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ON A FRACTIONAL DIFFERENTIAL INCLUSION WITH FOUR-POINT INTEGRAL BOUNDARY CONDITIONS

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Abstract. We study the existence of solutions for fractional differential inclusions of order $q \in (1, 2]$ with four-point integral boundary conditions. We establish Filippov type existence results in the case of nonconvex set-valued maps.

1 Introduction

This paper is concerned with the following boundary value problem

$$\begin{array}{l} D_c^q x(t) \in F(t, x(t)) \quad a.e.\ ([0,1]), \\ x(0) = \alpha \int_0^\xi x(s)ds, \quad x(1) = \beta \int_0^\eta x(s)ds, \end{array} \tag{1.1}$$

where $q \in (1, 2]$, D_c^q is the Caputo fractional derivative, $F : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $\alpha, \beta \in \mathbf{R}$ and $\xi, \eta \in (0, 1)$.

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations of fractional order ([12, 14] etc.). Recently several qualitative results for fractional differential inclusions were obtained in [2, 6, 7, 8] etc..

The present paper is motivated by a recent paper of Ahmad and Ntouyas ([1]) where existence results for problem (1.1) are established for convex as well as nonconvex set-valued maps. For the motivation, discussion on boundary conditions, examples and a consistent bibliography on these problems we refer to [1] and the references therein. The existence results in [1] are based on a nonlinear alternative of Leray-Schauder type and Covitz-Nadler contraction principle for set-valued maps.

The aim of our paper is to consider the situation when $F(.,.)$ has nonconvex values and to present two existence results for problem (1.1) which are Filippov type existence results for this problem.

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In our first approach we obtain an existence result by the application of the set-valued contraction principle in the space of derivatives of solutions instead of the space of solutions as in [1]. We note that the idea of applying the set-valued contraction principle due to Covitz and Nadler ([9]) in the space of derivatives of the trajectories belongs to Tallos ([11, 15]) and it was already used for similar results obtained for other classes of differential inclusions ([5, 6, 7]).

In our second approach we show that Filippov's ideas ([10]) can be suitably adapted in order to obtain the existence of solutions for problem (1.1). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([10]) consists in proving the existence of a solution starting from a given "quasi" or "almost" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries

In this short section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space and consider a set valued map T on X with nonempty values in X. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that:

$$d_H(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in X,$$

where $d_H(.,.)$ denotes the Pompeiu-Hausdorff distance. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\},\ d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y\in B} d(x, y)$.

The set-valued contraction principle ([9]) states that if X is complete, and $T : X \to \mathcal{P}(X)$ is a set valued contraction with nonempty closed values, then $T(.)$ has a fixed point, i.e. a point $z \in X$ such that $z \in T(z)$.

We denote by $Fix(T)$ the set of all fixed points of the set-valued map T. Obviously, $Fix(T)$ is closed.

Lemma 1. *([13]) Let X be a complete metric space and suppose that T_1, T_2 are λ-contractions with closed values in X. Then*

$$d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z\in X} d(T_1(z), T_2(z)).$$

Let $I = [0, 1]$, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(.)||_C = \sup_{t\in I} |x(t)|$ and $L^1(I, \mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u(.)||_1 = \int_0^1 |u(t)|dt$.

Definition 2. *([12]) a) The fractional integral of order $q > 0$ of a Lebesgue integrable function $f(.) : (0, \infty) \to \mathbf{R}$ is defined by*

$$I^q f(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(q) = \int_0^\infty t^{q-1}e^{-t}dt$.

b) The Caputo fractional derivative of order $q > 0$ of a function $f(.) : [0, \infty) \to \mathbf{R}$ is defined by

$$D_c^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{-q+n-1} f^{(n)}(s)ds,$$

where $n = [q] + 1$. It is assumed implicitly that $f(.)$ is n times differentiable whose n-th derivative is absolutely continuous.

We recall (e.g., [12]) that if $q > 0$ and $f(.) \in C(I, \mathbf{R})$ or $f(.) \in L^\infty(I, \mathbf{R})$ then $(D_c^q I^q f)(t) \equiv f(t)$.

