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FUZZY INITIAL VALUE PROBLEM

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Abstract. In this paper, we present the fuzzy solution concept of the initial value problem. After introducing a metric between two fuzzy vectors, we prove that the system of fuzzy equations has unique solution under some condition.

1 Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations was first introduced by Zadeh [11, 12], Dubois, and Prade [3]. We refer the reader to [4] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems including fuzzy metric spaces [8, 9], fuzzy differential equations [6], fuzzy linear systems [10], and fuzzy partial differential equations [5]. In this paper, we give a concept solution for initial value problem of fuzzy linear equations, we prove that the system of fuzzy equations has unique solution under some condition.

2 Preliminaries

Throughout this paper, the notation $\mathbb{R}^n$ denotes n-dimensional Euclidean space and the notation $\mathbb{R}^{n\times n}$ denotes the set of all $n\times n$ real matrices. The norm in the space $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$ is regarded as $\|\bullet\|_\infty$, that is

$$\|x\|_\infty = \max_{1\le i\le n} |x_i| \quad \text{for} \;\; x=(x_1,\dots,x_n)^T \in \mathbb{R}^n$$

and

$$\|A\|_\infty = \max_{1\le i\le n} \left|\sum_{j=1}^{n} a_{ij}\right| \quad \text{for} \;\; A=(a_{ij}) \in \mathbb{R}^{n\times n}$$

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Let $a = (a_1, \ldots, a_n)^T$ and $b = (b_1, \ldots, b_n)^T$. Define $a \leq b$ if and only if $a_j \leq b_j$ for $j = 1, 2, \ldots, n$. We denote

$$[a, b] = ([a_1, b_1], \ldots, [a_n, b_n])^T \qquad \text{if } a \leq b$$

and

$$\mathbf{I}^n(\mathbb{R}) = \Big\{ [a, b], \ a \leq b, \ a \in \mathbb{R}^n, \ b \in \mathbb{R}^n \Big\}$$

The addition, scale-multiplication and matrix-multiplication are defined as follows :

$$[a, b] + [u, v] = [a + u, v + b], \qquad [ta, tb] = \begin{cases} [ta, tb] & t \geq 0 \\ [tb, ta] & t < 0 \end{cases}$$
$$T[a, b] = \Big(\textstyle\sum_{j=1}^n t_{1j}[a_j, b_j], \ldots, \sum_{j=1}^n t_{nj}[a_j, b_j] \Big)$$

where $[a, b] \in \mathbf{I}^n(\mathbb{R})$, $[u, v] \in \mathbf{I}^n(\mathbb{R})$, $t \in \mathbb{R}$, $T = (t_{ij}) \in \mathbb{R}^{n \times n}$. A metric d, in the space $\mathbf{I}^n(\mathbb{R})$ is defined as

$$d\Big([a, b], [u, v] \Big) = \max \Big(\|a - u\|_\infty, \|b - v\|_\infty \Big)$$

for $[a, b] \in \mathbf{I}^n(\mathbb{R})$ and $[u, v] \in \mathbf{I}^n(\mathbb{R})$. Obviously, the above three operations within $\mathbf{I}^n(\mathbb{R})$ are closed respectively, and $\mathbf{I}^n(\mathbb{R})$ is a complete metric space with the metric d.
The α-cut of a fuzzy set u in $\mathbb{R}^n$ is denoted by $[u]^\alpha$ i.e.

$$[u]^\alpha = \Big\{ t \,|\, t \in \mathbb{R}^n, \ u(t) \geq \alpha \Big\}, \qquad \alpha > 0$$

Let

$$\mathbf{F}^n(\mathbb{R}) = \Big\{ u \,|\, u \text{ is a fuzzy set in } \mathbb{R}^n, \ [u]^\alpha \in \mathbf{I}^n(\mathbb{R}) \text{ and } [u]^1 \neq \emptyset \Big\}$$

