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MAXIMAL SUBGROUPS OF THE GROUP PSL(12,2)

Rauhi Ibrahim Elkhatib

Abstract. In this paper, We will find the maximal subgroups of the group PSL(12, 2) by Aschbacher's Theorem ([2]).

1 Introduction

The purpose of this research is to prove the following theorem:

Theorem 1. *Let G = PSL(12, 2). If H is a maximal subgroup of G, then H isomorphic to one of the following subgroups:*

1. *A group* $G_{(p)}$ *or* $G_{(10-\pi)}$*, stabilizing of a point or its dual, the stabilizer of a hyperplane. These are isomorphic to a group of form* $2^{11}.SL(11, 2)$;
2. *A group* $G_{(l)}$ *or* $G_{(9-\pi)}$*, stabilizing of a line or its dual, the stabilizer of a 9-space. These are isomorphic to a group of form* $2^{20}.(SL(2, 2) \times SL(10, 2))$;
3. *A group* $G_{(2-\pi)}$*, or* $G_{(8-\pi)}$*, stabilizing of a plane or its dual, the stabilizer of a 8-space. These are isomorphic to a group of form* $2^{27}.(SL(3, 2) \times SL(9, 2))$;
4. *A group* $G_{(3-\pi)}$*, or* $G_{(7-\pi)}$*, stabilizing of a 3-space or its dual, the stabilizer of a 7-space. These are isomorphic to a group of form* $2^{32}.(SL(4, 2) \times SL(8, 2))$;
5. *A group* $G_{(4-\pi)}$*, or* $G_{(6-\pi)}$*, stabilizing of a 4-space or its dual, the stabilizer of a 6-space. These are isomorphic to a group of form* $2^{35}.(SL(5, 2) \times SL(7, 2))$;
6. *A group* $G_{(5-\pi,5-\pi)}$*, stabilizing of a pair of 5-spaces. These are isomorphic to a group of form* $2^{36}.(SL(6, 2) \times SL(6, 2))$;
7. H_2*=PSL(3, 2)*$:S_4$ *a group preserving four mutually skew planes of PG(11, 2) and* H_2 *interchanges them;*

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8. *H_3=PSL(4, 2)*:*S_3 a group preserving three mutually skew 3-spaces of PG(11, 2) and H_3 interchanges them;*

9. *H_4=PSL(6, 2)*:*S_2 a group preserving two mutually skew 5-spaces of PG(11, 2) and H_4 interchanges them;*

10. *H_5=ΓL(2, 2^6), a group preserves six mutually skew lines of PG(11, 2^5) and H_5 interchanges them;*

11. *H_6=ΓL(3, 2^4), a group preserves four skew planes of PG(11, 2^4) and H_6 interchanges them;*

12. *H_7=ΓL(4, 2^3), a group preserves three skew 3-spaces of PG(11, 2^3) and H_7 interchanges them;*

13. *H_8=ΓL(6, 2^2), a group preserves two skew 5-spaces of PG(11, 2^2) and H_8 interchanges them;*

14. *H_{10}=PSL(3, 2) ∘PSL(4, 2);*

15. *Sp(12, 2);*

16. *PΓL(2, 11);*

17. *PΓL(2, 13);*

18. *PΓL(2, 25);*

19. *PΓL(3, 3).*

Through this research, ΓL(n, q) denote the group of all non-singular semi-linear transformation of a vector space V_n(q) of dimension n over a field F_q with q is a prime power. *The general linear group* GL(n, q), consisting of the set of all invertible n×n matrices. In fact, GL(n, q) is a subgroup of ΓL(n, q) consisting of all non-singular linear transformations of V_n(q). *The centre* Z of GL(n, q) is the set of all non-singular scalar matrices. The factor group $GL(n, q)/Z$ called *The projective general linear group* which is denoted by PGL(n, q). GL(n, q) has a normal subgroup SL(n, q), consisting of all matrices of determinant 1 called *the special linear group*. *The projective special linear group* PSL(n, q) is the quotient group $SL(n, q)/(Z \cap SL(n, q))$. PSL(n, q) is simple, except for PSL(2, 2) and PSL(2, 3).

PG(n-1, q) will denote *the projective space* of dimension n-1 associated with V_n(q). One, two and three- dimensional subspaces of V_n(q) will be called points, lines and planes respectively. An (n-1)-dimensional subspace shall be called *a hyperplane*.

A split extension (*a semidirect product*) A**:**B is a group G with a normal subgroup A and a subgroup B such that G = AB and A ∩ B = 1. *A non-split extension* A**.**B is a group G with a normal subgroup A and $G/A \cong B$, but with no subgroup B satisfying G = AB and $A \cap B = 1$. A group $G = A \circ B$ is *a central product* of its subgroups A and B if G = AB and [A, B], the commutator of A and B = {1}, in this case A and B are normal subgroups of G and $A \cap B = Z(G)$. If $A \cap B = \{1\}$, then $A \circ B = AB$.

G = PSL(12, 2) is a simple group of order

$$644176229278576214187891988140087941529600 0$$

thus $|G| = 2^{66}.3^8.5^3.7^4.11.13.17.23.31^2.73.89.127$. G acting as a doubly transitive permutation group on the points of the projective space PG(11, 2).

2 Aschbacher's Theorem

In this section, we will give some definitions before starting a brief description of Aschbacher's Theorem [2].

Definition 2. *Let V be a vector space of dimensional n over a finite field q, a subgroup H of GL(n, q) is called* reducible *if it stabilizes a proper nontrivial subspace of V. If H is not reducible, then it is called* irreducible. *If H is irreducible for all field extension F of* F_q*, then H is* absolutely irreducible. *An irreducible subgroup H of GL(n, q) is called* imprimitive *if there are subspaces* $V_1, V_2, \ldots, V_k$, $k = 2$, *of V such that* $V = V_1 \oplus \ldots \oplus V_k$ *and H permutes the elements of the set* $\{V_1, V_2, \ldots, V_k\}$ *among themselves. When H is not imprimitive then it is called* primitive.

Definition 3. *A group G = GL(n, q) is a* superfield group *of degree s if for some s divides n with s > 1, the group G may be embedded in* $\Gamma L(n/s, q^s)$.