Lemma 3. *([1]) For a given $f(.) \in C(I, \mathbf{R})$ the unique solution of the boundary value problem*

$$D_c^q x(t) = f(t),$$
$$x(0) = \alpha \int_0^\xi x(s)ds, \quad x(1) = \beta \int_0^\eta x(s)ds,$$

is given by

$$x(t) = \tfrac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds + \tfrac{\alpha}{\gamma\Gamma(q)}(\tfrac{2-\beta\eta^2}{2} + (\beta\eta-1)t) \int_0^\xi (\int_0^s (s-m)^{q-1}.$$
$$.f(m)dm)ds + \tfrac{\beta}{\gamma\Gamma(q)}(\tfrac{\alpha\xi^2}{2} + (1-\xi\alpha)t) \int_0^\eta (\int_0^s (s-m)^{q-1} f(m)dm)ds -$$
$$\tfrac{1}{\gamma\Gamma(q)}(\tfrac{\alpha\xi^2}{2} + (1-\xi\alpha)t) \int_0^1 (1-s)^{q-1} f(s)ds, \tag{2.1}$$

where

$$\gamma = \frac{1}{2}[(\alpha\xi - 1)(\beta\eta^2 - 2) - \alpha\xi^2(\beta\eta - 1)] \neq 0$$

Remark 4. *If we denote $A(t,s) = \frac{(t-s)^{q-1}}{\Gamma(q)} \chi_{[0,t]}(s)$, $B(t,s) = \frac{\alpha}{\gamma\Gamma(q)}(\frac{2-\beta\eta^2}{2}$ $+ (\beta\eta - 1)t)\frac{(\xi-s)^q}{q}\chi_{[0,\xi]}(s)$, $C(t,s) = \frac{\beta}{\gamma\Gamma(q)}(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t)\frac{(\eta-s)^q}{q}\chi_{[0,\eta]}(s)$, $D(t,s)$ $= -\frac{(1-s)^{q-1}}{\gamma\Gamma(q)}(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t)$ and $G(t,s) = A(t,s) + B(t,s) + C(t,s) + D(t,s)$, where $\chi_S(.)$ is the characteristic function of the set S, then the solution $x(.)$ in Lemma 3 may be written as*

$$x(t) = \int_0^1 G(t,s)f(s)ds, \tag{2.2}$$

Moreover, for any $t, s \in I$ we have

$$|G(t,s)| \leq \frac{1}{\Gamma(q)}[1 + \frac{|\alpha|}{|\gamma|}(|2 - \beta\eta^2| + |\beta\eta - 1|)\frac{\xi^q}{q} +$$

$$\frac{|\beta|}{|\gamma|}(\frac{|\alpha|\xi^2}{2}+|1-\xi\alpha|)\frac{\eta^q}{q}+\frac{1}{|\gamma|}(\frac{|\alpha|\xi^2}{2}+|1-\xi\alpha|)].$$

Since $q\in(1,2]$, *if we put* $\Lambda_1=|\alpha|(|2-\beta\eta^2|+2|\beta\eta-1|)\xi^q$ *and* $\Lambda_2=(|\alpha|\xi^2+2|1-\xi\alpha|)(|\beta|\eta^q+1)$ *we find that*

$$|G(t,s)|\leq\frac{1}{\Gamma(q)}(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|})=:M$$

Definition 5. *A function* $x(.)\in C(I,\mathbf{R})$ *with its Caputo derivative of order* q *existing on* $[0,1]$ *is a solution of problem (1.1) if there exists a function* $f(.)\in L^1(I,\mathbf{R})$ *such that* $f(t)\in F(t,x(t))$ *a.e.* (I) *and (2.1) is satisfied.*

3 The main results

We study first problem (1.1) with fixed point techniques. In order to do this we introduce the following hypothesis.

