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BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH FOUR-POINT INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we discuss the existence of solutions for a boundary value problem of second order fractional differential inclusions with four-point integral boundary conditions involving convex and non-convex multivalued maps. Our results are based on the nonlinear alternative of Leray Schauder type and some suitable theorems of fixed point theory.

1 Introduction

This paper is concerned with the existence of solutions of boundary value problems for fractional order differential inclusions with four-point integral boundary conditions. More precisely, in Section 3, we consider the following boundary value problem for fractional differential inclusions with four-point integral boundary conditions

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ x(0) = \alpha \displaystyle\int_0^\xi x(s)ds, \quad x(1) = \beta \int_0^\eta x(s)ds, \quad 0 < \xi, \eta < 1, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$.

Differential equations with fractional order have recently proved valuable tools in the modeling of many physical phenomena [10], [11], [12], [20], [21]. There has been a significant theoretical development in fractional differential equations in recent years; see [17], [18], [23], [24], [25].

In the last few years, there has been much attention focused on boundary value problems for fractional differential equations and inclusions, see [1], [4], [8], [9], [22] and the references therein. The existence of solutions for nonlocal boundary value

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problems has been considerably investigated in many publications such as [2], [3], [13].

The aim of our paper is to present existence results for the problem (1.1), when the right hand side is convex as well as nonconvex valued. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. The methods used are standard, however their exposition in the framework of problem (1.1) is new.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries

2.1 Multi-valued Analysis

Let us recall some basic definitions on multi-valued maps [14, 16].

For a normed space $(X, \|.\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0; 1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.
Let $C([0, 1])$ denote a Banach space of continuous functions from $[0, 1]$ into $\mathbb{R}$ with the norm $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$. Let $L^1([0, 1], \mathbb{R})$ be the Banach space of measurable functions $x : [0, 1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

Definition 1. *A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t,x)$ *is measurable for each* $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ *is upper semicontinuous for almost all* $t \in [0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) *for each* $\alpha > 0$, *there exists* $\varphi_\alpha \in L^1([0,T], \mathbb{R}^+)$ *such that*

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t)$$

for all $\|x\|_\infty \leq \alpha$ *and for a. e.* $t \in [0,T]$.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,1]\}.$$

Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $x, y \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.*

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Niemytzki operator associated with F.

Definition 3. *Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nymetzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Let (X, d) be a metric space induced from the normed space $(X; \|.\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a; b)$ and $d(a, B) = \inf_{b\in B} d(a; b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [16]).

Definition 4. *A multivalued operator $N : X \to P_{cl}(X)$ is called:*

(**a**) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \text{ for each } x, y \in X;$$

(**b**) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

The following lemmas will be used in the sequel.

Lemma 5. *([19]) Let X be a Banach space. Let $F : [0, T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0, 1], X)$ to $C([0, 1], X)$. Then the operator*

$$\Theta \circ S_F : C([0, 1], X) \to P_{cp,c}(C([0, 1], X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0, 1], X) \times C([0, 1], X)$.

Lemma 6. *([5]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, 1], \mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0, 1], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 7. *([7]) Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

2.2 Fractional Calculus

Let us recall some basic definitions of fractional calculus [17, 23, 25].

Definition 8. *For a continuous function $x : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$$^cD^q x(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} x^{(n)}(s) ds, \quad n-1 < q < n, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 9. *The Riemann-Liouville fractional integral of order q is defined as*

$$I^q x(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{x(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Lemma 10. *([17]) For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, \; i = 0, 1, 2, \ldots, n-1 \; (n = [q] + 1)$.

In view of Lemma 10, it follows that

$$I^q \, {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R}, \; i = 0, 1, 2, ..., n-1 \; (n = [q] + 1)$.

In order to define the solution of (1.1), we consider the following lemma.