Definition 4. *If the group G = GL(n,q) preserves a decomposition* $V = V_1 \otimes V_2$ *with* $dim(V_1) \neq dim(V_2)$ *then G is a* tensor product group.

Definition 5. *Suppose that* $n = r^m$ *and* $m > 1$. *If G = GL(n, q) preserves a decomposition* $V = V_1 \otimes \ldots \otimes V_m$ *with* $dim(V_i) = r$ *for* $1 = i = m$, *then G is a* tensor induced *group.*

Definition 6. *A group G = GL(n, q) is* a subfield group *if there exists a subfield* $F_{q_o} \subset F_q$ *such that G can be embedded in* $GL(n, q_o).Z$.

Definition 7. *A p-group G is called* special *if* $Z(G) = G'$ *and is called* extraspecial *if also* $|Z(G)| = p$.

Definition 8. *Let Z denote the group of scalar matrices of G. Then G is* almost simple modulo scalars *if there is a non-abelian simple group T such that* $T = G/Z = Aut(T)$, *the automorphism group of T.*

A classification of the maximal subgroups of GL(n, q) by Aschbacher's Theorem [2], which may be briefly summarized as follows:

Proposition 9. *(Aschbacher's Theorem): Let H be a subgroup of GL(n, q), $q = p^e$ with the center Z and V be the underlying n-dimensional vector space over a field q. If H is a maximal subgroup of GL(n, q), then one of the following holds:* C_1*:- H is a reducible group.*
C_2*:- H is an imprimitive group.*
C_3*:- H is a superfield group.*
C_4*:- H is a tensor product group.*
C_5*:- H is a subfield group.*
C_6*:- H normalizes an irreducible extraspecial or symplectic-type group.*
C_7*:- H is a tensor induced group.*
C_8*:- H normalizes a classical group in its natural representation.*
C_9*:- H is absolutely irreducible and H /(H∩Z)is almost simple.*

Note: The nine classes of Proposition 9 are not mutually exclusive.
To prove Theorem 1 by using Aschbacher's Theorem (Proposition 9), first, we will determine the maximal subgroups in the classes C_1 - C_8 of Proposition 9.

3 The maximal subgroups in the classes C_1 - C_8 of Proposition 9

3.1 The maximal subgroups of the class C_1

Let H be a reducible subgroup of G and W an invariant subspace of H. If we let d = dim(W), then 1 ≤ d≤ 12. Let $G_d = G_{(W)}$ denote the subgroup of G containing all elements fixing W as a whole and H ⊆ $G_{(W)}$. with a suitable choice of a basis, $G_{(W)}$ consists of all matrices of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A and C are d × d and (12-d) ×(12-d) non-singular matrices of determinant 1, where B is an arbitrary d×(12-d) matrix. G_d is isomorphic to a group of the form $2^{d(12-d)}$(SL(d, 2))×(SL(12-d, 2)). which give us the following reducible maximal subgroups of G:

1. A group $G_{(p)}$ or $G_{(10-\pi)}$, stabilizing of a point or its dual, the stabilizer of a hyperplane. These are isomorphic to a group of form $2^{11}.SL(11, 2)$.

2. A group $G_{(l)}$ or $G_{(9-\pi)}$, stabilizing of a line or its dual, the stabilizer of a 9-space. These are isomorphic to a group of form $2^{20}.(SL(2, 2) \times SL(10, 2))$.

3. A group $G_{(2-\pi)}$, or $G_{(8-\pi)}$, stabilizing of a plane or its dual, the stabilizer of a 8-space. These are isomorphic to a group of form $2^{27}.(SL(3, 2) \times SL(9, 2))$.

4. A group $G_{(3-\pi)}$, or $G_{(7-\pi)}$, stabilizing of a 3-space or its dual, the stabilizer of a 7-space. These are isomorphic to a group of form $2^{32}.(SL(4,2) \times\ SL(8,2))$.

5. A group $G_{(4-\pi)}$, or $G_{(6-\pi)}$, stabilizing of a 4-space or its dual, the stabilizer of a 6-space. These are isomorphic to a group of form $2^{35}.(SL(5,2) \times\ SL(7,2))$.

6. A group $G_{(5-\pi,5-\pi)}$, stabilizing of a pair of 5-spaces. These are isomorphic to a group of form $2^{36}.(SL(6,2) \times\ SL(6,2))$.

Which prove the points (1), (2), (3), (4), (5) and (6) of Theorem 1.

3.2 The maximal subgroups of the class C_2

If H is imprimitive, then H preserves a decomposition of V as a direct sum V = $V_1 \oplus \ldots \oplus V_t$, t > 1, into subspaces of V, each of dimension m = n/t, which are permuted transitively by H, thus C_2 are isomorphic to GL(m, q): S_t. Consequently, there are two imprimitive groups of C_2 in PSL(12, 2) which are:

1. H_1 = PSL(2, 2):S_6, a group preserving six mutually skew lines of PG(11, 2) and H_1 interchanges them.

2. H_2 = PSL(3, 2):S_4 a group preserving four mutually skew planes of PG(11, 2) and H_2 interchanges them.

3. H_3 = PSL(4, 2):S_3 a group preserving three mutually skew 3-spaces of PG(11, 2) and H_3 interchanges them.

4. H_4 = PSL(6, 2):S_2 a group preserving two mutually skew 5-spaces of PG(11, 2) and H_4 interchanges them.

But it shown in [14] that GL(k, 2):S_t is not maximal for k = 2. Thus H_1 is not a maximal subgroups of PSL(12, 2).
Which prove the points (7), (8) and (9) of Theorem 1.
Note: if q>2, then there exist in C_2 an imprimitive group $G_{(\Delta)}$ of order n!$(q-1)^{n-1}$ preserving a n-simplex points of PG(n-1, q) with coordinates in F_q and $G_{(\Delta)}$ interchanges them.

