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L_p- APPROXIMATION BY ITERATES OF CERTAIN SUMMATION-INTEGRAL TYPE OPERATORS

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Abstract. The present paper is a study of L_p- approximation in terms of higher order integral modulus of smoothness for an iterative combination due to Micchelli, of certain summation-integral type operators using the device of Steklov means.

1 Introduction

Let $H_\alpha[0,\infty)$ be the class of all locally integrable functions on $[0,\infty)$ and satisfying the growth condition

$$|f(t)| \leq M(1+t)^\alpha \ (M>0;\, \alpha>0;\, t\to\infty).$$

Then, for a function $f \in H_\alpha[0,\infty)$, Srivastava and Gupta [9] introduced a generalized family of linear positive operators

$$G_{n,c}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^\infty p_{n+c,k-1}(t;c)f(t)\,dt \qquad (1.1)$$
$$+ \quad p_{n,0}(x;c)\,f(0), \quad x\in[0,\infty),$$

where $p_{n,k}(x;c) = (-1)^k \dfrac{x^k}{k!}\phi_{n,c}^{(k)}(x)$ and $\{\phi_{n,c}\}_{n\in\mathbb{N}}$ be a sequence of functions defined on an interval $[0,b]$, $b>0$ having the following properties for every $n\in\mathbb{N}$, $k\in\mathbb{N}^0$ (the set of non-negative integers):
(i) $\phi_{n,c}\in C^\infty([a,b])$; (ii) $\phi_{n,c}(0)=1$;
(iii) $\phi_{n,c}$ is completely monotone i.e $(-1)^k\phi_{n,c}^{(k)}\geq 0$;
(iv) there exists an integer c such that $\phi_{n,c}^{(k+1)} = -n\phi_{n+c,c}^{(k)}, n>\max\{0,-c\}$.

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For $c=0$ and $\phi_{n,c}(x)=e^{-nx}$, the operators $G_{n,c}$ reduce to the Phillips operators (see e.g. [7],[8]).

For $c=1$ and $\phi_{n,c}(x)=(1+cx)^{-n/c}$, the operators $G_{n,c}$ reduce to a sequence of summation-integral type operators [3] which is almost similar to the sequence of operators introduced by Agrawal and Thamer [1]. The Bezier variant of these operators has been studied in [6].

Alteratively, we may rewrite the operators 1.1 as

$$G_{n,c}(f;x)=\int_0^\infty K_n(x,t;c)\,f(t)\,dt, \tag{1.2}$$

where

$$K_n(x,t;c)=n\sum_{k=1}^{\infty}p_{n,k}(x;c)p_{n+c,k-1}(t;c)+p_{n,0}(x;c)\,p_{n,0}(t;c)\,\delta(t),$$

$\delta(t)$ being the Dirac-delta function.

It turns out that the order of approximation by the operators 1.2 is, at best, $O(n^{-1})$, howsoever smooth the function may be. With the aim of improving the order of approximation by these operators, we use the iterative combination technique described in [2]. The iterative combination $T_{n,k}:H_\alpha[0,\infty)\to C^\infty(0,\infty)$ of the operators $G_{n,c}(f;x)$ is defined as

$$T_{n,k}(f(t);x)$$
$$=\big(I-(I-G_{n,c})^k\big)(f;x)=\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}G^r_{n,c}(f(t);x),$$

where $G^r_{n,c}$ denotes the $r-$th iterate of the operators $G_{n,c}$ and $G^0_{n,c}=I$.

In Section 2 of this paper we give some definitions and auxiliary results which will be needed to prove the main theorem. In Section 3 we obtain an estimate of error in L_p- approximation $(1\leqslant p<\infty)$ by the iterative combination $T_{n,k}(.;x)$ in terms of $2k-$th order integral modulus of smoothness of the function.

Throughout this paper, let $0<a_1<a_2<b_2<b_1<\infty$ and $I_i=[a_i,b_i]$, $i=1,2$.

2 Preliminaries

Let $f\in L_p[0,\infty)$, $1\le p<\infty$. Then, for sufficiently small $\eta>0$, the Steklov mean $f_{\eta,m}$ of $m-$th order corresponding to f is defined as follows:

$$f_{\eta,m}(t)=\eta^{-m}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}\Big(f(t)+(-1)^{m-1}\Delta^m_{\sum_{i=1}^m t_i}f(t)\Big)\prod_{i=1}^{m}dt_i,\ \ t\in I_1,$$

where Δ_h^m is the $m-$th order forward difference with step length h.

