

ÉTUDE GÉOMÉTRIQUE ET TOPOLOGIQUE DU  
FLOT GÉODÉSIQUE SUR LE GROUPE DES  
ROTATIONS

(GEOMETRIC AND TOPOLOGICAL STUDY OF  
THE GEODESIC FLOW ON THE ROTATION  
GROUP)

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*This article is dedicated to Professor Luc Haine on the occasion of his 60th birthday.*

**Abstract.** The aim of this survey paper is to investigate the algebraic complete integrability of Euler-Arnold's body description of the four dimensional rigid body, or equivalently of geodesics in  $SO(4)$  using left-invariant metrics that arise from inertia tensors, namely non-degenerate maps  $\Lambda : so(4) \rightarrow so(4)^* \cong so(4)$  together with the canonical inner product associated to the Killing form. Algebraic complete integrability is motivated by Arnold-Liouville's classical notion of complete integrability : one extends the value of space and time coordinates from  $\mathbb{R}$  to  $\mathbb{C}$ , and then the regular invariant manifolds are complex instead of real tori; in addition one demands such complex tori to be projective. Using different methods, as systematized by Adler-Haine-van Moerbeke-Mumford, to study the integrability of the geodesic flow on the rotation group, we will see that the linearization is carried on an abelian surface and each time a Prym variety appears related to this problem.

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## References

- [1] M. Adams, J. Harnad and E. Previato, *Isospectral Hamiltonian Flows in Finite and Infinite Dimensions. I. Generalized Moser Problem and Moment Maps into Loop Algebras*, Commun. Math. Phys., **117** (1988), 451-500. [MR0953833](#)(MR89k:58112). [Zbl 0659.58022](#).

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- [2] M. Adams, J. Harnad and J. Hurtubise, *Isospectral Hamiltonian Flows in Finite and Infinite Dimensions II. Integration of Flows*, Commun. Math. Phys., **134** (1990), 555-585. [MR1086744](#)(MR92a:58055). [Zbl 0717.58051](#).
- [3] M. Adler, *On a trace functional for pseudo-differential operators and the symplectic structure of the Korteweg-de Vries equation*, Invent. Math., **50** (1979), 219-248. [MR520927](#)(80i:58026) [Zbl 0393.35058](#).
- [4] M. Adler, P. and van Moerbeke, *Completely integrable systems, Euclidean Lie algebras and curves*, Adv. in Math., **38** (1980), 267-317. [MR597729](#)(MR83m:58041). [Zbl 0455.58017](#).
- [5] M. Adler, P. and van Moerbeke, *Linearization of Hamiltonian systems, Jacobi varieties and representation theory*, Adv. in Math., **38** (1980), 318-379. [MR0597729](#)(MR83m:58042). [Zbl 0455.58010](#).
- [6] M. Adler, P., van Moerbeke and P. Vanhaecke, *Algebraic integrability, Painlevé geometry and Lie algebras*, A series of modern surveys in mathematics, Volume **47**, Springer-Verlag, 2004. [MR2095251](#)(2006d:37106). [Zbl 1083.37001](#).
- [7] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of algebraic curves I*, Springer-Verlag, 1994. [MR0770932](#)(86h:14019).
- [8] V.I. Arnold, *Mathematical methods in classical mechanics*, Springer-Verlag, Berlin-Heidelberg- New York, 1978. [MR0690288](#)(MR57:14033b). [Zbl 0386.70001](#).
- [9] A.I. Belokolos and V.Z. Enol'skii, *Isospectral deformations of elliptic potentials*, Russ. Math. Surveys, **44** (1989), 155-156. [MR1040275](#)(91c:58046).
- [10] L.A. Dikii, *Hamiltonian systems connects with the rotation group*, Funct. Anal. Appl., **6** (1972), 83-84. [MR0312527](#)(47:1084). [Zbl 0288.58004](#).
- [11] P.A. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience 1978. [MR0507725](#)(80b:14001). [Zbl 0408.14001](#).
- [12] P.A. Griffiths, *Linearizing flows and a cohomological interpretation of Lax equations*, Amer. J. of Math., **107** (1985), 1445-1483. [MR0815768](#). [Zbl 0585.58028](#).
- [13] L. Haine, *Geodesic flow on  $SO(4)$  and Abelian surfaces*, Math. Ann., **263** (1983), 435-472. [MR0707241](#). [Zbl 0521.58042](#).
- [14] B. Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. in Math., **34** (1979), 195-338. [MR0550790](#)(MR82f:58045).

