Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 12 (2017), 179 – 192

ON THEORETICAL ASPECTS OF MIXTURE PROBLEMS

François Dubeau

Abstract. Mixture problems are basic but important problems in Operations Research. In this paper we consider variants of the basic linear mixture problem and indicate mathematical links between them.

1 Introduction

Mixture problems play an important role in Operations Research [2]. The basic mixture problem is a standard linear programming problem encountered in different fields like nutrition, metallurgy, milling, and others [3, 4, 6, 7, 8, 9, 10, 12]. After a short presentation of the basic problem in Section 2, we extend it in Section 3 by considering the problem of adding a *correction* to a free (no cost) given unfeasible mixture, called a *premix*. We consider least cost or least weight correction, so we are lead to consider a bi-criteria linear programming problem. In Section 4 we observe that we can obtain a single formulation which include the preceding two problems in terms of a bi-criteria linear program. In Section 5 we shortly recall the general form of a bi-criteria linear program together with the introduction of the *efficient set* or *Pareto set*. In section 6 we present mathematical links between feasible sets and efficient sets of the different formulations already presented in the preceding sections. We pay some attention to a specific geometric transformation and its effect on feasable sets, efficiency sets, and cones. In Section 7, we consider a correction to a non-free given unfeasible mixture to get a least unit cost *corrected mixture*, which is obtained by adding together the premix and the correction. It is solved by considering a linear-fractional program. Using the Charnes-Cooper's transformation we obtain an equivalent linear program to solve. The criteria of this linear program can be seen as a weighted-sum of two criteria of a linear problem. It

2010 Mathematics Subject Classification: 90C05; 90C08; 90C29; 90C31.
Keywords: Mixture problem; linear program; bi-criteria; Pareto set; linear-fractional program; geometric transformation.
This work was supported in part by the NSERC (Natural Sciences and Engineering Research Council of Canada)

follows that the variation of the unit cost of the corrected mixture in terms of the cost of the given premix is obtained from the Pareto set of a bi-criteria problem. A conclusion follows at the end of the paper.

2 Basic mixture problem

A *mixture* is an aggregate of two or more different *ingredients*. Each ingredient is characterized by *components* which are present in specific and fixed proportions. The basic mixture problem that we consider is to find a least unit cost mixture under conditions on components. These conditions are also called specifications on the components.

Let $x^o = (x_1^o, \ldots, x_n^o)^t$ be the column vector where x_i^o is the quantity of the i-th ingredients included in the mixture. Let us introduce two row vectors: $c = (c_1, \ldots, c_n)$, where $c_i \geq 0$ is usually called the unit cost of the i-th ingredient, and $u = (1, \ldots, 1)$, a vector of unit entries. Let

$$z(x^o) = cx^o = \sum_{i=1}^{n} c_i x_i^o$$

be the cost of a mixture of weight

$$w(x^o) = ux^o = \sum_{i=1}^{n} x_i^o.$$

The ratio $\dfrac{z(x^o)}{w(x^o)} = \dfrac{cx^o}{ux^o}$ is the unit cost of the mixture, which is the cost of a unit weight of the mixture.

To take into account of the constraints, the specifications and relations on specifications, we introduce the matrices B_s, B, et B_g, that are respectively a (m_s, n)-matrix, (m, n)-matrix, and (m_g, n)-matrix, and the corresponding colomn vectors β_s, le β, and β_g, of dimension respectively equal to m_s, m and m_g. The constraints are written as follows

$$\begin{cases} B_s x^o & \leq & \beta_s, \\ B x^o & = & \beta, \\ B_g x^o & \geq & \beta_g. \end{cases}$$

The set of unit weight feasible mixtures for the basic problem is

$$\mathcal{S}^o = \left\{ x^o \in \mathbb{R}^n \;\middle|\; \begin{array}{lcl} B_s x^o & \leq & \beta_s, \\ B x^o & = & \beta, \\ B_g x^o & \geq & \beta_g, \end{array} \quad ux^o = 1 \text{ and } x^o \geq 0 \right\}.$$

The mathematical formulation of the least unit cost mixture problem is given by

$$(P^o) \quad \begin{cases} & \min z(x^o) = cx^o \\ \text{subject to} & \\ & x^o \in \mathcal{S}^o. \end{cases}$$

Throughout the paper we assume that all components of vectors and matrices of data, or technical coefficients of the problem, are non negative.

