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GENERALIZED COMPATIBILITY IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we introduce the notion of generalized compatibility of a pair of mappings $F, G : X \times X \to X$, where (X, d) is a partially ordered metric space. We use this concept to prove a coupled coincidence point theorem for nonlinear contractions in partially ordered metric spaces. Our work extends the paper of Choudhury and Kundu [B.S. Choudhury and A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal. 73 (2010) 2524-2531]. Some examples are also given to illustrate the new concepts and the obtained result.

1 Introduction

Fixed point problems of contractive mappings in metric spaces endowed with a partial order have been studied by many authors (see [12, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 10, 13, 14]). In [12], some applications to matrix equations are presented and in [8, 11] some applications to ordinary differential equations are given. Bhaskar and Lakshmikantham [4] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and studied the problems of the uniqueness of a coupled fixed point in partially ordered metric spaces and applied their theorems to problems of the existence and uniqueness of solution for a periodic boundary value problem. In [9], Lakshmikantham and Ćirić introduced the concept of a coupled coincidence point for mappings $F : X \times X \to X$ and $g : X \to X$, and proved some nice coupled coincidence point theorems for nonlinear contractions in partially ordered metric spaces under the hypotheses that g is continuous and commutes with F. In 2011, Choudhury and Kundu [5] introduced the notion of compatibile mappings $F : X \times X \to X$ and $g : X \to X$, and obtained coupled coincidence point results under the hypotheses g is continuous and the pair $\{F, g\}$ is compatible.

In this paper, we consider mappings $F, G : X \times X \to X$, where (X, d) is a partially ordered metric space. We introduce a new concept of generalized compatibility of

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the pair $\{F, G\}$ and we prove a coupled coincidence point theorem for nonlinear contractions in partially ordered metric spaces. The presented theorem extends the recent result of Choudhury and Kundu [5] and some examples are also considered.

2 Mathematical preliminaries

Let $(X, \preceq)$ be a partially ordered set. The concept of a mixed monotone property of the mapping $F : X \times X \to X$ has been introduced by Bhaskar and Lakshmikantham in [4].

Definition 1. (see Bhaskar and Lakshmikantham [4]). *Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. Then the map F is said to have mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y; that is, for any $x, y \in X$,*

$$x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1 \preceq y_2 \text{ implies } F(x, y_2) \preceq F(x, y_1).$$

Lakshmikantham and Ćirić in [9] introduced the concept of a g-mixed monotone mapping.

Definition 2. (see Lakshmikantham and Ćirić [9]). *Let $(X, \preceq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. Then the map F is said to have mixed g-monotone property if $F(x, y)$ is monotone g-non-decreasing in x and is monotone g-non-increasing in y; that is, for any $x, y \in X$,*

$$gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y)$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x, y_2) \preceq F(x, y_1).$$

Definition 3. (see Bhaskar and Lakshmikantham [4]). *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if*

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Definition 4. (see Lakshmikantham and Ćirić [9]). *An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if*

$$F(x, y) = gx \text{ and } F(y, x) = gy.$$

Definition 5. (see Lakshmikantham and Ćirić [9]). *Let X be a non-empty set. Then we say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if*

$$g(F(x, y)) = F(gx, gy).$$

Lakshmikantham and Ćirić in [9] proved the following nice result.

Theorem 6. (see Lakshmikantham and Ćirić [9]). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(t) < t$ and $\lim_{r\to t^+} \phi(r) < t$ for each $t > 0$ and also suppose $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and*

$$d(F(x, y), F(u, v)) \leq \phi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right)$$

for all $x, y, u, v \in X$ with $gx \preceq gu$ and $gv \preceq gy$. Assume that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either F is continuous or X has the following properties:

1. *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
2. *if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$ then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point.

Choudhury and Kundu in [5] introduced the notion of compatibility.

Definition 7. (see Choudhury and Kundu [5]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if*

$$\lim_{n\to+\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n))) = 0$$

and

$$\lim_{n\to+\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n))) = 0,$$

whenever (x_n) and (y_n) are sequences in X, such that

$$\lim_{n\to+\infty} F(x_n, y_n) = \lim_{n\to+\infty} gx_n = x$$

and

$$\lim_{n\to+\infty} F(y_n, x_n) = \lim_{n\to+\infty} gy_n = y,$$

for all $x, y \in X$ are satisfied.

