cdot) = \log_e(\cdot)$ and that $[\alpha]$ denotes the integer part of α.

Definition 5. ([32]) *The Hadamard fractional integral of order α of a function $h : [1, T] \to \mathbb{E}$ is defined by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds, \quad \alpha > 0,$$

provided the integral exists.

Definition 6. ([32]) *For a function h given on the interval $[1, T]$, the Hadamard fractional derivative of order α of h is defined by*

$$({}^H D^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{h(s)}{s}ds,\ n-1 < \alpha < n,\ n = [\alpha]+1,$$

Here $[\alpha]$ denotes the integer part of α and $\log(\cdot) = \log_e(\cdot)$.

The following result, known as Mönch's fixed point theorem, will be used to prove our main results.

Theorem 7. ([37]) *Let K be a closed, convex subset of a Banach space $\mathbb{E}$, U be a relatively open subset of K, and $N : \overline{U} \mapsto \mathcal{P}(K)$. Assume that graph N is closed, N maps compact sets into relatively compact sets, and for some $x_0 \in U$, the following two conditions are satisfied:*

(i) $M \subset \overline{U}$, $M \subset conv(x_0 \cup N(M))$, and $\overline{M} = \overline{U}$ with C a countable subset of M, implies $\overline{M}$ is compact;

(ii) $x \notin (1-\lambda)x_0 + \lambda N(x)$ for all $x \in \overline{U} \setminus U$, $\lambda \in (0,1)$.

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

3 Main results

We begin this section with the definition of a solution to our problem (1.1)–(1.2).

Definition 8. *A function $y \in AC([1-r,T],\mathbb{R})$ is said to be a solution of (1.1)–(1.2), if there exists a function $v \in L^1([1,T],\mathbb{R})$, with $v(t) \in F(t,y_t)$ for a.e. $t \in [1,T]$, such that*

$$^{H}D^{\alpha}y(t) = v(t), \quad a.e. \quad t \in [1,T], \quad 0 < \alpha < 1,$$

and the function y satisfies condition (1.2).

Theorem 9. *Let $R > 0$, $B = \{x \in \mathbb{E} : \|x\| \le R\}$, and $U = \{x \in C(J,\mathbb{E}) : \|x\| \le R\}$, and assume the following conditions hold:*

(H1) $F : J \times \mathbb{E} \to \mathcal{P}_{cp,p}(\mathbb{E})$ is a Carathéodory multi-valued map;

(H2) There exists a function $p \in L^1(J,\mathbb{E})$ such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|v| : v(t) \in F(t,y)\} \le p(t)$$

for each $(t,y) \in J \times \mathbb{E}$ with $|y| \ge R$, and

$$\lim_{R \mapsto \infty} \inf \frac{\int_0^T p(t)dt}{R} = \delta < \infty;$$

(H3) There exists a Carathéodory function $\psi : J \times [1,2R] \mapsto \mathbb{R}_+$ such that

$$\beta(F(t,M)) \le \psi(t,\beta(M)) \text{ a.e. } t \in J \text{ and each } M \subset B;$$

(H4) The function $\varphi = 0$ is the unique solution in $C(J,[1,2R])$ of the inequality

$$\varphi(t) \le 2\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \psi(s,\varphi(s))\frac{ds}{s} \quad \text{for } t \in J.$$

Then the IVP (1.1)–(1.2) has at least one solution in $C(J,B)$, provided that

$$\delta < \frac{\Gamma(\alpha+1)}{(\log T)^{\alpha}}. \tag{3.1}$$

Proof. To transform the problem (1.1)–(1.2) into a fixed point problem, consider the multivalued operator

$$N(y)(t) = \Bigg\{ h \in C([1-r,T],\mathbb{R}) : h(t) = \left\{ \begin{array}{ll} \varphi(t), & \text{if } t \in [1-r,1] \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, & \text{if } t \in J \end{array} \right\} \text{ for } v \in S_{F,y} \Bigg\}.$$