3.3 The maximal subgroups of the class C_3

If H is (superfield group) a semilinear groups over extension fields of GF(q) of prime degree, then H acts on G as a group of semilinear automorphism of a (n/k)-dimensional space over the extension field GF(q^k), so H embeds in ΓL(n/k, q^k), for some prime number k dividing n. Consequently, there are four C_3 groups in PSL(12, 2) which are:

1. H_5 = ΓL(2, 2^6), a group preserves six mutually skew lines of PG(11, 2^5) and H_5 interchanges them.

2. H_6 = ΓL(3, 2^4), a group preserves four skew planes of PG(11, 2^4) and H_6 interchanges them.

3. H_7 = ΓL(4, 2^3), a group preserves three skew 3-spaces of PG(11, 2^3) and H_7 interchanges them.

4. H_8 = ΓL(6, 2^2), a group preserves two skew 5-spaces of PG(11, 2^2) and H_8 interchanges them.

Which prove the points (10), (11), (12) and (13) of Theorem 1.

Definition 10. A Singer cycle *of GL(n, q) is an element of order* q^n*-1.*

Remark 11. *([9, 13, 20]).*
If n is a prime number, then there exist a Singer cycles group $H = \Gamma L(1, q^n)$ *of order* $d^{-1}(q^n\text{-}1)/(q\text{-}1)$*, where d = gcd(n, q-1) and H is irreducible maximal subgroup of PSL(n, q) which it is the normalizer of the (cyclic) multiplicative group for GF(*q^n*). Consequently, there is no Singer cycle subgroup in PSL(12, 2), since 12 is not a prime number.*

3.4 The maximal subgroups of the class C_4

If H is a tensor product group, then H preserves a decomposition of V as a tensor product $V_1 \otimes V_2$, where $\dim(V_1) \neq \dim(V_2)$ of spaces of dimensions k, m > 1 over GF(q), and so H stabilize the tensor product decomposition $F^k \otimes F^m$, where n = km, k ≠ m. Thus, H is a subgroup of the central product of PSL(k, q)∘PSL(m, q). Consequently, there are two C_4 groups in PSL(12, 2) which are:

1. H_9 = PSL(2, 2)∘PSL(6, 2);

2. H_{10} = PSL(3, 2)∘PSL(4, 2);

but it shown in [14] that PSL(2, 2)∘PSL(k, 2) is not maximal for all k. Thus H_9 = PSL(2, 2)∘PSL(6, 2) is not a maximal subgroups of PSL(12, 2).
Which prove the point (14) of Theorem 1.

3.5 The maximal subgroups of the class C_5

If H is a subfield group, then H is the linear groups over subfields of GF(q) of prime index. Thus H can be embedded in GL(n, p^f).Z where e/f is prime number and q = p^e. Consequently, there are no C_5 groups in PSL(12, 2) since 2 is a prime number.

3.6 The maximal subgroups of the class C_6

For the dimension n = r^m, if r is prime number divides q-1, then H = r^{2m}**:**Sp(2m, r) is an extraspecial r-group of order r^{2m+1}, or if r = 2 and 4 divides q-1, then $H = 2^{2m}$**.**$O^{\in}(2m, 2)$ normalizes a 2-group of symplectic type of order 2^{2m+2}. Consequently, there are no C_6 groups in PSL(12, 2) since 12 is not prime power.

3.7 The maximal subgroups of the class C_7

If H is a tensor-induced, then H preserves a decomposition of V as $V_1 \otimes V_2 \otimes \ldots \otimes V_m$ where V_i are isomorphic and each V_i has dimension r > 1, n = dim V = r^m, and the set of V_i is permuted by H, so H stabilize the tensor product decomposition $F^r \otimes F^r \otimes \ldots \otimes F^r$, where F = F_q. Thus H/Z = PGL(r, q)**:**S_m. Consequently, there are no C_7 groups in PSL(12, 2) since 12 is not a proper power.

3.8 The maximal subgroups of the class C_8

If H normalizes a classical group in its natural representation, then H lies between a classical group and its normalizer in GL(n, q), so H preserves a classical form up to scalar multiplication. Thus H is a normalizer of such a subgroup PSL(n, q́), PSp(n, q́), PΩ(n, q́) or PSU(n, q́) for various q́ dividing q. But from [5], Sp(n, q) is a maximal subgroups of PSL(n, q). Consequently, In C_8, there are only Sp(12, 2) irreducible groups in PSL(12, 2) since 2 is not a square, and is even number.
Which proves the point (15) of Theorem 1.
Note: From [4] and [12], $O^-(12, 2)$ and $O^+(12, 2)$ are maximal subgroups of Sp(12, 2), then G contains subgroups isomorphic to $O^-(12, 2)$ and $O^+(12, 2)$ but these are not maximal in G. Thus $O^{\in}(12, 2) \subseteq$ Sp(12, 2) $\subseteq$ PSL(12, 2).

Finally, we will determine the maximal subgroups in class C_9 of Aschbacher's Theorem (Proposition 9):

4 The maximal subgroups of the class C_9

If H is absolutely irreducible and H /(H∩Z)is almost simple, then H is the normalizer of absolutely irreducible normal subgroup M of H which is non-abelian and simple group. Thus, to find the maximal subgroups of C_9, we will determine the maximal primitive subgroups H of G which have the property that a minimal normal subgroup M of H is non abelian group.

The following Corollary will play an important role in proving the main result of this section, (Theorem 37).

Corollary 12. *If M is a non abelian simple group of a primitive subgroup H of G, then M is isomorphic to one of the following groups:*

1. *A_{13};*
2. *A_{14};*
3. *PSL(2, 11);*
4. *PSL(2, 13);*
5. *PSL(2, 25);*
6. *PSL(3, 3);*
7. *PSU(2, 11);*
8. *PSU(2, 13);*
9. *Sp(12, 2);*
10. *$O^{\in}(12, 2)$, $\in = \{+, -\}$.*

Proof. let H be a primitive subgroup of G with a minimal normal subgroup M of H is not abelian. So, we will discuss the possibilities of a minimal normal subgroup M of H according to:

1. M contains transvections, (Section 4.1).
2. M does not contain any transvection, (Section 4.2).
3. M is doubly transitive, (Section 4.3).

□

4.1 Primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian is generated by transvections

To find the primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian is generated by transvections, we will use the following result of Mclaughlin [16]:

Proposition 13. *(Mclaughlin Theorem): Let H be a proper irreducible subgroup of SL(n, 2) generated by transvections. Then n>3 and H is Sp(n, 2), $O^{\in}(n, 2)$, S_{n+1} or S_{n+2}.*

In the following, we will discuss the different possibilities of Mclaughlin Theorem (Proposition 13), which will give us the following main result of Section 4.1.

