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# EXISTENCE OF MILD SOLUTIONS FOR NONLOCAL CAUCHY PROBLEM FOR FRACTIONAL NEUTRAL EVOLUTION EQUATIONS WITH INFINITE DELAY

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**Abstract**. In this article, we study the existence of mild solutions for nonlocal Cauchy problem for fractional neutral evolution equations with infinite delay. The results are obtained by using the Banach contraction principle. Finally, an application is given to illustrate the theory.

## 1 Introduction

The theory of fractional differential equations is emerging as an important area of investigation since it is richer in problems in comparison with corresponding theory of classical differential equations. In fact, such models can be considered as an efficient alternative to the classical nonlinear differential models to simulate many complex processes. Recently, it have been proved that the differential models involving derivatives of fractional order arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in many fields, for instance, physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so on. One can see the monographs of Kilbas et al. [13], Miller and Ross [16], Podlubny [21], Lakshmikantham et al. [14]. Recently, some authors focused on fractional functional differential equations in Banach spaces [1, 2, 5, 7–9, 15, 17–19, 22–24, 26–33].

There exist an extensive literature of differential equations with nonlocal conditions. The result concerning the existence and uniqueness of mild solutions to abstract Cauchy problems with nonlocal initial conditions was first formulated and proved by Byszewski, see [3, 4]. On the other hand, Hernandez, [10, 11], study the existence of mild, strong and classical solutions for the nonlocal neutral partial functional differential equation with unbounded delay. Since the appearance of this paper, several papers have addressed the issue of existence and uniqueness results for

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various types of nonlinear differential equations. In [19], Guérékata discussed the existence of mild solution for some fractional differential equations with nonlocal conditions. Related to this matter, we cite among others works, [6, 25]. Motivated by physical applications, Byszewski studied [4] the existence, uniqueness and continuous dependence on initial data of solutions to the nonlocal Cauchy problem

$$\begin{aligned} \dot{x}(t) &= Ax(t) + g(t, x_t), \quad t \in [\sigma, T] \\ x_0 &= \varphi + q(x_{t_1}, x_{t_2}, x_{t_3}, \cdots, x_{t_n}), \end{aligned}$$

where $A$ is the infinitesimal generator of a $C_0$-semigroup of linear operators; $t_i \in [\sigma, T]$; $x_t \in C([-r, 0] : X)$ and $q : C([-r, 0] : X)^n \to X$, $f : [\sigma, T] \times C([-r, 0] : X) \to X$ are appropriate functions. Recently, [32] Zhou studied the nonlocal Cauchy problem of the following form

$$\begin{aligned} {}^cD_t^q(x(t) - h(t, x_t)) + Ax(t) &= f(t, x_t), \quad t \in [0, b] \\ x_0(\vartheta) + g(x_{t_1}, x_{t_2}, x_{t_3}, \cdots, x_{t_n})(\vartheta) &= \varphi(\vartheta), \ \vartheta \in [-r, 0], \end{aligned}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $0 < q < 1$, $0 < t_1 < \cdots < t_n < a$, $a > 0$. $A$ is the infinitesimal generator of an analytic semigroup $T(t)_{t \geq 0}$ of operators on $E$, $f, h : [0, \infty) \times \mathcal{C} \to E$ and $g : \mathcal{C}^n \to \mathcal{C}$ are given functions satisfying some assumptions, $\varphi \in \mathcal{C}$ and define $x_t$ by $x_t(\vartheta) = x(t + \vartheta)$, for $\vartheta \in [-r, 0]$.