Clearly, the fixed points of N are solutions to (1.1)–(1.2). We shall show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C(J,B)$.* Let h_1, h_2 belong to $N(y)$; then there exist v_1, $v_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$h_i(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_i(s)}{s} ds,$$

for $i = 1,2$. Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [dv_1 + (1-d)v_2] \frac{ds}{s}.$$

Now $S_{F,y}$ is convex since F has convex values, so

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: *$N(M)$ is relatively compact for each compact set $M \subset \overline{U}$.* Let $M \subset \overline{U}$ be a compact set and let $\{h_n\}$ be any sequence of elements of $N(M)$. We will show that $\{h_n\}$ has a convergent subsequence by using the Arzelà-Ascoli theorem. Since $h_n \in N(M)$, there exist $y_n \in M$ and $v_n \in S_{F,y}$, $n = 1,2,\ldots$, such that

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s}. \tag{3.2}$$

Using Theorem 3 and the properties of the Kuratowski measure of noncompactness, we have

$$\beta(\{h_n(t)\}) \le 2 \left[\frac{1}{\Gamma(\alpha)} \int_1^t \beta\left(\left\{ \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} \right\} \right) ds \right]. \tag{3.3}$$

On the other hand, since $M(s)$ is compact in $\mathbb{E}$, the set $\{v_n(s) : n \geq 1\}$ is compact. Consequently, $\beta(\{v_n(s) : n \geq 1\}) = 0$ for a.e. $s \in J$. Furthermore,

$$\beta\left(\left\{\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_n(s)}{s}\right\}\right) = \left(\log\frac{t}{s}\right)^{\alpha-1}\beta(\{v_n(s) : n \geq 1\}) = 0,$$

for a.e. t, $s \in J$. Now (3.3) implies that $\{h_n(t) : n \geq 1\}$ is relatively compact in B for each $t \in J$. In addition, for each t_1, $t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned}
|h_n(t_2) - h_n(t_1)| &= \left|\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{v_n(s)}{s}ds\right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{v_n(s)}{s}ds\right| \\
&\leq \frac{p(t)}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{ds}{s} \\
&\quad + \frac{p(t)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n : n \geq 1\}$ is equicontinuous. Consequently, $N(M)$ is relatively compact in $C(J, B)$.

Step 3: *N has a closed graph.* Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. Now $h_n \in N(y_n)$ implies there exists $v_n \in S_{F,y}$ such that for each $t \in J$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}v_n(s)\frac{ds}{s}.$$

Consider the continuous linear operator $\theta : L^1(J, E) \mapsto C(J, E)$ defined by

$$\theta(v)(t) \mapsto h_n(t) = \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}v_n(s)\frac{ds}{s}.$$

Clearly, $\|h_n(t) - h(t)\| \to 0$ as $n \to \infty$. From Lemma 4 it follows that $\theta \circ S_{F,y}$ is a closed graph operator. Moreover, $h_n(t) \in \theta(S_{F,y_n})$. Since $y_n \to y$, Lemma 4 implies

$$h(t) = \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}v(s)\frac{ds}{s}.$$

Step 4: *$\overline{M}$ is compact.* Assume $M \subset \overline{U}$, $M \subset conv(\{0\} \cup N(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. By an argument similar to the one used in Step 2, we see that $N(M)$ is equicontinuous. Since $M \subset conv(\{0\} \cup N(M))$, we conclude that M is equicontinuous as well. To apply the Arzelà-Ascoli theorem, we need to show that $M(t)$ is relatively compact in $\mathbb{E}$ for each $t \in J$. Since $C \subset M \subset conv(\{0\} \cup N(M))$ and C is countable, we can find a countable set $H = \{h_n : n \geq 1\} \subset N(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s}.$$

From the fact that $M \subset C \subset conv(\{0\} \cup H))$, in view of Theorem 3, we have

$$\beta(M(t)) \leq \beta(C(t)) \leq \beta(H(t)) = \beta(\{h_n(t) : n \geq 1\}).$$

