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EXISTENCE RESULTS FOR NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS

Mouffak Benchohra and Jamal Eddine Lazreg

Abstract. In this paper, we establish the existence and uniqueness of solution for a class of initial value problem for implicit fractional differential equations with Caputo fractional derivative. The arguments are based upon the Banach contraction principle, Schauder' fixed point theorem and the nonlinear alternative of Leray-Schauder type. As applications, two examples are included to show the applicability of our results.

1 Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). Fractional differential equations has been studied by many researchers because they appear in various fields: physics, mechanics, engineering, control theory, rheology, electrochemistry, viscoelasticity, chaos and fractals, economics, etc. see for example [8, 17, 24, 25, 27], and references therein.

Several approaches to fractional derivatives exist: Riemann-Liouville, Hadamard, Grunwald-Letnikov, Weyl and Caputo etc... The Caputo fractional derivative is well suitable to the physical interpretation of initial conditions and boundary conditions. For more details of some recent theoretical results on fractional differential equations and their various applications, we refer the reader to books by Abbas *et al.* [3], Baleanu *et al.* [7], Kilbas *et al.* [20], Lakshmikantham *et al.* [21], and the papers by Abbas *et al.* [4], Agarwal *et al.* [1, 2], Babakhani and Daftardar-Gejji [5, 6], Belmekki and Benchohra [9], Benchohra *et al.* [11, 12, 13], Kilbas and Marzan [19], the references therein. More recently, some mathematicians have considered boundary value problems for fractional differential equations depending on the fractional derivative.

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In [26], by means of Schauder fixed-point theorem, Su and Liu studied the existence of nonlinear fractional boundary value problem involving Caputo's derivative:

$$^cD^{\alpha}u(t) = f(t, u(t), {}^cD^{\beta}u(t)), \text{ for each, } t \in (0,1), \ 1 < \alpha \leq 2, \ 0 < \beta \leq 1,$$

$$u(0) = u^{'}(1) = 0, \ or \ u^{'}(1) = u(1) = 0, or \ u(0) = u(1) = 0,$$

where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In [18], Hu and Wang investigated the existence of solution of the nonlinear fractional differential equation with integral boundary condition:

$$D^{\alpha}u(t) = f(t, u(t), D^{\beta}u(t)), \ t \in (0,1), \ 1 < \alpha \leq 2, \ 0 < \beta < 1,$$

$$u(0) = u_0, \ u(1) = \int_0^1 g(s)u(s)ds,$$

where D^{α} is the Riemann-Liouville fractional derivative, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, are continuous function and g be an integrable function. In [23], S. Murad and S. Hadid, by means of Schauder fixed-point theorem and the Banach contraction principle, considered the boundary value problem of the fractional differential equation:

$$D^{\alpha}y(t) = f(t, y(t), D^{\beta}y(t)), \ t \in (0,1), \ 1 < \alpha \leq 2, \ 0 < \beta < 1, \ 0 < \gamma \leq 1,$$

$$y(0) = 0, \ y(1) = I_0^{\gamma}y(s),$$

where D^{α} is the Riemann-Liouville fractional derivative, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a continuous function. In [15], A. G-Lakoud and R. Khaldi, studied the following boundary value problem of the fractional integral boundary conditions:

$$^cD^{q}y(t) = f(t, y(t), {}^cD^{p}y(t)), \ t \in (0,1), \ 1 < q \leq 2, \ 0 < p < 1,$$

$$y(0) = 0, \ y'(1) = \alpha I_0^{p}y(1),$$

where $^cD^{\alpha}$ is the Caputo fractional derivative, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a continuous function. In [10], Benchohra *et al.* studied the problem involving Caputo's derivative:

$$^cD^{\alpha}u(t) = f(t, u(t), {}^cD^{\alpha-1}u(t)), \text{ for each, } t \in J := [0,\infty), \ 1 < \alpha \leq 2,$$

$$u(0) = u_0, u \text{ is bounded on } J.$$

Motivated by the above cited works, the purpose of this paper, is to establish existence and uniqueness results to the following implicit fractional-order differential equation:

$$^cD^{\alpha}y(t) = f(t, y(t), {}^cD^{\alpha}y(t)), \text{ for each, } t \in J = [0,T], T > 0, \ 0 < \alpha \leq 1, \quad (1.1)$$

$$y(0) = y_0, \tag{1.2}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $y_0 \in \mathbb{R}$.

In this paper we present three results for the problem (1.1)-(1.2). The first one is based on the Banach contraction principle, the second one on Schauder's fixed point theorem, and the last one on the nonlinear alternative of Leray-Schauder type.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

Definition 1. *([20, 25]). The fractional (arbitrary) order integral of the function $h \in L^1([0, T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} h(s) ds,$$

where Γ is the gamma function.