Lemma 11. *For a given $g \in C[0,1]$, the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = g(t), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = \alpha \int_0^\xi x(s) ds, \quad x(1) = \beta \int_0^\eta x(s) ds, \quad 0 < \xi, \eta < 1, \end{cases}$$

is given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds \\ &+ \frac{\alpha}{\gamma\Gamma(q)} \left(\frac{2 - \beta\eta^2}{2} + (\beta\eta - 1)t \right) \int_0^\xi \left(\int_0^s (s-m)^{q-1} g(m) dm \right) ds \\ &+ \frac{\beta}{\gamma\Gamma(q)} \left(\frac{\alpha\xi^2}{2} + (1 - \xi\alpha)t \right) \int_0^\eta \left(\int_0^s (s-m)^{q-1} g(m) dm \right) ds \\ &- \frac{1}{\gamma\Gamma(q)} \left(\frac{\alpha\xi^2}{2} + (1 - \xi\alpha)t \right) \int_0^1 (1-s)^{q-1} g(s) ds, \end{aligned} \tag{2.2}$$

where

$$\gamma = \frac{1}{2}[(\alpha\xi - 1)(\beta\eta^2 - 2) - \alpha\xi^2(\beta\eta - 1)] \neq 0.$$

Proof. In view of Lemma 10, for some constants $c_0, c_1 \in \mathbb{R}$, we have

$$x(t) = I^q g(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)ds - c_0 - c_1 t. \qquad (2.3)$$

Using the boundary conditions, we find that

$$(\alpha\xi - 1)c_0 + \alpha\frac{\xi^2}{2}c_1 = \alpha A, \qquad (2.4)$$

$$(\beta\eta - 1)c_0 + \left(\frac{\beta\eta^2}{2} - 1\right)c_1 = \beta B - C, \qquad (2.5)$$

where

$$\begin{aligned} A &= \frac{1}{\Gamma(q)}\int_0^\xi \left(\int_0^s (s-m)^{q-1}g(m)dm\right)ds, \\ B &= \frac{1}{\Gamma(q)}\int_0^\eta \left(\int_0^s (s-m)^{q-1}g(m)dm\right)ds, \\ C &= \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}g(s)ds. \end{aligned}$$

Solving (2.4) and (2.5) for c_0 and c_1, we have that

$$c_0 = \frac{1}{\gamma}\left[\left(\frac{\alpha\beta\eta^2}{2} - \alpha\right)A - \frac{\alpha\beta\xi^2}{2}B + \frac{\alpha\xi^2}{2}C\right]$$

and

$$c_1 = \frac{1}{\gamma}\left[\beta(\alpha\xi - 1)B - (\alpha\xi - 1)C - \alpha(\beta\eta - 1)A\right].$$

Substituting the values of c_0 and c_1 in (2.3), we obtain (2.2). □

Definition 12. *A function $x \in C([0,1],\mathbb{R})$ with its Caputo derivative of order q existing on $[0,1]$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,1]$ and*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds \\ &+ \frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2} + (\beta\eta - 1)t\right)\int_0^\xi \left(\int_0^s (s-m)^{q-1}f(m)dm\right)ds \\ &+ \frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t\right)\int_0^\eta \left(\int_0^s (s-m)^{q-1}f(m)dm\right)ds \\ &- \frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1}f(s)ds. \end{aligned}$$

3 Main results

Theorem 13. *Assume that*

($\mathbf{H_1}$) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is Carathéodory and has convex values;*

($\mathbf{H_2}$) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in L^1([0,1],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|_\infty) \ \text{ for each}(t,x) \in [0,1]\times\mathbb{R};$$

($\mathbf{H_3}$) *there exists a number* $M > 0$ *such that*

$$\frac{M}{\frac{1}{\Gamma(q)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|}\right)\psi(M)\|p\|_{L^1}} > 1, \tag{3.1}$$

where

$$\Lambda_1 = |\alpha|(|2-\beta\eta^2| + 2|\beta\eta-1|)\xi^q,$$

and

$$\Lambda_2 = (|\alpha|\xi^2 + 2|1-\xi\alpha|)(|\beta|\eta^q + 1).$$

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Define an operator

$$\Omega(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds \\ +\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2} + (\beta\eta-1)t\right)\int_0^\xi\left(\int_0^s (s-m)^{q-1}f(m)dm\right)ds \\ +\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t\right)\int_0^\eta\left(\int_0^s (s-m)^{q-1}f(m)dm\right)ds \\ -\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1}f(s)ds, \ \ t\in[0,1] \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$. We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray- Schauder type. The proof consists of several steps. As a first step, we show that Ω is convex for each $x \in C([0,1],\mathbb{R})$. For that, let $h_1, h_2 \in \Omega$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,1]$, we have

$$\begin{aligned} h_i(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_i(s)ds \\ &\quad + \frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2} + (\beta\eta-1)t\right)\int_0^\xi\left(\int_0^s (s-m)^{q-1}f_i(m)dm\right)ds \end{aligned}$$

$$+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^\eta\left(\int_0^s(s-m)^{q-1}f_i(m)dm\right)ds$$
$$-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1(1-s)^{q-1}f_i(s)ds, \quad i=1,2.$$