Corollary 14. *If M is a proper irreducible subgroup of SL(12, 2) generated by transvections, then M isomorphic to symplectic group Sp(12, 2), orthogonal groups* $O^-(12, 2)$ *and* $O^+(12, 2)$*, symmetric groups* S_{13} *or* S_{14}*.*

Proof. From Mclaughlin Theorem (Proposition 13), M is isomorphic to one of the following groups: symplectic group Sp(12, 2), orthogonal groups $O^-(12, 2)$ and $O^+(12, 2)$, symmetric groups S_{13} or S_{14}.

1. From [5], the symplectic group Sp(12, 2) is a subgroup of G.

2. From [4] and [12], $O^-(12, 2)$ and $O^+(12, 2)$ are maximal subgroups of Sp(12, 2), then G contains subgroups isomorphic to $O^-(12, 2)$ and $O^+(12, 2)$ but these are not maximal in G. Thus $O^{\in}(12, 2) \subseteq \mathrm{Sp}(12, 2) \subseteq \mathrm{PSL}(12, 2)$.

3. $S_{13} \subset$ G, since, the irreducible 2-modular characters for S_{13} by GAP are:

   [[1, 1], [12, 1], [64, 2], [208, 1], [288, 1], [364, 2], [560, 1], [570, 1], [1572, 1], [1728, 1], [2208, 1], [2510, 1], [2848, 1], [3200, 1], [8008, 1], [8448, 1]].

   (gap> CharacterDegrees(CharacterTable("S13")mod 2);). But S_{13} is not a simple group.

4. $S_{14} \subset$ G, since, the irreducible 2-modular characters for S_{14} by GAP are:

   [[1, 1], [12, 1], [64, 2], [208, 1], [364, 1], [560, 2], [768, 1], [1300, 1], [2016, 1], [2510, 1], [3418, 1], [3808, 1], [4576, 1], [4704, 1], [10880, 1], [11648, 1], [13312, 1], [19240, 1], [23296, 1], [35840, 1]].

   (gap> CharacterDegrees(CharacterTable("S14)mod 2);). But S_{14} is not a simple group.

□

4.2 Primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian and does not contain transvections

In this section, we will consider a minimal normal subgroup M of H is not abelian and does not contain any transvections. The following corollary is the main result of Section 4.2.

Corollary 15. *If Y be a non - abelian simple subgroup of G which does not contain any transvection. Then Y is isomorphic to*

1. *PSL(2, 11);*
2. *PSL(2, 13);*
3. *PSL(2, 25);*

4. *PSL(3,3).*

Proof. We will prove Corollary 15 by series of Lemma 16 through Lemma 21 and Proposition 17. □

Lemma 16. *Let Y is a primitive subgroup of G such that Y does not contain any transvection. If S(2) be a 2-Sylow subgroup of Y, then S(2) contains no elementary abelian subgroup of order 8.*

Proof. A 2-Sylow subgroup of G can be represented by the set of all matrices of the form:

$$\begin{bmatrix}
1 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} \\
 & 1 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} & x_{20} & x_{21} \\
 & & 1 & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} & x_{29} & x_{30} \\
 & & & 1 & x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} & x_{37} & x_{38} \\
 & & & & 1 & x_{39} & x_{40} & x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\
 & & & & & 1 & x_{46} & x_{47} & x_{48} & x_{49} & x_{50} & x_{51} \\
 & & & & & & 1 & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\
 & & & & & & & 1 & x_{57} & x_{58} & x_{59} & x_{60} \\
 & & & & & & & & 1 & x_{61} & x_{62} & x_{63} \\
 & & & & & & & & & 1 & x_{64} & x_{65} \\
 & & & & & & & & & & 1 & x_{66} \\
 & & & & & & & & & & & 1
\end{bmatrix}$$

Where all entries are in F_2. Let Y is a primitive subgroup of G such that Y does not contain any transvection. If S(2) be a 2-Sylow subgroup of Y, then inside S(2), there exist only two elementary abelian subgroups of the form:-

$$\mathrm{A} = \left\{ \begin{bmatrix}
1 & \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{x}_4 & \mathrm{x}_5 & \mathrm{x}_6 & \mathrm{x}_7 & \mathrm{x}_8 & \mathrm{x}_9 & \mathrm{x}_{10} & \mathrm{x}_{11} \\
 & 1 & . & . & . & . & . & . & . & . & . & . \\
 & & 1 & . & . & . & . & . & . & . & . & . \\
 & & & 1 & . & . & . & . & . & . & . & . \\
 & & & & 1 & . & . & . & . & . & . & . \\
 & & & & & 1 & . & . & . & . & . & . \\
 & & & & & & 1 & . & . & . & . & . \\
 & & & & & & & 1 & . & . & . & . \\
 & & & & & & & & 1 & . & . & . \\
 & & & & & & & & & 1 & . & . \\
 & & & & & & & & & & 1 & . \\
 & & & & & & & & & & & 1
\end{bmatrix} \right\} \text{ and}$$

$$B = \left\{ \begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . & . & x_1 \\ & 1 & . & . & . & . & . & . & . & . & . & x_2 \\ & & 1 & . & . & . & . & . & . & . & . & x_3 \\ & & & 1 & . & . & . & . & . & . & . & x_4 \\ & & & & 1 & . & . & . & . & . & . & x_5 \\ & & & & & 1 & . & . & . & . & . & x_6 \\ & & & & & & 1 & . & . & . & . & x_7 \\ & & & & & & & 1 & . & . & . & x_8 \\ & & & & & & & & 1 & . & . & x_9 \\ & & & & & & & & & 1 & . & x_{10} \\ & & & & & & & & & & 1 & x_{11} \\ & & & & & & & & & & & 1 \end{bmatrix} \right\}$$

where the orders of A and B are equal to 2^{11}

A corresponds to transvections: $I + \begin{bmatrix} 1 \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \end{bmatrix} \begin{bmatrix} . & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} \end{bmatrix}$

And B corresponds to transvections: $I + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ . \end{bmatrix} \begin{bmatrix} . & . & . & . & . & . & . & . & . & . & 1 \end{bmatrix}$

Since S(2) does not contain any transvections, then both A and B must be the identity element. Then S(2) contains no elementary abelian subgroup of order 8. □

Proposition 17. *([1]) Let Y be a simple group. Assume that the 2-Sylow subgroup of Y contains no elementary abelian subgroup of order 8. Then Y is isomorphic to one of the following groups:* A_7, *PSL(2, q), PSL(3, q), PSU(3, q) with q odd or*

PSU(3, 4).