Motivated by the above works, in this article, we study the existence of mild solutions for nonlocal Cauchy problem for fractional neutral evolution equations with infinite delay modeled in the form

$$\begin{aligned} {}^cD_t^q(x(t) + f(t, x_t)) &= Ax(t) + g(t, x_t), \quad t \in [0, b] && (1.1) \\ x_0 &= \varphi + q(x_{t_1}, x_{t_2}, x_{t_3}, \cdots, x_{t_n}) \in \mathcal{B}, && (1.2) \end{aligned}$$

${}^cD^q$ is the Caputo fractional derivative of order $0 < q < 1$, $A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $T(t)$ on a Banach space $X$. The history $x_t : (-\infty, 0] \to X$ given by $x_t(\theta) = x(t + \theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, $0 < t_1 < t_2 < t_3 < \cdots < t_n \leq b$, $q : \mathcal{B}^n \to \mathcal{B}$ and $f, g : [0, b] \times \mathcal{B} \to X$ are appropriate functions.

## 2 Preliminaries

In this section, we first recall recent results in the theory of fractional differential equations and introduce some notations, definitions and lemmas which will be used throughout the papers [32, 33]. Let $A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{T(t)\}_{t \geq 0}$ of uniformly bounded linear operators on $X$. Let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of $A$. Then for $0 < \eta \leq 1$, it is possible to define the fractional power $A^\eta$ as a closed linear operator on its domain $D(A^\eta)$. For analytic semigroup $\{T(t)\}_{t \geq 0}$, the following properties will be used.

$(i)$ There is a $M \geq 1$ such that

$$M = \sup_{t \in [0,+\infty)} |T(t)| < \infty,$$

$(ii)$ for any $\eta \in (0,1]$, there exists a positive constant $C_\eta$ such that

$$|A^\eta T(t)| \leq \frac{C_\eta}{t^\eta}, \quad 0 < t \leq b.$$

We need some basic definitions and properties of the fractional calculus theory which which will be used for throughout this paper.

**Definition 1.** *The fractional integral of order* $\gamma$ *with the lower limit zero for a function* $f$ *is defined as*

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \ \gamma > 0,$$

*provided the right side is point-wise defined on* $[0, \infty)$*, where* $\Gamma(\cdot)$ *is the gamma function.*

**Definition 2.** *The Riemann-Liouville derivative of order* $\gamma$ *with the lower limit zero for a function* $f : [0, \infty) \to \mathbb{R}$ *can be written as*

$$^L D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \quad t > 0, \ n-1 < \gamma < n,$$

**Definition 3.** *The Caputo derivative of order* $\gamma$ *for a function* $f : [0, \infty) \to \mathbb{R}$ *can be written as*

$$^C D^\gamma f(t) =^L D^\gamma \Big( f(t) - \sum_{k=1}^{n-1} \frac{t^k}{k!} f^k(0) \Big), \quad t > 0, \ n-1 < \gamma < n,$$

**Remark 4.** **(i)** *If* $f(t) \in C^n[0, \infty)$*, then*

$$^C D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^n(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^n(t), \quad t > 0, \ n-1 < \gamma < n,$$

**(ii)** *The Caputo derivative of a constant is equal to zero.*

**(iii)** *If* $f$ *is an abstract function with values in* $X$*, then integrals which appear in Definitions 2 and 3 are taken in Bochner's sense.*

We will herein define the phase space $\mathcal{B}$ axiomatically, using ideas and notation developed in [12]. More precisely, $\mathcal{B}$ will denote the vector space of functions defined from $(-\infty, 0]$ into $X$ endowed with a seminorm denoted as $\| \cdot \|_{\mathcal{B}}$ and such that the following axioms hold:

(**A**) If $x : (-\infty, b) \to X$ is continuous on $[0, b]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, b]$ the following conditions hold:

- $(i)$ $x_t$ is in $\mathcal{B}$.
- $(ii)$ $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$.
- $(iii)$ $\|x_t\|_{\mathcal{B}} \leq K(t) \sup\{\|x(s)\| : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded, and $H, K(\cdot), M(\cdot)$ are independent of $x(\cdot)$.

(**A1**) For the function $x(\cdot)$ in (**A**), $x_t$ is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(**B**) The space $\mathcal{B}$ is complete.