It is clear, $\mathbf{F}^1(\mathbb{R})$, denoted by $\mathbf{F}(\mathbb{R})$ in short, is the set of all closed and convex fuzzy numbers. According to properties and representation theorems of closed and convex fuzzy numbers, we can easily obtain the following two lemmas

Lemma 1. *Let $u \in \boldsymbol{F}^n(\mathbb{R})$. Then, $u = (u_1, \ldots, u_n)^T$ and $[u]^\alpha = \Big([u_1]^\alpha, [u_2]^\alpha, \ldots, [u_n]^\alpha \Big)^T$, $\alpha > 0$, where $u_j \in \boldsymbol{F}(\mathbb{R})$ $(j = 1, 2, \ldots, n)$.*

Lemma 2. *Let u, $v \in \boldsymbol{F}^n(\mathbb{R})$, $\alpha > 0$, $a \in \mathbb{R}$. Then $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[au]^\alpha = a\,[u]^\alpha$.*

Let $Au = \Big(\displaystyle\sum_{j=1}^n a_{1j} u_j, \ldots, \sum_{j=1}^n a_{nj} u_j \Big)^T$ for $A \in \mathbb{R}^{n \times n}$, $u = (u_1, u_2, \ldots, u_n)^T \in \mathbf{F}^n(\mathbb{R})$ where the addition and the multiplication are defined by Zadeh's extension principle. It is easy to see $Au \in \mathbf{F}^n(\mathbb{R})$. By Lemmas 1 and 2, we have

$$[Au]^\alpha = A\,[u]^\alpha \quad \text{for } \ u \in \mathbf{F}^n(\mathbb{R}) \text{ and } \ A \in \mathbb{R}^{n \times n}$$

Let

$$[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha] = \left(\left[\underline{u}_1^\alpha, \overline{u}_1^\alpha\right], \ldots, \left[\underline{u}_n^\alpha, \overline{u}_n^\alpha\right] \right)^T$$

where $[u]^\alpha \in \mathbf{I}^n(\mathbb{R})$, $\underline{u}^\alpha \in \mathbb{R}^n$, $\overline{u}^\alpha \in \mathbb{R}^n$, $\underline{u}^\alpha \leq \overline{u}^\alpha$ and $\underline{u}_j^\alpha \in \mathbb{R}$, $\overline{u}_j^\alpha \in \mathbb{R}$, $\underline{u}_j^\alpha \leq \overline{u}_j^\alpha$. Define a mapping $\rho : \mathbf{F}^n(\mathbb{R}) \times \mathbf{F}^n(\mathbb{R}) \to \mathbb{R}$ as follows :

$$\rho(u, v) = \sup_{\alpha>0} d\left([u]^\alpha, [v]^\alpha\right) = \sup_{\alpha>0} \max_{1\leq j\leq n} \left(\left|\underline{u}_j^\alpha - \underline{v}_j^\alpha\right|, \left|\overline{u}_j^\alpha - \overline{v}_j^\alpha\right| \right)$$

We have

Lemma 3. *$\mathbf{F}^n(\mathbb{R})$ is a complete metric space with the metric ρ.*

Proof. According to Lemma 1, we obtain $\mathbf{F}^n(\mathbb{R}) = \mathbf{F}(\mathbb{R}) \times \cdots \times \mathbf{F}(\mathbb{R})$. $(\mathbf{F}(\mathbb{R}), d)$ is a complete metric space with the metric d (see [...]), where

$$d(x, y) = \sup_{\alpha>0} \left(\left|\underline{u}^\alpha - \underline{v}^\alpha\right|, \left|\overline{u}^\alpha - \overline{v}^\alpha\right| \right)$$

for $u \in \mathbf{F}(\mathbb{R})$ and $v \in \mathbf{F}(\mathbb{R})$. Moreover,

$$\begin{aligned} \rho(u, v) &= \sup_{\alpha>0} \max_{j} \left(\left|\underline{u}_j^\alpha - \underline{v}_j^\alpha\right|, \left|\overline{u}_j^\alpha - \overline{v}_j^\alpha\right| \right) \\ &= \max_{j} \sup_{\alpha>0} \left(\left|\underline{u}_j^\alpha - \underline{v}_j^\alpha\right|, \left|\overline{u}_j^\alpha - \overline{v}_j^\alpha\right| \right) = \max_{j} d\left(u_j, v_j\right) \end{aligned}$$

