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SPECTRUM OF A FAMILY OF OPERATORS

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Abstract. Having as start point the classic definitions of resolvent set and spectrum of a linear bounded operator on a Banach space, we introduce the resolvent set and spectrum of a family of linear bounded operators on a Banach space. In addition, we present some results which adapt to asymptotic case the classic results.

1 Introduction

Let X be a complex Banach space and $L(X)$ the Banach algebra of linear bounded operators on X. Let T be a linear bounded operator on X. The *norm* of T is

$$\|T\| = sup\{\|Tx\| \mid x \in X,\ \|x\| \leq 1\}.$$

The *spectrum* of an operator $T \in L(X)$ is defined as the set

$$Sp(T) = \mathbb{C} \backslash r(T),$$

where $r(T)$ is the *resolvent set* of T and consists in all complex numbers $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ is bijectiv on X.
It is an important fact that the *resolvent function* $\lambda \mapsto (\lambda I - T)^{-1}$ is an analytic function from $r(T)$ to $L(X)$ and for $\lambda \in r(T)$ we have

$$d(\lambda, r(T)) \geq \frac{1}{\left\|(\lambda I - T)^{-1}\right\|}.$$

Moreover, for $\lambda \in r(T)$, the *resolvent operator* $R(\lambda, T) \in L(X)$ is defined by the relation $R(\lambda, T) = (\lambda I - T)^{-1}$ and satisfied the *resolvent equation*

$$R(\lambda, T) - R(\mu, T) = (\mu - \lambda) R(\lambda, T) R(\mu, T),$$

for all $\lambda, \mu \in r(T)$. Therefore, in particular, $R(\lambda, T)$ and $R(\mu, T)$ commute.
We say that an infinite series of operators $\sum T_n$ is absolutely convergent if the series $\sum \|T_n\|$ is convergent in $L(X)$ and $\|\sum T_n\| \leq \sum \|T_n\|$.
If $\|T\| < 1$, then

$$(\lambda I - T)^{-1} = \sum T^n$$

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and it is absolutely convergent. A consequence of this is the fact that $r(T)$ is an open set of $\mathbb{C}$.

Theorem 1. *Theorem 1.1. Let $T \in L(X)$ be a linear bounded operator on X. Then $Sp\,(T)$ is a non-empty compact subset of C.*

The spectral radius of an operator $T \in L(X)$ is the positive number equal with $sup_{\lambda \in Sp(T)} |\lambda|$.

Theorem 2. *Let $T \in L(X)$. Then*

$$\sup_{\lambda \in Sp(T)} |\lambda| = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} .$$

Let Ω be an open neighborhood of $Sp\,(T)$ and let $H(\Omega)$ denote the space of all complex valued analytic functions defined on Ω. The application $f \mapsto f(T) : H(\Omega) \to L(X)$ defined by the relation

$$f\,(T) = \frac{1}{2\pi i} \int_{\gamma} f\,(\lambda)\, R\,(\lambda, T)\, d\lambda,$$

where γ is a contour which envelopes $Sp(T)$ in Ω, is called the holomorphic functional calculi of T.

Theorem 3. *Let $T \in L(X)$ and suppose that Ω is an open neighborhood of $Sp\,(T)$. Then, for all $f \in H(\Omega)$, we have*

$$f\,(Sp(T)) = Sp(f\,(T)).$$

We also remember that two operators $T, S \in L(X)$ are *quasinilpotent equivalent* if

$$\lim_{n \to \infty} \left\| (T-S)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n \to \infty} \left\| (S-T)^{[n]} \right\|^{\frac{1}{n}} = 0,$$

where $(T-S)^{[n]} = \sum_{k=0}^{n} (-1)^{n-k} C_k^n T^k S^{n-k}$, for any $n \in \mathbb{N}$.
The quasinilpotent equivalence relation is reflexive and symmetric. It is also transitive on $L(X)$.

Theorem 4. *Theorem 1.4. Let $T, S \in L(X)$ be two quasinilpotent equivalent operators. Then*

$$Sp\,(T) = Sp\,(S)\,.$$

2 Asymptotic equivalence and asymptotic quasinilpotent equivalence

Definition 5. *We say that two families of operators* $\{S_h\}$, $\{T_h\}$ $\subset L(X)$, *with* $h \in (0,1]$, *are asymptotic equivalent if*

$$\lim_{h\to 0} \|S_h - T_h\| = 0 \ .$$

Two families of operators $\{S_h\}$, $\{T_h\}$ $\subset L(X)$, *with* $h \in (0,1]$, *are asymptotic quasinilpotent equivalent if*

$$\lim_{n\to\infty} \limsup_{h\to 0} \left\| (S_h - T_h)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n\to\infty} \limsup_{h\to 0} \left\| (T_h - S_h)^{[n]} \right\|^{\frac{1}{n}} = 0.$$

Proposition 6. *The asymptotic (quasinilpotent) equivalence between two families of operators* $\{S_h\}$, $\{T_h\} \subset L(X)$ *is an equivalence relation (i.e. reflexive, symmetric and transitive) on* $L(X)$.

Proof. It is evidently that the asymptotic equivalence is reflexive and symmetric. Let $\{S_h\}$, $\{T_h\}$, $\{U_h\} \subset L(X)$ be families of linear bounded operators such that $\{S_h\}$, $\{T_h\}$ and $\{U_h\}$, $\{T_h\}$ are respectively asymptotic equivalent. Then

$$\begin{aligned} &\limsup_{h\to 0} \|S_h - U_h\| \\ &= \limsup_{h\to 0} \|S_h - T_h + T_h - U_h\| \le \lim_{h\to 0} \|S_h - T_h\| + \lim_{h\to 0} \|T_h - U_h\| \\ &= 0 \ . \end{aligned}$$

The asymptotic quasinilpotent equivalence is also reflexive and symmetric.
In order to prove that it is transitive, let $\{S_h\}$, $\{T_h\}$, $\{P_h\} \subset L(X)$ such that $\{T_h\}$, $\{P_h\}$ and $\{S_h\}$, $\{P_h\}$ be respectively asymptotic quasinilpotent equivalent. Then for any $\varepsilon > 0$ there exists a $n_\varepsilon \in \mathbb{N}$ such that

$$(T_h - P_h)^{[j]} < \varepsilon^j$$

and

$$(P_h - S_h)^{[n-j]} < \varepsilon^{n-j},$$

for every $j, n-j > n_\varepsilon$ and $h \in (0,1]$.
Taking

$$M_\varepsilon = \max_{1\le j\le n_\varepsilon} \left\{ \frac{\left\|(T_h - P_h)^{[j]}\right\|}{\varepsilon^j}, \frac{\left\|(P_h - S_h)^{[j]}\right\|}{\varepsilon^j}, 1 \right\}$$

we obtain

$$\left\|(T_h - P_h)^{[j]}\right\| < \varepsilon^j M_\varepsilon$$

and

$$\left\|(P_h - S_h)^{[j]}\right\| < \varepsilon^j M_\varepsilon,$$

for every $j \in \mathbb{N}$ and $h \in (0,1]$.
In view of above inequality and the following equality

$$(T-S)^{[n]} = \sum_{k=0}^{n} (-1)^{n-k} C_n^k (T-P)^{[k]} (P-S)^{[n-k]},$$

for every $n \in \mathbb{N}$ and $P \in L(X)$, it results that

$$\begin{aligned} \left\|(T_h - S_h)^{[n]}\right\| &\leq \sum_{k=0}^{n} C_n^k \left\|(T_h - P_h)^{[k]}\right\| \left\|(P_h - S_h)^{[n-k]}\right\| \\ &\leq \sum_{k=0}^{n} C_n^k \varepsilon^k \varepsilon^{n-k} {M_\varepsilon}^2 \\ &= (2\varepsilon)^n {M_\varepsilon}^2, \end{aligned}$$

for every $n \in \mathbb{N}$ and $h \in (0,1]$.
Therefore

$$\limsup_{h\to 0} \left\|(T_h - S_h)^{[n]}\right\| \leq (2\varepsilon)^n {M_\varepsilon}^2$$

and thus

$$\limsup_{h\to 0} \left\|(T_h - S_h)^{[n]}\right\|^{\frac{1}{n}} \leq 2\varepsilon {M_\varepsilon}^{2/n}.$$

Consequently

$$\lim_{n\to\infty} \limsup_{h\to 0} \left\|(T_h - S_h)^{[n]}\right\|^{\frac{1}{n}} \leq 2\varepsilon,$$

for any $\varepsilon > 0$.
Analogously we prove that $\lim_{n\to\infty} \limsup_{h\to 0} \left\|(S_h - T_h)^{[n]}\right\|^{\frac{1}{n}} = 0.$ □

Proposition 7. *Let* $\{S_h\}$, $\{T_h\} \subset L(X)$ *be asymptotic equivalent.*

i) *If* $\{S_h\}$ *is a bounded family of operators, then* $\{T_h\}$ *is also bounded and conversely;*

ii) $\{S_h\}$, $\{T_h\}$ *are asymptotic commuting (i.e.* $\lim_{h\to 0} \|S_h T_h - T_h S_h\| = 0$ *);*

iii) *Let* $\{U_h\} \subset L(X)$ *be a bounded family of operators such that*

$$\lim_{h\to 0} \|S_h U_h - U_h S_h\| = 0.$$

Then $\lim_{h\to 0} \|U_h T_h - T_h U_h\| = 0$.