Let us remark the following facts about this problem. When $\mathcal{S}^o$ is nonempty, it is a compact subset (closed and bounded) of $\mathbb{R}^n_+ = [0, +\infty)^n$. The set of values taken by the criteria for feasible solutions is given by

$$\mathcal{S}^o_c = \{ z(x^o) = cx^o \mid x^o \in \mathcal{S}^o \},$$

hence if $\mathcal{S}^o_c$ is nonempty it is a closed and bounded interval

$$[\min \{c_i | i = 1, \ldots, n\}, \max \{c_i | i = 1, \ldots, n\}] \subseteq \mathbb{R}_+ = [0, +\infty).$$

3 A first correction problem

The first extension to (P^o) concern the addition to a given non-feasible mixture, called a *premix*, of and amount of ingredients, called a *correction*, in such a way that the specifications of the *corrected mixture*, obtained by adding the premix and the correction together, satisfy all the constraints. In this first correction problem we do not consider the cost of the premix, we consider only the weight and the cost of the correction.

Let $x^a = (x^a_1, \ldots, x^a_n)^t$ be the column vector where x^a_i is the quantity of the i-th ingredients included in the correction. Let x^a_0 be the quantity of premix. For this problem we set $x^a_0 = 1$ because we consider a correction to one unit weight of premix.

The cost of the correction is given by

$$z^a_1(x^a) = z(x^a) = cx^a = \sum_{i=1}^{n} c_i x^a_i$$

and its weight by

$$z^a_2(x^a) = w(x^a) = ux^a = \sum_{i=1}^{n} x^a_i.$$

The ratio

$$\frac{z^a_1(x^a)}{z^a_2(x^a)} = \frac{z(x^a)}{w(x^a)} = \frac{cx^a}{ux^a}$$

is the unit cost of the correction, which is the cost of one unit weight of correction.

To take into account of the components of a unit weight of the premix, we add the following three column vectors : a m_s-column vector b_s, a m-column vector b, and a m_g-column vector b_g. Using the fact that $x_0^a = 1$, the set of feasible corrections to a unit weight of premix is given by

$$\mathcal{S}^a = \left\{ x^a \in \mathbb{R}^n \;\middle|\; \begin{array}{lcl} b_s + B_s x^a & \leq & \beta_s\,(1 + ux^a) \\ b + Bx^a & = & \beta\,(1 + ux^a) \\ b_g + B_g x^a & \geq & \beta_g\,(1 + ux^a) \end{array} \quad \text{and} \quad x^a \geq 0 \right\}.$$

We consider two criteria, the cost and the weight of the correction, that we would like to minimize. The mathematical formulation as a bi-criteria linear programming problem is then

$$(P^a) \quad \left\{ \begin{array}{lrcl} & \min z_1^a(x^a) & = & cx^a \\ & \min z_2^a(x^a) & = & ux^a \\ \text{subject to} & & & \\ & x^a \in \mathcal{S}^a. & & \end{array} \right.$$

It is important to point out that the total weight of the corrected mixture is $1 + ux^a = 1 + w(x^a)$ because we consider the correction of weight $w(x^a) = ux^a$ added to a unit weight $x_0^a = 1$ of premix.

The feasible set $\mathcal{S}^a \subseteq \mathbb{R}_+^n$ is a closed subset $\mathbb{R}_+^n$. If it is nonempty it might be bounded, and hence compact, or unbounded.

Let us use

$$z^a(x^a) = \begin{pmatrix} z_1^a(x^a) \\ z_2^a(x^a) \end{pmatrix} = \begin{pmatrix} cx^a \\ ux^a \end{pmatrix} = \begin{pmatrix} c \\ u \end{pmatrix} x^a = Cx^a$$

where C is the cost matrix

$$C = \begin{pmatrix} c \\ u \end{pmatrix}.$$

The set $\mathcal{S}_c^a$ of values taken by the criteria on the feasible set is

$$\mathcal{S}_c^a = \{\, z^a(x^a) = Cx^a \,|\, x^a \in \mathcal{S}^a \,\} = z^a\,(\mathcal{S}^a)\,.$$

Also $\mathcal{S}_c^a \subseteq \mathbb{R}_+^2$.