Definition 2. *([19]). For a function h given on the interval $[0, T]$, the Caputo fractional-order derivative of order α of h, is defined by*

$$({}^cD^\alpha h)(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$.

Lemma 3. *([22]) Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Then*

$$I^\alpha({}^cD^\alpha f(t)) = f(t) - \Sigma_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

We state the following generalization of Gronwall's lemma for singular kernels.

Lemma 4. *([16, 28]) Let $v : [0, T] \to [0, +\infty)$ be a real function and $w(.)$ is a nonnegative, locally integrable function on $[0, T]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t (t - s)^{-\alpha} v(s) ds.$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq w(t) + Ka \int_0^t (t - s)^{-\alpha} w(s) ds, \text{ for every } t \in [0, T].$$

Theorem 5. *([14]) (Banach's fixed point theorem). Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 6. *([14]) (Schauder's fixed point theorem). Let X be a Banach space. and C be a closed, convex and nonempty subset of X. Let $N : C \to C$ be a continuous mapping such that $N(C)$ is a relatively compact subset of X. Then N has at least one fixed point in C.*

Theorem 7. *([14]) ([Nonlinear Alternative of Leray-Schauder type). Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $N : \overline{U} \to C$ is a compact map. Then either,*

(i) N has a fixed point in $\overline{U}$; or

(ii) there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda N(u)$.

3 Existence of Solutions

Let us defining what we mean by a solution of problem (1.1)-(1.2).

Definition 8. *A function $u \in C^1(J, \mathbb{R})$ is said to be a solution of the problem (1.1)–(1.2) is u satisfied equation (1.1) and conditions (1.2) on J.*

For the existence of solutions for the problem (1.1)–(1.2), we need the following auxiliary lemmas:

Lemma 9. *Let a function $f(t, u, v) : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Then the problem (1.1)–(1.2) is equivalent to the problem:*

$$y(t) = y_0 + I^\alpha g(t) \tag{3.1}$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation:

$$g(t) = f(t, y_0 + I^\alpha g(t), g(t)).$$

Proof. If ${}^cD^\alpha y(t) = g(t)$ then $I^\alpha\ {}^cD^\alpha y(t) = I^\alpha g(t)$. So we obtain $y(t) = y_0 + I^\alpha g(t)$.

We are now in a position to state and prove our existence result for the problem (1.1)-(1.2) based on Banach's fixed point. First we list the following hypotheses:

Theorem 10. *Assume*

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq K|u-\bar{u}| + L|v-\bar{v}|$$

for any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$.

If

$$\frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} < 1, \tag{3.2}$$

then there exists a unique solution for IVP (1.1)-(1.2) on J.

Proof. The proof will be given in several steps. Transform the problem (1.1)-(1.2) into a fixed point problem. Define the operator $N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by:

$$N(y)(t) = y_0 + I^{\alpha} g(t), \tag{3.3}$$

where $g \in C(J,\mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y(t), g(t)).$$

Clearly, the fixed points of operator N are solutions of problem (1.1)-(1.2). Let $u, w \in C(J,\mathbb{R})$. Then for $t \in J$, we have

$$(Nu)(t) - (Nw)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(g(s) - h(s))ds,$$

where $g, h \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t, u(t), g(t)),$$

$$h(t) = f(t, w(t), h(t)),$$

Then, for $t \in J$

$$|(Nu)(t) - (Nw)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|g(s) - h(s)|ds. \tag{3.4}$$

By $(H2)$ we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, w(t), h(t))| \\ &\leq K|u(t) - w(t)| + L|g(t) - h(t)|. \end{aligned}$$

Thus

$$|g(t) - h(t)| \leq \frac{K}{1-L}|u(t) - w(t)|.$$

By (3.4) we have

$$\begin{aligned}|(Nu)(t)-(Nw)(t)| &\leq \frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s)-w(s)|ds\\ &\leq \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\|u-w\|_\infty.\end{aligned}$$

Then

$$\|Nu-Nw\|_\infty \leq \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\|u-w\|_\infty.$$

By (3.2), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is a unique solution of the problem (1.1)-(1.2).

Our next existence result is based on Schauder's fixed point theorem.

Theorem 11. *Assume (H1),(H2) and the following hypothesis holds.*

(H3) There exist $p, q, r \in C(J, \mathbb{R}_+)$ with $r^* = \sup_{t\in J} r(t) < 1$ such that

$$|f(t,u,w)| \leq p(t) + q(t)|u| + r(t)|w| \text{ for } t \in J \text{ and } u, w \in \mathbb{R}.$$

If

$$\frac{q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)} < 1, \tag{3.5}$$

where $p^* = \sup_{t\in J} p(t)$, and $q^* = \sup_{t\in J} q(t)$, then the IVP (1.1)-(1.2) has at least one solution.