We will proceed to determine which of these groups will satisfy the conditions of Proposition 17.

Lemma 18. $A_7 \not\subset G$.

Proof. Since the irreducible 2-modular characters for A_7 by GAP are:

$$[[1,1],[4,2],[6,1],[14,1],[20,1]]$$

(gap > CharacterDegrees (CharacterTable ("A7") mod 2)); And non of them of degree 11. □

Lemma 19. *If PSL(2, q) $\subset$ G, q odd, then q = 11, 13 or 25.*

Proof. PSL(2, q) has no projective representation in G of degree <(1/2)(q-1) ([15] and [18]) and (1/2)(q-1) > 12 for all odd q > 25. Hence we need only to consider the cases when q $\leq$ 25.

1. PSL(2, 3)$\not\subset$ G, since PSL(2, 3) is not simple.
2. PSL(2, 5) $\cong$ PSL(2, 2^2), The irreducible 2-modular characters for PSL(2, 5) by GAP are:

   $$[[1,1],[2,2],[4,1]],$$

   (gap > CharacterDegrees (CharacterTable (" L2(5) ") mod 2)); But non of them of degree 12. Therefore if PSL(2, 5) $\subset$ G, then it is reducible.
3. PSL(2, 7) $\cong$ PSL(3, 2), The irreducible 2-modular characters for PSL(2,7) by GAP are:

   $$[[1,1],[3,2],[8,1]],$$

   (gap > CharacterDegrees (CharacterTable (" L2(7) ") mod 2)); But non of them of degree 12. Therefore if PSL(2, 7) $\subset$ G, then it is reducible.
4. For PSL(2, 3^2) $\cong A_6$: The irreducible 2-modular characters for PSL(2, 3^2) by GAP are:

   $$[[1,1],[4,2],[8,2]].$$

   (gap > CharacterDegrees (CharacterTable (" L2(9) ") mod 2)); But non of them of degree 12. Therefore if PSL(2, 3^2) $\subset$ G, then it is reducible.
5. **PSL(2, 11)** $\subset$ **G**, since the irreducible 2-modular characters for PSL(2, 11) by GAP are:

   $$[[1,1],[5,2],[10,1],[12,2]].$$

   (gap > CharacterDegrees (CharacterTable ("L2(11) ") mod 2));

6. **PSL(2, 13)** $\subset$ **G**, since the irreducible 2-modular characters for PSL(2, 13) by GAP are:
$$[[1, 1], [6, 2], [12, 3], [14, 1]].$$
(gap > CharacterDegrees (CharacterTable (" L2(13) ") mod 2));

7. For PSL(2, 17): The irreducible 2-modular characters for PSL(2, 17) by GAP are:
$$[[1, 1], [8, 2], [16, 4]],$$
(gap > CharacterDegrees (CharacterTable (" L2(17) ") mod 2)); But non of them of degree 12. Therefore if PSL(2, 17) $\subset$ G, then it is reducible.

8. For PSL(2, 19): The irreducible 2-modular characters for PSL(2, 19) by GAP are:
$$[[1, 1], [9, 2], [18, 2], [20, 4]],$$
(gap > CharacterDegrees (CharacterTable (" L2(19) ") mod 2)); But non of them of degree 12. Therefore if PSL(2, 19) $\subset$ G, then it is reducible.

9. For PSL(2, 23): The irreducible 2-modular characters for PSL(2, 23) by GAP are:
$$[[1, 1], [11, 2], [22, 1], [24, 5]]$$
gap> CharacterDegrees(CharacterTable("PSL(2,23)")mod 2); But non of them of degree 12. Therefore if PSL(2, 23) $\subset$ G, then it is reducible.

10. **PSL(2, 25)** $\subset$ **G**, since the irreducible 2-modular characters for PSL(2, 25) by GAP are:
$$[[1, 1], [12, 2], [24, 6], [26, 1]]$$
gap> CharacterDegrees(CharacterTable("PSL(2,25)")mod 2);

□

Lemma 20. *If PSL(3, q) $\subset$ G, then q = 3.*

Proof. PSL(3, q) has no projective representation in G of degree $< q^{n-1} - 1 = q^2 - 1$ ([15] and [18]) and it is clear that q^2-1 > 13 for all q $\geq$ 4. Thus, we need to test PSL(3, 2) and PSL(3, 3) as primitive subgroups of G?

1. PSL(3, 2) $\not\subset$ G. Since PSL(3, 2) $\cong$ PSL(2, 7), and PSL(2, 7) $\not\subset$ G, [see Lemma 19].

2. **PSL(3, 3)** $\subset$ **G**, since the irreducible 2-modular characters for PSL(3, 3) by GAP are:
$$[1, 1], [12, 1], [16, 4], [26, 1]],$$
(gap > CharacterDegrees (CharacterTable (" PSL(3, 3) ") mod2));

□

Lemma 21. *PSU(3, q) $\not\subset$ G, for all q.*

Proof. PSU(3, q) has no projective representation in G of degree < q(q-1) [18], and it is clear that q(q-1) > 12 for all q $\geq$ 5. Thus, we need to test PSU(3, 2), PSU(3, 3) and PSU(3, 4) are primitive subgroups of G?

1. PSU(3, 2) is not simple.
2. PSU(3, 3) $\not\subset$ G, since the irreducible 2-modular characters for PSU(3, 3) by GAP are:
$$[1, 1], [6, 1], [14, 1], [32, 2]],$$
(gap> CharacterDegrees(CharacterTable("U3(3)")mod 2)). and non of these of degree 12.
3. PSU(3, 4) is not simple.