Proof. i) If $\{S_h\}$ is a bounded family of operators, then there is $\limsup_{h\to 0}\|S_h\| < \infty$. Since

$$\lim_{h\to 0}\|S_h - T_h\| = 0 \ ,$$

it follows that

$$\limsup_{h\to 0}\|T_h\| = \limsup_{h\to 0}\|T_h - S_h + S_h\| \\ \leq \lim_{h\to 0}\|S_h - T_h\| + \limsup_{h\to 0}\|S_h\| < \infty.$$

Therefore $\{T_h\}$ is a bounded family of operators.
Analogously we can prove that if $\{T_h\}$ is a bounded family of operators, than $\{S_h\}$ is a bounded family of operators.

ii)

$$\limsup_{h\to 0}\|S_hT_h - T_hS_h\| = \limsup_{h\to 0}\left\|S_hT_h - {S_h}^2 + {S_h}^2 - T_hS_h\right\| \leq \\ \limsup_{h\to 0}\|S_h\left(S_h - T_h\right)\| + \limsup_{h\to 0}\|\left(S_h - T_h\right)S_h\| \leq \\ 2\limsup_{h\to 0}\|S_h\|\,\|S_h - T_h\| \leq 0.$$

iii)

$$\limsup_{h\to 0}\|T_hU_h - U_hT_h\| = \\ \limsup_{h\to 0}\|T_hU_h - S_hU_h + S_hU_h - U_hS_h + U_hS_h - U_hT_h\| \leq \\ \limsup_{h\to 0}\|T_hU_h - S_hU_h\| + \limsup_{h\to 0}\|S_hU_h - U_hS_h\| + \limsup_{h\to 0}\|U_hS_h - U_hT_h\| \leq \\ 2\limsup_{h\to 0}\|U_h\|\,\|T_h - S_h\|\,.$$

Since $\{U_h\}$ is a bounded family of operators, then there is $\limsup_{h\to 0}\|U_h\| < \infty$. So

$$\lim_{h\to 0}\|U_hT_h - T_hU_h\| = 0.$$

□

Proposition 8. *Let $\{S_h\}\,,\ \{T_h\}\subset L(X)$ be two bounded families of operators such that $lim_{h\to 0}\|S_hT_h - T_hS_h\| = 0$. Then*

i) *$lim_{h\to 0}\|S_h^nT_h^m - T_h^mS_h^n\| = 0$, for any $n, m\in\mathbb{N}$;*

ii) *$lim_{h\to 0}\left\|(S_h - T_h)^{[n]}\right\| = lim_{h\to 0}\|(S_h - T_h)^n\|$, for any $n\in\mathbb{N}$;*

iii) $lim_{h\to 0}\left\|(S_hT_h)^n - S_h^nT_h^n\right\| \ = 0$, *for any* $n \in \mathbb{N}$.

Proof. i) We prove that $\lim_{h\to 0}\left\|S_h^nT_h - T_hS_h^n\right\| \ = 0$, for any $n \in \mathbb{N}$. For $n = 2$ we have

$$\limsup_{h\to 0}\left\|S_h^2T_h - T_hS_h^2\right\| \ = \\ \limsup_{h\to 0}\left\|S_h\left(S_hT_h\right) - S_h\left(T_hS_h\right) + \left(S_hT_h\right)S_h - (T_hS_h)S_h\right\| \ \le \\ 2\limsup_{h\to 0}\left\|(S_hT_h) - (T_hS_h)\right\|\left\|S_h\right\| \ = 0.$$

For $n = 3$ we have

$$\limsup_{h\to 0}\left\|S_h^3T_h - T_hS_h^3\right\| = \\ \limsup_{h\to 0}\left\|S_h\left(S_h^2T_h\right) - S_h\left(T_hS_h^2\right) + \left(S_hT_h\right)S_h^2 - (T_hS_h)S_h^2\right\| \ \le \\ \limsup_{h\to 0}\left\|S_h^2T_h - T_hS_h^2\right\|\left\|S_h\right\| \ + \lim_{h\to 0}\left\|S_hT_h - T_hS_h\right\|\left\|S_h^2\right\| \ = 0.$$

Considering relation $\lim_{h\to 0}\left\|S_h^nT_h - T_hS_h^n\right\| \ = 0$ true we prove that

$$\lim_{h\to 0}\left\|S_h^{n+1}T_h - T_hS_h^{n+1}\right\| \ = 0.$$

$$\limsup_{h\to 0}\left\|S_h^{n+1}T_h - T_hS_h^{n+1}\right\| \ = \\ \limsup_{h\to 0}\left\|S_h\left(S_h^nT_h\right) - S_h\left(T_hS_h^n\right) + \left(S_hT_h\right)S_h^n - (T_hS_h)S_h^n\right\| \ \le \\ \le \limsup_{h\to 0}\left\|S_h^nT_h - T_hS_h^n\right\|\left\|S_h\right\| \ + \limsup_{h\to 0}\left\|S_hT_h - T_hS_h\right\|\left\|S_h^n\right\| \ = 0.$$

Applying above relation to S_h^n and T_h, it follows that

$$\lim_{h\to 0}\left\|S_h^nT_h^m - T_h^mS_h^n\right\| \ = 0,$$

for every $n, m \in \mathbb{N}$.

ii) and iii) can be proved analogously i). □

Proposition 9. *Let* $\{S_h\}, \ \{T_h\} \ \subset L(X)$ *be two bounded families of operators.*

i) *If* $\{S_h\}, \ \{T_h\}$ *are asymptotic equivalent, then are asymptotic quasinilpotent equivalent.*

ii) *If* $\lim_{h\to 0}\|S_hT_h - T_hS_h\| = 0$ *and* $\lim_{n\to\infty}\limsup_{h\to 0}\left\|(S_h - T_h)^{[n]}\right\|^{\frac{1}{n}} = 0$, *then* $\{S_h\}$, $\{T_h\}$ *are asymptotic quasinilpotent equivalent, i.e.*

$$\lim_{n\to\infty}\limsup_{h\to 0}\left\|(T_h - S_h)^{[n]}\right\|^{\frac{1}{n}} = 0.$$

iii) *Let* $\{A_h\} \subset L(X)$ *be a bounded families of operators. If* $\{S_h\}$, $\{T_h\}$ *are asymptotic quasinilpotent equivalent and* $\lim_{h\to 0}\|S_hA_h - A_hS_h\| = 0$, *then it is not necessary that* $\lim_{h\to 0}\|T_hA_h - A_hT_h\| = 0$.

Proof. i) We prove that

$$\lim_{h\to 0}\left\|(S_h - T_h)^{[n]}\right\| = \lim_{h\to 0}\left\|(T_h - S_h)^{[n]}\right\| = 0,$$

for any $n \in \mathbb{N}$.

Since $(T-S)^{[n+1]} = T(T-S)^{[n]} - (T-S)^{[n]}S$, for any $n \in \mathbb{N}$, taking $n = 2$, it follows that

$$\limsup_{h\to 0}\left\|(T_h - S_h)^{[2]}\right\| = \limsup_{h\to 0}\|T_h(T_h - S_h) - (T_h - S_h)S_h\| \leq$$
$$\limsup_{h\to 0}\|T_h(T_h - S_h)\| + \limsup_{h\to 0}\|(T_h - S_h)S_h\| \leq$$
$$\limsup_{h\to 0}\|T_h\|\,\|(T_h - S_h)\| + \limsup_{h\to 0}\|(T_h - S_h)\|\ \|S_h\| \leq 0.$$

By induction, we prove that if $\lim_{h\to 0}\left\|(T_h - S_h)^{[n]}\right\| = 0$, then

$$\lim_{h\to 0}\left\|(T_h - S_h)^{[n+1]}\right\| = 0$$

$$\limsup_{h\to 0}\left\|(T_h - S_h)^{[n+1]}\right\| =$$
$$\limsup_{h\to 0}\left\|T_h(T_h - S_h)^{[n]} - (T_h - S_h)^{[n]}S_h\right\| \leq$$
$$\limsup_{h\to 0}\left\|T_h(T_h - S_h)^{[n]}\right\| + \limsup_{h\to 0}\left\|(T_h - S_h)^{[n]}S_h\right\| \leq$$
$$\limsup_{h\to 0}\|T_h\|\left\|(T_h - S_h)^{[n]}\right\| + \limsup_{h\to 0}\left\|(T_h - S_h)^{[n]}\right\|\ \|S_h\| \leq 0.$$

Similarly we can show that $\lim_{h\to 0}\left\|(S_h - T_h)^{[n]}\right\| = 0$, for any $n \in \mathbb{N}$.

When $n \to \infty$, we obtain

$$\lim_{n\to\infty}\limsup_{h\to 0}\left\|(S_h - T_h)^{[n]}\right\|^{\frac{1}{n}} = \lim_{n\to\infty}\limsup_{h\to 0}\left\|(T_h - S_h)^{[n]}\right\|^{\frac{1}{n}} = 0.$$

ii) We remember that for any two bounded linear operators T and S, we have

$$(T-S)^{[n]}=\sum_{k=0}^{n}(-1)^{n-k}C_k^nT^kS^{n-k}=(S-T)^{[n]}+\sum_{k=0}^{n-1}(-1)^{n-1-k}C_k^n(T^kS^{n-k}-S^{n-k}T^k),$$

where $n\in\mathbb{N}$.