Proof. Let the operator N defined in (3.3). We shall show that N satisfies the assumption of Schauder's fixed point theorem. The proof will be given in several steps.

Claim 1: *N is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, \mathbb{R})$. Then for each $t \in J$

$$|N(u_n)(t) - N(u)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g_n(s)-g(s)|ds, \tag{3.6}$$

where $g_n, g \in C(J, \mathbb{R})$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

By $(H2)$ we have

$$\begin{aligned} |g_n(t) - g(t)| &= |f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))| \\ &\leq K|u_n(t) - u(t)| + L|g_n(t) - g(t)|. \end{aligned}$$

Then

$$|g_n(t) - g(t)| \leq \frac{K}{1-L}|u_n(t) - u(t)|.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. and let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \leq \eta$ and $|g(t)| \leq \eta$, then, we have

$$\begin{aligned} (t-s)^{\alpha-1}|g_n(s) - g(s)| &\leq (t-s)^{\alpha-1}[|g_n(s)| + |g(s)|] \\ &\leq 2\eta(t-s)^{\alpha-1}. \end{aligned}$$

For each $t \in J$, the function $s \to 2\eta(t-s)^{\alpha-1}$ is integrable on $[0, t]$, then the Lebesgue Dominated Convergence Theorem and (3.6) imply that

$$|N(u_n)(t) - N(u)(t)| \to 0 \ as \ n \to \infty,$$

and hence

$$\|N(u_n) - N(u)\|_\infty \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Let

$$R \geq \frac{M|y_0| + p^*T^\alpha}{M - q^*T^\alpha},$$

where $M := (1 - r^*)\Gamma(\alpha + 1)$ and define

$$D_R = \{u \in C(J, \mathbb{R}) : \|u\|_\infty \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $C(J, \mathbb{R})$.

Claim 2: $N(D_R) \subset D_R$.

Let $u \in D_R$ we show that $Nu \in D_R$.
We have, for each $t \in J$

$$|Nu(t)| \leq |y_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|g(s)|ds. \tag{3.7}$$

By (H3) we have for each $t \in J$,

$$\begin{array}{rcl} |g(t)| & = & |f(t,u(t),g(t))| \\ & \leq & p(t)+q(t)|u(t)|+r(t)|g(t)| \\ & \leq & p(t)+q(t)R+r(t)|g(t)| \\ & \leq & p^*+q^*R+r^*|g(t)|. \end{array}$$

Then

$$|g(t)| \leq \frac{p^*+q^*R}{1-r^*} := \widetilde{M}.$$

Thus (3.7) implies that

$$\begin{array}{rcl} |Nu(t)| & \leq & |y_0| + \dfrac{p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)} + \dfrac{q^*RT^\alpha}{M} \\ & \leq & |y_0| + \dfrac{p^*T^\alpha}{M} + \dfrac{q^*RT^\alpha}{M} \\ & \leq & R. \end{array}$$

Then $N(D_R) \subset D_R$.

Claim 3: *$N(D_R)$ is relatively compact.*

Let $t_1, t_2 \in J,\ t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{array}{rcl} |N(u)(t_2)-N(u)(t_1)| & = & \left|\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]g(s)ds \right. \\ & & \left. +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t1}^{t_2}[(t_2-s)^{\alpha-1}g(s)ds\right| \\ & \leq & \dfrac{\widetilde{M}}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha + 2(t_2-t_1)^\alpha). \end{array}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem, we conclude that $N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is continuous and compact. As a consequence of Schauder's fixed point theorem ([14]), we deduce that N has a fixed point which is a solution of the problem $(1.1)-(1.2)$.

Our next existence result is based on Nonlinear alternative of Leray-Schauder type.

Theorem 12. *Assume (H1),(H2),(H3) hold. Then the IVP (1.1)-(1.2) has at least one solution.*

Proof. Consider the operator N defined in (3.3). We shall show that N satisfies the assumption of Leray-Schauder fixed point theorem. The proof will be given in several claims.

Claim 1: *Clearly N is continuous.*

Claim 2: *N maps bounded sets into bounded sets in $C(J,\mathbb{R})$.*

Indeed, it is enough to show that for any $\rho > 0$, there exist a positive constant ℓ such that for each $u \in B_\rho = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq \rho\}$, we have $\|N(u)\|_\infty \leq \ell$.