Using the concept of compatibility, Choudhury and Kundu proved the following interesting result.

Theorem 8. (see Choudhury and Kundu [5]). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $\phi : [0, +\infty) \to [0, +\infty)$ be such that $\phi(t) < t$ and $\lim_{r \to t^+} \phi(r) < t$ for all $t > 0$. Suppose $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property and satisfy*

$$d(F(x,y), F(u,v)) \leq \phi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right)$$

for all $x, y, u, v \in X$ with $gx \preceq gu$ and $gv \preceq gy$. Let $F(X \times X) \subseteq g(X)$, g be continuous and monotone increasing and F and g be compatible mappings. Also suppose either F is continuous or X has the following properties:

1. *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
2. *if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$ then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point.

Now, we introduce the following new concepts.
Let $(X, \preceq)$ be a partially ordered set endowed with a metric d. We consider two mappings $F, G : X \times X \to X$.

Definition 9. *F is said to be G-increasing with respect to $\preceq$ if for all $x, y, u, v \in X$, we have*

$$G(x,y) \preceq G(u,v) \text{ implies } F(x,y) \preceq F(u,v).$$

We present three examples illustrating Definition 9.

Example 10. *Let $X = (0, +\infty)$ endowed with the natural ordering of real numbers $\leq$. Define the mappings $F, G : X \times X \to X$ by*

$$F(x,y) = \ln(x+y) \quad \textit{and} \quad G(x,y) = x + y$$

for all $(x, y) \in X \times X$. Then F is G-increasing with respect to $\leq$.

Example 11. *Let $X = \mathbb{N}$ endowed with the partial order $\preceq$ defined by*

$$x, y \in X, \quad x \preceq y \quad \textit{if and only if} \quad y \textit{ divides } x.$$

Define the mappings $F, G : X \times X \to X$ by

$$F(x,y) = x^2 y^2 \quad \textit{and} \quad G(x,y) = xy$$

for all $(x, y) \in X \times X$. Then F is G-increasing with respect to $\preceq$.

Example 12. *Let X be the set of all subsets of $\mathbb{N}$. We endow X with the partial order $\preceq$ defined by*

$$A, B \in X, \quad A \preceq B \quad \textit{if and only if} \quad A \subseteq B.$$

Define the mappings $F, G : X \times X \to X$ by

$$F(A,B) = A \cup B \cup \{0\} \quad \textit{and} \quad G(A,B) = A \cup B$$

for all $A, B \in X$. Then F is G-increasing with respect to $\preceq$.

Definition 13. *An element $(x,y) \in X \times X$ is called a coupled coincidence point of F and G if*

$$F(x,y) = G(x,y) \quad \textit{and} \quad F(y,x) = G(y,x).$$

Example 14. *Let $X = \mathbb{R}$ and $F, G : X \times X \to X$ defined by*

$$F(x,y) = xy \quad \textit{and} \quad G(x,y) = \frac{2}{3}(x+y)$$

for all $x, y \in X$. Then $(0,0)$, $(1,2)$ and $(2,1)$ are coupled coincidence points of F and G.

Definition 15. *We say that the pair $\{F, G\}$ satisfies the generalized compatibility if*

$$\begin{cases} d(F(G(x_n,y_n),G(y_n,x_n)),G(F(x_n,y_n),F(y_n,x_n))) \to 0 & \textit{as} \;\; n \to +\infty; \\ d(F(G(y_n,x_n),G(x_n,y_n)),G(F(y_n,x_n),F(x_n,y_n))) \to 0 & \textit{as} \;\; n \to +\infty, \end{cases}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{cases} F(x_n,y_n) \to t_1 \quad G(x_n,y_n) \to t_1 & \textit{as} \;\; n \to +\infty; \\ F(y_n,x_n) \to t_2 \quad G(y_n,x_n) \to t_2 & \textit{as} \;\; n \to +\infty. \end{cases}$$

The following examples illustrate the concept of generalized compatibility.