Let $0\le\omega\le 1$. Then, for each $t\in[0,1]$, we have

$$\begin{aligned}&[\omega h_1+(1-\omega)h_2](t)\\ =\;& \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\theta(s)ds\\ &+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^\xi\left(\int_0^s(s-m)^{q-1}\theta(m)dm\right)ds\\ &+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^\eta\left(\int_0^s(s-m)^{q-1}\theta(m)dm\right)ds\\ &-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1(1-s)^{q-1}\theta(s)ds, \quad i=1,2,\end{aligned}$$

where $\theta(t)=\omega f_1(t)+(1-\omega)f_2(t)$. Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\omega h_1+(1-\omega)h_2\in\Omega(x)$.

Next, we show that $\Omega(x)$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number r, let $B_r=\{x\in C([0,1],\mathbb{R}):\|x\|_\infty\le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h\in\Omega(x), x\in B_r$, there exists $f\in S_{F,x}$ such that

$$\begin{aligned}h(t) =\;& \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s)ds\\ &+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^\xi\left(\int_0^s(s-m)^{q-1}f(m)dm\right)ds\\ &+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^\eta\left(\int_0^s(s-m)^{q-1}f(m)dm\right)ds\\ &-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1(1-s)^{q-1}f(s)ds.\end{aligned}$$

Then

$$\begin{aligned}|h(t)| =\;& \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}|f(s)|ds\\ &+\frac{|\alpha|}{2|\gamma|\Gamma(q)}(|2-\beta\eta^2|+2|\beta\eta-1|)\int_0^\xi\left(\int_0^s(s-m)^{q-1}|f(m)|dm\right)ds\\ &+\frac{|\beta|}{2|\gamma|\Gamma(q)}(|\alpha|\xi^2+|1-\xi\alpha|\int_0^\eta\left(\int_0^s(s-m)^{q-1}|f(m)|dm\right)ds\\ &+\frac{1}{2|\gamma|\Gamma(q)}(|\alpha|\xi^2+|1-\xi\alpha|\int_0^1(1-s)^{q-1}|f(s)|ds\end{aligned}$$

$$
\begin{aligned}
&\le \left(\frac{1}{\Gamma(q)} + \frac{|\alpha|(|2-\beta\eta^2| + 2|\beta\eta - 1|)\xi^q}{2|\gamma|\Gamma(q)} + \frac{|\beta|(|\alpha|\xi^2 + |1-\alpha\xi|)\eta^q}{2|\gamma|\Gamma(q)} \right.\\
&\quad \left. + \frac{|\alpha|\xi^2 + |1-\alpha\xi|}{2|\gamma|\Gamma(q)}\right) \int_0^1 p(s)\psi(\|x\|_\infty)ds \\
&\le \frac{\psi(\|x\|_\infty)}{\Gamma(q)} \left(1 + \frac{\Lambda_1 + \Lambda_2}{2|\gamma|}\right) \int_0^1 p(s)ds.
\end{aligned}
$$

Thus,

$$
\|h\|_\infty \le \frac{\psi(\|x\|_\infty)}{\Gamma(q)} \left(1 + \frac{\Lambda_1 + \Lambda_2}{2|\gamma|}\right) \int_0^1 p(s)ds,
$$

where we have used (H_2) and (H_3).
Now we show that Ω maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. For each $h \in \Omega(x)$, we obtain