Applying above relation to S_h şi T_h, when $h\to 0$, we obtain

$$\begin{aligned}&\limsup_{h\to 0}\left\|(T_h-S_h)^{[n]}\right\| =\\ &\limsup_{h\to 0}\left\|(S_h-T_h)^{[n]}-\sum_{k=0}^{n-1}(-1)^{n-1-k}C_k^n({T_h}^k{S_h}^{n-k}-{S_h}^{n-k}{T_h}^k)\right\| \le\\ &\limsup_{h\to 0}\left\|(S_h-T_h)^{[n]}\right\| +\limsup_{h\to 0}\left\|\sum_{k=0}^{n-1}(-1)^{n-1-k}C_k^n({T_h}^k{S_h}^{n-k}-{S_h}^{n-k}{T_h}^k)\right\| \le\\ &\limsup_{h\to 0}\left\|(S_h-T_h)^{[n]}\right\| +\sum_{k=0}^{n-1}C_k^n\limsup_{h\to 0}\left\|{T_h}^k{S_h}^{n-k}-{S_h}^{n-k}{T_h}^k\right\|.\end{aligned}$$

In view of Proposition 8 ii), it follows

$$\limsup_{h\to 0}\left\|(T_h-S_h)^{[n]}\right\| \le\limsup_{h\to 0}\left\|(S_h-T_h)^{[n]}\right\|,$$

for any $n\in\mathbb{N}$.

Analogously we can prove that $\limsup_{h\to 0}\left\|(S_h-T_h)^{[n]}\right\| \le\limsup_{h\to 0}\left\|(T_h-S_h)^{[n]}\right\|$.

iii) We suppose that the relation $\lim_{h\to 0}\|T_hA_h-A_hT_h\| =0$ is true. Then, taking $A_h=S_h$, for any $h\in(0,1]$, since

$$\lim_{h\to 0}\left\|{S_h}^2-{S_h}^2\right\| =0,$$

it follows

$$\lim_{h\to 0}\|S_hT_h-T_hS_h\| =0,$$

fact that is not true. □

Proposition 10. *Let $\{S_h\}$, $\{T_h\}$ $\subset L(X)$ be two asymptotic quasinilpotent equivalent families and $\{A_h\}$ $\subset L(X)$ a bounded family. Then*

i) *The families $\{S_h+A_h\}$, $\{T_h+A_h\}$ are asymptotic quasinilpotent equivalent;*

ii) *If $\{A_h\}$ $\subset L(X)$ is a bounded family such that $\lim_{h\to 0}\|S_hA_h-A_hS_h\|=0$ and $\lim_{h\to 0}\|T_hA_h-A_hT_h\| =0$, the families $\{S_hA_h\}$, $\{T_hA_h\}$ are asymptotic quasinilpotent equivalent.*

Proof. i) Since $\{S_h\}$, $\{T_h\}$ are asymptotic quasinilpotent equivalent, i.e.

$$\begin{aligned}\lim_{n\to\infty}\limsup_{h\to 0}\left\|(T_h-S_h)^{[n]}\right\|^{\frac{1}{n}} &= \lim_{n\to\infty}\limsup_{h\to 0}\left\|(S_h-T_h)^{[n]}\right\|^{\frac{1}{n}}\\ &= 0,\end{aligned}$$

then

$$\begin{aligned}&\lim_{n\to\infty}\limsup_{h\to 0}\left\|((T_h+A_h)-(S_h+A_h))^{[n]}\right\|^{\frac{1}{n}}\\ &\qquad = \lim_{n\to\infty}\limsup_{h\to 0}\left\|((S_h+A_h)-(T_h+A_h))^{[n]}\right\|^{\frac{1}{n}}\\ &\qquad = 0,\end{aligned}$$

so $\{S_h+A_h\}$, $\{T_h+A_h\}$ are asymptotic quasinilpotent equivalent.

ii) Since $\lim_{h\to 0}\|S_hA_h-A_hS_h\| = 0$ and $\lim_{h\to 0}\|T_hA_h-A_hT_h\| = 0$, taking into account Proposition 9, it follows

$$\begin{aligned}&\limsup_{h\to 0}\left\|(T_hA_h-S_hA_h)^{[n]}-(T_h-S_h)^{[n]}{A_h}^n\right\| =\\ &\limsup_{h\to 0}\left\|\sum_{k=0}^{n}(-1)^{n-k}C_n^k(T_hA_h)^k(S_hA_h)^{n-k}-\sum_{k=0}^{n}(-1)^{n-k}C_n^k{T_h}^k{S_h}^{n-k}{A_h}^n\right\| =\\ &\limsup_{h\to 0}\left\|\sum_{k=0}^{n}(-1)^{n-k}C_n^k((T_hA_h)^k(S_hA_h)^{n-k}-{T_h}^k{S_h}^{n-k}{A_h}^k{A_h}^{n-k})\right\| \le\\ &\sum_{k=0}^{n}C_n^k\ \limsup_{h\to 0}\|\,(T_hA_h)^k\,(S_hA_h)^{n-k}-{T_h}^k{A_h}^k{S_h}^{n-k}{A_h}^{n-k}+\\ &\qquad {T_h}^k{A_h}^k{S_h}^{n-k}{A_h}^{n-k}-{T_h}^k{S_h}^{n-k}{A_h}^k{A_h}^{n-k}\| \le\\ &\sum_{k=0}^{n}C_n^k\ \limsup_{h\to 0}\left\|(T_hA_h)^k(S_hA_h)^{n-k}-{T_h}^k{A_h}^k{S_h}^{n-k}{A_h}^{n-k}\right\|+\\ &\sum_{k=0}^{n}C_n^k\ \limsup_{h\to 0}\left\|{T_h}^k{A_h}^k{S_h}^{n-k}{A_h}^{n-k}-{T_h}^k{S_h}^{n-k}{A_h}^k{A_h}^{n-k}\right\| \le\\ &\sum_{k=0}^{n}C_n^k\ \limsup_{h\to 0}\|(T_hA_h)^k(S_hA_h)^{n-k}-(T_hA_h)^k{S_h}^{n-k}{A_h}^{n-k}+\\ &\qquad (T_hA_h)^k{S_h}^{n-k}{A_h}^{n-k}-{T_h}^k{A_h}^k{S_h}^{n-k}{A_h}^{n-k}\|+\\ &\sum_{k=0}^{n}C_n^k\ \limsup_{h\to 0}\left\|{T_h}^k\right\|\left\|{A_h}^k{S_h}^{n-k}-{S_h}^{n-k}{A_h}^k)\right\|\left\|{A_h}^{n-k}\right\| \le\end{aligned}$$

$$\leq \sum_{k=0}^{n} C_n^k \limsup_{h\to 0} \left\| (T_hA_h)^k (S_hA_h)^{n-k} - (T_hA_h)^k {S_h}^{n-k} {A_h}^{n-k} \right\| +$$
$$\sum_{k=0}^{n} C_n^k \limsup_{h\to 0} \left\| (T_hA_h)^k {S_h}^{n-k} {A_h}^{n-k} - {T_h}^k {A_h}^k {S_h}^{n-k} {A_h}^{n-k} \right\| \leq$$
$$\sum_{k=0}^{n} C_n^k \limsup_{h\to 0} \left\| (T_hA_h)^k \right\| \left\| (S_hA_h)^{n-k} - {S_h}^{n-k} {A_h}^{n-k} \right\| +$$
$$\sum_{k=0}^{n} C_n^k \limsup_{h\to 0} \left\| (T_hA_h)^k - {T_h}^k {A_h}^k {S_h}^{n-k} {A_h}^{n-k} \right\| \left\| {S_h}^{n-k} \right\| \left\| {A_h}^{n-k} \right\| = 0.$$

Having in view that $\{S_h\}$, $\{T_h\}$ are asymptotic quasinilpotent equivalent and taking into account the above relation, it results

$$\begin{aligned} & \lim_{n\to\infty} \limsup_{h\to 0} \left\| (T_hA_h - S_hA_h)^{[n]} \right\|^{\frac{1}{n}} \\ = \;& \lim_{n\to\infty} \limsup_{h\to 0} \left\| (T_h - S_h)^{[n]} {A_h}^n \right\|^{\frac{1}{n}} \\ \leq \;& \lim_{n\to\infty} \left(\limsup_{h\to 0} \left\| (T_h - S_h)^{[n]} \right\| \left\| {A_h}^n \right\| \right)^{\frac{1}{n}} \\ \leq \;& \lim_{n\to\infty} \lim_{h\to 0} \left\| (T_h - S_h)^{[n]} \right\|^{\frac{1}{n}} \limsup_{h\to 0} \left\| A_h \right\| = 0. \end{aligned}$$

Analogously we can prove that $\lim_{n\to\infty} \limsup_{h\to 0} \left\| (S_hA_h - T_hA_h)^{[n]} \right\|^{\frac{1}{n}} = 0.$ □