Lemma 1. *For the function $f_{\eta,m}$, we have*
(a) $f_{\eta,m}$ has derivatives up to order m over I_1; $f_{\eta,m}^{(m-1)} \in AC(I_1)$ and $f_{\eta,m}^{(m)}$ exists a.e and belongs to $L_p(I_1)$;
(b) $\|f_{\eta,m}^{(r)}\|_{L_p(I_2)} \leq C_r\eta^{-r}\omega_r(f,\eta,I_1), r = 1,2,...,m$;
(c) $\|f - f_{\eta,m}\|_{L_p(I_2)} \leq C_{m+1}\ \omega_m(f,\eta,I_1)$;
(d) $\|f_{\eta,m}\|_{L_p(I_2)} \leq C_{m+2}\|f\|_{L_p[0,\infty]}$;
(e) $\|f_{\eta,m}^{(m)}\|_{L_p(I_2)} \leq C_{m+3}\ \|f\|_{L_p(I_1)}$,
where $C_i's$ are certain constants that depend on i but are independent of f and η.

Proof. Using (Theorem 18.17, [5]) or ([10], pp. 163-165), the proof of this lemma easily follows. Hence details are omitted.

□

Lemma 2. *[9] For $m \in \mathbb{N}^0$, the $m-$th order moment for the operators $G_{n,c}$ is defined as*

$$\mu_{n,m}(x;c) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^{\infty} p_{n+c,k-1}(t;c)(t-x)^m\,dt + (-x)^m p_{n,0}(x;c)$$

then

$$\mu_{n,0}(x;c) = 1,\ \ \mu_{n,1}(x;c) = \frac{cx}{n-c}$$

and

$$\mu_{n,2}(x;c) = \frac{x(1+cx)(2n-c)+(1+3cx)cx}{(n-c)(n-2c)}.$$

Also, the following recurrence relation holds

$$\begin{aligned}[n-c(m+1)]\mu_{n,m+1}(x;c) &= x(1+cx)[\mu_{n,m}^{(1)}(x,c) + 2m\mu_{n,m-1}(x;c)] \\ &+ [(1+2cx)m + cx]\mu_{n,m}(x;c).\end{aligned}$$

Consequently,
(i) $\mu_{n,m}(x;c)$ is a polynomial in x of degree m;
(ii) for every $x \in [0,\infty)$, $\mu_{n,m}(x;c) = O\left(n^{-[(m+1)/2]}\right)$,
where $[\beta]$ denotes the integer part of β.

Remark 3. *It is easily shown that for each $k > 0$ and for every $x \in [0,\infty)$*

$$G_{n,c}(|t-x|^k;x) = O(n^{-k/2}). \tag{2.1}$$

For every $m \in \mathbb{N}^0$, the $m-$th order moment $\mu_{n,m}^{[p]}(x;c)$ for the operator $G_{n,c}^p$ is defined as $\mu_{n,m}^{[p]}(x;c) = G_{n,c}^p((t-x)^m;x)$, we denote $\mu_{n,m}^{[1]}(x;c) = \mu_{n,m}(x;c)$.

Lemma 4. *There holds the recurrence relation*

$$\mu_{n,m}^{[r+1]}(t;c)=\sum_{j=0}^{m}\sum_{i=0}^{m-j}\binom{m}{j}\frac{1}{i!}D^i\left(\mu_{n,m-j}^{[r]}(t;c)\right)\mu_{n,i+j}(t;c).$$

Proof. We can write

$$\begin{aligned}\mu_{n,m}^{[r+1]}(t;c) &= G_{n,c}^{r+1}\left((u-t)^m;t\right)\\ &= G_{n,c}\left(G_{n,c}^{r}\left((u-t)^m;x\right);t\right)\\ &= \sum_{j=0}^{m}\binom{m}{j}G_{n,c}\left((x-t)^jG_{n,c}^{r}\left((u-x)^{m-j};x\right);t\right). \qquad (2.2)\end{aligned}$$

Since $G_{n,c}^{r}\left((u-x)^{m-j};x\right)$ is a polynomial in x of degree $m-j$, by Taylor's expansion we can write it as

$$G_{n,c}^{r}\left((u-x)^{m-j};x\right)=\sum_{i=0}^{m-j}\frac{(x-t)^i}{i!}D^i\left(\mu_{n,m-j}^{[r]}(t,c)\right). \qquad (2.3)$$