for $u = (u_1, u_2, \ldots, u_n)^T \in \mathbf{F}^n(\mathbb{R})$ and $v = (v_1, v_2, \ldots, v_n)^T \in \mathbf{F}^n(\mathbb{R})$. Hence, $(\mathbf{F}^n(\mathbb{R}), \rho)$ is a complete metric space. □

Lemma 4. *$\rho\left(Au, Av\right) \leq \|A\|_\infty \rho\left(u, v\right)$ for u, $v \in \mathbf{F}^n(\mathbb{R})$, $A \in \mathbb{R}^{n\times n}$*

Proof. According to the operation principle within $\mathbf{I}^n(\mathbb{R})$, Lemmas 1 and 2, for each $\alpha > 0$, we have

$$\begin{aligned} [Au]^\alpha = A\,[u]^\alpha = A\,[\underline{u}^\alpha, \overline{u}^\alpha] &= A\left([\underline{u}_1^\alpha, \overline{u}_1^\alpha], \ldots, [\underline{u}_n^\alpha, \overline{u}_n^\alpha]\right)^T \\ &= \left(\sum_{j=1}^n a_{1j}\left[\underline{u}_j^\alpha, \overline{u}_j^\alpha\right], \ldots, \sum_{j=1}^n a_{nj}\left[\underline{u}_j^\alpha, \overline{u}_j^\alpha\right] \right)^T \\ &= \left(\left[\sum_{j=1}^n a_{1j} s_1(u_j), \sum_{j=1}^n a_{1j} s_1^*(u_j) \right], \ldots, \left[\sum_{j=1}^n a_{nj} s_n(u_j), \sum_{j=1}^n a_{nj} s_n^*(u_j) \right] \right)^T \end{aligned}$$

where

$$s_i\left(u_j\right) = \begin{cases} \underline{u}_j^\alpha & a_{ij} \geq 0 \\ \overline{u}_j^\alpha & a_{ij} \leq 0 \end{cases}$$

and

$$s_i^*(u_j) = \underline{u}_j^\alpha + \overline{u}_j^\alpha - s_i(u_j), \quad j = 1, 2, \ldots, n \quad i = 1, 2, \ldots, n$$

Replacing u and u_j $(j = 1, 2, \ldots, n)$ by v and v_j $(j = 1, 2, \ldots, n)$ in the above equalities, we can obtain a similar result. Therefore,

$$\begin{aligned} d([Au]^\alpha, [Av]^\alpha) &= \max\left(\max_{1\le i\le n} \left| \sum_{j=1}^n a_{1j}(s_i(u_j) - s_i(v_j)) \right|, \max_{1\le i\le n} \left| \sum_{j=1}^n a_{1j}(s_i^*(u_j) - s_i^*(v_j)) \right| \right) \\ &= \max\left(\sum_{j=1}^n |a_{ij}| \right) \max_{1\le i\le n} \left(\left|\underline{u}_j^\alpha - \underline{v}_j^\alpha\right|, \left|\overline{u}_j^\alpha - \overline{v}_j^\alpha\right| \right) \\ &= \|A\|_\infty \rho(u, v) \end{aligned}$$

which implies that

$$\rho(Au, Av) = \sup_{\alpha>0} d\Big([Au]^\alpha, [Av]^\alpha \Big) \le \|A\|_\infty \rho(u, v)$$

The proof is completed. □

Lemma 5. *$\rho(u + w, v + w) = \rho(u, v)$ for u, v, $w \in \boldsymbol{F}^n(\mathbb{R})$*

Proof.