$$
\begin{aligned}
|h(t'') - h(t')| &= \left| \frac{1}{\Gamma(q)} \int_0^{t''} (t''^{q-1} f(s)ds - \frac{1}{\Gamma(q)} \int_0^{t'} (t'^{q-1} f(s)ds \right.\\
&\quad + \frac{\alpha}{\gamma\Gamma(q)} (\beta\eta - 1)(t'' - t') \int_0^{\xi} \left(\int_0^s (s-m)^{q-1} f(m)dm \right) ds \\
&\quad + \frac{\beta}{\gamma\Gamma(q)} (1 - \xi\alpha)(t'' - t') \int_0^{\eta} \left(\int_0^s (s-m)^{q-1} f(m)dm \right) ds \\
&\quad \left. - \frac{1}{\gamma\Gamma(q)} (1 - \xi\alpha)(t'' - t') \int_0^1 (1-s)^{q-1} f(s)ds. \right| \\
&\le \left| \frac{1}{\Gamma(q)} \int_0^{t'} [(t''^{q-1} - (t'^{q-1}] p(s)\psi(\|x\|_\infty)ds \right| \\
&\quad + \left| \frac{1}{\Gamma(q)} \int_{t'}^{t''} (t''^{q-1} p(s)\psi(\|x\|_\infty)ds \right| \\
&\quad + \left| \frac{\alpha}{\gamma\Gamma(q)} (\beta\eta - 1)(t'' - t') \int_0^{\xi} \left(\int_0^s (s-m)^{q-1} p(m)\psi(\|x\|_\infty)dm \right) ds \right| \\
&\quad + \left| \frac{\beta}{\gamma\Gamma(q)} (1 - \xi\alpha)(t'' - t') \int_0^{\eta} \left(\int_0^s (s-m)^{q-1} p(m)\psi(\|x\|_\infty)dm \right) ds \right| \\
&\quad + \left| \frac{1}{\gamma\Gamma(q)} (1 - \xi\alpha)(t'' - t') \int_0^1 (1-s)^{q-1} p(s)\psi(\|x\|_\infty)ds \right|.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t'' - t' \to 0$. As Ω satisfies the above three assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω has a closed graph. Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_n(s)ds \\ &+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^\xi\left(\int_0^s (s-m)^{q-1}f_n(m)dm\right)ds \\ &+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^\eta\left(\int_0^s (s-m)^{q-1}f_n(m)dm\right)ds \\ &-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1}f_n(s)ds. \end{aligned}$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_*(s)ds \\ &+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^\xi\left(\int_0^s (s-m)^{q-1}f_*(m)dm\right)ds \\ &+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^\eta\left(\int_0^s (s-m)^{q-1}f_*(m)dm\right)ds \\ &-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1}f_*(s)ds. \end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds \\ &+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^\xi\left(\int_0^s (s-m)^{q-1}f(m)dm\right)ds \\ &+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^\eta\left(\int_0^s (s-m)^{q-1}f(m)dm\right)ds \\ &-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1}f(s)ds. \end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n(t)-h_*(t)\| \\ = &\left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(f_n(s)-f_*(s))ds\right. \end{aligned}$$

$$+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^{\xi}\left(\int_0^s(s-m)^{q-1}(f_n(m)-f_*(m))dm\right)ds$$
$$+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{\eta}\left(\int_0^s(s-m)^{q-1}(f_n(m)-f_*(m))dm\right)ds$$
$$-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1(1-s)^{q-1}(f_n(s)-f_*(s))ds\Bigg\|\to 0.$$

as $n\to\infty$. Thus, it follows by Lemma 5 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$\begin{aligned}
h_*(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f_*(s)ds\\
&+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^{\xi}\left(\int_0^s(s-m)^{q-1}f_*(m)dm\right)ds\\
&+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{\eta}\left(\int_0^s(s-m)^{q-1}f_*(m)dm\right)ds\\
&-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1(1-s)^{q-1}f_*(s)ds,
\end{aligned}$$

for some $f_*\in S_{F,x_*}$.

Finally, we discuss a priori bounds on solutions. Let x be a solution of (1.1). Then there exists $f\in L^1([0,1],\mathbb{R})$ with $f\in S_{F,x}$ such that, for $t\in[0,1]$, we have

$$\begin{aligned}
x(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s)ds\\
&+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^{\xi}\left(\int_0^s(s-m)^{q-1}f(m)dm\right)ds\\
&+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{\eta}\left(\int_0^s(s-m)^{q-1}f(m)dm\right)ds\\
&-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1(1-s)^{q-1}f(s)ds.
\end{aligned}$$

In view of (H_2), and using the computations in second step above, for each $t\in[0,1]$, we obtain

$$\begin{aligned}
|x(t)| &\le \frac{1}{\Gamma(q)}\int_0^1 f(s)ds+\frac{(|\alpha|(|2-\beta\eta^2|+2|\beta\eta-1|)\xi^q}{2\gamma\Gamma(q)}\int_0^1 f(s)ds\\
&+\frac{|\beta|(|\alpha|\xi^2+|1-\xi\alpha|)\eta^q}{2|\gamma|\Gamma(q)}\int_0^1 f(s)ds+\frac{|\alpha|\xi^2+|1-\xi\alpha|}{2|\gamma|\Gamma(q)}\int_0^1 f(s)ds\\
&\le \frac{1}{\Gamma(q)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|}\right)\int_0^1 f(s)ds
\end{aligned}$$

$$\leq \frac{1}{\Gamma(q)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|}\right)\psi(\|x\|_\infty)\int_0^1 p(s)ds.$$