**Example 5. The Phase Space $C_r \times L^p(h, X)$.**

*Let $r \geq 0$, $1 \leq p < \infty$ and $h : (-\infty, -r] \to \mathbb{R}$ be a non-negative, measurable function which satisfies the conditions $(g-5)-(g-6)$ in the terminology of [12]. Briefly, this means that $g$ is locally integrable and there exists a non-negative, locally bounded function $\eta(\cdot)$ on $(-\infty, 0]$ such that $h(\xi + \theta) \leq \eta(\xi)h(\theta)$ for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(h, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that $\varphi$ is continuous on $[-r, 0]$ and is Lebesgue measurable, and $h\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(h, X)$ defined by*

$$\|\varphi\|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0]\} + \left(\int_{-\infty}^{-r} h(\theta)\|\varphi(\theta)\|^p d\theta\right)^{1/p}.$$

*The space $\mathcal{B} = C_r \times L^p(h, X)$ satisfies the axioms (**A**), (**A$_1$**) and (**B**). Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $K(t) = 1 + \left(\int_{-t}^{0} h(\theta)d\theta\right)^{1/2}$ and $M(t) = \eta(-t)^{1/2}$, for $t \geq 0$ (see [12, Theorem 1.3.8] for details).*

For additional details concerning phase space we refer the reader to [12].

The following lemma will be used in the proof of our main results.

**Lemma 6.** *[32, 33] The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties:*

(**i**) *For any fixed $t \geq 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are linear and bounded operators, i.e., for any $x \in X$,*

$$\|\mathscr{T}(t)x\| \leq M\|x\| \quad \text{and} \quad \|\mathscr{S}(t)x\| \leq \frac{qM}{\Gamma(1+q)}\|x\|.$$

(**ii**) *$\{\mathscr{T}(t), t \geq 0\}$ and $\{\mathscr{S}(t), t \geq 0\}$ are strongly continuous.*

(**iii**) *For every $t > 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are also compact operators if $T(t)$, $t > 0$ is compact.*

## 3 Existence Results

In this section we study the existence of mild solutions of the system (1.1)-(1.2). In order to define the concept of mild solution for the system (1.1)-(1.2), by comparison with the fractional differential equations given in [32, 33], we associate system (1.1)-(1.2) to the integral equation

$$\begin{aligned} x(t) = &\mathscr{T}(t)(\varphi(0) + f(0,\varphi) + q(x_{t_1}, x_{t_2}, x_{t_3}, \cdots, x_{t_n})(0)) - f(t, x_t) \\ &- \int_0^t (t-s)^{q-1} A \mathscr{S}(t-s) f(s, x_s) ds + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) g(s, x_s) ds, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &\mathscr{T}(t) = \int_0^\infty \xi_q(\theta) T(t^q \theta) d\theta, \quad \mathscr{S}(t) = q \int_0^\infty \theta \xi_q(\theta) T(t^q\theta) d\theta, \\ &\xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \varpi_q\left(\theta^{-\frac{1}{q}}\right) \geq 0, \\ &\varpi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \theta \in (0,\infty), \end{aligned}$$

and $\xi_q$ is a probability density function defined on $(0,\infty)$, that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0,\infty) \quad \text{and} \quad \int_0^\infty \xi_q(\theta) d\theta = 1.$$

In the sequel we introduce the following assumptions.

**($\mathbf{H_1}$)** $q : \mathcal{B}^n \to \mathcal{B}$ is continuous and exist positive constants $L_i(q)$ such that

$$\|q(\psi_1, \psi_2, \psi_3, \cdots, \psi_n) - q(\varphi_1, \varphi_2, \varphi_3, \cdots, \varphi_n)\| \leq \sum_{i=1}^n L_i(q) \|\psi_i - \varphi_i\|_{\mathcal{B}},$$

for every $\psi_i, \varphi_i \in B_r[0, \mathcal{B}]$.