Example 16. *Let $X = \mathbb{R}$ endowed with the standard metric $d(x,y) = |x-y|$ for all $x, y \in X$. Define $F, G : X \times X \to X$ by*

$$F(x,y) = x^2 - y^2 \quad \textit{and} \quad G(x,y) = x^2 + y^2$$

for all $x, y \in X$. Let (x_n) and (y_n) two sequences in X such that

$$\begin{cases} F(x_n,y_n) \to t_1 \quad G(x_n,y_n) \to t_1 & \textit{as} \;\; n \to +\infty; \\ F(y_n,x_n) \to t_2 \quad G(y_n,x_n) \to t_2 & \textit{as} \;\; n \to +\infty. \end{cases}$$

We can prove easily that $t_1 = t_2 = 0$ and

$$\begin{cases} d(F(G(x_n,y_n),G(y_n,x_n)),G(F(x_n,y_n),F(y_n,x_n))) \to 0 & \textit{as} \;\; n \to +\infty; \\ d(F(G(y_n,x_n),G(x_n,y_n)),G(F(y_n,x_n),F(x_n,y_n))) \to 0 & \textit{as} \;\; n \to +\infty. \end{cases}$$

Then the pair $\{F, G\}$ satisfies the generalized compatibility.

Example 17. *Let (X,d) be a metric space, $F : X \times X \to X$ and $g : X \to X$. Define the mapping $G : X \times X \to X$ by*

$$G(x,y) = gx, \ \forall\,(x,y) \in X \times X.$$

It is easy to show that if $\{F,g\}$ is compatible, then $\{F,G\}$ satisfies the generalized compatibility.

3 Main result

First, denote by Φ be the set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying

(i) φ is non-decreasing,

(ii) $\varphi(t) < t$ for all $t > 0$,

(iii) $\lim_{r\to t^+} \varphi(r) < t$ for all $t > 0$.

Lemma 18. *Let $\varphi \in \Phi$ and (u_n) be a given sequence such that $u_n \to 0^+$ as $n \to +\infty$. Then, $\varphi(u_n) \to 0^+$ as $n \to +\infty$.*

Proof. Let $\varepsilon > 0$. Since $u_n \to 0^+$ as $n \to +\infty$, there exists $N \in \mathbb{N}$ such that

$$0 \le u_n < \varepsilon \text{ for all } n \ge N.$$

Using (i) and (ii), we get

$$\varphi(u_n) \le \varphi(\varepsilon) < \varepsilon \text{ for all } n \ge N.$$

Thus we proved that $\varphi(u_n) \to 0^+$ as $n \to +\infty$. ∎

Theorem 19. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F,G : X \times X \to X$ be two mappings such that F is G-increasing with respect to $\preceq$, and satisfy*

$$d(F(x,y),F(u,v)) \le \varphi\left(\frac{d(G(x,y),G(u,v)) + d(G(y,x),G(v,u))}{2}\right), \tag{3.1}$$

for all $x,y,u,v \in X$, with $G(x,y) \preceq G(u,v)$ and $G(v,u) \preceq G(y,x)$, where $\varphi \in \Phi$. Suppose that for any $x,y \in X$, there exist $u,v \in X$ such that

$$\begin{cases} F(x,y) = G(u,v) \\ F(y,x) = G(v,u). \end{cases} \tag{3.2}$$

Suppose that G is continuous and has the mixed monotone property, and the pair $\{F,G\}$ satisfies the generalized compatibility. Also suppose either F is continuous or X has the following properties:

(a) if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,

(b) if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all n.

If there exist $x_0, y_0 \in X$ such that $G(x_0, y_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq G(y_0, x_0)$, then F and G have a coupled coincidence point.

Proof. Let $x_0, y_0 \in X$ such that $G(x_0, y_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq G(y_0, x_0)$ (such points exist by hypothesis). Thanks to (3.2), there exists $(x_1, y_1) \in X \times X$ such that

$$F(x_0, y_0) = G(x_1, y_1) \quad \text{and} \quad F(y_0, x_0) = G(y_1, x_1).$$

Continuing this process, we can construct two sequences (x_n) and (y_n) in X such that

$$F(x_n, y_n) = G(x_{n+1}, y_{n+1}), \quad F(y_n, x_n) = G(y_{n+1}, x_{n+1}), \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

We will show that for all $n \in \mathbb{N}$, we have

$$G(x_n, y_n) \preceq G(x_{n+1}, y_{n+1}) \quad \text{and} \quad G(y_{n+1}, x_{n+1}) \preceq G(y_n, x_n). \tag{3.4}$$