3 Spectrum of a family of operators

Let be the sets

$$C_b\left((0,1],\ B\left(X\right)\right) =$$
$$\{\varphi : (0,1] \to B\left(X\right) |\, \varphi\left(h\right) = T_h \text{ such that } \varphi \textit{ is } \text{countinous and bounded}\} =$$
$$\left\{ \{T_h\}_{h\in(0,1]} \subset B(X) \middle| \{T_h\}_{h\in(0,1]} \text{ is a bounded family, i.e. } \underset{h\in(0,1]}{sup} \|T_h\| < \infty \right\}$$

and

$$C_0\left((0,1],\ B\left(X\right)\right) = \left\{ \varphi \in C_b\left((0,1],\ B\left(X\right)\right) |\, \lim_{h\to 0} \|\varphi(h)\| = 0 \right\} =$$
$$\left\{ \{T_h\}_{h\in(0,1]} \subset B(X) \middle| \lim_{h\to 0} \|T_h\| = 0 \right\}.$$

$C_b((0,1],\ B(X))$ is a Banach algebra non-commutative with norm

$$\|\{T_h\}\| = sup_{h\in(0,1]}\|T_h\|,$$

and $C_0((0,1],\ B(X))$ is a closed bilateral ideal of $C_b((0,1],\ B(X))$. Therefore the quotient algebra $C_b((0,1],\ B(X))/C_0((0,1],\ B(X))$, which will be called from now B_∞, is also a Banach algebra with quotient norm

$$\left\|\{\dot{T_h}\}\right\| = inf_{\{U_h\}_{h\in(0,1]}\in C_0((0,1],\ B(X))}\|\{T_h\}+\{U_h\}\| = inf_{\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}}\|\{S_h\}\|.$$

Then

$$\left\|\{\dot{T_h}\}\right\| = inf_{\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}}\|\{S_h\}\| \le \|\{S_h\}\| = sup_{h\in(0,1]}\|S_h\|,$$

for any $\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}$. Moreover,

$$\left\|\{\dot{T_h}\}\right\| = inf_{\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}}\|\{S_h\}\| = inf_{\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}} sup_{h\in(0,1]}\|S_h\|.$$

If two bounded families $\{T_h\}_{h\in(0,1]}$, $\{S_h\}_{h\in(0,1]}\subset B(X)$ are asymptotically equivalent, then $\lim_{h\to0}\|S_h-T_h\|=0$, i.e. $\{T_h-S_h\}_{h\in(0,1]}\in C_0((0,1],\ B(X))$.
Let $\{T_h\}_{h\in(0,1]}$, $\{S_h\}_{h\in(0,1]}\in C_b((0,1],\ B(X))$ be asymptotically equivalent. Then

$$\limsup_{h\to0}\|S_h\| = \limsup_{h\to0}\|T_h\|.$$

Since

$$\limsup_{h\to0}\|S_h\| \le \sup_{h\in(0,1]}\|S_h\|,$$

results that

$$\limsup_{h\to0}\|S_h\| = inf_{\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}}\limsup_{h\to0}\|S_h\| \le$$
$$inf_{\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}}\sup_{h\in(0,1]}\|S_h\| = \left\|\{\dot{T_h}\}\right\|,$$

for any $\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}$.
In particular

$$\lim_{h\to0}\lim\|T_h\| \le \left\|\{\dot{T_h}\}\right\| \le \|\{T_h\}\| = \sup_{h\in(0,1]}\|T_h\|.$$

Definition 11. *We call the resolvent set of a family of operators*

$$\{S_h\} \in C_b((0,1],\ B(X))$$

the set

$$r(\{S_h\}) = \{\lambda\in\mathbb{C}|\,\exists\{\mathcal{R}(\lambda,S_h)\}\in C_b((0,1],\ B(X)),\ \lim_{h\to0}\|(\lambda I-S_h)\mathcal{R}(\lambda,S_h)-I\| =$$
$$\lim_{h\to0}\|\mathcal{R}(\lambda,S_h)(\lambda I-S_h)-I\| = 0\}$$

We call the spectrum of a family of operators $\{S_h\}\in C_b((0,1],\ B(X))$ *the set*

$$Sp(\{S_h\}) = \mathbb{C}\backslash r(\{S_h\}).$$

Remark 12. **i)** *If* $\lambda \in r(S_h)$ *for any* $h \in (0,1]$*, then* $\lambda \in r(\{S_h\})$. *Therefore* $\bigcap_{h\in(0,1]} r(S_h) \subseteq r(\{S_h\})$*;*

ii) *If* $\lambda \in Sp(\{S_h\})$*, then* $|\lambda| \leq \limsup_{h\to 0} \|S_h\|\}$ *;*

iii) *If* $\|S_h\| < |\lambda|$ *for any* $h \in (0,1]$*, then* $\lambda \in r(\{S_h\})$*;*

iv) $r(\{S_h\})$ *is an open set of* C *and* $Sp(\{S_h\})$ *is a compact set of* C*.*

Proof. iv) Let $\lambda \in r(\{S_h\})$. From Definition 11, it follows that there is $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0,1], B(X))$ such that

$$\lim_{h\to 0} \|(\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I\| = \lim_{h\to 0} \|\mathcal{R}(\lambda, S_h)(\lambda I - S_h) - I\| = 0.$$

Let $\mu \in D(\lambda, \frac{1}{\limsup_{h\to 0}\|\mathcal{R}(\lambda,S_h)\|})$. So

$$|\lambda - \mu| < \frac{1}{\limsup_{h\to 0} \|\mathcal{R}(\lambda, S_h)\|}.$$

According to ii), it follows $1 \in r(\{(\lambda-\mu)\mathcal{R}(\lambda, S_h)\})$, therefore there is

$$\{\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h))\} \in C_b((0,1], B(X))$$

such that

$$\lim_{h\to 0} \|(I - (\lambda-\mu)\mathcal{R}(\lambda, S_h))\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h)) - I\| =$$
$$\lim_{h\to 0} \|\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h))(I - (\lambda-\mu)\mathcal{R}(\lambda, S_h)) - I\| = 0.$$

Having in view the above relation, it results

$$\limsup_{h\to 0} \|(\mu I - S_h)\mathcal{R}(\lambda, S_h)\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h)) - I\| =$$
$$\limsup_{h\to 0} \|(\lambda I - S_h)\mathcal{R}(\lambda, S_h)\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h)) -$$
$$(\lambda-\mu)\mathcal{R}(\lambda, S_h)\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h)) - I\| =$$
$$\limsup_{h\to 0} \|((\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I)\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h)) + \mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h)) -$$
$$(\lambda-\mu)\mathcal{R}(\lambda, S_h)\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h)) - I\| \leq$$
$$\limsup_{h\to 0} \|((\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I)\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h))\| +$$
$$\limsup_{h\to 0} \|\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h)) - (\lambda-\mu)\mathcal{R}(\lambda, S_h)\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h)) - I\| \leq$$
$$\limsup_{h\to 0} \|((\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I)\| \|\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h))\| +$$
$$\lim_{h\to 0} \|(I - (\lambda-\mu)\mathcal{R}(\lambda, S_h))\mathcal{R}(1, (\lambda-\mu)\mathcal{R}(\lambda, S_h)) - I\| = 0,$$

so $\mu \in r(\{S_h\})$, for every $\mu \in D(\lambda, \frac{1}{\limsup_{h\to 0}\|\mathcal{R}(\lambda,S_h)\|})$.

Therefore, for any $\lambda \in r(\{S_h\})$, there is an open disk $D(\lambda, \frac{1}{\limsup_{h\to 0}\|\mathcal{R}(\lambda,S_h)\|})$ such that $D(\lambda, \frac{1}{\limsup_{h\to 0}\|\mathcal{R}(\lambda,S_h)\|}) \subset r(\{S_h\})$.

If $\{S_h\}$ is a bounded family, from ii) we have

$$|\lambda| \leq \limsup_{h\to 0}\|S_h\| < \infty,$$

for any $\lambda \in Sp(\{S_h\})$, so $Sp(\{S_h\})$ is a compact set. □

Proposition 13. *Let* $\{S_h\} \in C_b((0,1], B(X))$ *be a family of operators and* $\lambda \in r(\{S_h\})$. *Then, for any* $\{\mathcal{R}(\lambda,S_h)\} \in C_b((0,1], B(X))$ *such that*

$$\mathop{lim}_{h\to 0}\|(\lambda I - S_h)\mathcal{R}(\lambda,S_h) - I\| = \lim_{h\to 0}\|\mathcal{R}(\lambda,S_h)(\lambda I - S_h) - I\| = 0,$$

we have

$$\lim_{h\to 0}\|S_h\mathcal{R}(\lambda,S_h) - \mathcal{R}(\lambda,S_h)S_h\| = 0.$$

Proof. Let $\lambda \in r(\{S_h\})$ and $\{\mathcal{R}(\lambda,S_h)\} \in C_b((0,1], B(X))$ such that $\lim_{h\to 0}\|(\lambda I - S_h)\mathcal{R}(\lambda,S_h) - I\| = \lim_{h\to 0}\|\mathcal{R}(\lambda,S_h)(\lambda I - S_h) - I\| = 0.$

Using this relation we have

$$\limsup_{h\to 0}\|S_h\mathcal{R}(\lambda,S_h) - \mathcal{R}(\lambda,S_h)S_h\| =$$
$$\limsup_{h\to 0}\|\mathcal{R}(\lambda,S_h)(\lambda I - S_h) - (\lambda I - S_h)\mathcal{R}(\lambda,S_h)\| =$$
$$\limsup_{h\to 0}\|\mathcal{R}(\lambda,S_h)(\lambda I - S_h) - I + I - (\lambda I - S_h)\mathcal{R}(\lambda,S_h)\| \leq$$
$$\lim_{h\to 0}\|(\lambda I - S_h)\mathcal{R}(\lambda,S_h) - I\| + \lim_{h\to 0}\|\mathcal{R}(\lambda,S_h)(\lambda I - S_h) - I\| = 0$$