**($\mathbf{H_2}$)** The function $f(\cdot)$ is $(-A)^\vartheta$-valued, $f : I \times \mathcal{B} \to [D((-A)^{-\vartheta})]$, the function $g(\cdot)$ is defined on $g : I \times \mathcal{B} \to X$, and there exist positive constants $L_f$ and $L_g$ such that for all $(t_i, \psi_j) \in I \times \mathcal{B}$,

$$\begin{aligned} \|(-A)^\vartheta f(t_1, \psi_1) - (-A)^\vartheta f(t_2, \psi_2)\| &\leq L_f(|t_1 - t_2| + \|\psi_1 - \psi_2\|_{\mathcal{B}}), \\ \|g(t_1, \psi_1) - g(t_2, \psi_2)\| &\leq L_g(|t_1 - t_2| + \|\psi_1 - \psi_2\|_{\mathcal{B}}). \end{aligned}$$

**Remark 7.** *In the rest of this section, $M_b$ and $K_b$ are the constants $M_b = \sup_{s\in[0,b]} M(s)$, $K_b = \sup_{s\in[0,b]} K(s)$, and $N_{(-A)^\vartheta f}$, $N_f$, $N_g$ represent the supreme of the functions $(-A)^\vartheta f$, $f$ and $g$ on $[0,b] \times B_r[0,\mathcal{B}]$.*

**Theorem 8.** *Let conditions* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *be hold. If*

$$\begin{aligned}\rho \;=\; & \Bigg[(Mb+K_bMH)\parallel \varphi \parallel_{\mathcal{B}} +(M_b+K_bM)N_q+(K_b+1)N_f \\ & +\frac{K_bN_{(-A)^{\beta}f}\Gamma(1+\beta)C_{1-\beta}b^{q\beta}}{\beta\Gamma(1+\beta q)}+\frac{K_bN_gMq}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q_1)}\Bigg] < r\end{aligned}$$

*and*

$$\Lambda = \max\left\{M_b\left(M_b\sum_{i=1}^{n}L_i(q)+K_b\theta\right), K_b\left(M_b\sum_{i=1}^{n}L_i(q)+K_b\theta\right)\right\} < 1,$$

*where*

$$\begin{aligned}\theta = & \Bigg(M\sum_{i=1}^{n}L_i(q)+L_f\Big((M+1)\|(-A)^{-\vartheta}\|+\frac{\Gamma(1+\beta)C_{1-\beta}b^{q\beta}}{\beta\Gamma(1+\beta q)}\Big) \\ & +\frac{Mq}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q_1)}\Bigg).\end{aligned}$$

*Then there exists a mild solution of the system* (1.1)-(1.2) *on* $I$.

*Proof.* Consider the space $S(b)=\{x:(-\infty,b]\to X : x_0\in\mathcal{B}; x\in C([0,b]:X)\}$ endowed with the norm

$$\parallel x\parallel_{S(b)}:= M_b\parallel x_0\parallel_{\mathcal{B}} +K_b\parallel x\parallel_b .$$

Let $Y=B_r[0,S(b)]$, we define the operator $\Gamma : Y\to S(b)$ by

$$\begin{aligned}\Gamma x(t) = & \mathscr{T}(t)(\varphi(0)+f(0,\varphi)+q(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_n})(0))-f(t,x_t) \\ & -\int_0^t(t-s)^{q-1}A\mathscr{S}(t-s)f(s,x_s)ds+\int_0^t(t-s)^{q-1}\mathscr{S}(t-s)g(s,x_s)ds, \\ (\Gamma u)_0 = & \;\varphi+q(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_n}).\end{aligned}$$

for $t\in[0,b]$.

Using an similar argument on the proof of Theorem 3.1 in [10], we will prove the $\Gamma$ is continuous. Next we will prove that $\Gamma(Y)\subset Y$.