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<http://www.utgjiu.ro/math/sma>

- [15] S. Kowalewski, *Sur le problème de la rotation d'un corps solide autour d'un point fixe*, Acta Math., **12** (1889), 177-232. [Zbl 21.0935.01](#).
- [16] P. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure and Appl. Math., **21** (1968), 467-490. [MR0235310](#). [Zbl 0162.41103](#).
- [17] A. Lesfari, *Abelian surfaces and Kowalewski's top*, Ann. Scient. Éc. Norm. Sup., Paris, 4e série, **21** (1988), 193-223. [MR0956766\(89k:58125\)](#). [Zbl 0667.58019](#).
- [18] A. Lesfari, *Geodesic flow on  $SO(4)$ , Kac-Moody Lie algebra and singularities in the complex  $t$ -plane*, Publ. Mat., Barc., **43**(1) (1999), 261-279. [MR1697525\(2000f:37078\)](#). [Zbl 0968.35010](#).
- [19] A. Lesfari, *Le système différentiel de Hénon-Heiles et les variétés Prym*, Pacific J. Math., **212**(1) (2003), 125-132. [MR2016973](#). [Zbl 1070.37040](#).
- [20] A. Lesfari, *Prym varieties and applications*, J. Geom. Phys., **58**(9) (2008), 1063-1079. [MR2451270](#). [Zbl 1149.14045](#).
- [21] A. Lesfari, *Introduction à la géométrie algébrique complexe*, Hermann, Paris 2015. [Zbl 1327.14001](#).
- [22] S.V. Manakov, *A remark on the integration of the Euler equations of the dynamics of an  $n$ -dimensional rigid body*, Functional Anal. Appl, **10** (1976), 93-94. [MR0455031](#). [Zbl 03535638](#).
- [23] A.S. Mishchenko and A.T. Fomenko, *The Euler equations on finite-dimensional Lie groups*, Math. USSR-Izv, **42** (1978), 371-389. [MR0482832](#).
- [24] J. Moser, *Geometry of quadrics and spectral theory*, Chern Sympos., Springer-Verlag, pp. 147-188, 1980. [MR0609560](#). [Zbl 0455.58018](#).
- [25] J. Moser, *Various aspects of integrable Hamiltonian systems*, Progress in Math. 8, 223-289, Birkhäuser-Verlag, 1980. [MR0589592](#).
- [26] D. Mumford, *Prym varieties I*, in Contributions to Analysis (L.V. Ahlfors, I. Kra, B. Maskit, L. Nirenberg, eds.), Academic Press, New-York, 325-350, 1974. [MR0379510](#). [Zbl 316.14010](#).
- [27] D. Mumford, *Tata Lectures on Theta II*, Progress in Math., Birkhäuser, Boston, 1984. [MR0742776\(MR86b:14017\)](#). [Zbl 0549.14014](#).
- [28] W. Symes, *Systems of Toda type, inverse spectral problems and representation theory*, Invent. Math., **59** (1980), 13-51. [MR0575079](#). [Zbl 0474.58009](#).
- [29] P. van Moerbeke and D. Mumford, *The spectrum of difference operators and algebraic curves*, Acta Math., **143** (1979), 93-154. [MR0533894](#). [Zbl 0502.58032](#).

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- [30] P. Vanhaecke, *Integrable systems in the realm of algebraic geometry*, Lecture Notes in Math., 1638, Springer-Verlag, Second edition 2001. [MR1850713](#). [Zbl 0997.37032](#).

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