□

4.3 Primitive subgroups H of G which have the property that a minimal normal subgroup of H which is not abelian is doubly transitive group

In this section, we will consider a minimal normal subgroup M of H is not abelian and is doubly transitive group: The following Corollary will be the main result of this section:

Corollary 22. *If M is a non abelian simple group of doubly transitive group H of G, then M is isomorphic to one of the following groups:*

1. *A_{13};*
2. *A_{14};*
3. *PSL(2, 11);*
4. *PSL(2, 13);*
5. *PSL(2, 25);*
6. *PSL(3, 3);*
7. *PSU(2, 11);*
8. *PSU(2, 13).*

Proof. Since every doubly transitive group is a primitive group [3], then we will use the classification of doubly transitive groups (Proposition 23). And we will prove Corollary 22 by series of Lemma 24 through Lemma 36 and Proposition 23. □

Proposition 23. *([8, 17]). If Y be a doubly transitive group, then Y has a simple normal subgroup M^*, and $M^* \subseteq Y \subseteq Aut(M^*)$, where M^* as follows:*

1. A_n, $n \geq 5$;
2. *PSL(d, q), $d \geq 2$, where $(d, q) \neq (2, 2), (2, 3)$;*
3. *PSU(3, q), $q > 2$;*
4. *the Suzuki group Sz(q), $q = 2^{2m+1}$ and $m > 0$;*
5. *the Ree group Re(q), $q = 3^{2m+1}$ and $m > 0$;*
6. *Sp(2n, 2), $n \geq 3$;*
7. *PSL(2, 11);*
8. *Mathieu groups M_n, n = 11, 12, 22, 23, 24;*
9. *HS (Higman-Sims group);*
10. *CO_3 (Conway's smallest group).*

In the following, we will discuss the different possibilities of Proposition 23:

Lemma 24. *If $A_n \subset G$, then n = 13 or 14.*

Proof. From [19], An for all n > 8, has a unique faithful 2-modular representation of least degree, this degree being (n-1) if n is odd and (n-2) if n is even, so, the 2-modular representation of least degree is greater than 12 for all n = 15. Thus $A_n \not\subset$ G for any n = 15.

1. $A_5 \not\subset$ G: since the irreducible 2-modular characters for A_5 by GAP are:

$$[[1, 1], [2, 2], [4, 1]]$$

(gap > CharacterDegrees (CharacterTable ("A5") mod 2));

2. $A_6 \not\subset$ G: since the irreducible 2-modular characters for A_6 by GAP are:

$$[[1, 1], [4, 2], [8, 2]]$$

(gap > CharacterDegrees (CharacterTable ("A6") mod 2));

3. $A_7 \not\subset$ G: since the irreducible 2-modular characters for A_7 by GAP are:

$$[[1,1],[4,2],[6,1],[14,1],[20,1]]$$

(gap > CharacterDegrees (CharacterTable ("A7") mod 2));

4. $A_8 \not\subset$ G: since the irreducible 2-modular characters for A_8 by GAP are:

$$[[1,1],[4,2],[6,1],[14,1],[20,2],[64,1]]$$

(gap > CharacterDegrees (CharacterTable ("A8") mod 2));

5. $A_9 \not\subset$ G: since the irreducible 2-modular characters for A_9 by GAP are:

$$[[1,1],[8,3],[20,2],[26,1],[48,1],[78,1],[160,1]]$$

(gap > CharacterDegrees (CharacterTable ("A9") mod 2));

6. $A_{10} \not\subset$ G: since the irreducible 2-modular characters for A_{10} by GAP are:
[[1, 1], [8, 1], [16, 1], [26, 1], [48, 1], [64, 2], [160, 1], [198, 1], [200, 1], [384, 2]]
(gap > CharacterDegrees (CharacterTable ("A10") mod 2)).

7. $A_{11} \not\subset$G: since the irreducible 2-modular characters for A_{11} by GAP are:
[[1, 1], [10, 1], [16, 2], [44, 1], [100, 1], [144, 1], [164, 1], [186, 1], [198, 1], [416, 1], [584, 2], [848, 1]]
(gap > CharacterDegrees (CharacterTable ("A11") mod 2));

8. $A_{12} \not\subset$ G: since the irreducible 2-modular characters for A_{12} by GAP are:
[[1, 1], [10, 1], [16, 2], [44, 1], [100, 1], [144, 2], [164, 1], [320, 1], [416, 1], [570, 1], [1046, 1], [1184, 2], [1408, 1], [1792, 1], [5632,1].
(gap > CharacterDegrees (CharacterTable ("A12") mod 2));

9. $A_{13} \subset$ G: since the irreducible 2-modular characters for A_{13} by GAP are:
[[1, 1], [12, 1], [32, 2], [64, 1], [144, 2], [208, 1], [364, 2], [560, 1], [570, 1], [1572, 1], [1728, 1], [2208, 1], [2510, 1], [2848, 1], [3200, 1], [4224, 2], [8008, 1]]
(gap > CharacterDegrees (CharacterTable ("A13") mod 2));

10. $A_{14} \subset$ G: since the irreducible 2-modular characters for A_{14} by GAP are:
[[1, 1], [12, 1], [64, 2], [208, 1], [364, 1], [384, 2], [560, 2], [1300, 1], [2016, 1], [2510, 1], [3418, 1], [3808, 1], [4576, 1], [4704, 1], [6656, 2], [10880, 1], [11648, 1], [17920, 2], [19240, 1], [23296, 1]]
(gap> CharacterDegrees(CharacterTable("A14")mod 2);)

□

Lemma 25. *If PSL(2, q) $\subset$ G, then q = 11, 13 or q = 25.*

Proof. We have two cases:
Case 1: q is even:
PSL(2, q) has no projective representation in G of degree < (1/d)(q-1), d = g,c.d(2, q-1) ([15] and [18]) and (q-1) >12 for all even q $\geq$ 16. Also,

1. PSL(2, 2) not simple.

2. PSL(2, 4) $\not\subset$ G, since the irreducible 2-modular characters for PSL(2, 4) by GAP are:

   $$[[1, 1], [2, 2], [4, 1]],$$

   (gap > CharacterDegrees (CharacterTable (" L2(4) ") mod 2)); and non of these of degree 12.

3. PSL(2, 8) $\not\subset$ G, since the irreducible 2-modular characters for PSL(2, 8) by GAP are:

   $$[[1, 1], [2, 3], [4, 3], [8, 1]],$$

   (gap > CharacterDegrees (CharacterTable (" L2(4) ") mod 2)); and non of these of degree 12.