4 A single formulation : (P^o) and (P^a) together

It is possible to write down a single formulation which allows us to get the information about the two preceding problems (P^o) and (P^a). Let us consider a correction given by $x^s = (x_1^s, \ldots, x_n^s)^t \in \mathbb{R}_+^n$ to an arbitrary weight $x_0^s \geq 0$ of premix. The total weight of the corrected mixture is then given by $x_0^s + ux^s = x_0^s + w(x^s)$.

The problem is formulated for a unit weight of corrected mixture, the correction and the premix added together. The two criteria that we consider are the cost and the weight of the correction. Hence we have the following bi-criteria linear programming problem

$$(P^s) \quad \left\{ \begin{array}{llllll} & \min z_1^s(x_0^s, x^s) & = & z(x^s) & = & cx^s \\ & \min z_2^s(x_0^s, x^s) & = & w(x^s) & = & ux^s \\ \text{subject to} & & & & & \\ & b_s x_0^s + B_s x^s & \leq & \beta_s & & \\ & b x_0^s + B x^s & = & \beta & & \\ & b_g x_0^s + B_g x^s & \geq & \beta_g & & \\ & x_0^s + ux^s & = & 1 & & \\ & x_0^s \geq 0 & \text{and} & x^s \geq 0. & & \end{array} \right.$$

Let us note by $\mathcal{S}^s$ the set of feasible solutions (x_0^s, x^s) to this problem. It is a subset of $\mathbb{R}_+ \times \mathbb{R}_+^n$. In the case that $\mathcal{S}^s$ is nonempty it is a compact (closed and bounded) set.

In the decision space, let us use

$$z^s(x_0^s, x^s) = \begin{pmatrix} z_1^s(x_0^s, x^s) \\ z_2^s(x_0^s, x^s) \end{pmatrix} = \begin{pmatrix} cx^s \\ ux^s \end{pmatrix} = \begin{pmatrix} c \\ u \end{pmatrix} x^s = Cx^s$$

where C is the matrix

$$C = \begin{pmatrix} c \\ u \end{pmatrix}.$$

Let $\mathcal{S}_c^s$ be the set of values taken by the criteria on the feasible solution set

$$\mathcal{S}_c^s = \{ z^s(x_0^s, x^s) = Cx^s \,|\, (x_0^s, x^s) \in \mathcal{S}^s \} = z^s(\mathcal{S}^s).$$

Since $z_2^s(x_0^s, x^s) \in [0,1]$, we have that $\mathcal{S}_c^s \subseteq \mathbb{R}_+ \times [0,1]$.

5 Bi-criteria problem

The general form of a bi-criteria linear problem is [11]

$$(P) \quad \left\{ \begin{array}{ll} & \min z_1(x) \\ & \min z_2(x) \\ \text{subject to} & \\ & x \in \mathcal{S} \end{array} \right.$$

where $z_1(x)$ and $z_2(x)$ are linear functions, and $\mathcal{S}$ is the set of feasible solutions of the problem as a polyhedral subset of $\mathbb{R}^N$ for an appropriate dimension N.

Let

$$z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$$

and let us note $\mathcal{S}_c$ the set of values taken by the criteria on the feasible set

$$\mathcal{S}_c = \{\, z(x) \,|\, x \in \mathcal{S} \,\} = z\,(\mathcal{S})\,.$$

A feasible solution $x \in \mathcal{S}$ of this bicriteria problem is *efficient* if and only if there exists no other feasible solution $\bar{x} \in \mathcal{S}$ such that (i) $z_i(\bar{x}) \leq z_i(x)$ for each $i \in \{1,2\}$, and (ii) $z_j(\bar{x}) < z_j(x)$ for at least one $j \in \{1,2\}$. In other words, there is no other solution which can improve one criteria without deteriorare the other. The set of efficient feasible solutions in the decision space is called *efficiency set* or *Pareto set* and is noted $\mathcal{E}$. The efficiency set, or Pareto set, in the criteria spaces is the set

$$\mathcal{E}_c = \{\, z(x) \,|\, x \in \mathcal{E} \,\} = z\,(\mathcal{E})\,.$$

Moreover we have the following characterization for $\mathcal{E}_c$:

$$z \in \mathcal{E}_c \quad \text{if and only if} \quad \left[z - \mathbb{R}^2_+\right] \cap \mathcal{S}_c = \{z\}\,.$$