$$\begin{aligned} \rho(u + w, v + w) &= \sup_{\alpha>0} d([u + w]^\alpha, [v + w]^\alpha) \\ &= \sup_{\alpha>0} d([u]^\alpha + [w]^\alpha, [v]^\alpha + [w]^\alpha) \\ &= \sup_{\alpha>0} d([\underline{u}^\alpha + \underline{w}^\alpha, \overline{u}^\alpha + \overline{w}^\alpha], [\underline{v}^\alpha + \underline{w}^\alpha, \overline{v}^\alpha + \overline{w}^\alpha]) \\ &= \sup_{\alpha>0} \max(\|\underline{u}^\alpha - \underline{v}^\alpha\|_\infty, \|\overline{u}^\alpha - \overline{v}^\alpha\|_\infty) \\ &= \sup_{\alpha>0} d([u]^\alpha, [v]^\alpha) = \rho(u, v) \end{aligned}$$

□

In the following, we discuss the solution of fuzzy initial value problem

$$\partial_t u + \Lambda \partial_x u = Cu + D \qquad u(x, 0) = u_0(x) \tag{2.1}$$

with $\Lambda = \text{diag}\Big(\lambda_1, \ldots, \lambda_n\Big)$ a square matrix for which its terms are fuzzy numbers, C a real square matrix and D a fuzzy function.

3 Fuzzy Initial Value Problem

We consider the initial value problem (2.1)

Definition 6. *The system (2.1) is called hyperbolic in the sense that the eigenvalues* λ_i *satify*

$$\lambda_1 \succ \lambda_2 \succ \cdots \succ \lambda_n$$

i.e.

$$\underline{\lambda}_i > \overline{\lambda}_j \quad \text{for all} \quad i < j \quad \text{with} \quad \lambda_i = \left[\underline{\lambda}_i, \overline{\lambda}_i\right]$$

Along a characteristic C_i of the ith family

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + \lambda_i \frac{\partial u}{\partial x} = \sum_k c_{ik} u_k + d_i$$

The ith backward characteristic C_i through a point (X, T) has an equation

$$x = X - \lambda_i (T - t) = \gamma_i (t, X, T) \tag{3.1}$$

obtained by solving the differential equation

$$\frac{dx}{dt} = \lambda_i$$

with $x = X$ for $t = T$. Then integrate along C_i

$$u_i (X, T) = u_0^i (X, T) + \int_0^T h_i (t, X, T)\, dt \tag{3.2}$$

with

$$u_0^i (X, T) = u_0 (\gamma_i (0, X, T))$$

and

$$h_i (t, X, T) = \sum_k c_{ik} u_k + d_i$$

where in the integrand x has to be replaced by $\gamma_i (t, X, T)$. Formula (3.2) resembles a system of integral equations except that the domain of integration is different for each component of u. We write (3.2) symbolically as

$$u = W + Su \tag{3.3}$$

where W is the vector (considered known) with components

$$W_i (X, T) = u_0^i (X, T) + \int_0^T d_i (\gamma_i (t, X, T), t)\, dt$$

and S is the linear operator taking a vector u with components $u_k(x,t)$ into a vector $w = Su$ with component

$$w_i(X,T) = \int_0^T \sum_k c_{ik} \cdot u_k(\gamma_i(t,X,T), t)\, dt$$

Lemma 7. *For $u, v \in \boldsymbol{F}^n(\mathbb{R})$, $C \in \mathbb{R}^{n\times n}$, we have*

$$\rho(Cu, Cv) \leq \|C\|_\infty \rho(u,v)$$

Proof. According to the operation principle within $\mathbf{I}^n(\mathbb{R})$, Lemmas 1 and 2, for each $\alpha > 0$, we have

$$\begin{aligned} [Cu]^\alpha &= C[u]^\alpha = C[\underline{u}^\alpha, \overline{u}^\alpha] \\ &= C([\underline{u}_1^\alpha, \overline{u}_1^\alpha], \ldots, [\underline{u}_n^\alpha, \overline{u}_n^\alpha]) \\ &= \left(\sum_{j=1}^n c_{1j}[\underline{u}_j^\alpha, \overline{u}_j^\alpha], \ldots, \sum_{j=1}^n c_{nj}[\underline{u}_j^\alpha, \overline{u}_j^\alpha]\right) \\ &= (C_1, \ldots, C_n)^T \end{aligned}$$

