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UNIFORMLY CONTINUOUS FUNCTIONS ON NON-HAUSDORFF GROUPOIDS

Mădălina Roxana Buneci

Abstract. The purpose of this paper is to study the notion of uniform continuity introduced in [1]. For a locally compact (not necessarily Hausdorff) groupoid endowed with pre-Haar systems (in the sense of [1] adapted to non-Hausdorff case) we prove that the space of bounded compactly supported functions which are left and right uniformly continuous on fibres can be made into a *-algebra and endowed with a (reduced) C^*-norm. The advantage of working with uniformly continuous on fibres functions is the fact that even if the groupoid does not admit a continuous Haar system, various C^*-algebras can be associated with it.

1 Terminology and notation

We begin this section by recalling basic definitions, notation and terminology from groupoids. A groupoid is like a group with multiplication only partially defined. More precisely, a *groupoid* is a set G, together with a distinguished subset $G^{(2)} \subset G \times G$ (called the *set of composable pairs*), and two maps:

$$(x, y) \to xy \ \left[: G^{(2)} \to G\right] \text{ (product map)}$$

$$x \to x^{-1} \ [: G \to G] \text{ (inverse map)}$$

such that the following relations are satisfied:

1. If $(x, y) \in G^{(2)}$ and $(y, z) \in G^{(2)}$, then $(xy, z) \in G^{(2)}$, $(x, yz) \in G^{(2)}$ and $(xy)\, z = x\, (yz)$.
2. $\left(x^{-1}\right)^{-1} = x$ for all $x \in G$.
3. For all $x \in G$, $\left(x, x^{-1}\right) \in G^{(2)}$, and if $(z, x) \in G^{(2)}$, then $(zx)\, x^{-1} = z$.
4. For all $x \in G$, $\left(x^{-1}, x\right) \in G^{(2)}$, and if $(x, y) \in G^{(2)}$, then $x^{-1}(xy) = y$.

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The maps r and d on G, defined by the formula $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the *range* and the *source* (*domain*) maps. It follows easily from the definition that they have a common image called the *unit space* of G, which is denoted $G^{(0)}$. Its elements are *units* in the sense that $xd(x) = r(x)x = x$. It is useful to note that a pair (x, y) lies in $G^{(2)}$ precisely when $d(x) = r(y)$, and that the cancellation laws hold (e.g. $xy = xz$ iff $y = z$).

The *fibres* of the range and the source maps are denoted by $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively, and their intersection is denoted by $G^u_v = G^u \cap G_v$, $u, v \in G^{(0)}$. The graphs of the equivalence relations induced by r, respectively d are:

$$\begin{aligned} G_r \times_r G &= \{(x, y) : r(x) = r(y)\} \\ G_d \times_d G &= \{(x, y) : d(x) = d(y)\}. \end{aligned}$$

A *topological groupoid* consists of a groupoid G and a topology compatible with the groupoid structure. This means that:

(1) $x \to x^{-1}$ $[: G \to G]$ is continuous.

(2) (x, y) $\left[: G^{(2)} \to G\right]$ is continuous where $G^{(2)}$ has the induced topology from $G \times G$.

Obviously, if G is a topological groupoid, then the inverse map $x \mapsto x^{-1}$ is a homeomorphism from G to G and the maps r and d are continuous. Moreover, the maps r and d are identification maps, since they have the inclusion $G^{(0)} \hookrightarrow G$ as a right inverse. If u is a unit then $x \mapsto x^{-1}$ is a homeomorphism from G^u to G_u. If $u \sim v$ are two equivalent units and x is such that $r(x) = u$ and $d(x) = v$ then $y \mapsto xy$ is a homeomorphism from G^v to G^u, $y \mapsto yx$ is a homeomorphism from G_u to G_v and $y \mapsto xyx^{-1}$ is an isomorphism of topological groups from G^v_v to G^u_u.

If $G^{(0)}$ is Hausdorff, then $G^{(2)}$ is closed in $G \times G$, being the set where two continuous maps (to a Hausdorff space) coincide. If G is Hausdorff then $G^{(0)}$ is closed in G, being the image of the map $x \mapsto xx^{-1}$ $[: G \to G]$ whose square is itself.

If G is a topological group such that the singleton sets are closed, then G is Hausdorff. However for general topological groupoids G this is no longer true. But topological groupoids for which the points are closed and which are non-Hausdorff occur in many important examples of foliations such as Reeb foliations.

We are concerned with topological groupoids whose topology is (not necessarily Hausdorff) locally compact. Lots of natural examples of groupoids have such a topology. Let us establish terminology and let us recall a few basic facts about locally compact groupoids.

Definition 1. *A topological space X is said to be* locally compact *if every point $x \in X$ has a compact Hausdorff neighborhood.*

Any locally compact space X in the sense of Definition 1 is locally Hausdorff. Therefore every singleton subset of X is closed and the diagonal in $X \times X$ is locally closed.