Direct calculation gives that $(t-s)^{q-1}\in L^{\frac{1}{1-q_1}}[0,t]$, for $t\in J$ and $q_1\in[0,q)$. Let $a=\frac{q-1}{1-q_1}\in(-1,0)$. By using Holder inequality, and $(\mathbf{H_2})$, according to [32, 33], we have

$$\begin{aligned}\int_0^t|(t-s)^{q-1}g(s,x_s)|ds & \leq \left(\int_0^t(t-s)^{\frac{q-1}{1-q_1}}ds\right)^{1-q_1}N_g \\ & \leq \frac{N_g}{(1+a)^{1-q_1}}b^{(1+a)(1-q_1)}. \qquad (3.2)\end{aligned}$$

From the inequality (3.2) and Lemma 3.1, we obtain the following inequality [32, 33]

$$\begin{aligned}\int_0^t |(t-s)^{q-1}\mathscr{S}(t-s)g(s,x_s)|ds &\le \frac{Mq}{\Gamma(1+q)}\int_0^t |(t-s)^{q-1}g(s,x_s)|ds\\ &\le \frac{N_gMq}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q_1)}.\end{aligned} \tag{3.3}$$

According to [33], we obtain the following relation:

$$\begin{aligned}\int_0^t |(t-s)^{q-1}A\mathscr{S}(t-s)f(s,x_s)|ds &\le \int_0^t |(t-s)^{q-1}A^{1-\beta}\mathscr{S}(t-s)A^{\beta}f(s,x_s)|ds\\ &\le \frac{N_{(-A)^{\beta}f}\Gamma(1+\beta)C_{1-\beta}b^{q\beta}}{\beta\Gamma(1+\beta q)}.\end{aligned} \tag{3.4}$$

Let $x \in Y$ and $t \in [0,b]$, we observe from axiom (**A**) of the phase spaces, we obtain that $\| x_t \|_{\mathcal{B}} \le K_b \| x \|_b + M_b \| x_0 \|_{\mathcal{B}} \le r$ this implies that $x_t \in B_r[0,\mathcal{B}]$, and this case

$$\begin{aligned}\|\Gamma x(t)\| &\le \|\mathscr{T}(t)\|(\|\varphi(0)\| + \|f(0,\varphi)\| + \|q(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_n})(0)\|)\\ &\quad +\|f(t,x_t)\| + \int_0^t (t-s)^{q-1}\|A\mathscr{S}(t-s)\|\|f(s,x_s)\|ds\\ &\quad + \int_0^t (t-s)^{q-1}A\mathscr{S}(t-s)\|g(s,x_s)\|ds\\ &\le M(H\|\varphi\|_{\mathcal{B}} + N_f + N_q) + N_f + \frac{N_{(-A)^{\beta}f}\Gamma(1+\beta)C_{1-\beta}b^{q\beta}}{\beta\Gamma(1+\beta q)}\\ &\quad +\frac{N_gMq}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q_1)}.\end{aligned} \tag{3.5}$$

and

$$\|(\Gamma u)_0\| \le \|\varphi\|_{\mathcal{B}} + N_q. \tag{3.6}$$

From (3.5)-(3.6), we have that

$$\begin{aligned}\| \Gamma x(t) \|_{S(b)} &\le M_b \| (\Gamma x)_0 \|_{\mathcal{B}} + K_b \| x \|_b\\ &\le (Mb + K_bMH) \| \varphi \|_{\mathcal{B}} + (M_b + K_bM)N_q + (K_b+1)N_f\\ &\quad + \frac{K_bN_{(-A)^{\beta}f}\Gamma(1+\beta)C_{1-\beta}b^{q\beta}}{\beta\Gamma(1+\beta q)} + \frac{K_bN_gMq}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q_1)}\\ &= \rho < r.\end{aligned} \tag{3.7}$$

which prove that $\Gamma(x) \in Y$.

