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DETERMINATION TEMPERATURE OF A HEAT EQUATION FROM THE FINAL VALUE DATA

Tuan H. Nguyen*, Tri V. Phan, H. Vu and Hoa V. Ngo

Abstract. We introduce the truncation method for solving a backward heat conduction problem. For this method, we give the stability analysis with new error estimates. Meanwhile, we investigate the roles of regularization parameters in these two methods. These estimates prove that our method is effective.

1 Introduction

In this paper,we consider the problem of finding the temperature $u(x,t)$, $(x,t) \in (0,\pi) \times [0,T]$, such that

$$\begin{cases} u_{xx} = u_t, (x,t) \in (0,\pi) \times (0,T) \\ u(0,t) = u(\pi,t) = 0, t \in (0,T) \\ u(x,T) = g(x), (x,t) \in (0,\pi) \times (0,T) \end{cases} \tag{1.1}$$

where $g(x)$ is given. The problem is called the backward heat problem, the backward Cauchy problem, or the final value problem. As is known, the problem is severely ill-posed; i.e., solutions do not always exist, and in the case of existence, these do not depend continuously on the given data. Physically, g can only be measured, there will be measurement errors, and we would actually have as data some function $g^\epsilon \in L^2(0,\pi)$, for which $\|g^\epsilon - g\| \le \epsilon$ where the constant $\epsilon > 0$ represents a bound on the measurement error, $\|.\|$denotes the L^2-norm. Notice the reader that the problem (1.1) is investigated in some papers of Clark and Oppenheimer [2], Denche and Bessila [4], ChuLiFu [3, 6, 5] Tautenhahn[12], et al . The nonhomogeneous case of (1.1) has been considered by Trong et al in [9, 11]. However, in those paper, the error estimates are only established in logarithmic form, i.e.,

$$\|u(.,t) - v^\epsilon(.,t)\| \le C_1 \frac{1}{\ln(\frac{1}{\epsilon^r})}, \quad r > 0. \tag{1.2}$$

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Corresponding authors at Division of Applied Mathematics, Ton Duc Thang University.

Although there are many papers on the backward heat equation, but there are rarely works gave the error estimates in the Holder type i.e.,

$$\|u(.,t) - v^{\epsilon}(.,t)\| \leq C\epsilon^{k}, \quad k > 0. \tag{1.3}$$

where C is the constant depend on u, k is a constant is not depend on t, u . It is easy to see that ϵ^k converges to zero more quickly than the logarithmic terms. So, the major object of this paper is to provide truncation regularization method to established the Holder estimates such as (1.3). This type of method is also applied to solve the backward heat in the unbouded region (See [6]).

2 Error estimates with the main results

Suppose the Problem (1.1) has an exact solution

$$u \in C([0,T]; H_0^1(0,\pi)) \cap C^1((0,T); L^2(0,\pi)),$$

then u can be formulated in the frequency domain

$$u(x,t) = \sum_{m=1}^{\infty} e^{-(t-T)m^2} g_m \sin(mx) \tag{2.1}$$

where

$$g_m = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(mx)dx,$$

and $< .,. >$ is the inner product in $L^2(0,\pi)$.

From (2.1), we note that $e^{(T-t)m^2}$ tends to infinity as m tends to infinity, then in order to guarantee the convergence of solution u given by (2.1), the coefficient $< u, \sin mx >$ must decay rapidly. Usually such a decay is not likely to occur for the measured data g^{ϵ}. Therefore, a natural way to obtain a stable approximation solution u is to eliminate the high frequencies and consider the solution u for $m < N$, where N is a positive integer. We define the truncation regularized solution as follows

$$u_N^{\epsilon}(x,t) = \sum_{m=1}^{N} e^{-(t-T)m^2} g_m \sin(mx) \tag{2.2}$$

and

$$v_N^{\epsilon}(x,t) = \sum_{m=1}^{N} e^{-(t-T)m^2} g_m^{\epsilon} \sin(mx) \tag{2.3}$$

where the positive integer N plays the role of the regularization parameter.