Hypothesis. (i) $F(.,.):I\times\mathbf{R}\to\mathcal{P}(\mathbf{R})$ *has nonempty closed values and for every* $x\in\mathbf{R}$, $F(.,x)$ *is measurable.*

(ii) *There exists* $L(.)\in L^1(I,\mathbf{R}_+)$ *such that for almost all* $t\in I, F(t,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t,x),F(t,y))\leq L(t)|x-y|\quad\forall\ x,y\in\mathbf{R}.$$

(iii) $d(0,F(t,0))\leq L(t)$ *a.e.* (I)

Denote $L_0:=\int_0^1 L(s)ds$.

Theorem 6. *Assume that Hypothesis is satisfied and* $ML_0<1$. *Let* $y(.)\in C(I,\mathbf{R})$ *be such that* $y(0)=\alpha\int_0^\xi y(s)ds$, $y(1)=\beta\int_0^\eta y(s)ds$ *and there exists* $p(.)\in L^1(I,\mathbf{R}_+)$ *with* $d(D_c^qy(t),F(t,y(t)))\leq p(t)$ *a.e.* (I).

Then for every $\varepsilon>0$ *there exists* $x(.)\in C(I,\mathbf{R})$ *a solution of problem (1.1) satisfying for all* $t\in I$

$$|x(t)-y(t)|\leq\frac{M}{1-ML_0}\int_0^1 p(t)dt+\varepsilon.$$

Proof. For $u(.)\in L^1(I,\mathbf{R})$ define the following set-valued maps

$$M_u(t)=F(t,\int_0^1 G(t,s)u(s)ds),\quad t\in I,$$

$$T(u)=\{\phi(.)\in L^1(I,\mathbf{R});\quad \phi(t)\in M_u(t)\quad a.e.\ (I)\}.$$

It follows from Lemma 3 that $x(.)$ is a solution of problem (1.1) if and only if $D_c^qx(.)$ is a fixed point of $T(.)$.

We shall prove first that $T(u)$ is nonempty and closed for every $u \in L^1(I, \mathbf{R})$. The fact that the set valued map $M_u(.)$ is measurable is well known. For example the map $t \to \int_0^1 G(t,s)u(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [4]. Since the values of F are closed with the measurable selection theorem (Theorem III.6 in [4]) we infer that $M_u(.)$ admits a measurable selection ϕ. One has

$$|\phi(t)| \leq d(0, F(t,0)) + d_H(F(t,0), F(t, \int_0^1 G(t,s)u(s)ds)) \leq$$

$$\leq L(t)(1 + M \int_0^1 |u(s)|ds),$$

which shows that $\phi \in L^1(I, \mathbf{R})$ and $T(u)$ is nonempty.

On the other hand, the set $T(u)$ is also closed. Indeed, if $\phi_n \in T(u)$ and $||\phi_n - \phi||_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T(u)$.

We show next that $T(.)$ is a contraction on $L^1(I, \mathbf{R})$.

Let $u, v \in L^1(I, \mathbf{R})$ be given and $\phi \in T(u)$. Consider the following set-valued map

$$H(t) = M_v(t) \cap \{x \in \mathbf{R}; \quad |\phi(t) - x| \leq L(t)| \int_0^1 G(t,s)(u(s) - v(s))ds|\}.$$

From Proposition III.4 in [4], $H(.)$ is measurable and from Hypothesis ii) $H(.)$ has nonempty closed values. Therefore, there exists $\psi(.)$ a measurable selection of $H(.)$. It follows that $\psi \in T(v)$ and according with the definition of the norm we have

$$||\phi - \psi||_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 L(t)(\int_0^1 |G(t,s)|.|u(s) - v(s)|ds)dt$$

$$= \int_0^1 (\int_0^1 L(t)|G(t,s)|dt)|u(s) - v(s)|ds \leq ML_0||u - v||_1.$$

We deduce that

$$d(\phi, T(v)) \leq ML_0||u - v||_1.$$

Replacing u by v we obtain

$$d_H(T(u), T(v)) \leq ML_0||u - v||_1,$$

thus $T(.)$ is a contraction on $L^1(I, \mathbf{R})$.