From (2.2) and (2.3), we get the required result. □

Lemma 5. *For $k,l\in\mathbb{N}$, there holds $T_{n,k}\left((u-t)^l;t\right)=O(n^{-k})$.*

Proof. For $k=1$, the result holds from Lemma 2. Let us assume that it is true for a certain k, then by the definition of $T_{n,k}$ we get

$$\begin{aligned}T_{n,k+1}\left((u-t)^l;t\right) &= \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k+1}{r}G_{n,c}^{r}((u-t)^l;t)\\ &= T_{n,k}\left((u-t)^l;t\right)\\ &+ \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k}{r-1}G_{n,c}^{r}((u-t)^l;t)=I_1+I_2, \text{say}. \qquad (2.4)\end{aligned}$$

Next, by Lemma 4

$$\begin{aligned}
I_2 &= \sum_{r=0}^{k}(-1)^r\binom{k}{r}\mu_{n,l}^{[r+1]}(t;c)\\
&= \mu_{n,l}(t;c)-\sum_{j=1}^{l}\sum_{i=0}^{l-j}\binom{l}{j}\frac{1}{i!}\left[D^iT_{n,k}\left((u-t)^{l-j};t\right)\right]\mu_{n,i+j}(t;c)\\
&\qquad -\sum_{i=0}^{l}\frac{1}{i!}\left[D^iT_{n,k}\left((u-t)^{l};t\right)\right]\mu_{n,i}(t;c)\\
&= -\sum_{j=1}^{l-1}\sum_{i=0}^{l-j}\binom{l}{j}\frac{1}{i!}\left[D^iT_{n,k}\left((u-t)^{l-j};t\right)\right]\mu_{n,i+j}(t;c)\\
&\qquad -\sum_{i=1}^{l}\frac{1}{i!}\left[D^iT_{n,k}\left((u-t)^{l};t\right)\right]\mu_{n,i}(t)-T_{n,k}((u-t)^l;t).
\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5) and then using Lemma 2, we get

$$T_{n,k+1}\left((u-t)^l;t\right)=O\left(n^{-(k+1)}\right).$$

Thus, the result holds for $k+1$. Hence the lemma is proved by induction for all $k\in\mathbb{N}$. □

Lemma 6. *For $p\in\mathbb{N}, m\in\mathbb{N}^0$ and each $t\in[0,\infty)$, we have*

$$\mu_{n,m}^{[p]}(t;c)=O\left(n^{-[(m+1)/2]}\right). \tag{2.6}$$

Proof. For $p=1$, the result follows from Lemma 2. Suppose (2.6) is true for a certain p.Then $\mu_{n,m-j}^{[p]}(t;c)=O\left(n^{-[(m+1)/2]}\right)$, $\forall\ 0\le j\le m$. Also, $\mu_{n,m-j}^{[p]}(t;c)$ is a polynomial in t of degree $m-j$, therefore, we have

$$D^i\left(\mu_{n,m-j}^{[p]}(t;c)\right)=O\left(n^{-[(m-j+1)/2]}\right),\ \ \forall\ 0\le i\le m-j.$$

Now, applying Lemma 4,

$$\begin{aligned}
\mu_{n,m}^{[p+1]}(t;c) &= \sum_{j=0}^{m}\sum_{i=0}^{m-j}O\left(n^{-[(m-j+1)/2]}\right).O\left(n^{-[(i+j+1)/2]}\right)\\
&= O\left(n^{-[(m+1)/2]}\right).
\end{aligned}$$

Hence, the lemma follows by induction on p. □

Lemma 7. *[4] Let $1 \leq p < \infty, f \in L_p(a,b], f^{(k)} \in AC[a,b]$ and $f^{(k+1)} \in L_p[a,b]$ then*

$$\|f^{(j)}\|_{L_p[a,b]} \leq K_j(\|f^{(k+1)}\|_{L_p[a,b]} + \|f\|_{L_p[a,b]}), j = 1,2,3....k,$$

where $K_j's$ are certain constants depending only on j,k,p,a and b.