Definition 1. *Let* $0 = q < \infty$. *By* $H^q(0,\pi)$ *we denote the space of all functions* $g \in L^2(0,\pi)$ *with the property*

$$\sum_{m=1}^{\infty}(1+m^2)^q|g_m|^2 < \infty,$$

where $g_m = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(mx)dx$. *We also define the norm of* $H^q(0,\pi)$ *as follows*

$$\|g\|^2_{H^q(0,\pi)} = \sum_{m=1}^{\infty}(1+m^2)^q|g_m|^2.$$

Lemma 2. *The problem* (1.1) *has a unique solution* u *if and only if*

$$\sum_{m=1}^{\infty} e^{2Tm^2} g_m^2 < \infty. \tag{2.4}$$

Lemma 3. *The solution* u_N^{ϵ} *given in* (2.2) *depends continuously on* g *in* $L^2(0,\pi)$. *Furthermore, we have*

$$\|v_N^{\epsilon}(x,t) - u_N^{\epsilon}(x,t)\| \le e^{(T-t)N^2}\epsilon.$$

Theorem 4. *Assume that there exists the positive numbers* A_1 *such that* $\|u(.,0)\|^2 \le A_1$. *Let us* $N = [p]$ *where* $[.]$ *denotes the largest integer part of a real number with*

$$p = \sqrt{\frac{1}{T}\ln(\frac{1}{\epsilon})},$$

then the following convergence estimate holds for every $t \in [0,T]$

$$\|v_N^{\epsilon}(x,t) - u(x,t)\| \le (\sqrt{A_1}+1)\epsilon^{\frac{t}{T}}. \tag{2.5}$$

Remark 5. *From Theorem 4, we find that* v_N^{ϵ} *is an approximation of exact solution* u. *The approximation error depends continuously on the measurement error for fixed* $0 < t \le T$. *However, as* $t \to 0$ *the accuracy of regularized solution becomes progressively lower. This is a common thing in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution. To retain the continuous dependence of the solution at* $t = 0$, *we introduce a stronger a priori assumption.*

Theorem 6. *Assume that there exists the positive numbers* q, A_2 *such that*

$$\|u(.,t)\|_{H^q(0,\pi)} < A_2.$$

Let us $N = [p]$ *where* $[.]$ *denotes the largest integer part of a real number with*

$$p = \sqrt{\frac{1}{T+\alpha}\ln(\frac{1}{\epsilon})}$$

for $k > 0$*. Then the following convergence estimate holds*

$$\|v_N^\epsilon(x,t) - u(x,t)\| \le A_2\left(\frac{1}{T+\alpha}\ln(\frac{1}{\epsilon})\right)^{-\frac{q}{2}} + \epsilon^{\frac{t+\alpha}{T+\alpha}}. \tag{2.6}$$

for every $t \in [0,T]$*.*

Remark 7. *1. Denche and Bessila in [4], Trong and his group [8, 9, 11] gave the error estimates in the form*

$$\|v^\epsilon(.,t) - u(.,t)\| \le \frac{C_1}{1+\ln\frac{T}{\epsilon}}. \tag{2.7}$$

In recently, Chu-Li Fu and his coauthors [3, 5, 6] gave the error estimates as follows

$$\|v^{\epsilon,\delta}(.) - u(.)\| \le \frac{\delta}{2\sqrt{\epsilon}} + \max\{\left(\frac{4T}{\ln\frac{1}{\epsilon}}\right)^{\frac{p}{2}}, \epsilon^{\frac{1}{2}}\}. \tag{2.8}$$