We consider next the following set-valued maps

$$F_1(t,x) = F(t,x) + p(t)[-1,1], \quad (t,x) \in I \times \mathbf{R},$$

$$M_u^1(t) = F_1(t, \int_0^1 G(t,s)u(s)ds),$$

$$T_1(u) = \{\psi(.) \in L^1(I, \mathbf{R}); \quad \psi(t) \in M_u^1(t) \quad a.e.\ (I)\}, \quad u(.) \in L^1(I, \mathbf{R}).$$

Obviously, $F_1(.,.)$ satisfies Hypothesis 3.1.

Repeating the previous step of the proof we obtain that T_1 is also a ML_0-contraction on $L^1(I, \mathbf{R})$ with closed nonempty values.

We prove next the following estimate

$$d_H(T(u), T_1(u)) \leq \int_0^1 p(t)dt. \tag{3.1}$$

Let $\phi \in T(u)$ and define

$$H_1(t) = M_u^1(t) \cap \{z \in \mathbf{R}; \quad |\phi(t) - z| \leq p(t)\}.$$

With the same arguments used for the set valued map $H(.)$, we deduce that $H_1(.)$ is measurable with nonempty closed values. Hence let $\psi(.)$ be a measurable selection of $H_1(.)$. It follows that $\psi \in T_1(u)$ and one has

$$||\phi - \psi||_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 p(t)dt.$$

As above we obtain (3.1).

We apply Lemma 1 and we infer that

$$d_H(Fix(T), Fix(T_1)) \leq \frac{1}{1 - ML_0} \int_0^1 p(t)dt.$$

Since $v(.) = D_c^q y(.) \in Fix(T_1)$ it follows that for any $\varepsilon > 0$ there exists $u(.) \in Fix(T)$ such that

$$||v - u||_1 \leq \frac{1}{1 - ML_0} \int_0^1 p(t)dt + \frac{\varepsilon}{M}.$$

We define $x(t) = \int_0^1 G(t,s)u(s)ds$, $t \in I$ and we have

$$|x(t) - y(t)| \leq \int_0^1 |G(t,s)|.|u(s) - v(s)|ds \leq \frac{M}{1 - ML_0} \int_0^1 p(t)dt + \varepsilon$$

which completes the proof. □

The assumption in Theorem 6 is satisfied, in particular, for $y(.) = 0$ and thus, via Hypothesis (iii), with $p(.) = L(.)$. We obtain the following consequence of Theorem 6.

Corollary 7. *Assume that Hypothesis is satisfied and* $ML_0 < 1$. *Then for every* $\varepsilon > 0$ *there exists* $x(.)$ *a solution of problem* (1.1) *satisfying for all* $t \in I$

$$|x(t)| \leq \frac{ML_0}{1 - ML_0} + \varepsilon. \tag{3.2}$$

Remark 8. *The existence result in Corollary 7 extends Theorem 15 in [1]. The approach in [1], apart from the requirement that the values of* $F(.,.)$ *are compact, does not provides a priori bounds as in (3.2).*

We present next the main result of this paper.

Theorem 9. *Assume that Hypothesis (i), (ii) is satisfied and* $ML_0 < 1$. *Let* $y(.) \in C(I, \mathbf{R})$ *be such that* $y(0) = \alpha \int_0^\xi y(s)ds$, $y(1) = \beta \int_0^\eta y(s)ds$ *and there exists* $p(.) \in L^1(I, \mathbf{R}_+)$ *with* $d(D_c^q y(t), F(t, y(t))) \leq p(t)$ *a.e.* (I).

