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A NOTE OF ZUK'S CRITERION

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Abstract. Zuk's criterion give us a condition for a finitely generated group to have Property (T): the smallest non - zero eigenvalue of Laplace operator Δ_μ corresponding to the simple random walk on $\mathcal{G}$(S) satisfies $\lambda_1(\mathcal{G}) > \frac{1}{2}$. We present here two examples that prove that this condition cannot be improved.

Definition 1. *(see [1] and [2])*

i) A random walk or Markov kernel on a non-empty set X is a kernel with non-negative values $\mu : X \times X \to \mathbb{R}_+$ such that:

$$\sum_{y \in X} \mu(x, y) = 1, \forall x \in X.$$

ii) A stationary measure for a random walk μ is a function $\nu : X \to \mathbb{R}_+^*$ such that :

$$\nu(x)\mu(x, y) = \nu(y)\mu(y, x), \forall x, y \in X.$$

Example 2. *Let $\mathcal{G}$ =(X,E) be a locally finite graph. For x,y $\in$ X, set*

$$\mu(x, y) = \begin{cases} \dfrac{1}{deg(x)} & if (x, y) \in E \\ 0 & otherwise \end{cases} \tag{0.1}$$

and deg(x) =card $\{y \in X | (x, y) \in E\}$ is the degree of a vertex $x \in X$.
μ is called simple random walk on X and ν is a stationary measure for μ.

Consider the Hilbert space:

$$\Omega^0_{\mathbb{C}}(X) = \{f : X \to \mathbb{C} | \sum_{x \in X} |f(x)|^2 \nu(x) < \infty\}$$

The Laplace operator Δ_μ on $\Omega^0_{\mathbb{C}}(X)$ is definited by $(\Delta_\mu f)(x) = f(x) - \sum_{x \sim y} f(y)\mu(x, y)$.

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Let Γ be a group generated by a finite set S. We assume that e $\notin$ S and S = S^{-1} (S is simetric).

The graph $\mathcal{G}(S)$ associated to S has vertex set S and the set of edges is the set of pairs $(s,t) \in S \times S$ such that $s^{-1}t \in S$.

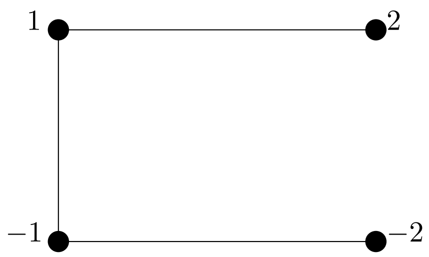
Theorem 3. *(Zuk's criterion)(see [3])*

Let Γ be a group generated by a finite set S with $e \notin S$. Let $\mathcal{G}(S)$ be the graph associated to S. Assume that $\mathcal{G}(S)$ is connected and that the smallest non-zero eigenvalue of the Laplace operator Δ_μ corresponding to the simple random walk on $\mathcal{G}(S)$ satisfies $\lambda_1(\mathcal{G}(S)) > \frac{1}{2}$.

Then Γ has Property (T).

We prove that the condition $\lambda_1(\mathcal{G}(S)) > \frac{1}{2}$ cannot be improved, using two examples.

Example 4. *Consider $S = \{\ 1, -1, 2, -2\}$ a generating set of the group $\mathbb{Z}$ and let $\mathcal{G}(S)$ be the finite graph associated to S. Then the graph $\mathcal{G}(S)$ is the graph:*



Since the Laplace operator Δ_μ is defined by:

$$(\Delta_\mu f)(x) = f(x) - \sum_{x \sim y} f(y)\mu(x,y),$$

and

$$\mu(x,y) = \begin{cases} \dfrac{1}{deg(x)} & if(x,y) \in S \times S \\ 0 & otherwise \end{cases} \tag{0.2}$$

Then the matrix of the Laplace operator Δ_μ with respect to the basis $\{\ \delta_s | s \in S\ \}$

is the following matrix:

$$A = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -1 & 1 \end{pmatrix} \tag{0.3}$$

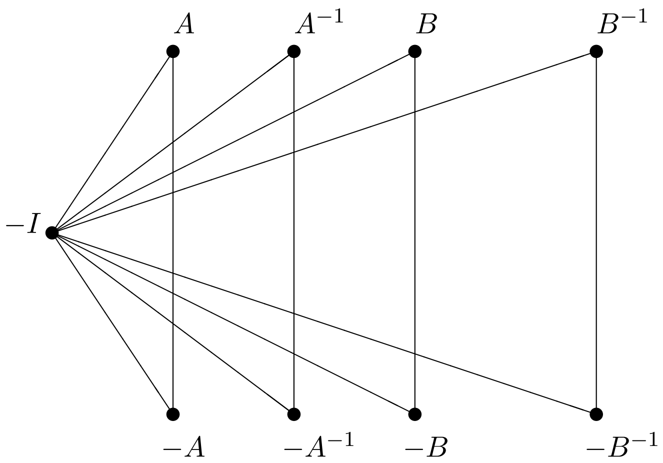
Then $det(A-\alpha I_4) = (1-\alpha)^2[(1-\alpha)^2 - \frac{3}{4}] - \frac{1}{2}(1-\alpha)^2 + \frac{1}{4} = 0$ $\Rightarrow \alpha \in \{0, \frac{1}{2}, \frac{3}{2}, 2\} \Rightarrow \lambda_1(\mathcal{G}(S)) = \frac{1}{2}$.

But $\mathbb{Z}$ does not have Property (T).(see [1])

Example 5. *The group $SL_2(\mathbb{Z})$ is generated by the matrices* $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We consider the following generating set of the group $SL_2(\mathbb{Z})$: $S = \{-I, A, B, -A, -B, A^{-1}, B^{-1}, -A^{-1}, -B^{-1}\}$.

The graph $\mathcal{G}(S)$ is:



Then the matrix of Laplace operatorΔ_μ with respect to the basis $\{\delta_s | s \in S\}$ is the following matrix:

$$A = \begin{pmatrix} 1 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix} \quad (0.4)$$

Computing $\det(A - \alpha I_9) = [(1-\alpha)^2 - \frac{1}{4}]^3(\frac{3}{2} - \alpha)(\alpha^2 - \frac{5}{2}\alpha) = 0 \Rightarrow$
$\Rightarrow \alpha \in \{0, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\} \Rightarrow \lambda_1(\mathcal{G}(S)) = \frac{1}{2}$.

But $SL_2(\mathbb{Z})$ does not have Property (T). (see [1])

These two examples shows that $\frac{1}{2}$ is the best constant in Zuk's criterion and cannot be improved.

References

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