If $q = 2$*, the error (2.6) is the same order as these above results.*
2. Notice that the first term of the right hand side of (2.6) is the logarithmic form, and the second term is a power, so the order of (2.6) is also logarithmic order. Suppose that $E_\epsilon = \|v^\epsilon - u\|$ *be the error of the exact solution and the approximate solution. In most of results concerning the backward heat, then optimal error between is of the logarithmic form. It means that*

$$E_\epsilon \le C\left(\ln\frac{T}{\epsilon}\right)^{-q}$$

where $q > 0$*. The error order of logarithmic form is investigated in many recent papers, such as [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. This often occurs in the boundary error estimate for ill-posed problems. To retain the Holder order in* $[0,T]$*, we introduce the following Theorem with different priori assumption.*

Theorem 8. *Assume that there exists the positive numbers* β, A_3 *such that*

$$\frac{\pi}{2}\sum_{m=1}^{\infty} e^{2\beta m^2} u_m^2(t) < A_3^2. \tag{2.9}$$

Let us $N = [p]$ where $[.]$ denotes the largest integer part of a real number with

$$p = \sqrt{\frac{1}{T+\beta}\ln(\frac{1}{\epsilon})}$$

then the following convergence estimate holds

$$\|v_N^\epsilon(x,t) - u(x,t)\| \le \left(A_3 + \epsilon^{\frac{t}{T+\beta}}\right)\epsilon^{\frac{\beta}{T+\beta}}. \tag{2.10}$$

for every $t \in [0,T]$.

Remark 9. *1. The condition (2.9) is not verifiable. Hence, we can check it by replacing the conditions of f and g. Thus, we have*

$$\sum_{m=1}^{\infty} e^{2\beta m^2} u_m^2(t) = \sum_{m=1}^{\infty} e^{2\beta m^2} e^{-2(t-T)m^2} g_m^2.$$

Hence, we can replace (2.9) by the different condition

$$\sum_{m=1}^{\infty} e^{2(T+\beta)m^2} g_m^2 < \infty.$$

2. Notice the reader that the error (2.10) $(\beta > 0)$ is the order of Holder type for all $t \in [0,T]$. It is easy to see that the convergence rate of ϵ^p, $(0 < p)$ is more quickly than the logarithmic order $\left(\ln(\frac{1}{\epsilon})\right)^{-q}$ $(q > 0)$ when $\epsilon \to 0$. Comparing (2.10) with the results in [2, 3, 4, 5, 6] and our recent results in Electronic Journal of Differential, we can see that the method in the present paper gives the better approximation. This proves that our method is effective.

3 Proof of the main results

Proof of Lemma 2

Proof. Suppose the Problem (1.1) has an exact solution $u \in C([0,T]; H_0^1(0,\pi)) \cap C^1((0,T); L^2(0,\pi))$, then u can be formulated in the frequency domain

$$u(x,t) = \sum_{m=1}^{\infty} e^{-(t-T)m^2} g_m \sin(mx). \tag{3.1}$$

This implies that

$$u_m(0) = e^{Tm^2} g_m. \tag{3.2}$$

Then

$$\|u(.,0)\|^2 = \sum_{m=1}^{\infty} e^{2Tm^2} g_m^2 < \infty.$$

If we get (2.4), then define $v(x)$ be as the function

$$v(x) = \sum_{m=1}^{\infty} e^{Tm^2} g_m \sin mx \in L^2(0,\pi).$$

Consider the problem

$$\begin{cases} u_t - u_{xx} = 0, \\ u(0,t) = u(\pi,t) = 0, \quad t \in (0,T) \\ u(x,0) = v(x), \quad x \in (0,\pi) \end{cases} \tag{3.3}$$

It is clear to see that (3.3) is the direct problem so it has a unique solution u. We have

$$u(x,t) = \sum_{m=1}^{\infty} e^{-tm^2} < v(x), \sin mx > \sin mx \tag{3.4}$$

Let $t = T$ in (3.4), we have

$$\begin{aligned} u(x,T) &= \sum_{m=1}^{\infty} e^{-Tm^2} e^{Tm^2} g_m \sin mx \\ &= \sum_{m=1}^{\infty} g_m \sin mx = g(x). \end{aligned}$$

Hence, u is the unique solution of (1.1).