Consequently, we have

$$\frac{\|x\|_\infty}{\frac{1}{\Gamma(q)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|}\right)\psi(\|x\|_\infty)\|p\|_{L^1}} \leq 1.$$

In view of (H_3), there exists M such that $\|x\|_\infty \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\|_\infty < M+1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [15], we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [5] for lower semi-continuous maps with decomposable values.

Theorem 14. *Assume that $(H_2), (H_3)$ and the following conditions hold:*

$(\mathbf{H_4})$ *$F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

- **(a)** *$(t,x) \longmapsto F(t,x)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable,*
- **(b)** *$x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$;*

$(\mathbf{H_5})$ *for each $\sigma > 0$, there exists $\varphi_\sigma \in L^1([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \leq \varphi_\sigma(t) \text{ for all } \|x\|_\infty \leq \sigma \text{ and for a.e.} t \in [0,1].$$

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. It follows from (H_4) and (H_5) that F is of l.s.c. type. Then from Lemma 6, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \quad 0<t<1, \quad 1<q\leq 2, \\ x(0) = \alpha\int_0^\xi x(s)ds, \quad x(1) = \beta\int_0^\eta x(s)ds, \quad 0<\xi,\eta<1. \end{cases} \tag{3.2}$$

Observe that if x is a solution of (3.2), then x is a solution to the problem (1.1). In order to transform the problem (3.2) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned}\overline{\Omega}x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(x(s))ds \\ &+ \frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^{\xi}\left(\int_0^s (s-m)^{q-1} f(x(m))dm\right)ds \\ &+ \frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{\eta}\left(\int_0^s (s-m)^{q-1} f(x(m))dm\right)ds \\ &- \frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1} f(x(s))ds.\end{aligned}$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 13. So we omit it. This completes the proof. □

Now we prove the existence of solutions for the problem (1.1) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [7].

Theorem 15. *Assume that the following conditions hold:*

($\mathbf{H_6}$) *$F : [0,1]\times\mathbb{R} \to P_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0,1] \to P_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$.*

($\mathbf{H_7}$) *$H_d(F(t,x), F(t,\bar{x})) \le m(t)|x-\bar{x}|$ for almost all $t \in [0,1]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in L^1([0,1], \mathbb{R}^+)$ and $d(0, F(t,0)) \le m(t)$ for almost all $t \in [0,1]$.*

Then the boundary value problem (1.1) has at least one solution on $[0,1]$ if

$$\frac{1}{\Gamma(q)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|}\right)\|m\|_{L^1} < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,1],\mathbb{R})$ by the assumption (H_6), so F has a measurable selection (see Theorem III.6 [6]). Now we show that the operator Ω satisfies the assumptions of Lemma 7. To show that $\Omega(x) \in P_{cl}((C[0,1],\mathbb{R}))$ for each $x \in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\ge 0} \in \Omega(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,1],\mathbb{R})$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$u_n(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} v_n(s)ds$$

$$+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^{\xi}\left(\int_0^s (s-m)^{q-1}v_n(m)dm\right)ds$$
$$+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{\eta}\left(\int_0^s (s-m)^{q-1}v_n(m)dm\right)ds$$
$$-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{1}(1-s)^{q-1}v_n(s)ds.$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) \to u(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v(s)ds \\ &+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^{\xi}\left(\int_0^s (s-m)^{q-1}v(m)dm\right)ds \\ &+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{\eta}\left(\int_0^s (s-m)^{q-1}v(m)dm\right)ds \\ &-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{1}(1-s)^{q-1}v(s)ds. \end{aligned}$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $\gamma < 1$ such that

$$H_d(\Omega(x),\Omega(\bar{x})) \le \gamma\|x-\bar{x}\|_\infty \quad \text{for each} \quad x,\bar{x} \in C([0,1],\mathbb{R}).$$