We shall use the mathematical induction. Since $G(x_0, y_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq G(y_0, x_0)$, and as $G(x_1, y_1) = F(x_0, y_0)$ and $G(y_1, x_1) = F(y_0, x_0)$, we have

$$G(x_0, y_0) \preceq G(x_1, y_1) \quad \text{and} \quad G(y_1, x_1) \preceq G(y_0, x_0).$$

Thus (3.4) holds for $n = 0$. Suppose now that (3.4) holds for some fixed $n \in \mathbb{N}$. Since F is G-increasing with respect to $\preceq$, we have

$$G(x_{n+1}, y_{n+1}) = F(x_n, y_n) \preceq F(x_{n+1}, y_{n+1}) = G(x_{n+2}, y_{n+2})$$

and

$$F(y_{n+1}, x_{n+1}) = G(y_{n+2}, x_{n+2}) \preceq F(y_n, x_n) = G(y_{n+1}, x_{n+1}).$$

Thus we proved that (3.4) holds for all $n \in \mathbb{N}$.

For all $n \in \mathbb{N}$, denote

$$\delta_n = d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1})). \tag{3.5}$$

We can suppose that $\delta_n > 0$ for all $n \in \mathbb{N}$, if not, (x_n, y_n) will be a coincidence point and the proof is finished. We claim that for any $n \in \mathbb{N}$, we have

$$\delta_{n+1} \leq 2\varphi\left(\frac{\delta_n}{2}\right). \tag{3.6}$$

Since $G(x_n, y_n) \preceq G(x_{n+1}, y_{n+1})$ and $G(y_n, x_n) \succeq G(y_{n+1}, x_{n+1})$, letting $x = x_n$, $y = y_n$, $u = x_{n+1}$ and $v = y_{n+1}$ in (3.1), and using (3.3), we get

$$\begin{aligned} d(G(x_{n+1}, y_{n+1}), G(x_{n+2}, y_{n+2})) &= d(F(x_n, y_n), F(x_{n+1}, y_{n+1})) \\ &\leq \varphi\left(\frac{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))}{2}\right) \\ &= \varphi\left(\frac{\delta_n}{2}\right). \end{aligned} \tag{3.7}$$

Similarly, since $G(y_{n+1}, x_{n+1}) \preceq G(y_n, x_n)$ and $G(x_{n+1}, y_{n+1}) \succeq G(x_n, y_n)$, we have

$$\begin{aligned} d(G(y_{n+2}, x_{n+2}), G(y_{n+1}, x_{n+1})) &= d(F(y_{n+1}, x_{n+1}), F(y_n, x_n)) \\ &\leq \varphi\left(\frac{d(G(y_{n+1}, x_{n+1}), G(y_n, x_n)) + d(G(x_{n+1}, y_{n+1}), G(x_n, y_n))}{2}\right) \\ &= \varphi\left(\frac{\delta_n}{2}\right). \end{aligned} \tag{3.8}$$

Summing (3.7) to (3.8) yields (3.6).

From (3.6), since $\varphi(t) < t$ for all $t > 0$, it follows that the sequence (δ_n) is monotone decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to+\infty} \delta_n = \delta^+.$$

If possible, let $\delta > 0$. Taking the limit as $n \to +\infty$ in (3.6) and using $\lim_{r\to t^+} \varphi(r) < t$ for all $t > 0$, we obtain

$$\delta = \lim_{n\to+\infty} \delta_n \leq 2 \lim_{n\to+\infty} \varphi\left(\frac{\delta_{n-1}}{2}\right) = 2 \lim_{\delta_{n-1}\to\delta^+} \varphi\left(\frac{\delta_{n-1}}{2}\right) < 2\frac{\delta}{2} = \delta,$$

which is a contradiction. Thus $\delta = 0$, that is,

$$\lim_{n\to+\infty} d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1})) = \lim_{n\to+\infty} \delta_n = 0. \tag{3.9}$$