Lemma 8. *Let $f \in L_p[0,\infty)$, $1 \leq p \leq \infty$ and $[a,b] \subset [0,\infty)$. Then, for n sufficiently large we have*

$$\|G_{n,c}(f,.)\|_{L_p[a,b]} \leq C\,\|f\|_{L_p[0,\infty)}.$$

Proof. First, we consider the case $p = 1$. Let φ be the characteristic function of the interval $[a,b]$. Then, we have

$$\begin{aligned}
\|G_{n,c}(f,.)\|_{L_p[a,b]} &= \int_a^b \left| \int_0^\infty W_n(u,t) f(u)\,du \right| dt \\
&= \int_a^b \left| \int_0^\infty W_n(u,t)\varphi(u) f(u)\,du \right| dt \\
&\quad + \int_a^b \left| \int_0^\infty W_n(u,t)(1-\varphi(u)) f(u)\,du \right| dt \\
&= F_1 + F_2 \ \text{ say.}
\end{aligned}$$

In view of Fubini's theorem and Lemma 2, we have

$$\begin{aligned}
F_1 &= \int_a^b \left(\int_0^\infty W_n(u,t)\varphi(u)|f(u)|\,du \right) dt \\
&\leq \int_a^b \left(\int_a^b W_n(u,t)\,dt \right) |f(u)|\,du \\
&\leq C \int_a^b |f(u)|\,du \\
&\leq C\,\|f\|_{L_1[a,b]}.
\end{aligned}$$

Again applying Fubini's theorem and Lemma 2, we obtain

$$\begin{aligned}
F_2 &= \int_a^b \left(\int_0^\infty W_n(u,t)(1-\varphi(u))|f(u)|\,du \right) dt \\
&\leq C\,\delta^{-2m} \int_0^\infty \left(\int_a^b W_n(u,t)(u-t)^{2m}\,dt \right) |f(u)|\,du \\
&\leq C\,\delta^{-2m} n^{-m} \|f\|_{L_1[0,\infty)} \to 0 \ \text{ as, } \ n \to \infty.
\end{aligned}$$

Next, for $p = \infty$, we have

$$\begin{aligned} \|G_{n,c}(f,.)\|_\infty &= \left\| \int_0^\infty W_n(u,t) f(u)\, du \right\| \\ &\leq \|f\|_\infty \int_0^\infty W_n(u,t)\, du \\ &\leq \|f\|_\infty. \end{aligned}$$

Thus, the lemma is established for the values $p = 1$ and $p = \infty$. Therefore, in view of the Riesz-Thorin interpolation theorem, the lemma is proved for $1 \leq p \leq \infty$. □

Corollary 9. *Using an induction on $r \in \mathbb{N}$, it follows that*

$$\|G^r_{n,c}(f,.)\|_{L_p[a,b]} \leq C \, \|f\|_{L_p[a,b]},$$

for all $1 \leq p \leq \infty$.
Consequently, $\|T_{n,k}(f)\|_{L_p[a,b]} \leq C \, \|G^r_{n,c}(f,.)\|_{L_p[a,b]} \leq C \, \|G_{n,c}(f,.)\|_{L_p[a,b]} \leq C \, \|f\|_{L_p[a,b]}$.

3 Main results

Theorem 10. *If $p > 1, f \in L_p[0,\infty), f$ has derivatives of order $2k$ on I_1 with $f^{(2k-1)} \in AC(I_1)$ and $f^{(2k)} \in L_p(I_1)$, then for all n sufficiently large*

$$\|T_{n,k}(f,.) - f(.)\|_{L_p(I_2)} \leq M_1 n^{-k} \left(\|f^{(2k)}\|_{L_p(I_1)} + \|f\|_{L_p[0,\infty)} \right). \tag{3.1}$$

Moreover, if $f \in L_1[0,\infty)$, f has derivatives of order $(2k-1)$ on I_1 with $f^{(2k-2)} \in AC(I_1)$ and $f^{(2k-1)} \in BV(I_1)$, then for all n sufficiently large

$$\|T_{n,k}(f,.) - f(.)\|_{L_1(I_2)} \leq M_2 \, n^{-k} \left(\|f^{(2k-1)}\|_{BV(I_1)} + \|f^{(2k-1)}\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)} \right), \tag{3.2}$$

where M_1 and M_2 are certain constants independent of f and n.