□

Proof of Lemma 3

Proof. Let u_N^ϵ and w_N^ϵ be two solutions of (2.2) corresponding to the final values g and h. From (2.2), we have

$$u_N^\epsilon(x,t) = \sum_{m=1}^{N} e^{-(t-T)m^2} g_m \sin(mx) \quad 0 \le t \le T, \tag{3.5}$$

$$w_N^\epsilon(x,t) = \sum_{m=1}^{N} e^{-(t-T)m^2} h_m \sin(mx) \quad 0 \le t \le T, \tag{3.6}$$

where

$$g_m = \frac{2}{\pi}\int_0^\pi g(x)\sin(mx)dx, \quad h_m = \frac{2}{\pi}\int_0^\pi h(x)\sin(mx)dx.$$

This follows that

$$\begin{aligned}\|u_N^\epsilon(.,t) - w_N^\epsilon(.,t)\|^2 &= \frac{\pi}{2}\sum_{m=1}^{N}\left|e^{(T-t)m^2}(g_m - h_m)\right|^2 \\ &\le \frac{\pi}{2}e^{2(T-t)N^2}\sum_{m=1}^{N}|g_m - h_m|^2 \\ &\le e^{2(T-t)N^2}\|g-h\|^2. \end{aligned} \tag{3.7}$$

Hence

$$\|u_N^\epsilon(.,t) - w_N^\epsilon(.,t)\| \le e^{(T-t)N^2}\|g-h\|. \tag{3.8}$$

This completes the proof theorem.
Since (3.8) and the condition $\|g^\epsilon - g\| \le \epsilon$, we have

$$\|v_N^\epsilon(x,t) - u_N^\epsilon(x,t)\| \le e^{(T-t)N^2}\epsilon. \tag{3.9}$$

□

Proof of Theorem 4

Proof. Since (2.2), we have

$$\begin{aligned} u(x,t) - u_N^\epsilon(x,t) &= \sum_{m=N}^{\infty} e^{-(t-T)m^2} g_m \sin(mx) \\ &= \sum_{m=N}^{\infty} < u(x,t), \sin mx > \sin mx. \end{aligned}$$

Thus, using the inequality $(a+b)^2 \le 2(a^2+b^2)$ and Holder inequality, we have

$$\begin{aligned}\|u(.,t) - u_N^\epsilon(.,t)\|^2 &= \frac{\pi}{2}\sum_{m=N}^{\infty} e^{-2(t-T)m^2} g_m^2 \\ &= \frac{\pi}{2}\sum_{m=N}^{\infty} e^{-2tm^2} u_m^2(0) \\ &\le e^{-2tN^2}\|u(.,0)\|^2 \\ &\le e^{-2tN^2}A_1. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10)then

$$\begin{aligned} \|v_N^\epsilon(x,t) - u(x,t)\| &\leq \|v_N^\epsilon(.,t) - u_N(.,t)\| + u_N(.,t) - u(.,t)\| \\ &\leq e^{-tN^2}\sqrt{A_1} + e^{(T-t)N^2}\epsilon. \end{aligned}$$

From

$$N = \sqrt{\frac{1}{T}\ln(\frac{1}{\epsilon})}$$

then the following convergence estimate holds

$$\|v_N^\epsilon(x,t) - u(x,t)\| \leq \epsilon^{\frac{t}{T}}\left(\sqrt{A_1}+1\right).$$