We shall prove that $((G(x_n, y_n), G(y_n, x_n)))$ is a Cauchy sequence in $X \times X$ endowed with the metric η defined by

$$\eta((x, y), (u, v)) = d(x, u) + d(y, v)$$

for all $(x, y), (u, v) \in X \times X$. We argue by contradiction. Suppose that $((G(x_n, y_n), G(y_n, x_n)))$ is not a Cauchy sequence in $(X \times X, \eta)$. Then, there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $(m(k))$ and $(n(k))$ such that for all positive integer k with $n(k) > m(k) > k$, we have

$$\begin{cases}\eta(((Gx_{m(k)}, Gy_{m(k)}), (Gy_{m(k)}, Gx_{m(k)})), ((Gx_{n(k)}, Gy_{n(k)}), (Gy_{n(k)}, Gx_{n(k)}))) > \varepsilon, \\ \eta(((Gx_{m(k)}, Gy_{m(k)}), (Gy_{m(k)}, Gx_{m(k)})), ((Gx_{n(k)-1}, Gy_{n(k)-1}), (Gy_{n(k)-1}, Gx_{n(k)-1}))) \leq \varepsilon. \end{cases} \tag{3.10}$$

By definition of the metric η, we have

$$d_k = d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{n(k)}, Gy_{n(k)})) + d((Gy_{m(k)}, Gx_{m(k)}), (Gy_{n(k)}, Gx_{n(k)})) > \varepsilon \tag{3.11}$$

and

$$d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{n(k)-1}, Gy_{n(k)-1})) + d((Gy_{m(k)}, Gx_{m(k)}), (Gy_{n(k)-1}, Gx_{n(k)-1})) \leq \varepsilon. \tag{3.12}$$

Further from (3.11) and (3.12), for all $k \geq 0$, we have

$$\begin{aligned}\varepsilon < d_k \leq & d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{n(k)-1}, Gy_{n(k)-1})) + d((Gx_{n(k)-1}, Gy_{n(k)-1}), (Gx_{n(k)}, Gy_{n(k)})) \\ & + d((Gy_{m(k)}, Gx_{m(k)}), (Gy_{n(k)-1}, Gx_{n(k)-1})) + d((Gy_{n(k)-1}, Gx_{n(k)-1}), (Gy_{n(k)}, Gx_{n(k)})) \\ \leq & \varepsilon + \delta_{n(k)-1}.\end{aligned}$$

Taking the limit as $k \to +\infty$ in the above inequality, we have by (3.9),

$$\lim_{k \to +\infty} d_k = \varepsilon^+. \tag{3.13}$$

Again, for all $k \geq 0$, we have

$$\begin{aligned} d_k = & d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{n(k)}, Gy_{n(k)})) + d((Gy_{m(k)}, Gx_{m(k)}), (Gy_{n(k)}, Gx_{n(k)})) \\ \leq & d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{m(k)+1}, Gy_{m(k)+1})) + d((Gx_{m(k)+1}, Gy_{m(k)+1}), (Gx_{n(k)+1}, Gy_{n(k)+1})) \\ & + d((Gx_{n(k)+1}, Gy_{n(k)+1}), (Gx_{n(k)}, Gy_{n(k)})) + d((Gy_{m(k)}, Gx_{m(k)}), (Gy_{m(k)+1}, Gx_{m(k)+1})) \\ & + d((Gy_{m(k)+1}, Gx_{m(k)+1}), (Gy_{n(k)+1}, Gx_{n(k)+1})) + d((Gy_{n(k)+1}, Gx_{n(k)+1}), (Gy_{n(k)}, Gx_{n(k)})) \\ = & d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{m(k)+1}, Gy_{m(k)+1})) + d((Gy_{m(k)}, Gx_{m(k)}), (Gy_{m(k)+1}, Gx_{m(k)+1})) \\ & + d((Gx_{n(k)+1}, Gy_{n(k)+1}), (Gx_{n(k)}, Gy_{n(k)})) + d((Gy_{n(k)+1}, Gx_{n(k)+1}), (Gy_{n(k)}, Gx_{n(k)})) \\ & + d((Gx_{m(k)+1}, Gy_{m(k)+1}), (Gx_{n(k)+1}, Gy_{n(k)+1})) + d((Gy_{m(k)+1}, Gx_{m(k)+1}), (Gy_{n(k)+1}, Gx_{n(k)+1})). \end{aligned}$$

Hence, for all $k \geq 0$,

$$\begin{aligned} d_k \leq & \delta_{m(k)} + \delta_{n(k)} \\ & + d((Gx_{m(k)+1}, Gy_{m(k)+1}), (Gx_{n(k)+1}, Gy_{n(k)+1})) + d((Gy_{m(k)+1}, Gx_{m(k)+1}), (Gy_{n(k)+1}, Gx_{n(k)+1})). \end{aligned} \tag{3.14}$$