Proof. First, assume that $p > 1$. Then, by our hypothesis, for $t \in I_2$ and $u \in I_1$

$$f(u) = \sum_{j=0}^{2k-1} f^{(j)}(t) \frac{(u-t)^j}{j!} + \frac{1}{(2k-1)!} \int_t^u (u-w)^{2k-1} f^{(2k)}(w) dw.$$

Hence,

$$\begin{aligned} f(u) &= \sum_{j=0}^{2k-1} \frac{(u-t)^j}{j!} f^{(j)}(t) \\ &+ \frac{1}{2k-1!} \int_t^u (u-w)^{2k-1} \phi(u) f^{(2k)}(w) dw \\ &+ F(u,t)(1-\phi(u)), \end{aligned} \tag{3.3}$$

where $\phi(u)$ is the characteristic function of I_1 and for all $u \in [0,\infty)$ and $t \in I_2$

$$F(u,t) = f(u) - \sum_{j=0}^{2k-1} \frac{(u-t)^j}{j!} f^{(j)}(t).$$

Operating $T_{n,k}$ on both sides of 3.3, we have

$$\begin{aligned} T_{n,k}(f,t) - f(t) &= \sum_{j=1}^{2k-1} \frac{f^{(j)}(t)}{j!} T_{n,k}((u-t)^j, t) \\ &+ \frac{1}{(2k-1)!} T_{n,k}\left(\int_t^u (u-w)^{2k-1} \phi(u) f^{(2k)}(w)dw, t\right) \\ &+ T_{n,k}(F(u,t)(1-\phi(u),t) \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say.} \end{aligned}$$

In view of Lemma 5 and [4]

$$\begin{aligned} \|\Sigma_1\|_{L_p(I_2)} &\le C_1\, n^{-k} \left(\sum_{j=1}^{2k-1} \|f^{(j)}(t)\|_{L_p(I_2)} \right) \\ &\le C_2\, n^{-k} \left(\|f\|_{L_p(I_2)} + \|f^{(2k)}\|_{L_p(I_2)} \right). \end{aligned}$$

To estimate Σ_2, let h_f be the Hardy-Littlewood majorant [11] of $f^{(2k)}$ on I_1. Use of Hölder's inequality and 2.1 leads to

$$\begin{aligned} J_1 &:= \left| G_{n,c}\left(\phi(u) \int_t^u (u-w)^{2k-1} f^{(2k)}(w)dw, t \right) \right| \\ &\le G_{n,c}\left(\phi(u) \left| \int_t^u (u-w)^{2k-1} f^{(2k)}(w)dw \right|, t \right) \\ &\le G_{n,c}\left(\phi(u) \left| \int_t^u |u-w|^{2k-1} |f^{(2k)}(w)| dw \right|, t \right) \\ &\le G_{n,c}\left(\phi(u)(u-t)^{2k} |h_f(u)|, t \right) \\ &\le \left(G_{n,c}\left(|u-t|^{2kq} \phi(u), t \right) \right)^{1/q} . \left(G_{n,c}\left(|h_f(u)|^p \phi(u), t \right) \right)^{1/p} \\ &\le C_3\, n^{-k} \left(\int_{a_1}^{b_1} K_n(x,t;c) |h_f(u)|^p du \right)^{1/p}. \end{aligned}$$

Using Fubini's theorem, we get

$$\begin{aligned}\|J_1\|^p_{L_p(I_2)} &\leq C_3\, n^{-kp}\int_{a_2}^{b_2}\int_{a_1}^{b_1} K_n(x,t;c)|h_f(u)|^p du\, dt\\ &\leq C_3\, n^{-kp}\int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} K_n(x,t;c)dt\Big)|h_f(u)|^p du\\ &\leq C_3\, n^{-kp}\int_{a_2}^{b_2}|h_f(u)|^p du\\ &\leq C_3\, n^{-kp}\|h_f\|^p_{L_p(I_1)}\\ &\leq C_4\, n^{-kp}\|f^{(2k)}\|^p_{L_p(I_1)}.\end{aligned}$$

Consequently, $\|\Sigma_2\|_{L_p(I_2)} \leq C_5\, n^{-k}\|f^{(2k)}\|_{L_p(I_1)}$.
For $u \in [0,\infty)\setminus[a_1,b_1], t\in I_2$ there exists a $\delta>0$ such that $|u-t|\geq\delta$. Thus

$$\begin{aligned}|G_{n,c}(F(u,t)(1-\phi(u));t)| &\leq \delta^{-2k}G_{n,c}(|F(u,t)|(u-t)^{2k};t)\\ &= \delta^{-2k}\left[G_{n,c}(|f(u)|(u-t)^{2k};t)+\sum_{j=0}^{2k-1}\frac{|f^{(j)}(t)|}{j!}G_{n,c}(|u-t|^{2k+j};t)\right]\\ &= J_2+J_3, \text{ say.}\end{aligned}$$