□

Proposition 14. *(resolvent equation - asymptotic) Let* $\{S_h\} \in C_b((0,1], B(X))$ *be a bounded family and* $\lambda, \mu \in r(\{S_h\})$. *Then*

$$\lim_{h\to 0}\|\mathcal{R}(\lambda,S_h) - \mathcal{R}(\mu,S_h) - (\mu - \lambda)\mathcal{R}(\lambda,S_h)\mathcal{R}(\mu,S_h)\| = 0.$$

Proof. Since $\{\mathcal{R}(\lambda,S_h)\}$ and $\{\mathcal{R}(\lambda,S_h)\}$ are bounded, we have

$$\limsup_{h\to 0}\|\mathcal{R}(\lambda,S_h) - \mathcal{R}(\mu,S_h) - (\mu - \lambda)\mathcal{R}(\lambda,S_h)R(\mu,S_h)\| =$$
$$\limsup_{h\to 0}\|\mathcal{R}(\lambda,S_h)(I - \mu\mathcal{R}(\mu,S_h)) - (I - \lambda\mathcal{R}(\lambda,S_h))\mathcal{R}(\mu,S_h)\| =$$
$$\limsup_{h\to 0}\|\mathcal{R}(\lambda,S_h)(I - (\mu I - S_h)\mathcal{R}(\mu,S_h)) - (I - \mathcal{R}(\lambda,S_h)(\lambda I - S_h))\mathcal{R}(\mu,S_h)\| \leq$$

$$\leq \limsup_{h\to 0} \|\mathcal{R}(\lambda, S_h)(I-(\mu I - S_h)\mathcal{R}(\mu, S_h))\| +$$
$$\limsup_{h\to 0} \|(I-\mathcal{R}(\lambda, S_h)(\lambda I - S_h))\mathcal{R}(\mu, S_h)\| \leq$$
$$\limsup_{h\to 0} \|\mathcal{R}(\lambda, S_h)\| \, \|I-(\mu I - S_h)\mathcal{R}(\mu, S_h)\| +$$
$$\limsup_{h\to 0} \|I-\mathcal{R}(\lambda, S_h)(\lambda I - S_h)\| \, \|\mathcal{R}(\mu, S_h)\| \leq 0.$$

□

Corollary 15. *Let $\{S_h\} \in C_b((0,1], B(X))$ be a bounded family and $\lambda, \mu \in r(\{S_h\})$ be not-equal. Then*
$$\lim_{h\to 0} \|\mathcal{R}(\lambda, S_h)\mathcal{R}(\mu, S_h) - \mathcal{R}(\mu, S_h)\mathcal{R}(\lambda, S_h)\| = 0.$$

Proof.
$$\limsup_{h\to 0} \|\mathcal{R}(\lambda, S_h)\mathcal{R}(\mu, S_h) - \mathcal{R}(\mu, S_h)\mathcal{R}(\lambda, S_h)\| =$$
$$\frac{1}{|\lambda-\mu|}\limsup_{h\to 0} \|(\lambda-\mu)\mathcal{R}(\lambda, S_h)\mathcal{R}(\mu, S_h) + (\mu-\lambda)\mathcal{R}(\mu, S_h)\mathcal{R}(\lambda, S_h)\| =$$
$$\frac{1}{|\lambda-\mu|}\limsup_{h\to 0} \| [\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h) - (\mu-\lambda)\mathcal{R}(\lambda, S_h)\mathcal{R}(\mu, S_h)] +$$
$$[\mathcal{R}(\mu, S_h) - \mathcal{R}(\lambda, S_h) - (\lambda-\mu)\mathcal{R}(\mu, S_h)\mathcal{R}(\lambda, S_h)] \| \leq$$
$$\frac{1}{|\lambda-\mu|}\lim_{h\to 0} \|[\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h) - (\mu-\lambda)\mathcal{R}(\lambda, S_h)\mathcal{R}(\mu, S_h)]\| +$$
$$\frac{1}{|\lambda-\mu|}\lim_{h\to 0} \|[\mathcal{R}(\mu, S_h) - \mathcal{R}(\lambda, S_h) - (\lambda-\mu)\mathcal{R}(\mu, S_h)\mathcal{R}(\lambda, S_h)]\| = 0.$$

□

Proposition 16. *Let $\{S_h\} \in C_b((0,1], B(X))$ be a bounded family. If $\lambda \in r(\{S_h\})$ and $\{\mathcal{R}_i(\lambda, S_h)\} \in C_b((0,1], B(X))$, $i = \overline{1,2}$ such that*
$$\lim_{h\to 0} \|(\lambda I - S_h)\mathcal{R}_i(\lambda, S_h) - I\| = \lim_{h\to 0} \|\mathcal{R}_i(\lambda, S_h)(\lambda I - S_h) - I\| = 0$$
for $i = \overline{1,2}$, then
$$\lim_{h\to 0} \|\mathcal{R}_1(\lambda, S_h) - \mathcal{R}_2(\lambda, S_h)\| = 0.$$

Proof. Let $\lambda \in r(\{S_h\})$ and $\{\mathcal{R}_i(\lambda, S_h)\} \in C_b((0,1], B(X))$, $i = \overline{1,2}$, such that
$$\lim_{h\to 0} \|(\lambda I - S_h)\mathcal{R}_i(\lambda, S_h) - I\| = \lim_{h\to 0} \|\mathcal{R}_i(\lambda, S_h)(\lambda I - S_h) - I\| = 0$$
Therefore
$$\limsup_{h\to 0} \|\mathcal{R}_1(\lambda, S_h) - \mathcal{R}_2(\lambda, S_h)\| =$$

$$\limsup_{h\to 0}\|\mathcal{R}_1(\lambda,S_h)-\mathcal{R}_2(\lambda,S_h)-(\lambda-\lambda)\mathcal{R}_1(\lambda,S_h)\mathcal{R}_2(\lambda,S_h)\| =$$

$$\limsup_{h\to 0}\|\mathcal{R}_1(\lambda,S_h)(I-\lambda\mathcal{R}_2(\lambda,S_h))-(I-\lambda\mathcal{R}_1(\lambda,S_h))\mathcal{R}_2(\lambda,S_h)\| =$$

$$\limsup_{h\to 0}\|\mathcal{R}_1(\lambda,S_h)(I-(\lambda I-S_h)\mathcal{R}_2(\lambda,S_h))-(I-\mathcal{R}_1(\lambda,S_h)(\lambda I-S_h))\mathcal{R}_2(\lambda,S_h)\| \leq$$

$$\limsup_{h\to 0}\|\mathcal{R}_1(\lambda,S_h)(I-(\lambda I-S_h)\mathcal{R}_2(\lambda,S_h))\|+$$

$$\limsup_{h\to 0}\|(I-\mathcal{R}_1(\lambda,S_h)(\lambda I-S_h))\mathcal{R}_2(\lambda,S_h)\| \leq$$

$$\limsup_{h\to 0}\|\mathcal{R}_1(\lambda,S_h)\|\,\|I-(\lambda I-S_h)\mathcal{R}_2(\lambda,S_h)\|+$$

$$+\limsup_{h\to 0}\|I-\mathcal{R}_1(\lambda,S_h)(\lambda I-S_h)\|\,\|\mathcal{R}_2(\lambda,S_h)\| \leq 0$$

□

Proposition 17. *Let* $\{S_h\} \in C_b((0,1], B(X))$ *be a bounded family,* $\lambda\in r(\{S_h\})$ *and* $\{\mathcal{R}(\lambda,S_h)\}\in C_b((0,1], B(X))$ *such that*

$$\mathit{lim}_{h\to 0}\|(\lambda I-S_h)\mathcal{R}(\lambda,S_h)-I\| = \mathit{lim}_{h\to 0}\|\mathcal{R}(\lambda,S_h)(\lambda I-S_h)-I\| = 0.$$

If $\{R_h\} \in C_b((0,1], B(X))$ *is a bounded family such that it is asymptotic equivalent with* $\{\mathcal{R}(\lambda,S_h)\}\in C_b((0,1], B(X))$*, then*

$$\lim_{h\to 0}\|(\lambda I-S_h)R_h-I\| = \lim_{h\to 0}\|R_h(\lambda I-S_h)-I\| = 0.$$

Proof. Let $\lambda\in r(\{S_h\})$. It results

$$\limsup_{h\to 0}\|(\lambda I-S_h)R_h-I\| =$$

$$= \limsup_{h\to 0}\|(\lambda I-S_h)R_h-(\lambda I-S_h)\mathcal{R}(\lambda,S_h)+(\lambda I-S_h)\mathcal{R}(\lambda,S_h)-I\| \leq$$

$$\limsup_{h\to 0}\|(\lambda I-S_h)R_h-(\lambda I-S_h)\mathcal{R}(\lambda,S_h)\|+\limsup_{h\to 0}\|(\lambda I-S_h)\mathcal{R}(\lambda,S_h)-I\| \leq$$

$$\leq \limsup_{h\to 0}\|\lambda I-S_h\|\,\|R_h-\mathcal{R}(\lambda,S_h)\| \leq 0.$$