From (3.1), (3.4) and (3.11), for all $k \geq 0$, we have

$$d((Gx_{m(k)+1}, Gy_{m(k)+1}), (Gx_{n(k)+1}, Gy_{n(k)+1})) = d((Fx_{m(k)}, Fy_{m(k)}), (Fx_{n(k)}, Fy_{n(k)}))$$
$$\leq \varphi\Big(\frac{d(G(x_{m(k)}, y_{m(k)}), G(x_{n(k)}, y_{n(k)})) + d(G(y_{m(k)}, x_{m(k)}), G(y_{n(k)}, x_{n(k)}))}{2}\Big)$$
$$= \varphi\left(\frac{d_k}{2}\right). \tag{3.15}$$

Also, from (3.1), (3.4) and (3.11), for all $k \geq 0$, we have

$$d((Gy_{m(k)+1}, Gx_{m(k)+1}), (Gy_{n(k)+1}, Gx_{n(k)+1})) = d((Fy_{m(k)}, Fx_{m(k)}), (Fy_{n(k)}, Fx_{n(k)}))$$
$$\leq \varphi\Big(\frac{d(G(y_{m(k)}, x_{m(k)}), G(y_{n(k)}, x_{n(k)})) + d(G(x_{m(k)}, y_{m(k)}), G(x_{n(k)}, y_{n(k)}))}{2}\Big)$$
$$= \varphi\left(\frac{d_k}{2}\right). \tag{3.16}$$

Putting (3.15) and (3.16) in (3.14), we get

$$d_k \leq \delta_{m(k)} + \delta_{n(k)} + 2\varphi\left(\frac{d_k}{2}\right).$$

Letting $k \to +\infty$ in the above inequality and using (3.9) and (3.13), we obtain

$$\varepsilon \leq 2 \lim_{k\to+\infty} \varphi\left(\frac{d_k}{2}\right) = 2 \lim_{d_k\to\varepsilon^+} \varphi\left(\frac{d_k}{2}\right) < 2\frac{\varepsilon}{2} = \varepsilon, \tag{3.17}$$

which is a contradiction. Thus we proved that $((G(x_n, y_n), G(y_n, x_n)))$ is a Cauchy sequence in $(X \times X, \eta)$, which implies that $((G(x_n, y_n))$ and $(G(y_n, x_n))$ are Cauchy sequences in (X, d).

Now, since (X, d) is complete, there exist $x, y \in X$ such that

$$\lim_{n\to+\infty} G(x_n, y_n) = \lim_{n\to+\infty} F(x_n, y_n) = x \text{ and } \lim_{n\to+\infty} G(y_n, x_n) = \lim_{n\to+\infty} F(y_n, x_n) = y. \tag{3.18}$$

Since the pair $\{F, G\}$ satisfies the generalized compatibility, from (3.18), we get

$$\lim_{n\to+\infty} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) = 0 \tag{3.19}$$

and

$$\lim_{n\to+\infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) = 0. \tag{3.20}$$

Suppose that F is continuous.
For all $n \geq 0$, we have

$$\begin{aligned} d(G(x,y),F(G(x_n,y_n),G(y_n,x_n))) &\leq d(G(x,y),G(F(x_n,y_n),F(y_n,x_n))) \\ &\quad +d(G(F(x_n,y_n),F(y_n,x_n)),F(G(x_n,y_n),G(y_n,x_n))). \end{aligned}$$

Taking the limit as $n \to +\infty$, using (3.18), (3.19) and the fact that F and G are continuous, we have

$$G(x,y) = F(x,y). \tag{3.21}$$

Similarly, using (3.18), (3.20) and the fact that F and G are continuous, we have

$$G(y,x) = F(y,x). \tag{3.22}$$

Thus, we proved that (x,y) is a coupled coincidence point of F and G.