Thus, PSL(2, q) $\not\subset$ G for all q is even.
Case 2: q is odd:
If PSL(2, q) $\subset$ G, q is odd, then q = 11, 13 or 25. [see Lemma 19]. □

Lemma 26. *PSL(n, 2) $\not\subset$ G for all n.*

Proof. PSL(n, 2) has no projective representation in G of degree < q^{n-1}-1 = 2^{n-1}-1 ([15] and [18]), and it is clear that 2^{n-1}-1 > 12 for all n > 4. Thus, we need to test PSL(2, 2), PSL(3, 2) and PSL(4, 2) are primitive subgroups of G?

1. PSL(2, 2) is not simple.

2. PSL(3, 2) $\not\subset$G. Since PSL(3, 2) $\cong$ PSL(2, 7), and PSL(2, 7) $\not\subset$ G [see Lemma 19].

3. PSL(4, 2) $\not\subset$G. Since PSL(4, 2) $\cong$ A_8, and A_8 $\not\subset$ G [see Lemma 19].

□

Lemma 27. *If PSL(n, q) $\subset$ G, then (n, q) = (2, 11), (2, 13), (2, 25) or (3, 3).*

Proof. PSL(n, q) has no projective representation in G of degree < (q^{n-1}-1) ([15] and [18]), which > 12 for all for all q $\geq$ 3 and n $\geq$ 4. Thus, we need to test PSL(2, q), PSL(3, q) and PSL(n, 2) as primitive subgroups of G?

1. If PSL(2, q) $\subset$ G, then q = 11 , 13 or 25 [see Lemma 25].
2. If PSL(3, q) $\subset$ G, then q = 3 [see Lemma 20].
3. PSL(n, 2) $\not\subset$ G for all n [see Lemma 26].

□

Lemma 28. *If PSU(2, q) $\subset$ G, then q = 11 or 13.*

Proof. PSU(2, q) $\subseteq$ PGL(2, q). But PGL(2, q) has no projective representation in G of degree < (q-1), provided q $\neq$ 9 [18], which > 12 for all q > 13. Thus, we need to test PSU(2, 2), PSU(2, 3), PSU(2, 4), PSU(2, 5), PSU(2, 7), PSU(2, 9), PSU(2, 11) and PSU(2, 13) are primitive subgroups of G?

1. PSU(2, 2) is not simple.
2. PSU(2, 3) is not simple.
3. PSU(2, 4) $\not\subset$ G, since the irreducible 2-modular characters for PSU(2, 4) by GAP are:

   $$[[1,1],[2,2],[4,1]],$$

   (gap> CharacterDegrees(CharacterTable("U2(4)")mod 2)) and there is non of degree 12.
4. PSU(2, 5) $\not\subset$ G, since the irreducible 2-modular characters for PSU(2, 5) by GAP are:

   $$[[1,1],[2,2],[4,1]],$$

   (gap> CharacterDegrees(CharacterTable("U2(5)")mod 2)). and there is non of degree 12.
5. PSU(2, 7) $\not\subset$ G, since the irreducible 2-modular characters for PSU(2, 7) by GAP are:

   $$[[1,1],[3,2],[8,1]],$$

   (gap> CharacterDegrees(CharacterTable("U2(7)")mod 2)). and there is non of degree 12.
6. PSU(2, 9) $\not\subset$ G, since the irreducible 2-modular characters for PSU(2, 9) by GAP are:

   $$[[1,1],[4,2],[8,2]],$$

   (gap> CharacterDegrees(CharacterTable("U2(9)")mod 2)). and there is non of degree 12.

7. **PSU(2, 11)** $\subset$ **G,** since the irreducible 2-modular characters for PSU(2, 11) by GAP are:
$$[[1,1],[5,2],[10,1],[12,2]],$$
(gap> CharacterDegrees(CharacterTable("U2(11)")mod 2)).

8. **PSU(2, 13)** $\subset$ **G**, since the irreducible 2-modular characters for PSU(2, 13) by GAP are:
$$[[1,1],[6,2],[12,3],[14,1]],$$
(gap> CharacterDegrees(CharacterTable("U2(13)")mod 2)).

□

Lemma 29. *PSU(n, 2)* $\not\subset$ *G, for all n.*

Proof. PSU(n, q), n $\geq$ 3, has no projective representation in G of degree < q(q^{n-1}-1)/(q+1) if n is odd, and PSU(n, q), n $\geq$ 3, has no projective representation in G of degree < (q^n-1)/(q+1) if n is even ([15] and [18]), thus the minimal projective degree for PSU(n, 2) is > 12 for all n $\geq$ 6. Thus, we need to test PSU(2, 2) , PSU(3, 2), PSU(4, 2) and PSU(5, 2) are primitive subgroups of G?

1. PSU(2, 2^2) is not simple.

2. PSU(3, 2^2) is not simple.

3. PSU(4, 2) $\not\subset$ G. Since the irreducible 2-modular characters for PSU(4, 2) by GAP are:
$$[[1,1],[4,2],[6,1],[14,1],[20,2],[64,1]],$$
(gap> CharacterDegrees(CharacterTable("U4(2)") mod 2)). and non of these of degree 12.

4. PSU(5, 2) $\not\subset$ G, since the irreducible 2-modular characters for PSU(5, 2) by GAP are:
$$[[1,1],[5,2],[10,2],[24,1],[40,4],[74,1],[160,2],[280,2],[1024,1]],$$
(gap> CharacterDegrees(CharacterTable("U5(2)") mod 2)). and non of these of degree 12.

□

Lemma 30. *if PSU(n, q)* $\subset$ *G, then n = 2, q = 11 or 13.*

Proof. PSU(n, q), n $\geq$ 3, has no projective representation in G of degree < q(q^{n-1}-1)/(q+1) if n is odd, and PSU(n, q) , n $\geq$ 3, has no projective representation in G of degree < (q^n-1)/(q+1) if n is even ([15] and [18]), thus, the minimal projective degree is > 11 for all n > 3 and q $\geq$ 3. Thus, we need to test PSU(n, 2), PSU(2, q) and PSU(3, q) are primitive subgroups of G?

1. PSU(n, 2) $\not\subset$ G [see Lemma 29].
2. PSU(2, q) $\subset$ G, for q = 11 or 13 [see Lemma 28].
3. PSU(3, q) $\not\subset$ G [see Lemma 21].