For $u \in B_\rho$, we have, for each $t \in J$,

$$|Nu(t)| \leq |y_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(t)| ds. \tag{3.8}$$

By (H3) we have for each $t \in J$,

$$\begin{aligned} |g(t)| &= |f(t, u(t), g(t))| \\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\leq p(t) + q(t)\rho + r(t)|g(t)| \\ &\leq p^* + q^*\rho + r^*|g(t)|. \end{aligned}$$

Then

$$|g(t)| \leq \frac{p^* + q^*\rho}{1 - r^*} := M^*.$$

Thus (3.8) implies that

$$|Nu(t)| \leq |y_0| + \frac{M^* T^\alpha}{\Gamma(\alpha+1)}.$$

Thus

$$\|Nu\|_\infty \leq |y_0| + \frac{M^* T^\alpha}{\Gamma(\alpha+1)} := l.$$

Claim 3: *Clearly, N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.*

We conclude that $N : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ is continuous and completely continuous.

Claim 4: *A priori bounds.*

We now show there exists an open set $U \subseteq C(J,\mathbb{R})$ with $u \neq \lambda N(u)$, for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in C(J,\mathbb{R})$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $t \in J$, we have

$$u(t) = \lambda y_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds.$$

This implies by (H2) that for each $t \in J$ we have

$$|u(t)| \leq |y_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s)| ds. \tag{3.9}$$

And, by (H3) we have for each $t \in J$,

$$\begin{aligned} |g(t)| &= |f(t,u(t),g(t))| \\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\leq p^* + q^*|u(t)| + r^*|g(t)|. \end{aligned}$$

Thus

$$|g(t)| \leq \frac{1}{1-r^*}(p^* + q^*|u(t)|).$$

Hence

$$|u(t)| \leq |y_0| + \frac{p^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} + \frac{q^*}{(1-r^*)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s)| ds.$$

Then Lemma 4 implies that for each $t \in J$

$$|u(t)| \leq \left(|y_0| + \frac{p^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} \right) \left(1 + \frac{K q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} \right).$$

Thus

$$\|u\|_\infty \leq \left(|y_0| + \frac{p^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} \right) \left(1 + \frac{K q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} \right) := \overline{M}. \tag{3.10}$$

Let

$$U = \{u \in C(J,\mathbb{R}) : \|u\|_\infty < \overline{M} + 1\}.$$

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of Leray-Schauder's theorem ([14]), we deduce that N has a fixed point u in $\overline{U}$ which is a solution to (1.1)-(1.2).

4 Examples

Example 1. Consider the following Cauchy problem

$$ {}^cD^{\frac{1}{2}}y(t) = \frac{1}{2e^{t+1}(1+|y(t)|+|{}^cD^{\frac{1}{2}}y(t)|)}, \text{ for each, } t \in [0,1], \tag{4.1}$$

$$ y(0) = 1. \tag{4.2}$$

Set

$$ f(t,u,v) = \frac{1}{2e^{t+1}(1+|u|+|v|)}, \quad t \in [0,1], \ u,v \in \mathbb{R}$$

Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$:

$$ |f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{2e}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K = L = \frac{1}{2e}$.
It remains to show that condition (3.2) is satisfied. Indeed, we have

$$ \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)} = \frac{1}{(2e-1)\Gamma(\frac{3}{2})} < 1.$$

It follows from Theorem 10 that the problem (4.1)-(4.2) as a unique solution.

Example 2. Consider the following Cauchy problem

$$ {}^cD^{\frac{1}{2}}y(t) = \frac{(2+|y(t)|+|{}^cD^{\frac{1}{2}}y(t)|)}{2e^{t+1}(1+|y(t)|+|{}^cD^{\frac{1}{2}}y(t)|)}, \text{ for each, } t \in [0,1], \tag{4.3}$$

$$ y(0) = 1. \tag{4.4}$$

Set

$$ f(t,u,v) = \frac{(2+|u|+|v|)}{2e^{t+1}(1+|u|+|v|)}, \quad t \in [0,1], \ u,v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$ |f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{2e}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K = L = \frac{1}{2e}$. Also, we have,

$$ |f(t,u,v)| \leq \frac{1}{2e^{t+1}}(2+|u|+|v|).$$

Thus condition (H3) is satisfied with $p(t) = \frac{1}{e^{t+1}}$ and $q(t) = r(t) = \frac{1}{2e^{t+1}}$.

And condition

$$\frac{q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} = \frac{1}{(2e-1)\Gamma(\frac{3}{2})} < 1,$$

is satisfied with $T = 1$, $\alpha = \frac{1}{2}$, and $q^* = r^* = \frac{1}{2e}$.

It follows from Theorem 11 that the problem (4.3)-(4.4) has at least one solution.

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Mouffak Benchohra
Laboratory of Mathematics, University of Sidi Bel-Abbès
P. O. 89, Sidi Bel-Abbès 22000, Algérie.
and
Department of Mathematics, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
e-mail: benchohra@yahoo.com

Jamal Eddine Lazreg
Laboratory of Mathematics, University of Sidi Bel-Abbès
P. O. 89, Sidi Bel-Abbès 22000, Algérie.
e-mail: lazregjamal@yahoo.fr