Then there exists $x(.) \in C(I, \mathbf{R})$ *a solution of problem* (1.1) *satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{M}{1 - ML_0} \int_0^1 p(t)dt. \tag{3.3}$$

Proof. The set-valued map $t \to F(t, y(t))$ is measurable with closed values and

$$F(t, y(t)) \cap \{D_c^q y(t) + p(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows (e.g., Theorem 1.14.1 in [3]) that there exists a measurable selection $f_1(t) \in F(t, y(t))$ *a.e.* (I) such that

$$|f_1(t) - D_c^q y(t)| \leq p(t) \quad a.e.\ (I) \tag{3.4}$$

Define $x_1(t) = \int_0^1 G(t, s) f_1(s)ds$ and one has

$$|x_1(t) - y(t)| \leq M \int_0^1 p(t)dt.$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_0^1 G(t, s) f_n(s)ds, \quad t \in I, \tag{3.5}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad a.e.\ (I),\ n \geq 1, \tag{3.6}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)|x_n(t) - x_{n-1}(t)| \quad a.e.\ (I), n \geq 1. \tag{3.7}$$

If this construction is realized then from (3.4)-(3.7) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq \int_0^1 |G(t, t_1)|.|f_{n+1}(t_1) - f_n(t_1)|dt_1 \leq$$

$$M\int_0^1 L(t_1)|x_n(t_1)-x_{n-1}(t_1)|dt_1 \leq M\int_0^1 L(t_1)\int_0^1 |G(t_1,t_2)|.$$

$$|f_n(t_2)-f_{n-1}(t_2)|dt_2 \leq M^2\int_0^1 L(t_1)\int_0^1 L(t_2)|x_{n-1}(t_2)-x_{n-2}(t_2)|dt_2dt_1$$

$$\leq (M)^n\int_0^1 L(t_1)\int_0^1 L(t_2)...\int_0^1 L(t_n)|x_1(t_n)-y(t_n)|dt_n...dt_1 \leq$$

$$\leq (ML_0)^n M\int_0^1 p(t)dt.$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$, hence converging uniformly to some $x(.) \in C(I,\mathbf{R})$. Therefore, by (3.7), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$.

Moreover, one has

$$\begin{array}{l} |x_n(t)-y(t)| \leq |x_1(t)-y(t)| + \sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)| \leq \\ M\int_0^1 p(t)dt + \sum_{i=1}^{n-1}(M\int_0^1 p(t)dt)(ML_0)^i = \frac{M\int_0^1 p(t)dt}{1-ML_0}. \end{array} \tag{3.8}$$

On the other hand, from (3.4), (3.7) and (3.8) we obtain for almost all $t \in I$

$$\begin{array}{l} |f_n(t)-D_c^q y(t)| \leq \sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)| + \\ +|f_1(t)-D_c^q y(t)| \leq L(t)\frac{M\int_0^1 p(t)dt}{1-ML_0} + p(t). \end{array}$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.) \in L^1(I,\mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.5), (3.6) we deduce that $x(.)$ is a solution of (1.1). Finally, passing to the limit in (3.8) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (3.5)-(3.7). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I,\mathbf{R})$ and $f_n(.) \in L^1(I,\mathbf{R})$, $n = 1,2,...N$ satisfying (3.5), (3.7) for $n = 1,2,...N$ and (3.6) for $n = 1,2,...N-1$. The set-valued map $t \to F(t,x_N(t))$ is measurable. Moreover, the map $L(.)|x_N(.)-x_{N-1}(.)|$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t \in I$

$$F(t,x_N(t)) \cap \{f_N(t) + L(t)|x_N(t)-x_{N-1}(t)|[-1,1]\} \neq \emptyset.$$

Theorem 1.14.1 in [3] yields that there exist a measurable selection $f_{N+1}(.)$ of $F(.,x_N(.))$ such that

$$|f_{N+1}(t)-f_N(t)| \leq L(t)|x_N(t)-x_{N-1}(t)| \quad a.e.\ (I).$$

We define $x_{N+1}(.)$ as in (3.5) with $n = N+1$. Thus $f_{N+1}(.)$ satisfies (3.6) and (3.7) and the proof is complete. □

Remark 10. *Obviously, Theorem 9 extends Theorem 6. We do not suppose that* $d(0, F(t, 0)) \leq L(t)$ *a.e.* (I) *and the estimate in (3.3) is better than the one in Theorem 6.*

Even if Theorem 9 improves Theorem 6, we chosen to present both results; on one hand because the methods used in their proofs are different and on the other hand to show that there exists situations when the fixed point approaches are less powerful.

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