By considering the weighted-sum of the two criteria using a parameter $\lambda \in [0,1]$, and the problem

$$(P_\lambda) \quad \begin{cases} & \min z_\lambda(x) = (1-\lambda) z_1(x) + \lambda z_2(x) \\ \text{subjet to} & \\ & x \in \mathcal{S}, \end{cases}$$

we can use the following relation for the Pareto set $\mathcal{E}$ [11]

$$\mathcal{E} = \bigcup_{\lambda \in (0,1)} \arg\min \{ z_\lambda(x) \mid x \in \mathcal{S} \}\,.$$

The efficient set $\mathcal{E}$ being a union of a finite number of faces of $\mathcal{S}$, it follows that $\mathcal{E}_c$ is of the same form since it is the image of $\mathcal{E}$ by a linear transformation. As a subset of $\mathbb{R}^2$, $\mathcal{E}_c$ is formed by a finite number of pairewise segments connected by their endpoints, also called *efficient vertices* or *efficient extreme points*. It is a polygonal line. See [5] for a complete description of $\mathcal{E}_c$.

Let us remark that for the mixture problems we consider $\mathcal{S} \subseteq \mathbb{R}^N_+$, $z(x) \geq 0$ for all $x \in \mathbb{R}^N_+$, and consequently $\mathcal{S}_c \subseteq \mathbb{R}^2_+$.

6 Links between (P^o), (P^a) and (P^s)

6.1 Links between $\mathcal{S}^o$, $\mathcal{S}^a$ and $\mathcal{S}^s$

There exists a simple relation between the feasible sets of the two problems $\mathcal{S}^o$ and $\mathcal{S}^a$. These sets are two closed subsets of $\mathbb{R}^N$. When $\mathcal{S}^o$ is nonempty, it is a bounded (and compact) set because

$$ux^o = \sum_{n=1}^{N} x_n^o = 1 \quad \text{and} \quad x_n^o \geq 0 \text{ for } n = 1, \ldots, N.$$

When $\mathcal{S}^a$ is nonempty, it can be bounded (and hence compact) or unbounded. The next result indicates a link between these situations. All situations are reported in Table 1.

Theorem 1. [Link between $\mathcal{S}^o$ and $\mathcal{S}^a$]. *If $\mathcal{S}^a$ is nonempty, we have*

- *$\mathcal{S}^a$ is bounded (hence compact) if and only if si $\mathcal{S}^o$ is empty, or*
- *$\mathcal{S}^a$ is unbounded (hence not compact) if and only if si $\mathcal{S}^o$ is nonempty.*

Proof. If $\mathcal{S}^a$ is nonempty, take $0 \neq x^o \in \mathcal{S}^o$ and also $x^a \in \mathcal{S}^a$. For $t \geq 0$ set $x^a(t) = x^a + tx^o$. We verify directly that $x^a(t) \in \mathcal{S}^a$ and hence $\mathcal{S}^a$ is unbounded. Conversely, if $\mathcal{S}^a$ is unbounded, there are $x^a \in \mathcal{S}^a$ and $x^o \in \mathbb{R}^n$ such that $ux^o = 1$ and $x^a(t) = x^a + tx^o \in \mathcal{S}^a$ (for all $t \geq 0$). Then we verify that $x^o \in \mathcal{S}^o$ by direct substitution of $x^a(t)$ in the constraints of (P^a), dividing by t, and letting t going to infinity. □

Theorem 2. [Link between $\mathcal{S}^o$ and $\mathcal{S}^s$]. *We have*

$$\{0\} \times \mathcal{S}^o = \mathcal{S}^s \cap \left[\{0\} \times \mathbb{R}_+^n\right].$$

Proof. Because $x \in \mathcal{S}^o$ if and only if $(0, x) \in \mathcal{S}^s$. □

To obtain the link between $\mathcal{S}^a$ and $\mathcal{S}^s$ we use a geometric transformation between $\mathbb{R}_+^n$ and $(0, +\infty) \times \mathbb{R}_+^n$ which is analyzed in the next two results.