In order to prove that $\Gamma$ satisfies a Lipschitz condition, $u, v \in Y$. If $t \in [0, b]$ we see that

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\| \\
&\leq \|\mathscr{T}(t)(q(u_{t_1}, u_{t_2}, u_{t_3}, \cdots, u_{t_n})(0) - q(v_{t_1}, v_{t_2}, v_{t_3}, \cdots, v_{t_n})(0))\| \\
&\quad + \|(-A)^{-\vartheta}\| \|(-A)^{\vartheta} f(0, u_0) - (-A)^{\vartheta} f(0, v_0)\| \\
&\quad + \|(-A)^{-\vartheta}\| \|(-A)^{\vartheta} f(t, u_t) - (-A)^{\vartheta} f(t, v_t)\| \\
&\quad + \int_0^t (t-s)^{q-1} \|(-A)^{1-\vartheta} \mathscr{S}(t-s)\| \|(-A)^{\vartheta} f(s, u_s) - (-A)^{\vartheta} f(s, v_s)\| ds \\
&\quad + \int_0^t t-s)^{q-1} \|\mathscr{S}(t-s)\| \|g(s, u_s) - g(s, v_s)\| ds \\
&\leq M \sum_{i=1}^{n} L_i(q) K_b \|u_{t_i} - v_{t_i}\|_{\mathcal{B}} + (M+1)\|(-A)^{-\vartheta}\| L_f (K_b \|u - v\|_b + M_b \|u_0 - v_0\|_{\mathcal{B}}) \\
&\quad + L_f (\|u - v\|_b + M_b \|u_0 - v_0\|_{\mathcal{B}}) \frac{\Gamma(1+\beta) C_{1-\beta} b^{q\beta}}{\beta \Gamma(1+\beta q)} \\
&\quad + M L_g (K_b \|u - v\|_b + M_b \|u_0 - v_0\|_{\mathcal{B}}) \frac{Mq}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)}
\end{aligned}$$

$$\begin{aligned}
&\leq M_b \Bigg( \sum_{i=1}^{n} L_i(q) + L_f \Big( (M+1)\|(-A)^{-\vartheta}\| + \frac{\Gamma(1+\beta) C_{1-\beta} b^{q\beta}}{\beta \Gamma(1+\beta q)} \Big) \\
&\quad + \frac{Mq}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)} \Bigg) \|u_0 - v_0\|_{\mathcal{B}} \\
&\quad + K_b \Bigg( \sum_{i=1}^{n} L_i(q) + L_f \Big( (M+1)\|(-A)^{-\vartheta}\| + \frac{\Gamma(1+\beta) C_{1-\beta} b^{q\beta}}{\beta \Gamma(1+\beta q)} \Big) \\
&\quad + \frac{Mq}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)} \Bigg) \|u - v\|_b \\
&\leq M_b \theta \|u_0 - v_0\|_{\mathcal{B}} + K_b \theta \|u - v\|_b.
\end{aligned}$$

On the other hand, a simple calculus prove that

$$\|(\Gamma u)_0 - (\Gamma v)_0\| \leq \sum_{i=1}^{n} L_i(q) M_b \|u_0 - v_0\|_{\mathcal{B}} + K_b \|u - v\|_b.$$

Finally we see that

$$
\begin{aligned}
\|\Gamma u-\Gamma v\|_{S(b)} &\leq M_b \parallel (\Gamma u)_0-(\Gamma v)_0 \parallel_{\mathcal{B}} + K_b \parallel \Gamma u-\Gamma v \parallel_b \\
&\leq M_b\left(M_b\sum_{i=1}^{n}L_i(q)+\theta\right)\|u_0-v_0\|+K_b\left(M_b\sum_{i=1}^{n}L_i(q)+\theta\right)\|u-v\|_{\mathcal{B}} \\
&\leq \Lambda\|u-v\|_{S(b)}, \qquad (3.8)
\end{aligned}
$$

which infer that $\Gamma$ is a contraction on $Y$. Clearly a fixed point of $\Gamma$ is the unique mild solution of the nonlocal problem (1.1)-(1.2). The proof is complete.