Analogously we can prove that $\lim_{h\to 0}\|R_h(\lambda I-S_h)-I\| = 0$. □

Proposition 18. *Let* $\{S_h\} \in C_b((0,1], B(X))$*,* $\lambda\in r(\{S_h\})$ *and* $\{\mathcal{R}(\lambda,S_h)\}\in C_b((0,1], B(X))$ *such that*

$$\lim_{h\to 0}\|(\lambda I-S_h)\mathcal{R}(\lambda,S_h)-I\| = \lim_{h\to 0}\|\mathcal{R}(\lambda,S_h)(\lambda I-S_h)-I\| = 0.$$

Then

$$\limsup_{h\to 0}\|\mathcal{R}(\lambda,S_h)\|\neq 0\ .$$

Proof. Suppose that $\limsup_{h\to 0} \|\mathcal{R}(\lambda, S_h)\| = 0$. Since

$$1 = \|I\| \leq \|(\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I\| + \|(\lambda I - S_h)\mathcal{R}(\lambda, S_h)\|$$

And taking into account that $\{S_h\} \in C_b((0,1], B(X))$, it follows that

$$1 \leq \limsup_{h\to 0} \|(\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I\| + \limsup_{h\to 0} \|(\lambda I - S_h)\mathcal{R}(\lambda, S_h)\| \leq$$

$$\limsup_{h\to 0} \|\lambda I - S_h\| \|\mathcal{R}(\lambda, S_h)\| \leq \left(|\lambda| + \limsup_{h\to 0} \|S_h\|\right) \limsup_{h\to 0} \|\mathcal{R}(\lambda, S_h)\| = 0,$$

contradiction. □

Proposition 19. *Let $\{S_h\} \in C_b((0,1], B(X))$. If $\lambda, \mu \in r(\{S_h\})$ such that there are $\{\mathcal{R}(\lambda, S_h)\}$, $\{\mathcal{R}(\mu, S_h)\} \in C_b((0,1], B(X))$ with property*

$$\lim_{h\to 0} \|\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h)\| = 0,$$

then $\lambda = \mu$.

Proof. For $\lambda \in r(\{S_h\})$ let $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0,1], B(X))$ such that

$$\lim_{h\to 0} \|(\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I\| = \lim_{h\to 0} \|\mathcal{R}(\lambda, S_h)(\lambda I - S_h) - I\| = 0$$

and for $\mu \in r(\{S_h\})$ let $\{\mathcal{R}(\mu, S_h)\} \in C_b((0,1], B(X))$ such that

$$\lim_{h\to 0} \|(\mu I - S_h)\mathcal{R}(\mu, S_h) - I\| = \lim_{h\to 0} \|\mathcal{R}(\mu, S_h)(\mu I - S_h) - I\| = 0.$$

If

$$\lim_{h\to 0} \|\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h)\| = 0,$$

Having in view Proposition 17, we obtain

$$\lim_{h\to 0} \|(\mu I - S_h)\mathcal{R}(\lambda, S_h) - I\| = \lim_{h\to 0} \|\mathcal{R}(\lambda, S_h)(\mu I - S_h) - I\| = 0.$$

Hence

$$\limsup_{h\to 0} \|(\lambda I - S_h)\mathcal{R}(\lambda, S_h) - (\mu I - S_h)\mathcal{R}(\lambda, S_h)\| \leq$$

$$\limsup_{h\to 0} \|(\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I\| + \lim_{h\to 0} \|(\mu I - S_h)\mathcal{R}(\lambda, S_h) - I\| = 0.$$

Therefore

$$|\lambda - \mu| \limsup_{h\to 0} \|\mathcal{R}(\lambda, S_h)\| = 0$$

And according to Proposition 18 ($\limsup_{h\to 0} \|R(\lambda, S_h)\| \neq 0$) it follows $\lambda = \mu$. □

Lemma 20. *If two bounded families* $\{S_h\}$, $\{T_h\}$ $\in C_b((0,1],\ B(X))$ *are asymptotic equivalent and there is* $\{R_h(\lambda)\} \in C_b((0,1],\ B(X))$ *such that*

$$\lim_{h\to 0} \|(\lambda I - S_h) R_h(\lambda) - I\| = \lim_{h\to 0} \|R_h(\lambda)(\lambda I - S_h) - I\| = 0,$$

then

$$\lim_{h\to 0} \|(\lambda I - T_h) R_h(\lambda) - I\| = \lim_{h\to 0} \|R_h(\lambda)(\lambda I - T_h) - I\| = 0.$$

Proof. Since the two families $\{S_h\}$, $\{T_h\}$ $\in C_b((0,1],\ B(X))$ are asymptotically equivalent, i.e. $\lim_{h\to 0} \|S_h - T_h\| = 0$, we have

$$\limsup_{h\to 0} \|(\lambda I - T_h) R_h(\lambda) - I\| =$$

$$= \limsup_{h\to 0} \|(\lambda I - T_h) R_h(\lambda) - (\lambda I - S_h) R_h(\lambda) + (\lambda I - S_h) R_h(\lambda) - I\| \leq$$

$$\limsup_{h\to 0} \|(\lambda I - T_h) R_h(\lambda) - (\lambda I - S_h) R_h(\lambda)\| + \limsup_{h\to 0} \|(\lambda I - S_h) R_h(\lambda) - I\| =$$

$$\limsup_{h\to 0} \|T_h R_h(\lambda) - S_h R_h(\lambda)\| \leq \limsup_{h\to 0} \|T_h - S_h\| \ \|R_h(\lambda)\| \leq 0.$$

□

Remark 21. *Since* $B_\infty = C_b((0,1],\ B(X)) / C_0((0,1],\ B(X))$ *is a Banach algebra, then make sense*

$$r\left(\{\dot{S_h}\}\right) = \left\{\lambda \in \mathbb{C} | \ \exists \{\dot{R_h}\} \in \ B_\infty \ \ a.\hat{\imath}.\ \left(\lambda\{\dot{I}\} - \{\dot{S_h}\}\right)\{\dot{R_h}\} = \{\dot{I}\} = \{\dot{R_h}\}\left(\lambda\{\dot{I}\} - \{\dot{S_h}\}\right)\right\}$$

and

$$Sp\left(\{\dot{S_h}\}\right) = \mathbb{C} \setminus r\left(\{\dot{S_h}\}\right).$$

Let $\{S_h\}$ $\in C_b((0,1],\ B(X))$ *and* $\lambda \in r(\{S_h\})$. *Fie* $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0,1],\ B(X))$ *such that*

$$\lim_{h\to 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h\to 0} \|\mathcal{R}(\lambda, S_h)(\lambda I - S_h) - I\| = 0.$$

By Proposition 17, it results that for any $\{\mathcal{R}'(\lambda, S_h)\} \in \{\mathcal{R}(\dot{\lambda}, S_h)\}$, *we have*

$$\lim_{h\to 0} \left\|(\lambda I - S_h) \mathcal{R}'(\lambda, S_h) - I\right\| = \lim_{h\to 0} \left\|\mathcal{R}'(\lambda, S_h)(\lambda I - S_h) - I\right\| = 0.$$

Moreover, for every $\{S'_h\} \in \{\dot{S_h}\}$, *by Lemma 20 we have*

$$\lim_{h\to 0} \left\|\left(\lambda I - S'_h\right) \mathcal{R}(\lambda, S_h) - I\right\| = \lim_{h\to 0} \left\|\mathcal{R}(\lambda, S_h)\left(\lambda I - S'_h\right) - I\right\| = 0.$$

Therefore every representative of class $\{\mathcal{R}(\dot{\lambda}, S_h)\} \in$ B_∞ *is an "inverse" for any representative of class* $\{\dot{S_h}\}$.

Theorem 22. *Let* $\{S_h\} \in C_b((0,1],\ B(X))$. *Then*

$$Sp\left(\{\dot{S_h}\}\right) = Sp(\{S_h\}).$$

Proof. Let $\lambda \in r\left(\{\dot{S_h}\}\right)$. Then there is $\{\dot{R_h}\} \in\ B_\infty$ such that

$$\left(\lambda\{\dot{I}\} - \{\dot{S_h}\}\right)\{\dot{R_h}\} = \{\dot{I}\} = \{\dot{R_h}\}\left(\lambda\{\dot{I}\} - \{\dot{S_h}\}\right).$$

Taking into account the algebraic relations of the Banach algebra B_∞, it results

$$\{\dot{I}\} = \left(\lambda\{\dot{I}\} - \{\dot{S_h}\}\right)\{\dot{R_h}\} = \ \{\lambda I \dot{-} S_h\}\{\dot{R_h}\} = \{(\lambda I - \dot{S_h})R_h\}.$$

Therefore $\{(\lambda I - S_h)\, R_h - I\} \in B_\infty$, i.e.

$$\lim_{h\to 0} \|(\lambda I - S_h)\, R_h - I\| \ = 0.$$

Analogously we can show that $\lim_{h\to 0} \|R_h\,(\lambda I - S_h) - I\| \ = 0$. Then $\lambda \in r\left(\{S_h\}\right)$. Conversely, let $\lambda \in r\left(\{S_h\}\right)$. Then there is $\{\mathcal{R}(\lambda, S_h)\} \in C_b\left((0,1]\,,\ B\,(X)\right)$ such that

$$\lim_{h\to 0} \|(\lambda I - S_h)\,\mathcal{R}\,(\lambda, S_h) - I\| \ = \lim_{h\to 0} \|\mathcal{R}\,(\lambda, S_h)\,(\lambda I - S_h) - I\| \ = 0.$$