Now in view of the fact that $v_n(s) \in M(s)$, applying (3.3), we have

$$\begin{aligned}\beta(M(t)) &\leq 2\left[\frac{1}{\Gamma(\alpha)} \int_1^t \beta\left(\left\{\left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{1}{s} : n \geq 1\right\}\right) ds\right] \\ &\leq 2\left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \beta(M(s)) \frac{ds}{s}\right] \\ &\leq 2\left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \psi(s, \beta(M(s))) \frac{ds}{s}\right].\end{aligned}$$

Also, the function φ given by $\varphi(t) = \alpha(M(t))$ belongs to $C(J, [1, 2R])$. Consequently, by $(H3)$, $\varphi = 0$; that is, $\beta(M(t)) = 0$ for all $t \in J$. Thus, by the Arzelà-Ascoli theorem, M is relatively compact in $C(J, B)$.

Step 5: *N has a fixed point.* Let $h \in N(y)$ with $y \in U$. To see that $N(U) \subset U$, suppose this is not the the case. Then there would exist a function $y \in U$ with $\|N(y)\|_{\mathcal{P}} > R$ and

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s}$$

for some $v \in S_{F,y}$. On the other hand,

$$R \leq \|N(y)\|_{\mathcal{P}} \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v(s)| \frac{ds}{s} \leq \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \int_1^t p(s) ds.$$

Dividing both sides by R and taking the $\liminf R \to \infty$, we conclude that

$$\left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] \delta \geq 1,$$

which contradicts (3.1). Hence $N(U) \subset U$.

As a consequence of Steps 1–5 and Theorem 7, we conclude that N has a fixed point $y \in C(J, B)$ that in turn is a solution of the problem (1.1)–(1.2). □

4 An example

In this section we apply the main result in this paper, Theorem 9 above, to the fractional differential inclusion

$$^{H}D^{\alpha}y(t) \in F(t, y_t) \quad \text{for a.e. } t \in J = [1, T],\ 0 < \alpha \le 1, \tag{4.1}$$

$$y(t) = \varphi(t), \quad t \in [1 - r, 1], \tag{4.2}$$

where $F : [1 - r, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, and $\varphi \in C([1 - r, 1], \mathbb{R})$ with $\varphi(1) = 0$. Set

$$F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \le v \le f_2(t, y)\}$$

where $f_1,\ f_2 : [1 - r, T] \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [1 - r, T]$, $f_1(t, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and $f_2(t, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). We also assume that there is a function $p \in L^1(J, \mathbb{R}))$ such that

$$\begin{aligned} \|F(t, u)\|_{\mathcal{P}} &= \sup\{|v| : v(t) \in F(t, y)\} \\ &= \max(|f_1(t, y)|, |f_2(t, y)| \le p(t) \quad \text{for } t \in [1 - r, T] \quad \text{and} \quad y \in \mathbb{R}. \end{aligned}$$

It is clear that F is compact and convex valued and is upper semi-continuous.

We take $C(s)$ to be the space of linear functions, i.e., we will choose $\varphi(t) = \beta(C(t))$ such that

$$\beta(u(s)) = \frac{u(s)}{2}$$

where

$$u(s) = as, \quad a > 0, \quad \text{and} \quad \frac{2}{a} \le s \le \frac{4R}{a}.$$

For each $(t, y) \in J \times \mathbb{R}$ with $|y| \ge R$ we have

$$\lim_{R \mapsto \infty} \inf \frac{\int_0^T p(t)dt}{R} = \delta < \infty.$$

Finally, we assume that there exists a Carathéodory function $\psi : J \times [1, 2R] \mapsto \mathbb{R}_+$ such that

$$\beta(F(t, M)) \le \psi(t, \beta(M)),\ a.e.\ t \in J \text{ and each } M \subset B,$$

and $\varphi = 0$ is the unique solution in $C(J, [1, 2R])$ of the inequality

$$\varphi(t) \le 2\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \psi(s, \varphi(s)) \frac{ds}{s}$$

for $t \in J$. Since all the conditions of Theorem 9 are satisfied, problem (4.1)–(4.2) has at least one solution y on $[1 - r, e]$.

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