Hölder's inequality and 2.1 get us

$$\begin{aligned}|J_2| &\leq \delta^{-2k}(G_{n,c}(|f(u)|^p;t)^{1/p}(G_{n,c}(|u-t|^{2kq};t)^{1/q}\\ &\leq C_6\, n^{-k}(G_{n,c}(|f(u)|^p;t)^{1/p}.\end{aligned}$$

Again, applying Fubini's theorem, we get $|J_2|\leq C_7\, n^{-k}\|f\|_{L_p[0,\infty)}$. Moreover, using 2.1 and [4] we obtain

$$\begin{aligned}\|J_3\|_{L_p(I_2)} &\leq C_8\, n^{-k}\sum_{j=0}^{2k-1}\|f^{(j)}\|_{L_p(I_2)}\\ &\leq C_8\, n^{-k}\Big(\|f\|_{L_p(I_2)}+\|f^{(2k)}\|_{L_p(I_2)}\Big).\end{aligned}$$

Combining the estimates of J_2 and J_3, we get

$$\|\Sigma_3\|_{L_p(I_2)} \leq C_9\, n^{-k}\Big[\|f\|_{L_p[0,\infty)}+\|f^{(2k)}\|_{L_p(I_2)}\Big].$$

Hence the result 3.1 follows.

Now, assume $p = 1$, then by the assumptions on f, for almost all $t \in I_2$ and for all $u \in I_1$,

$$\begin{aligned} f(u) &= \sum_{j=0}^{2k-1} \frac{(u-t)^j}{j!} f^{(j)}(t) \\ &+ \frac{1}{(2k-1)!} \int_t^u (u-w)^{2k-1} \phi(u) df^{(2k-1)}(w) \\ &+ F(u,t)(1-\phi(u)), \end{aligned} \tag{3.4}$$

where $\phi(u)$ denotes the characteristic function of I_1 and $F(u,t)$ is defined as

$$F(u,t) = f(u) - \sum_{j=0}^{2k-1} \frac{(u-t)^j}{j!} f^{(j)}(t),$$

for almost all $t \in I_2$ and for all $u \in [0, \infty)$. Thus

$$\begin{aligned} T_{n,k}(f,t) - f(t) &= \sum_{j=1}^{2k-1} \frac{f^{(j)}(t)}{j!} T_{n,k}((u-t)^j, t) \\ &+ \frac{1}{(2k-1)!} T_{n,k}\left(\int_t^u (u-w)^{2k-1} \phi(u) df^{(2k-1)}(w), t \right) \\ &+ T_{n,k}(F(u,t)(1-\phi(u), t) \\ &= \Gamma_1 + \Gamma_2 + \Gamma_3, \text{ say.} \end{aligned}$$

Applying Lemma 4 and [4] we obtain

$$\|\Gamma_1\|_{L_1(I_2)} \leq C_1 n^{-k} \Big(\|f\|_{L_1(I_2)} + \|f^{(2k-1)}\|_{L_1(I_2)} \Big).$$

Furthermore,

$$\begin{aligned} K &:= \|G_{n,c}\left(\int_t^u (u-w)^{2k-1} \phi(u) df^{(2k-1)}(w), t \right) \|_{L_1(I_2)} \\ &\leq \int_{a_2}^{b_2} \int_{a_1}^{b_1} K_n(x,t;c) |u-t|^{2k-1} \left| \int_t^u |df^{(2k-1)}(w)| \right| du\, dt. \end{aligned}$$

For each n there exists a non-negative integer $r = r(n)$ such that

$$r\, n^{-1/2} < \max(b_1 - a_2, b_2 - a_1) \leq (r+1)\, n^{-1/2}.$$

Then, we have

$$
\begin{aligned}
K &\leq \sum_{l=0}^{r}\int_{a_2}^{b_2}\left\{\int_{t+ln^{-1/2}}^{t+(l+1)n^{-1/2}}\phi(u)K_n(x,t;c)|u-t|^{2k-1}\right.\\
&\quad \left(\int_t^{t+(l+1)n^{-1/2}}\phi(w)|df^{(2k-1)}(w)|\right)du\\
&\quad +\int_{t-(l+1)n^{-1/2}}^{t-ln^{-1/2}}\phi(u)K_n(x,t;c)|u-t|^{2k-1}\\
&\quad \left.\left(\int_{t-(l+1)n^{-1/2}}^{t}\phi(w)|df^{(2k-1)}(w)|\right)du\right\}dt.
\end{aligned}
$$