Theorem 3. [Link between $\mathcal{S}^a$ and $\mathcal{S}^s$].
(A) *Consider the following transformation*

$$T^{a \to s} : \mathcal{S}^a \to \mathcal{S}^s \cap \left[(0, +\infty) \times \mathbb{R}_+^n\right]$$

defined by

$$T^{a \to s}(x^a) = \left(\frac{1}{1 + ux^a}, \frac{x^a}{1 + ux^a}\right) = (x_0^s, x^s) \in \mathcal{S}^s$$

for $x^a \in \mathcal{S}^a$. *We have*

$$T^{a\to s}\left(\mathcal{S}^a\right) = \mathcal{S}^s \cap \left[(0,+\infty)\times \mathbb{R}^n_+\right].$$

(B) *Conversely, consider the following transformation*

$$T^{s\to a} : \mathcal{S}^s \cap \left[(0,+\infty)\times \mathbb{R}^n_+\right] \to \mathcal{S}^a$$

well defined by

$$T^{s\to a}(x_0^s, x^s) = \frac{x^s}{x_0^s} = x^a \in \mathcal{S}^a$$

because $x_0^s > 0$. *We have*

$$T^{s\to a}\left(\mathcal{S}^s \cap \left[(0,+\infty)\times \mathbb{R}^n_+\right]\right) = \mathcal{S}^a.$$

Proof. The two functions are well defined. A direct verification leads to

$$T^{s\to a}\circ T^{a\to s}(x^a) = x^a \quad \text{for any} \quad x^a \in \mathcal{S}^a,$$

and

$$T^{a\to s}\circ T^{s\to a}(x_0^s, x^s) = (x_0^s, x^s) \quad \text{for any} \quad (x_0^s, x^s) \in \mathcal{S}^s \cap \left[(0,+\infty)\times \mathbb{R}^n_+\right].$$

□

6.2 Links between $\mathcal{S}_c^o$, $\mathcal{S}_c^a$ and $\mathcal{S}_c^s$

Theorem 4. [Link between $\mathcal{S}_c^o$ and $\mathcal{S}_c^s$]. *We have*

$$\mathcal{S}_c^o \times \{1\} = \mathcal{S}_c^s \cap \left[\mathbb{R}_+ \times \{1\}\right].$$

Proof. Direct consequence of Theorem 2. □

A second geometric transformation is now used to analyze the link between the efficient sets. For $x^a \in \mathcal{S}^a$ and $(x_0^s, x^s) = T^{a\to s}(x^a)$ we have

$$z_1^s(x_0^s, x^s) = z_1^s\left(T^{a\to s}(x^a)\right) = \frac{cx^a}{1+ux^a} = \frac{z_1^a(x^a)}{1+z_2^a(x^a)}$$

and

$$z_2^s(x_0^s, x^s) = z_2^s\left(T^{a\to s}(x^a)\right) = \frac{ux^a}{1+ux^a} = \frac{z_2^a(x^a)}{1+z_2^a(x^a)}.$$

Conversely, for $(x_0^s, x^s) \in \mathcal{S}^s \cap \left[(0,+\infty)\times \mathbb{R}^n_+\right]$ and $x^a = T^{s\to a}(x_0^s, x^s)$, we have

$$z_1^a(x^a) = z_1^a\left(T^{s\to a}(x_0^s, x^s)\right) = \frac{cx^s}{1-ux^s} = \frac{z_1^s(x_0^s, x^s)}{1-z_2^s(x_0^s, x^s)}$$

and

$$z_2^a(x^a) = z_2^a\left(T^{s\to a}(x_0^s, x^s)\right) = \frac{ux^s}{1-ux^s} = \frac{z_2^s(x_0^s, x^s)}{1-z_2^s(x_0^s, x^s)}.$$

It follows we have the bijections

$$T_c : \mathbb{R}_+^2 \to \mathbb{R}_+ \times [0,1)$$

defined by

$$T_c(z_1, z_2) = \left(\frac{z_1}{1+z_2}, \frac{z_2}{1+z_2}\right).$$

and its inverse

$$T_c^{-1} : \mathbb{R}_+ \times [0,1) \to \mathbb{R}_+^2$$

given by

$$T_c^{-1}(z_1, z_2) = \left(\frac{z_1}{1-z_2}, \frac{z_2}{1-z_2}\right).$$

As a consequence we have the next result.