Let $\{R_h\} \in \{\mathcal{R}(\dot{\lambda}, S_h)\}$. Then

$$\lim_{h\to 0} \|(\lambda I - S_h)\, R_h - I\| \ = \lim_{h\to 0} \|R_h\,(\lambda I - S_h) - I\| \ = 0$$

and $\{(\lambda I - S_h)\, R_h - I\}\,,\ \{R_h\,(\lambda I - S_h) - I\} \in B_\infty$, i.e.

$$\left(\lambda\{\dot{I}\} - \{\dot{S_h}\}\right)\{\dot{R_h}\} = \{(\lambda I - \dot{S_h})R_h\} = \{\dot{I}\}$$

and

$$\{\dot{R_h}\}\left(\lambda\{\dot{I}\} - \{\dot{S_h}\}\right) = \{R_h(\lambda \dot{I} - S_h)\} = \{\dot{I}\}.$$

Therefore $\lambda \in r\left(\{\dot{S_h}\}\right)$. □

Remark 23. *Let* $\{S_h\}\ \in C_b\left((0,1]\,,\ B\,(X)\right)$ *and* $\lambda \in r(\{S_h\})$. *Then there is* $\{\mathcal{R}(\lambda, S_h)\} \in C_b\left((0,1]\,,\ B\,(X)\right)$ *such that*

$$\lim_{h\to 0} \|(\lambda I - S_h)\,\mathcal{R}\,(\lambda, S_h) - I\| \ = \lim_{h\to 0} \|\mathcal{R}\,(\lambda, S_h)\,(\lambda I - S_h) - I\| \ = 0.$$

if and only if

$$\left(\lambda\{\dot{I}\} - \{\dot{S_h}\}\right)\{\mathcal{R}\,(\dot{\lambda}, S_h)\} = \{\dot{I}\} = \{\mathcal{R}\,(\dot{\lambda}, S_h)\}\left(\lambda\{\dot{I}\} - \{\dot{S_h}\}\right).$$

Proposition 24. *Let* $\{S_h\}\,,\ \{T_h\}\ \in C_b\left((0,1]\,,\ B\,(X)\right)$ *be two families. If*

$$\lim_{h\to 0} \|T_h S_h - S_h T_h\| \ = 0,$$

then $\lim_{h\to 0} \|R(\lambda, T_h)S_h - S_h R(\lambda, T_h)\| \ = 0$*, for any* $\lambda \in r(\{T_h\})$.

Proof. If $\lambda \in r(\{T_h\})$, then there is $\{\mathcal{R}(\lambda, T_h)\} \in C_b((0,1], B(X))$ such that
$$\lim_{h\to 0} \|(\lambda I - T_h)\mathcal{R}(\lambda, T_h) - I\| = \lim_{h\to 0} \|\mathcal{R}(\lambda, T_h)(\lambda I - T_h) - I\| = 0.$$
Therefore
$$\lim_{h\to 0} \|T_h S_h - S_h T_h\| = 0 \Leftrightarrow \{\dot{S_h}\}\{\dot{T_h}\} = \{\dot{T_h}\}\{\dot{S_h}\} \Leftrightarrow$$
$$\{\dot{S_h}\}\{\mathcal{R}(\lambda,\dot{T}_h)\} = \{\mathcal{R}(\lambda,\dot{T}_h)\}\{\dot{S_h}\} \Leftrightarrow \lim_{h\to 0} \|R(\lambda, T_h)S_h - S_h R(\lambda, T_h)\| = 0.$$
□

Remark 25. *i) Let $\{S_h\}, \{T_h\} \in C_b((0,1], B(X))$ such that S_h is asymptotically equivalent with T_h, $\forall h \in (0,1]$. Then*
$$Sp(T_h) = Sp(S_h), \forall h \in (0,1].$$
ii) Let $\{S_h\}, \{T_h\} \in C_b((0,1], B(X))$ be asymptotically equivalent. Then
$$Sp(\{T_h\}) = Sp(\{S_h\}).$$

Theorem 26. *Let $\{S_h\}, \{T_h\} \in C_b((0,1], B(X))$ be two asymptotic quasinilpotent equivalent families. Then*
$$Sp(\{T_h\}) = Sp(\{S_h\}).$$

Proof. Let $\lambda \in r(\{\dot{T}_h\})$. Then there is $\{\mathcal{R}(\lambda,\dot{T}_h)\} \in B_\infty$ such that
$$\left(\lambda\{\dot{I}\} - \{\dot{T}_h\}\right)\{\mathcal{R}(\lambda,\dot{T}_h)\} = \{\mathcal{R}(\lambda,\dot{T}_h)\}\left(\lambda\{\dot{I}\} - \{\dot{T}_h\}\right) = \{\dot{I}\}.$$

Since B_∞ is a Banach algebra, the map $\lambda \mapsto \{\mathcal{R}(\lambda,\dot{T}_h)\} : r(\{\dot{T}_h\}) \to B_\infty$ is analytic. Let $D_1 = \{\lambda \in \mathbb{C} | \, |\lambda - \lambda_0| \le r_1\} \subset r(\{\dot{T}_h\})$ and $D_0 = \{\lambda \in \mathbb{C} | \, |\lambda - \lambda_0| \le r_0\}$ with $r_1 > r_0$.
Set
$$\{\dot{R_n}(\lambda)\} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\{\mathcal{R}(\lambda,\dot{T}_h)\}, \forall n \in \mathbb{N},$$
and
$$\{\dot{R}(\lambda)\} = \sum_{n\in N} \frac{(-1)^n}{n!}\left\{(S_h - T_h)^{[n]}\right\}^{\cdot}\{\dot{R_n}(\lambda)\}.$$
If we set $M_1 = sup_{\mu\in D_1}\left\|\{\mathcal{R}(\mu,\dot{T}_h)\},\right\|$, it follows that $\left\|\{\dot{R_n}(\lambda)\}\right\| \le \frac{r_1 M_1}{(r_1 - r_0)^{n+1}}$.
Deriving the relation $\left(\lambda\{\dot{I}\} - \{\dot{T}_h\}\right)\{\mathcal{R}(\lambda,\dot{T}_h)\} = \{\dot{I}\}$ by n times, we obtain
$$\left(\lambda\{\dot{I}\} - \{\dot{T}_h\}\right)\frac{d^n}{d\lambda^n}\{\mathcal{R}(\lambda,\dot{T}_h)\} = -n\frac{d^{n-1}}{d\lambda^{n-1}}\{\mathcal{R}(\lambda,\dot{T}_h)\}.$$
Moreover, since
$$\left\{(T_h - S_h)^{[n+1]}\right\}^{\cdot} = \left\{T_h(T_h - S_h)^{[n]} - (T_h - S_h)^{[n]}S_h\right\}^{\cdot} =$$
$$= \{\dot{T_h}\}\left\{(T_h - S_h)^{[n]}\right\}^{\cdot} - \left\{(T_h - S_h)^{[n]}\right\}^{\cdot}\{\dot{S_h}\}, \forall n \in \mathbb{N},$$

we have

$$\left(\lambda\{\dot{I}\}-\{\dot{S_h}\}\right)\{R\dot{(}\lambda)\}=\{\lambda I\dot{-}S_h\}\sum_{n\in N}\frac{(-1)^n}{n!}\left\{(S_h\dot{-}T_h)^{[n]}\right\}\{R_n\dot{(}\lambda)\}=$$

$$=\sum_{n\in N}\frac{(-1)^n}{n!}\{\lambda I\dot{-}S_h\}\left\{(S_h\dot{-}T_h)^{[n]}\right\}\{R_n\dot{(}\lambda)\}=$$

$$=\sum_{n\in N}\frac{(-1)^n}{n!}\left\{(\lambda I-S_h)\dot{(}S_h-T_h)^{[n]}\right\}\{R_n\dot{(}\lambda)\}=$$

$$=\sum_{n\in N}\frac{1}{n!}\left\{\left(((\lambda I-S_h)-(\lambda I-T_h))^{[n+1]}+((\dot{\lambda}I-S_h)-(\lambda I-T_h))^{[n]}(\lambda I-T_h)\right)\right\}\{R_n\dot{(}\lambda)\}=$$

$$=\sum_{n=0}\frac{(-1)^{n+1}}{n!}\left\{(S_h-\dot{T_h})^{[n+1]}\right\}\{R_n\dot{(}\lambda)\}+\left(\lambda\{\dot{I}\}-\{\dot{T_h}\}\right)\{\mathcal{R}\,(\dot{\lambda},T_h)\}-$$

$$-\sum_{n=1}\frac{(-1)^n}{n-1!}\left\{(S_h\dot{-}T_h)^{[n]}\right\}\left(\lambda\{\dot{I}\}-\{\dot{T_h}\}\right)\frac{d^n}{d\lambda^n}\{\mathcal{R}\,(\dot{\lambda},T_h)\}=$$

$$=\sum_{n=0}\frac{(-1)^{n+1}}{n!}\left\{(S_h-\dot{T_h})^{[n+1]}\right\}\{R_n\dot{(}\lambda)\}+\left(\lambda\{\dot{I}\}-\{\dot{T_h}\}\right)\{\mathcal{R}\,(\dot{\lambda},T_h)\}-$$

$$-\sum_{n=1}\frac{(-1)^n}{n-1!}\left\{(S_h\dot{-}T_h)^{[n]}\right\}\frac{d^{n-1}}{d\lambda^{n-1}}\{\mathcal{R}\,(\dot{\lambda},T_h)\}=$$

$$=\sum_{n=0}\frac{(-1)^{n+1}}{n!}\left\{(S_h-\dot{T_h})^{[n+1]}\right\}\{R_n\dot{(}\lambda)\}+\left(\lambda\{\dot{I}\}-\{\dot{T_h}\}\right)\{\mathcal{R}\,(\dot{\lambda},T_h)\}-$$

$$-\sum_{n=1}\frac{(-1)^n}{n-1!}\left\{(S_h\dot{-}T_h)^{[n]}\right\}\{R_{n-1}\dot{(}\lambda)\}.$$