□

Proof of Theorem 6

Proof. Since (2.2), we have

$$\begin{aligned} u(x,t) - u_N^\epsilon(x,t) &= \sum_{m=N}^{\infty} e^{-(t-T)m^2} g_m \sin(mx) \\ &= \sum_{m=N}^{\infty} < u(x,t), \sin mx > \sin mx. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|u(.,t) - u_N^\epsilon(.,t)\|^2 &= \frac{\pi}{2}\sum_{m=N}^{\infty} m^{-2q} m^{2q} u_m^2(t) \\ &\leq N^{-2q}\frac{\pi}{2}\sum_{m=1}^{\infty} m^{2q} u_m^2(t) \\ &\leq N^{-2q}\frac{\pi}{2}\sum_{m=1}^{\infty}(1+m^2)^q u_m^2(t) \\ &\leq N^{-2q}\frac{\pi}{2}A_2^2. \end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11) then

$$\begin{aligned} \|v_N^\epsilon(x,t) - u(x,t)\| &\leq \|v_N^\epsilon(.,t) - u_N(.,t)\| + u_N(.,t) - u(.,t)\| \\ &\leq N^{-q}A_2 + e^{(T-t)N^2}\epsilon. \end{aligned}$$

From

$$N = \sqrt{\frac{1}{T+\alpha}\ln(\frac{1}{\epsilon})}$$

then the following convergence estimate holds

$$\|v_N^\epsilon(x,t) - u(x,t)\| \leq \left(\frac{1}{T+\alpha}\ln(\frac{1}{\epsilon})\right)^{-\frac{q}{2}} A_2 + \epsilon^{\frac{t+\alpha}{T+\alpha}}.$$

□

Proof of Theorem 8

Proof. Since (2.2), we have

$$\begin{aligned} u(x,t) - u_N^\epsilon(x,t) &= \sum_{m=N}^{\infty} e^{-(t-T)m^2} g_m \sin(mx) \\ &= \sum_{m=N}^{\infty} < u(x,t), \sin mx > \sin mx. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|u(.,t) - u_N^\epsilon(.,t)\|^2 &= \frac{\pi}{2}\sum_{m=N}^{\infty} e^{-2\beta m^2} e^{2\beta m^2} u_m^2(t) \\ &\leq \frac{\pi}{2} e^{-2\beta N^2} \sum_{m=N}^{\infty} e^{2\beta m^2} u_m^2(t) \\ &\leq \frac{\pi}{2} e^{-2\beta N^2} \sum_{m=1}^{\infty} e^{2\beta m^2} u_m^2(t) \leq e^{-2\beta N^2} A_3^2. \end{aligned} \tag{3.12}$$

Combining (3.9) and (3.12), we get

$$\begin{aligned} \|v_N^\epsilon(x,t) - u(x,t)\| &\leq \|v_N^\epsilon(.,t) - u_N(.,t)\| + u_N(.,t) - u(.,t)\| \\ &\leq e^{-\beta N^2} A_3 + e^{(T-t)N^2}\epsilon. \end{aligned}$$

From

$$N = \left[\sqrt{\frac{1}{T+\beta}\ln(\frac{1}{\epsilon})}\right]$$

then the following convergence estimate holds

$$\|v_N^\epsilon(x,t) - u(x,t)\| \leq \epsilon^{\frac{\beta}{T+\beta}} A_3 + \epsilon^{\frac{t+\beta}{T+\beta}} = \epsilon^{\frac{\beta}{T+\beta}}\left(A_3 + \epsilon^{\frac{t}{T+\beta}}\right).$$

□

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Tuan H. Nguyen
Division of Applied Mathematics,
Ton Duc Thang University,
Nguyen Huu Tho Street, District 7,
Hochiminh City, Vietnam.
e-mail: tuanhuy_bs@yahoo.com

Tri V. Phan
Division of Applied Mathematics,
Ton Duc Thang University,
Nguyen Huu Tho Street, District 7,
Hochiminh City, Vietnam.
e-mail: phanvantri82@gmail.com

Hoa V. Ngo
Division of Applied Mathematics,
Ton Duc Thang University,
Nguyen Huu Tho Street, District 7,
Hochiminh City, Vietnam.
n.v.hoamath@gmail.com

H. Vu
Division of Applied Mathematics,
Ton Duc Thang University,
Nguyen Huu Tho Street, District 7,
Hochiminh City, Vietnam.
e-mail: hovumath@gmail.com