Theorem 5. *[Link between $\mathcal{S}_c^a$ and $\mathcal{S}_c^s$]. We have* $T_c\left(\mathcal{S}_c^a\right) = \mathcal{S}_c^s \cap \left[\mathbb{R}_+ \times [0,1)\right].$ □

6.3 Links between $\mathcal{E}_c^a$ and $\mathcal{E}_c^s$

To establish a correspondance between the efficiency sets $\mathcal{E}_c^a$ and $\mathcal{E}_c^s$ we use the next lemma which relate the preference cones of the two problems.

Lemma 6. *Restricted to $\mathbb{R}_+^2$, the transformation by T_c of the translated preference cone given by*

$$(z_1, z_2) - \mathbb{R}_+^2$$

is

$$T_c\big((z_1, z_2) - \mathbb{R}_+^2\big) = T_c\,(z_1, z_2) - C\big((1,0), (0,1) - T_c\,(z_1, z_2)\big). \quad \square$$

Using this lemma, we obtain the next result.

Theorem 7. *The sets $\mathcal{E}_c^a$ and $\mathcal{E}_c^s$ are related by the following expressions*

$$T_c(\mathcal{E}_c^a) = \left\{\ (z_1, z_2) \in \mathcal{E}_c^s \ \middle|\ \left[(z_1, z_2) - C\big((1,0), (0,1) - T_c\,(z_1, z_2)\big)\right] \cap \mathcal{S}_c^s = \{(z_1, z_2)\}\right\}.$$

and

$$\mathcal{E}_c^a = T_c^{-1}\Big(\left\{\ (z_1, z_2) \in \mathcal{E}_c^s \ \middle|\ \left[(z_1, z_2) - C\big((1,0), (0,1) - T_c\,(z_1, z_2)\big)\right] \cap \mathcal{S}_c^s = \{(z_1, z_2)\}\right\}\Big). \quad \square$$

$\mathcal{S}^o$	$\mathcal{S}^a = \mathcal{S}^f$	$\mathcal{S}^s = \widetilde{\mathcal{S}}^f$
empty	empty	empty
empty	nonempty closed and bounded (compact)	nonempty closed and bounded (compact) $\nexists (x_0^s, x^s) \in \mathcal{S}^s$ such that $x_0^s = 0$
nonempty closed and bounded (compact)	empty	nonempty closed and bounded (compact) $\forall (x_0^s, x^s) \in \mathcal{S}^s$ we have $x_0^s = 0$
nonempty closed and bounded (compact)	nonempty closed and unbounded	nonempty closed and bounded (compact) $\exists (x_0^s, x^s) \in \mathcal{S}^s$ such that $x_0^s = 0$ $\exists (x_0^s, x^s) \in \mathcal{S}^s$ such that $x_0^s > 0$

Table 1: Relations between the feasible sets $\mathcal{S}^o$, $\mathcal{S}^a$, and $\mathcal{S}^s$.

7 A second correction problem

7.1 Problem formulation : a linear-fractional program

As for the first correction problem we consider a correction given by $x^f = (x_1^f, \ldots, x_n^f) \in \mathbb{R}_+^n$ to a quantity of premix of weight x_0^f. For the problem we consider, it is a unit weight $x_0^f = 1$ of premix that we want to correct, and now the premix might have a nonzero unit cost $c_0 \geq 0$.

Because we consider a correction to one unit weight of premix $x_0^f = 1$, the cost of the corrected mixture is

$$z_1^f(x^f) = c_0 x_0^f + c x^f = c_0 + z(x^f),$$

and the weight of this corrected mixture is given by

$$z_2^f(x^f) = x_0^f + u x^f = 1 + w(x^f).$$

Then the ratio $z^f(x^f) = \dfrac{z_1^f(x^f)}{z_2^f(x^f)}$ is the unit cost of the corrected mixture.

Since we would like to minimize the unit cost of the corrected mixture, the formulation is a linear problem with a fractional criteria given by

$$
(P^f)\quad \begin{cases} \min z^f(x^f) & = & \dfrac{z_1^f(x^f)}{z_2^f(x^f)} \\ \text{subject to} & & \\ b_s + B_s x^f & \leq & \beta_s\left(1+ux^f\right) \\ b + Bx^f & = & \beta\left(1+ux^f\right) \\ b_g + B_g x^f & \geq & \beta_g\left(1+ux^f\right) \\ x^f & \geq & 0. \end{cases}
$$

$\mathcal{S}^f$ will be the set of feasible solutions for this problem. We immediately observe that $\mathcal{S}^f = \mathcal{S}^a$.