Now, suppose that (a) and (b) hold.
By (3.4) and (3.18), we have $(G(x_n,y_n))$ is non-decreasing sequence, $G(x_n,y_n) \to x$ and $(G(y_n,x_n))$ is non-increasing sequence, $G(y_n,x_n) \to y$ as $n \to +\infty$. Then by (a) and (b), for all $n \in \mathbb{N}$, we have

$$G(x_n,y_n) \preceq x \quad \text{and} \quad G(y_n,x_n) \succeq y. \tag{3.23}$$

Since the pair $\{F,G\}$ satisfies the generalized compatibility and G is continuous, by (3.19) and (3.20), we have

$$\begin{aligned} \lim_{n\to+\infty} G(G(x_n,y_n),G(y_n,x_n)) &= G(x,y) \\ &= \lim_{n\to+\infty} G(F(x_n,y_n),F(y_n,x_n)) \\ &= \lim_{n\to+\infty} F(G(x_n,y_n),G(y_n,x_n)) \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \lim_{n\to+\infty} G(G(y_n,x_n),G(x_n,y_n)) &= G(y,x) \\ &= \lim_{n\to+\infty} G(F(y_n,x_n),F(x_n,y_n)) \\ &= \lim_{n\to+\infty} F(G(y_n,x_n),G(x_n,y_n)). \end{aligned} \tag{3.25}$$

Now, we have

$$\begin{aligned} d(G(x,y),F(x,y)) &\leq d(G(x,y),G(G(x_{n+1},y_{n+1}),G(y_{n+1},x_{n+1}))) \\ &\quad +d(G(F(x_n,y_n),F(y_n,x_n)),F(x,y)). \end{aligned}$$

Letting $n \to +\infty$ in the above inequality and using (3.24), we get

$$\begin{aligned} d(G(x,y),F(x,y)) &\leq \lim_{n\to+\infty} d(G(F(x_n,y_n),F(y_n,x_n)),F(x,y)) \\ &= \lim_{n\to+\infty} d(F(G(x_n,y_n),G(y_n,x_n)),F(x,y)). \end{aligned}$$

Since G has the mixed monotone property, it follows from (3.23) that

$$G(G(x_n, y_n), G(y_n, x_n)) \preceq G(x, y) \quad \text{and} \quad G(G(y_n, x_n), G(x_n, y_n)) \succeq G(y, x).$$

Then, using (3.1), (3.24), (3.25) and Lemma 18, we get

$$d(G(x,y), F(x,y)) \\ \leq \lim_{n\to+\infty} \varphi\left(\frac{d(G(G(x_n,y_n),G(y_n,x_n)),G(x,y)) + d(G(G(y_n,x_n),G(x_n,y_n)),G(y,x))}{2}\right) = 0.$$

Then we get

$$G(x, y) = F(x, y).$$

Similarly, we can show that

$$G(y, x) = F(y, x).$$

Thus we proved that (x, y) is a coupled coincidence point of F and G.

This completes the proof of the Theorem 19. □

Now, we deduce an analogous result to Theorem 8 of Choudhury and Kundu [5]. At first, we introduce the following definition.

Definition 20. *Let $(X, \preceq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. We say that F is g-increasing with respect to $\preceq$ if for any $x, y \in X$,*

$$gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y)$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x, y_1) \preceq F(x, y_2).$$

Corollary 21. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F is g-increasing with respect to $\preceq$, and satisfy*

$$d(F(x, y), F(u, v)) \leq \varphi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right),$$

for all $x, y, u, v \in X$, with $gx \preceq gu$ and $gv \preceq gy$, where $\varphi \in \Phi$. Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and monotone increasing with respect to $\preceq$, and the pair $\{F, g\}$ is compatible. Also suppose either F is continuous or X has the following properties:

(a) if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,

(b) if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all n.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then F and g have a coupled coincidence point, that is, there exists $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Proof. Taking $G : X \times X \to X$, $(x, y) \mapsto G(x, y) = gx$ in Theorem 19, we obtain Corollary 21. □

Now, we present an example to illustrate our obtained result given by Theorem 19.

Example 22. *Let $X = [0, 1]$ endowed with the natural ordering of real numbers. We endow X with the standard metric $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete metric space. Define the mappings $G, F : X \times X \to X$ by*

$$G(x,y) = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases} \quad \text{and} \quad F(x,y) = \begin{cases} \frac{x-y}{3} & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}.$$

Let us prove that F is G-increasing.
Let $(x, y), (u, v) \in X \times X$ with $G(x, y) \leq G(u, v)$. We consider the following cases.
Case-1: $x < y$.
In this case, we have $F(x, y) = 0 \leq F(u, v)$.
Case-2: $x \geq y$.
If $u \geq v$, we get

$$G(x,y) \leq G(u,v) \Rightarrow x - y \leq u - v \Rightarrow \frac{x-y}{3} \leq \frac{u-v}{3} \Rightarrow F(x,y) \leq F(u,v).$$