□

# 4 An example

In this section, we consider an application of our abstract results. At first we introduce the required technical framework. In the rest of this section, $X = L^2([0,\pi])$, $\mathcal{B} = C_0 \times L^p(g,X)$ is the space introduced in Example 5 and $A : D(A) \subseteq X \to X$ is the operator defined by $Ax = x''$, with domain $D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$. The operator $A$ is the infinitesimal generator of an analytic semigroup on $X$. Then

$$A = -\sum_{i=1}^{\infty} n^2\langle x, e_n\rangle e_n, \quad x \in D(A),$$

where $e_n(\xi) = \left(\frac{2}{\pi}\right)^{1/2}\sin(n\xi)$, $0 \leq \xi \leq \pi$, $n = 1,2,\cdots$. Clearly $A$ generates a compact semigroup $T(t)$, $t > 0$ in $X$ and is given by

$$T(t)x = \sum_{i=1}^{\infty} e^{-n^2t}\langle x, e_n\rangle e_n, \quad \text{for every } x \in X.$$

Consider the following fractional partial differential system

$$
\begin{aligned}
&\frac{\partial^\alpha}{\partial t^\alpha}\Big(u(t,\xi)+\int_{-\infty}^{t}\int_0^{\pi} b(t-s,\eta,\xi)u(s,\eta)d\eta ds\Big) \\
&\qquad = \frac{\partial^2}{\partial\xi^2}u(t,\xi)+\int_{-\infty}^{t} a_0(s-t)u(s,\xi)ds, \quad (t,\xi) \in I\times[0,\pi],
\end{aligned}
\tag{4.1}
$$

$$u(t,0) = u(t,\pi) = 0,\ t \in [0,b], \tag{4.2}$$

$$u(\theta,\xi) = \phi(\theta,\xi)+\sum_{i=0}^{n} L_i u(t_i+\xi), \theta \leq 0,\ \xi \in [0,\pi]. \tag{4.3}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is a Caputo fractional partial derivative of order $0<\alpha<1$,$n$ is a positive integer, $0<t_i<a$, $L_i, i=1,2,\dots n$, are fixed numbers.

In the sequel, we assume that $\varphi(\theta)(\xi)=\phi(\theta,\xi)$ is a function in $\mathcal{B}$ and that the following conditions are verified.

(i) The functions $a_0:\mathbb{R}\to\mathbb{R}$ are continuous and $L_g := \left(\int_{-\infty}^0 \frac{(a_0(s))^2}{g(s)}ds\right)^{1/2}<\infty$.

(ii) The functions $b(s,\eta,\xi)$, $\frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s,\eta,\pi)=b(s,\eta,0)=0$ for all $(s,\eta)$ and

$$L_f := \max\{(\int_0^\pi\int_{-\infty}^0\int_0^\pi g^{-1}(\theta)\Big(\frac{\partial^i}{\partial\xi^i}b(\theta,\eta,\xi)\Big)^2 d\eta d\theta d\xi)^{1/2} : i=0,1\}<\infty.$$

Defining the operators $f,g: I\times\mathcal{B}\to X$ by

$$f(\psi)(\xi)=\int_{-\infty}^0\int_0^\pi b(s,\eta,\xi)\psi(s,\eta)d\eta ds,$$
$$g(\psi)(\xi)=\int_{-\infty}^0 a_0(s)\psi(s,\xi)ds.$$

Under the above conditions we can represent the system (4.1)-(4.3) into the abstract system (1.1)-(1.2). Moreover, $f,g$ are bounded linear operators with $\|f(\cdot)\|_{\mathcal{L}(\mathcal{B},X)}\le L_f$, $\|g(\cdot)\|_{\mathcal{L}(\mathcal{B},X)}\le L_g$. Therefore, $(\mathbf{H_1})$ and $(\mathbf{H_2})$ are fulfill. Therefore, all the conditions of Theorem 8 are satisfied. The following result is a direct consequence of Theorem 8.

**Proposition 9.** *For $b$ sufficiently small there exist a mild solutions of (4.1)-(4.3).*

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