Therefore $\lambda\in r(\{S_h\})$.
Analogously we can prove the other inclusion. By Theorem 22, it results that

$$Sp(\{T_h\})=Sp(\{\dot{T_h}\})=Sp(\{\dot{S_h}\})=Sp(\{S_h\}).$$

□

Theorem 27. *Let* $\{T_h\}\in C_b\left((0,1]\,,\ B\left(X\right)\right)$ *and* Ω *be an open set which contains* $\bigcup_{h\in(0,1]}Sp(T_h)$. *Then for any analytic function* $f:\Omega\to\mathbb{C}$ *we have*

$$Sp\left(\{f\left(T_h\right)\}\right)=f(Sp\left(\{T_h\}\right)).$$

Proof. If $\{T_h\} \in C_b((0,1],\ B(X))$, then there is a $M < \infty$ such that $\|T_h\| \leq M$, $\forall h \in (0,1]$. Therefore $Sp(T_h) \subset D(0,M)$ $\forall h \in (0,1]$, so that $\bigcup_{h\in(0,1]} Sp(T_h)$ is a bounded set.

"$\supseteq$" Let $f : \Omega \to \mathbb{C}$ be an analytic function and $\lambda \in Sp(\{T_h\})$. For $\xi \in \Omega$, we define the function

$$g(\xi) = \begin{cases} \frac{f(\xi)-f(\lambda)}{\xi-\lambda}, & \xi \neq \lambda \\ f'(\lambda), & \xi = \lambda \end{cases}.$$

Hence $g : \Omega \to \mathbb{C}$ is analytic and

$$f(T_h) - f(\lambda) I = g(T_h)(T_h - \lambda I) = (T_h - \lambda I) g(T_h),$$

for any $h \in (0,1]$.

We suppose that $f(\lambda) \in r(\{f(T_h)\})$. Then there is $\{\mathcal{R}(f(\lambda), f(T_h))\} \subset B(X)$ such that

$$\lim_{h\to 0} \|(f(\lambda) I - f(T_h))\mathcal{R}(f(\lambda), f(T_h)) - I\| =$$

$$\lim_{h\to 0} \|\mathcal{R}(f(\lambda), f(T_h))(f(\lambda) I - f(T_h)) - I\| = 0.$$

Having in view the last relation, we have

$$\lim_{h\to 0} \|(T_h - \lambda I) g(T_h)\mathcal{R}(f(\lambda), f(T_h)) - I\| =$$

$$\lim_{h\to 0} \|\mathcal{R}(f(\lambda), f(T_h)) g(T_h)(T_h - \lambda I) - I\| = 0.(*)$$

Since

$$g(T_h) T_h = T_h g(T_h),$$

for any $h \in (0,1]$, according to the properties of holomorphic functional calculi it follows

$$g(T_h) f(T_h) = f(T_h) g(T_h),$$

for every $h \in (0,1]$. Applying Proposition 21, we obtain

$$\lim_{h\to 0} \|g(T_h)\mathcal{R}(f(\lambda), f(T_h)) - \mathcal{R}(f(\lambda), f(T_h)) g(T_h)\| = 0.$$

Hence

$$\lim_{h\to 0} \|g(T_h)\mathcal{R}(f(\lambda), f(T_h))(T_h - \lambda I) - I\| =$$

$$\lim_{h\to 0} \|g(T_h)\mathcal{R}(f(\lambda), f(T_h))(T_h - \lambda I) - \mathcal{R}(f(\lambda), f(T_h)) g(T_h)(T_h - \lambda I) +$$

$$+\mathcal{R}(f(\lambda), f(T_h)) g(T_h)(T_h - \lambda I) - I\| \leq$$

$$\leq \lim_{h\to 0} \|g(T_h)\mathcal{R}(f(\lambda), f(T_h)) - \mathcal{R}(f(\lambda), f(T_h)) g(T_h)\| \|T_h - \lambda I\| +$$

$$+\lim_{h\to 0} \|\mathcal{R}(f(\lambda), f(T_h)) g(T_h)(T_h - \lambda I) - I\| = 0.(**)$$

From (*) and (**), it results

$$\lim_{h\to 0} \|(T_h - \lambda I) g(T_h)\mathcal{R}(f(\lambda), f(T_h)) - I\| =$$

$$\lim_{h\to 0}\|g(T_h)\mathcal{R}(f(\lambda),\ f(T_h))(T_h-\lambda I)-I\| \ =0,$$

so $\lambda\in r(\{T_h\})$, contradiction with $\lambda\in Sp(\{T_h\})$. Therfore $f(\lambda)\in\ Sp(\{f(T_h)\})$.
"$\subseteq$" Let $\lambda\in\ Sp(\{f(T_h)\})$. If $\lambda\notin f(Sp(\{T_h\}))$, then $\lambda\neq f(\xi)$ for any $\xi\in Sp(\{T_h\})$. Let Ω' an open neighborhood $\bigcup_{h\in(0,1]}Sp(T_h)$ and

$$h(\xi)=\frac{1}{f(\xi)-\lambda},$$

for every $\xi\in\Omega'$. Then h is an analytic function and applying the holomorphic functional calculi, we obtain

$$h(T_h)(f(T_h)-\lambda I)=(f(T_h)-\lambda I)h(T_h)=I,$$

for any $h\in(0,1]$. Therefore $\lambda\in r(f(T_h))$, for any $h\in(0,1]$. Since $\bigcap_{h\in(0,1]}r(f(T_h))\subseteq r(\{f(T_h)\})$ (Remark 12 i)), it follows $\lambda\in r(\{f(T_h)\})$, contradiction with $\lambda\in Sp(\{f(T_h)\})$. Hence $\lambda\in f(Sp(\{T_h\}))$.

□

Definition 28. *A family $\{U_h\}\subset L(X)$ is calling asymptotic quasinilpotent operator if*

$$\lim_{n\to\infty}\limsup_{h\to 0}\|{U_h}^n\|^{\frac{1}{n}} \ =0.$$

Theorem 29. *A family $\{U_h\}\in C_b((0,1],\ B(X))$ is an asymptotic quasinilpotent operator if and only if $Sp(\{U_h\})=\{0\}$.*

Proof. Let $\{U_h\}\in C_b((0,1],\ B(X))$ be an asymptotic quasinilpotent operator. Then $\{U_h\}$ is asymptotically spectral equivalent with $\{0\}_{h\in(0,1]}\in C_b((0,1],\ B(X))$. By Theorem 26 it follows that

$$Sp(\{U_h\})=Sp(\{0\})=\{0\}.$$

Consequently, suppose that $Sp(\{U_h\})=\{0\}$. By Theorem 22, we have

$$Sp\left(\{\dot{U_h}\}\right)=Sp(\{U_h\})=\{0\}.$$

Then the spectral radius of $\{\dot{U_h}\}$, which we will call from now $r_{sp}\left(\{\dot{U_h}\}\right)$, is zero. Since

$$r_{sp}\left(\{\dot{U_h}\}\right)=\lim_{n\to\infty}\left\|\left(\{\dot{U_h}\}\right)^n\right\|^{\frac{1}{n}},$$

it follows that

$$\lim_{n\to\infty}\left\|\left(\{\dot{U_h}\}\right)^n\right\|^{\frac{1}{n}} \ =0.$$

But, on the other hand, we have

$$\lim_{n\to\infty} \left\| \left(\{\dot{U_h}\}\right)^n \right\|^{\frac{1}{n}} = \lim_{n\to\infty} inf_{\{U_h\}\in\{\dot{U_h}\}} \|\{U_h\}^n\|^{\frac{1}{n}} =$$

$$\lim_{n\to\infty} inf_{\{U_h\}\in\{\dot{U_h}\}} \|\{U_h{}^n\}\|^{\frac{1}{n}} = \lim_{n\to\infty} inf_{\{U_h\}\in\{\dot{U_h}\}} sup_{h\in(0,1]} \|U_h{}^n\|^{\frac{1}{n}} \geq$$

$$\lim_{n\to\infty} inf_{\{U_h\}\in\{\dot{U_h}\}} \limsup_{h\to 0} \|U_h{}^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \limsup_{h\to 0} \|U_h{}^n\|^{\frac{1}{n}} .$$

By the above relations, we obtain

$$\lim_{n\to\infty} \limsup_{h\to 0} \|U_h{}^n\|^{\frac{1}{n}} = 0,$$

so that $\{U_h\}$ is an asymptotic quasinilpotent operator. □

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