where

$$C_i = \left[\sum_{j=1}^n c_{ij} s_1(u_j), \sum_{j=1}^n c_{ij} s_1^*(u_j)\right] \tag{3.4}$$

$$s_i(u_j) = \begin{cases} \underline{u}_j^\alpha & c_{ij} \leq 0 \\ \overline{u}_j^\alpha & c_{ij} \leq 0 \end{cases}$$

and

$$s_i^*(u_j) = \underline{u}_j^\alpha + \overline{u}_j^\alpha - s_i(u_j)$$

for $i, j = 1, \ldots, n$
Replacing u and u_j $(j = 1, 2, \ldots, n)$by v and v_j in the above equalities, we can obtain a similar result. Therefore for

$$\begin{aligned} A &= \max_{1\leq i\leq n} \left|\sum_{j=1}^n c_{1j}(s_i(u_j) - s_i(v_j))\right| \\ B &= \max_{1\leq i\leq n} \left|\sum_{j=1}^n c_{1j}(s_i^*(u_j) - s_i^*(v_j))\right| \end{aligned}$$

$$\begin{aligned} d([Cu]^\alpha, [Cv]^\alpha) &= \max(A, B) \\ &\leq \|C\|_\infty \max_{1\leq i\leq n} (|\underline{u}_i^\alpha - \underline{v}_i^\alpha|, |\overline{u}_i^\alpha - \overline{v}_i^\alpha|) \\ &= \|C\|_\infty \rho(u,v) \end{aligned}$$

which implies that

$$\rho\left(Cu, Cv\right) = \sup_{\alpha>0} d\left([Cu]^{\alpha}, [Cv]^{\alpha}\right) \leq \|C\|_{\infty}\, \rho\left(u, v\right)$$

The proof is completed. □

Lemma 8. *For u, v, $z \in \boldsymbol{F}^n(\mathbb{R})$, we have*

$$\rho\left(u+z, v+z\right) = \rho\left(u, v\right)$$

Proof.

$$\begin{aligned}
\rho\left(u+z, v+z\right) &= \sup_{\alpha>0} d\left([u+z]^{\alpha}, [v+z]^{\alpha}\right) \\
&= \sup_{\alpha>0} d\left([u]^{\alpha} + [z]^{\alpha}, [v]^{\alpha} + [z]^{\alpha}\right) \\
&= \sup_{\alpha>0} \max\left(\|\underline{u}^{\alpha} - \underline{v}^{\alpha}\|_{\infty}, \|\overline{u}^{\alpha} - \overline{v}^{\alpha}\|_{\infty}\right) \\
&= \sup_{\alpha>0} d\left([u]^{\alpha}, [v]^{\alpha}\right) = \rho\left(u, v\right)
\end{aligned}$$

□

Since, the norm $\|S\|$ of the operator S obviously is bounded by the constant

$$q = T\,\|C\|_{\infty}$$

In the following, the solution of a system of fuzzy linear equation $u = W + Su$. can be regarded as the fixed point of a mapping :

$$u \to gu = W + Su$$

Theorem 9. *The mapping g has unique fixed point within $\boldsymbol{F}^n(\mathbb{R})$ if $q < 1$ (i.e. for T sufficiently small)*

Proof. By Lemmas 1 and 2, we know

$$\rho\left(gu, gv\right) \leq q\rho\left(u, v\right)$$

holds well for u, $v \in \mathbf{F}^n(\mathbb{R})$. Hence the mapping g is contractive with respect to the metric ρ.
Lemma 3 shows that $(\mathbf{F}^n(\mathbb{R}), \rho)$ is a complete metric space, therefore, there uniquely exists a point $u^* \in \mathbf{F}^n(\mathbb{R})$ such that $gu^* = W + Su^* = u^*$, which completes the proof. □

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