If $u < v$, we get

$$G(x,y) \leq G(u,v) \Rightarrow 0 \leq x - y \leq 0 \Rightarrow x = y \Rightarrow F(x,y) = 0 \leq F(u,v).$$

Thus we proved that F is G-increasing.
Let us prove that for any $x, y \in X$, there exist $u, v \in X$ such that

$$\begin{cases} F(x,y) = G(u,v) \\ F(y,x) = G(v,u) \end{cases}.$$

Let $(x, y) \in X \times X$ be fixed. We consider the following cases:
Case-1: $x = y$.
In this case, $F(x, y) = 0 = G(x, y)$ and $F(y, x) = 0 = G(y, x)$.
Case-2: $x > y$.
In this case, we have

$$F(x,y) = \frac{x-y}{3} = G(x/3, y/3) \quad \text{and} \quad F(y,x) = 0 = G(y/3, x/3).$$

Case-3: $x < y$.
In this case, we have

$$F(x,y) = 0 = G(x/3, y/3) \quad \text{and} \quad F(y,x) = \frac{y-x}{3} = G(y/3, x/3).$$

G is continuous and has the mixed monotone property.
Clearly G is continuous. Let $(x,y) \in X \times X$ be fixed. Suppose that $x_1, x_2 \in X$ are such that $x_1 < x_2$. We distinguish the following cases.
Case-1: $x_1 < y$.
In this case, we have $G(x_1, y) = 0 \leq G(x_2, y)$.
Case-2: $x_2 > x_1 \geq y$.
In this case, we have

$$G(x_1, y) = x_1 - y \leq x_2 - y = G(x_2, y).$$

Similarly, we can show that if $y_1, y_2 \in X$ are such that $y_1 < y_2$, then $G(x, y_1) \geq G(x, y_2)$.
Now, we prove that the pair $\{F, G\}$ satisfies the generalized compatibility hypothesis.
Let (x_n) and (y_n) be two sequences in X such that

$$t_1 = \lim_{n\to+\infty} G(x_n, y_n) = \lim_{n\to+\infty} F(x_n, y_n)$$

and

$$t_2 = \lim_{n\to+\infty} G(y_n, x_n) = \lim_{n\to+\infty} F(y_n, x_n).$$

Then obviously, $t_1 = t_2 = 0$. It follows easily that

$$\lim_{n\to+\infty} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) = 0$$

and

$$\lim_{n\to+\infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) = 0.$$

There exists $(x_0, y_0) \in X \times X$ such that $G(x_0, y_0) \leq F(x_0, y_0)$ and $G(y_0, x_0) \geq F(y_0, x_0)$.
We have

$$G(0, 1/2) = 0 = F(0, 1/2) \quad \text{and} \quad G(1/2, 0) = 1/2 \geq 1/6 = F(1/2, 0).$$

Now, let $\varphi : [0, +\infty) \to [0, +\infty)$ be defined as

$$\varphi(t) = \frac{2t}{3} \text{ for all } t \geq 0.$$

Clearly $\varphi \in \Phi$. Let us prove that inequality (3.1) is satisfied for all $x, y, u, v \in X$, with $G(x,y) \preceq G(u,v)$ and $G(v,u) \preceq G(y,x)$.

Let $x, y, u, v \in X$, with $G(x,y) \preceq G(u,v)$ and $G(v,u) \preceq G(y,x)$. We have

$$\begin{aligned} d(F(x,y),F(u,v)) &= |F(x,y)-F(u,v)| \\ &= \frac{1}{3}|G(x,y)-G(u,v)| \\ &= \frac{2}{3}\left(\frac{|G(x,y)-G(u,v)|}{2}\right) \\ &\leq \frac{2}{3}\left(\frac{|G(x,y)-G(u,v)|+|G(y,x)-G(v,u)|}{2}\right) \\ &= \varphi\left(\frac{d(G(x,y),G(u,v))+d(G(y,x),G(v,u))}{2}\right). \end{aligned}$$

Then, inequality (3.1) is satisfied.

Now, all the required hypotheses of Theorem 19 are satisfied. Thus we deduce the existence of a coupled coincidence point of F and G. Here, $(0,0)$ is a coupled coincidence point of F and G.

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