□

Lemma 31. *Sz(q)* $\not\subset$ *G,* $q = 2^{2m+1}$ *and* $m > 0$*.*

Proof. The irreducible 2-modular characters for Suzuki groups by GAP are:

$$[[1, 1], [4, 3], [16, 3], [64, 1]]$$

(gap > CharacterDegrees (CharacterTable (" Sz(8) ") mod 2)); and non of these of degree 12, thus Sz(q) $\not\subset$ G. □

Lemma 32. *Re(q)* $\not\subset$ *G,* $q = 3^{2m+1}$*.*

Proof. The irreducible 2-modular characters for Ree group Re(q) by GAP are:

$$[[1, 1], [702, 1], [741, 2], [2184, 2], [13832, 6], [16796, 1], [18278, 1], [19684, 6], [26936, 3]]$$

(gap> CharacterDegrees (CharacterTable (" R(27) ") mod 2)); and non of these of degree 12, thus Re(q) $\not\subset$ G. □

Lemma 33. *PSp(2n, 2)* $\not\subset$ *G for all* $n \geq 3$*.*

Proof. From ([15] and [18]), PSp(2n, q), n $\geq$ 2 has no projective representation in G of degree < $(1/2)q^{n-1}(q^{n-1}$ - 1)(q-1) if q is even. And since q = 2, then $(1/2)q^{n-1}(q^{n-1}$ - 1)(q-1) > 12 for all n $\geq$ 4. Thus, we need to test PSp(6, 2) is a primitive subgroups of G? The irreducible 2-modular characters for PSp(6, 2) by GAP are:

$$[1, 1], [6, 1], [8, 1], [14, 1], [48, 1], [64, 1], [112, 1], [512, 1]]$$

(gap> CharacterDegrees(CharacterTable("S6(2)")mod 2); and non of these of degree 12, thus PSp(6, 2) $\not\subset$ G. □

Lemma 34. *The Mathieu groups* $M_n \not\subset G$*, for all n = 11, 12, 22, 23 and 24.*

Proof.

1. $M_{11} \not\subset$ G, since the irreducible 2-modular characters for Mathieu group M_{11} by GAP are:

$$[[1, 1], [10, 1], [16, 2], [44, 1]],$$

(gap > CharacterDegrees (CharacterTable (" M11 ") mod 2)); and non of these of degree 12.

2. $M_{12} \not\subset$ G, since the irreducible 2-modular characters for Mathieu group M_{12} by GAP are:

$$[[1,1],[10,1],[16,2],[44,1],[144,1]],$$

(gap > CharacterDegrees (CharacterTable (" M12 ") mod 2)); and non of these of degree 12.

3. $M_{22} \not\subset$ G, since the irreducible 2-modular characters for Mathieu group M_{22} by GAP are:

$$[[1,1],[10,2],[34,1],[70,2],[98,1]],$$

(gap > CharacterDegrees (CharacterTable (" M22 ") mod 2)). and non of these of degree 12.

4. $M_{23} \not\subset$ G, since the irreducible 2-modular characters for Mathieu group M_{23} by GAP are:

$$[[1,1],[11,2],[44,2],[120,1],[220,2],[252,1],[896,2]]$$

gap> CharacterDegrees(CharacterTable("M23")mod 2); and non of these of degree 12.

5. $M_{24} \not\subset$ G, since the irreducible 2-modular characters for Mathieu group M_{24} by GAP are:

$$[[1,1],[11,2],[44,2],[120,1],[220,2],[252,1],[320,2],[1242,1],[1792,1]].$$

gap> CharacterDegrees(CharacterTable("M24")mod 2); and non of these of degree 12.

□

Lemma 35. *HS (Higman-Sims group)* $\not\subset$ *G;*

Proof. The minimal degrees of faithful representations of the Higman-Sims group over F_2 is 20, which is greater than 12 [7]. □

Lemma 36. CO_3 *(Conway's smallest group)* $\not\subset$ *G;*

Proof. The minimal degrees of faithful representations of the CO_3 over F_2 is 22, which is greater than 12 [7]. □

Now, we will determine the maximal primitive groups of the class C_9:

Theorem 37. *: If H is a maximal primitive subgroup of G which has the property that a minimal normal subgroup M of H is not abelian group, then H is isomorphic to one of the following subgroups of G:*

1. *P*Γ*L(2, 11),*
2. *P*Γ*L(2, 13),*
3. *P*Γ*L(2, 25),*
4. *P*Γ*L(3, 3),*

Proof. We will prove this theorem by finding the normalizers of the groups of Corollary (12) and determine which of them are maximal:

1. The normalizer of A_{13} is the symmetric group S_{13} which is not simple group, also, he normalizer of A_{14} is the symmetric group S_{14} which is not simple group [19].

2. The normalizer of PSL(2, 11) is PΓL(2, 11) ([10], [13], [21] and [22]). Thus PΓL(2, 11) is a maximal primitive subgroup of G.

3. The normalizer of PSL(2, 13) is PΓL(2, 13) ([10], [13], [21] and [22]). Thus PΓL(2, 13) is a maximal primitive subgroup of G.

4. The normalizer of PSL(2, 25) is PΓL(2, 25) ([10], [13], [21] and [22]). Thus PΓL(2, 25) is a maximal primitive subgroup of G

5. The normalizer of PSL(3, 3) is PΓL(3, 3) ([10], [13], [21] and [22]). Thus PΓL(3, 3) is a maximal primitive subgroup of G.

6. The normalizer of PSU(2, 11) is PGL(2, 11) ([10], [13], [21] and [22]). But PGL(2, 11) $\subset$ PΓL(2, 11), thus PGL(2, 11) is not a maximal primitive subgroup of G. similarly the normalizer of PSU(2, 13) is PGL(2, 13), but PGL(2, 13) $\subset$ PΓL(2, 13), thus PGL(2, 13) is not a maximal primitive subgroup of G

□

Theorem 37 prove the points (16), (17), (18), (19) of Theorem 1, and this complete the proof of Theorem 1.

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Rauhi Ibrahim Elkhatib

Dept. of Mathematics, Faculty of Applied Science, Thamar University, Yemen.

P.O. Box: 12559.

e-mail: rauhie@yahoo.com