7.2 Transformed problem

The problem (P^f) is an example of a linear-fractional problem. Under the assumption that $\mathcal{S}^f$ is nonempty and bounded (compact), the Charnes-Cooper's transformation [1]

$$\eta_0^f = \frac{1}{1+ux^f},$$

and

$$\eta^f = \frac{x^f}{1+ux^f}$$

transforms the linear fractional problem (P^f) into an equivalent linear problem

$$
(\widetilde{P}^f)\quad \begin{cases} \min \tilde{z}^f(\eta_0^f, \eta^f) & = & c_0\eta_0^f + c\eta^f \\ \text{subject to} & & \\ b_s\eta_0^f + B_s\eta^f & \leq & \beta_s \\ b\eta_0^f + B\eta^f & = & \beta \\ b_g\eta_0^f + B_g\eta^f & \geq & \beta_g \\ \eta_0^f + u\eta^f & = & 1 \\ \eta_0^f \geq 0 & \text{and} & \eta^f \geq 0. \end{cases}
$$

Let $\widetilde{\mathcal{S}}^f$ be the feasible set for this problem. Let us observe that $\widetilde{\mathcal{S}}^f = \mathcal{S}^s$. Also $(\widetilde{P}^f)$ is of the same type as (P^o), because the variable η_0^f can be considered as a variable like the other n variables.

The assumption that $\mathcal{S}^f = \mathcal{S}^a$ is a nonempty and bounded set implies, from Theorem 1, that (P^o) has no solution. Consequently (P^f) has a solution, and $(\widetilde{P}^f)$ has no solution (η_0^f, η^f) with $\eta_0^f = 0$, because such a solution correspond to a solution of (P^o). It follows that any solution of $(\widetilde{P}^f)$ is such that $\eta_0^f > 0$. Hence the corresponding solution of (P^f) is given by

$$x_i^f = \frac{\eta_i^f}{\eta_0^f} \quad \text{for} \quad i = 0, \dots, n.$$

In the case that the feasible set $\mathcal{S}^f = \mathcal{S}^a$ is nonempty and unbounded, $(\widetilde{P}^f)$ can have a solution (η_0^f, η^f) with $\eta_0^f > 0$. Then the solution of (P^f) is yet given by the preceding formula. But it can also happend that $(\widetilde{P}^f)$ has a solution (η_0^f, η^f) with $\eta_0^f = 0$, in that case $x^o = x^f = \eta^f$ is a solution of the original problem (P^o). In this case we define $x_0^f = 0$.

7.3 Parametric analysis

To obtain the variation of the criteria, the unit cost of the corrected mixture, with respect to the unit cost c_0 of the premix, we consider a parametric analysis of the criteria with respect to the parameter c_0. We use a decomposition of the criteria of $(\widetilde{P}^f)$ in two parts

$$\tilde{z}^f(\eta) = \tilde{z}_1^f(\eta) + c_0 \tilde{z}_2^f(\eta) = \left(\sum_{i=1}^{n} c_i \eta_i \right) + c_0 \eta_0,$$

and consider the following bi-criteria problem

$$(\widetilde{P}^{fs}) \quad \left\{ \begin{array}{lrcl} & \min \tilde{z}_1^f(\eta) & = & \sum_{i=1}^n c_i \eta_i \\ & \min \tilde{z}_2^f(\eta) & = & \eta_0 \\ \text{subject to} & & & \\ & \eta \in \widetilde{\mathcal{S}}^f. & & \end{array} \right.$$

The Pareto curve of this problem, obtained from the weighted-sum criterion

$$\tilde{z}_\lambda^f(\eta) = (1 - \lambda) \tilde{z}_1^f(\eta) + \lambda \tilde{z}_2^f(\eta),$$

and the correspondance

$$\lambda = \frac{c_0}{1 + c_0} \quad \text{or else} \quad c_0 = \frac{\lambda}{1 - \lambda}$$

allow us to find an expression for the optimal value function $z_*^f(c_0) = \min_{x^f \in \mathcal{S}^f} z_f(x^f)$ which is an increasing continuous concave piecewise linear function with respect to c_0.