Let $x,\bar{x} \in C([0,1],\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t,x(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v_1(s)ds \\ &+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^{\xi}\left(\int_0^s (s-m)^{q-1}v_1(m)dm\right)ds \\ &+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{\eta}\left(\int_0^s (s-m)^{q-1}v_1(m)dm\right)ds \\ &-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{1}(1-s)^{q-1}v_1(s)ds. \end{aligned}$$

By (H_7), we have

$$H_d(F(t,x),F(t,\bar{x})) \le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w \in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w| \le m(t)|x(t)-\bar{x}(t)|, \quad t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [6]), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} v_2(s)ds \\ &+ \frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2} + (\beta\eta - 1)t\right)\int_0^\xi \left(\int_0^s (s-m)^{q-1} v_2(m)dm\right) ds \\ &+ \frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t\right)\int_0^\eta \left(\int_0^s (s-m)^{q-1} v_2(m)dm\right) ds \\ &- \frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1} v_2(s)ds. \end{aligned}$$

Thus,

$$\begin{aligned} &|h_1(t) - h_2(t)| \\ \leq\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|v_1(s) - v_2(s)|ds \\ &+ \frac{|\alpha|(|2-\beta\eta^2| + 2|\beta\eta - 1|)\xi^q}{2\gamma\Gamma(q)}\int_0^\xi \left(\int_0^s (s-m)^{q-1}|v_1(m) - v_2(m)|dm\right) ds \\ &+ \frac{|\beta|(|\alpha|\xi^2 + |1-\xi\alpha|)\eta^q}{2|\gamma|\Gamma(q)}\int_0^\eta \left(\int_0^s (s-m)^{q-1}|v_1(m) - v_2(m)|dm\right) ds \\ &+ \frac{|\alpha|\xi^2 + |1-\xi\alpha|}{2|\gamma|\Gamma(q)}\int_0^1 (1-s)^{q-1}|v_1(s) - v_2(s)|ds \\ \leq\ & \frac{1}{\Gamma(q)}\left(1 + \frac{\Lambda_1 + \Lambda_2}{2|\gamma|}\right)\int_0^1 m(s)\|x - \overline{x}\|ds. \end{aligned}$$

Hence,

$$\|h_1 - h_2\|_\infty \leq \frac{1}{\Gamma(q)}\left(1 + \frac{\Lambda_1 + \Lambda_2}{2|\gamma|}\right)\|m\|_{L^1}\|x - \overline{x}\|_\infty.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} H_d(\Omega(x), \Omega(\bar{x})) &\leq \gamma\|x - \bar{x}\|_\infty \\ &\leq \frac{1}{\Gamma(q)}\left(1 + \frac{\Lambda_1 + \Lambda_2}{2|\gamma|}\right)\|m\|_{L^1}\|x - \overline{x}\|_\infty. \end{aligned}$$

Since Ω is a contraction, it follows by Lemma 7 that Ω has a fixed point x which is a solution of (1.1). This completes the proof. □

Example 16. *Consider the nonlocal fractional inclusion boundary value problem*

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t,x(t)), \quad t \in [0,1], \\ x(0) = \displaystyle\int_0^{1/3} x(s)ds, \quad x(1) = \int_0^{2/3} x(s)ds, \end{cases} \tag{3.3}$$

where $q = 3/2, \ \alpha = 1, \ \beta = 1, \ \xi = 1/3, \ \eta = 2/3$ *and* $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a multivalued map given by*

$$x \to F(t,x) = \left[\frac{|x|^3}{|x|^3+3} + t^3 + 5, \frac{|x|}{|x|+1} + t + 1 \right].$$

For $f \in F$, we have

$$|f| \leq \max \left(\frac{|x|^3}{|x|^3+3} + t^3 + 5, \frac{|x|}{|x|+1} + t + 1 \right) \leq 7, \quad x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq 7 = p(t)\psi(\|x\|_\infty), \quad x \in \mathbb{R},$$

with $p(t) = 1, \ \psi(\|x\|_\infty) = 7$. Further, using the condition

$$\frac{M}{\frac{1}{\Gamma(q)}\left(1 + \frac{\Lambda_1+\Lambda_2}{2|\gamma|}\right)\psi(M)\|p\|_{L^1}} > 1,$$

we find that $M > \frac{14}{29\sqrt{3\pi}}(20+26\sqrt{2}+68\sqrt{3})$. Clearly, all the conditions of Theorem 13 are satisfied. So there exists at least one solution of the problem (3.3) on $[0,1]$.

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