Let $\varphi_{t,m_1,m_2}(w)$ denote the characteristic function of the interval

$$[t-m_1n^{-1/2},t+m_2n^{-1/2}],$$

where m_1,m_2 are non-negative integers. Now proceeding along lines of ([11], p. 70) we obtain after using Lemma 2 and Fubini's theorem:

$$
\begin{aligned}
K &\leq C_2\,n^{-(2k+1)/2}\left\{\sum_{l=1}^{r}l^{-4}\left(\int_{a_1}^{b_1}\left(\int_{w-(l+1)n^{-1/2}}^{w}dt\right)|df^{(2k-1)}(w)|\right.\right.\\
&+\int_{a_1}^{b_1}\left(\int_{w}^{w+(l+1)n^{-1/2}}dt\right)|df^{(2k-1)}(w)|\Bigg)\\
&+\left.\int_{a_1}^{b_1}\left(\int_{w-n^{-1/2}}^{w+n^{-1/2}}dt\right)|df^{(2k-1)}(w)|\right\}\\
&\leq C_3\,n^{-k}\|f^{(2k-1)}(w)\|_{BV(I_1)}.
\end{aligned}
$$

Hence, $\|\Gamma_2\|_{L_1(I_2)}\leq C_4\,n^{-k}\|f^{(2k-1)}\|_{BV(I_1)}$, where C_4 is a constant which depends on k.

For all $u\in[0,\infty)\setminus[a_1,b_1]$ and all $t\in I_2$, we can choose a $\delta>0$ such that $|u-t|\geq\delta$. Therefore

$$
\begin{aligned}
\|G_{n,c}((F(u,t)(1-\phi(u));t)\|_{L_1(I_2)} &\leq \int_{a_2}^{b_2}\int_0^{\infty}K_n(x,t;c)|f(u)|(1-\phi(u))\,du\,dt\\
&+\sum_{i=0}^{2k-1}\frac{1}{i!}\int_{a_2}^{b_2}\int_0^{\infty}K_n(x,t;c)|f^{(i)}(t)||u-t|^i(1-\phi(u))\,du\,dt\\
&= \Gamma_4+\Gamma_5,\text{ say.}
\end{aligned}
$$

For sufficiently large u, there exist positive constants R_0 and C_5 such that

$$\frac{(u-t)^{2k}}{u^{2k}+1}>C_5,\ \ \forall\,u\geq R_0,\,t\in I_2.$$

By Fubini's theorem

$$\begin{aligned}\Gamma_4 &= \left(\int_0^{R_0}\int_{a_2}^{b_2}+\int_{R_0}^{\infty}\int_{a_2}^{b_2}\right)K_n(x,t;c)|f(u)|(1-\phi(u))\,dt\,du\\ &= \Gamma_6+\Gamma_7, \text{ say.}\end{aligned}$$

Next, by using Lemma 2 we have

$$\Gamma_6 \leq C_6\, n^{-k}\left(\int_0^{R_0}|f(u)|du\right)$$

and

$$\begin{aligned}\Gamma_7 &\leq \frac{1}{C_5}\int_{R_0}^{\infty}\int_{a_2}^{b_2}K_n(x,t;c)\frac{(u-t)^{2k}}{u^{2k}+1}|f(u)|\,dt\,du\\ &\leq C_7\, n^{-k}\left(\int_{R_0}^{\infty}|f(u)|du\right),\ u \text{ is sufficiently large.}\end{aligned}$$

Hence, $\Gamma_4 \leq C_8\, n^{-k}\|f\|_{L_1[0,\infty)}$. Further, using 2.1 and [4] we get

$$\Gamma_5 \leq C_9\, n^{-k}\Big(\|f\|_{L_1(I_2)}+\|f^{(2k-1)}\|_{L_1(I_2)}\Big).$$

Combining the estimates of Γ_4 and Γ_5, we have

$$\Gamma_3 \leq C_{10}\, n^{-k}\Big(\|f\|_{L_1[0,\infty)}+\|f^{(2k-1)}\|_{L_1(I_2)}\Big).$$

Finally, combining the estimates of $\Gamma_1-\Gamma_3$, we get 3.2.