Since

$$z(x^f) = cx^f = \sum_{i=1}^{n} c_i x_i^f = \left\{ \begin{array}{lcll} \frac{1}{\eta_0} \sum_{i=1}^n c_i \eta_i & = & \frac{1}{\eta_0} \tilde{z}_1^f(\eta) & \text{if} \quad \eta_0 > 0 \text{ (or } x_0^f = 1), \\ \sum_{i=1}^n c_i \eta_i & = & \tilde{z}_1^f(\eta) & \text{if} \quad \eta_0 = 0 \text{ (or } x_0^f = 0), \end{array} \right.$$

and

$$w(x^f) = ux^f = \sum_{n=1}^{N} x_n^f = \begin{cases} \frac{1}{\eta_0} \sum_{i=1}^{n} \eta_i & = & \frac{1}{\eta_0}(1-\eta_0) & \text{if} & \eta_0 > 0 \text{ (or } x_0^f = 1), \\ \sum_{i=1}^{n} \eta_i & = & 1 & \text{if} & \eta_0 = 0 \text{ (or } x_0^f = 0), \end{cases}$$

we have

$$z_*^f(c_0) = \frac{c_0 x_0^f + z(x^f)}{x_0^f + w(x^f)}$$

which is valid not only for $x_0^f = 1$ but also for $x_0^f = 0$.

8 Conclusion

In this short paper we have considered different mixtures problems and establish links between them. We started from the basic mixture problem and considered corrections to a premix, a non feasable mixture. Two situations were considered, firstly for a free premix and secondly for a nonfree premix. Bi-criteria formulations helped us to analyse these problems, and we have obtained the variation of the unit cost of the corrected mixture via their Pareto sets. Finally the geometric transformations introduced in this paper could be analyzed from a geometric point of view.

References

[1] A. Charnes and W.W. Cooper, *Programming with linear fractional functionals*, Naval Research Logistics Quarterly, **9**(1962), 181-186. MR0152370. Zbl 0127.36901.

[2] G.B. Dantzig. *Linear Programming and Extensions*, Princeton Press, Princeton, New Jersey, USA, 1963. MR0201189. Zbl 0108.33103.

[3] G.B. Dantzig, *The diet problem*, Interfaces, **20**(1990), 43-47.

[4] F. Dubeau, P.-O. Julien, and C. Pomar, *Étude de cas : un porcin dans un environnement sain*, INFOR, **46**(2008), 199-217.

[5] F. Dubeau and A. Kadri, *Computation and visualization of the Pareto set in the criterion space for the bicriteria linear programming problem*, International Journal of Mathematics and Computation, **15(2)**(2012), 1-15. MR2877371. Zbl 0127.36901.

[6] F. Dubeau, M.E. Ntigura Habingabwa, *On best mixture problems in metallurgy: a case study*, Mathematical Problems in Engineering, Volume **2017**(2017), Article ID 2742540, 11 pages.

[7] S.G. Garille and S.I. Gass, *Stigler's diet problem revisited*, Operations Research, **49**(2001), 1-13.

[8] É. Joannopoulos, F. Dubeau, J.-P. Dussault and C. Pomar, *Chapter 18 Diet problems*, in Handbook of Operations Research in Agriculture and the Agri-Food Industry, L.M. Plà Aragonés (ed.), International Series in Operations Research & Management Science, Vol. **224**(2015), 397-417.

[9] L.M. Lancaster, *The evolution of the diet model in managing food systems*, Interfaces, **22**(1992), 59-68.

[10] L.M. Lancaster, *The history of the application of mathematical programming to menu planning*, European Journal of Operational Research, **57**(1992), 339-347.

[11] R.E. Steuer, *Multiple Criteria Optimization*, Wiley, New York, USA, 1986. MR0836977. Zbl 0663.90085.

[12] G.J. Stigler, *The cost of subsistance*, Journal of Farm Economics, **27**(1945), 303-314.

François Dubeau
Département de mathématiques, Université de Sherbrooke,
2500 Boulevard de l'Université,
Sherbrooke (QC), Canada, J1K2R1.
francois.dubeau@usherbrooke.ca
https://www.usherbrooke.ca/mathematiques/personnel/professeurs/professeurs/francois-dubeau/

License