□

In our next result, we estimate the error in the L_p- approximation in terms of the $2k-$th order integral modulus of smoothness of the function.

Theorem 11. *If $p\geq 1, f\in L_p[0,\infty)$. Then, for all n sufficiently large there holds*

$$\|T_{n,k}(f;.)-f\|_{L_p(I_2)} \leq C_k\left(\omega_{2k}\Big(f,\frac{1}{\sqrt{n}},p,I_1\Big)+n^{-k}\|f\|_{L_p[0,\infty)}\right), \tag{3.5}$$

where C_k is a constant independent of f and n.

Proof. Let $f_{\eta,2k}(t)$ be the Steklov mean of $2k-$th order corresponding to $f(t)$ over I_1, where $\eta>0$ is sufficiently small and $f_{\eta,2k}(t)$ is defined to be zero outside I_1. Then, we have

$$\begin{aligned}\|T_{n,k}(f,.)-f\|_{L_p(I_2)} &\leq \|T_{n,k}(f-f_{\eta,2k},.)\|_{L_p(I_2)}\\ &+ \|T_{n,k}(f_{\eta,2k},.)-f_{\eta,2k}\|_{L_p(I_2)}+\|f_{\eta,2k}-f\|_{L_p(I_2)}\\ &= K_1+K_2+K_3,\ \text{say.}\end{aligned}$$

In view of property (c) of Steklov mean, we get

$$K_3 \leq M_1 \, \omega_{2k}(f, \eta, p, I_1).$$

It is well known that

$$\|f_{\eta,2k}^{(2k-1)}\|_{BV(I_3)} = \|f_{\eta,2k}^{(2k-1)}\|_{L_1(I_3)}.$$

By virtue of Theorem 10 ($p \geq 1$), we have

$$\begin{aligned} K_2 &\leq M_2 \, n^{-k} \left(\|f_{\eta,2k}^{(2k)}\|_{L_p(I_3)} + \|f_{\eta,2k}\|_{L_p[0,\infty)} \right) \\ &\leq M_3 \, n^{-k} \left(\eta^{-2k} \omega_{2k}(f, \eta, p, I_1) + \|f\|_{L_p[0,\infty)} \right), \end{aligned}$$

in view of the properties (b) and (d) of Lemma 1.
To estimate K_1, let $\phi(t)$ be the characteristic function of I_3. Then

$$\begin{aligned} G_{n,c}\,(f - f_{\eta,2k}(t), x) &= G_{n,c}\,(\phi(t)(f - f_{\eta,2k})(t), x) \\ &+ G_{n,c}\,((1 - \phi(t))(f - f_{\eta,2k})(t), x) \\ &= K_4 + K_5, \text{ say.} \end{aligned}$$

Clearly, the following inequality is true for $p = 1$, the truth of the same for $p > 1$ follows from Hölder's inequality

$$\int_{a_2}^{b_2} |K_4|^p dx \leq \int_{a_2}^{b_2} \int_{a_3}^{b_3} K_n(x, t; c)|(f - f_{\eta,2k})(t))|^p dt\, dx.$$

Now, applying Fubini's theorem, we get

$$\begin{aligned} \int_{a_2}^{b_2} |K_4|^p dx &\leq \int_{a_3}^{b_3} \int_{a_2}^{b_2} K_n(x, t; c)|(f - f_{\eta,2k})(t))|^p dx\, dt \\ &\leq \|f - f_{\eta,2k}\|^p_{L_p(I_3)}. \end{aligned}$$

Hence, $\|K_4\|_{L_p(I_2)} \leq \|f - f_{\eta,2k}\|_{L_p(I_3)}$.
Proceeding similarly, for all $p \geq 1$, we get

$$\|K_5\|_{L_p(I_2)} \leq M_3 \, n^{-k} \|f - f_{\eta,2k}\|_{L_p[0,\infty)}.$$

Consequently, by the property (c) of Lemma 1, we obtain

$$K_1 \leq M_4 \left(\omega_{2k}(f, \eta, p, I_1) + n^{-k} \|f\|_{L_p[0,\infty)} \right).$$

Choosing $\eta = n^{-\frac{1}{2}}$ and combining the estimates of $K_1 - K_3$, we obtain the required result. □

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