Definition 2. *A topological groupoid G is said to be locally compact if it is locally compact (in the sense of Definition 1) as a topological space.*

Definition 3. *We say that a topological groupoid G has Hausdorff fibres if for every $u \in G^{(0)}$, G^u (and hence G_u) is a Hausdorff subspace.*

Let us state some consequences (proved by J.L. Tu [8, p. 569]) of the assumption of locally compactness of a topological groupoid:

(1) If the groupoid G is locally compact, then it is locally Hausdorff. Therefore every singleton subset of G is closed and consequently, for every $u \in G^{(0)}$, $G^u = r^{-1}(\{u\})$ and $G_u = d^{-1}(\{u\})$ are closed in G, and hence locally compact.

(2) If G is a locally compact groupoid, then $G^{(0)}$ is locally closed in G, hence locally compact.

(3) If G is a locally compact groupoid and if $G^{(0)}$ is Hausdorff, then for every $u \in G^{(0)}$, G^u and G_u are Hausdorff. Thus G is a locally compact groupoid with Hausdorff fibres.

2 Uniform continuity on locally compact non-Hausdorff groupoids

It is well known that any continuous function with compact support defined on a locally compact group is uniformly continuous. Let us show that in a suitable sense (introduced in [1]) the property of locally compact groups generalizes to locally compact groupoids.

Definition 4. *([1, Definition 3.1/p. 39]) Let G be a topological groupoid and E be a Banach space. The function $h : G \to E$ is said to be* left uniformly continuous on fibres *if and only if for each $\varepsilon > 0$ there is a neighborhood W of the unit space $G^{(0)}$ such that:*

$$|h(x) - h(y)| < \varepsilon \ \text{ for all } (x, y) \in G_r \times_r G,\ y^{-1}x \in W.$$

The function $h : G \to E$ is said to be right uniformly continuous on fibres *if and only if for each $\varepsilon > 0$ there is a neighborhood W of the unit space $G^{(0)}$ such that:*

$$|h(x) - h(y)| < \varepsilon \ \text{ for all } (x, y) \in G_d \times_d G,\ xy^{-1} \in W.$$

It is easy to see that h is right uniformly continuous on fibres if and only if for each $\varepsilon > 0$ there is a neighborhood W of the unit space such that:

$$|h(sy) - h(y)| < \varepsilon \ \text{ for all } s \in W \text{ and } y \in G^{d(s)}$$

and that h is left uniformly continuous on fibres if and only if for each $\varepsilon > 0$ there is a neighborhood W of the unit space such that:

$$|h(ys) - h(y)| < \varepsilon \text{ for all } s \in W \text{ and } y \in G_{r(s)}.$$

Let us notice that a left or/and right uniformly continuous on fibres function need not be continuous. For instance, if G is a topological groupoid with open unit space $G^{(0)}$ (an r-discrete groupoid in the sense of [5, Definition I.2.6/p. 18]), then any function on G is left or right uniformly continuous on fibres in the sense of the preceding definition.

Remark 5. *Let G be a topological groupoid.*

1. *If $f : G \to \mathbb{C}$ is left (respectively, right) uniformly continuous on fibres , then for all $u \in G^{(0)}$, $f|_{G^u}$ (respectively, $f|_{G_u}$) is continuous. Indeed, let u be a unit, $x_0 \in G^u$ and $\varepsilon > 0$. If f is left uniformly continuous on fibres, then there is a neighborhood W_ε of $G^{(0)}$ such that:*

   $$|f(x) - f(y)| < \varepsilon \text{ for all } (x, y) \in G_r \times_r G,\ x^{-1}y \in W_\varepsilon.$$

   Since $y \mapsto x_0 y$ is a homeomorphism from $G^{d(x_0)}$ to G^u, it follows that $x_0 W_\varepsilon = x_0\left(W_\varepsilon \cap G^{d(x_0)}\right)$ is an open subset of G^u containing x_0. But if $y \in x_0 W_\varepsilon$, then $x_0^{-1} y \in W_\varepsilon$ and therefore $|f(x_0) - f(y)| < \varepsilon$. Thus $f|_{G^u}$ is continuous in x_0. Similarly, if f is right uniformly continuous on fibres, then $f|_{G_u}$ is continuous.

2. *If $f, g : G \to \mathbb{C}$ are left (respectively, right) uniformly continuous on fibres, then $|f|$, $\overline{f}$, $f + g$ are left (respectively, right) uniformly continuous on fibres. If $f, g : G \to \mathbb{C}$ are left (respectively, right) uniformly continuous on fibres bounded functions, then fg is a left (respectively, right) uniformly continuous on fibres bounded function.*

Lemma 6. *Let G be a topological groupoid, let K be a compact set such that $r(K)$ has a Hausdorff neighborhood in $G^{(0)}$, let S be a topological space, let W be an open subset of S and let $f : G^{(2)} \to S$ be a function which is continuous in every point of $(U \times K) \cap G^{(2)}$, where U is an open set in G. Then*

$$L = \left\{ s \in U : f(s, y) \in W \text{ for all } y \in K \cap G^{d(s)} \right\}$$

is an open subset of G.

Proof. Let $A \subset G^{(0)}$ be a Hausdorff neighborhood of $r(K)$. Let $s_0 \in U$ and $y \in K$ with $d(s_0) = r(y)$. Since $f(s_0, y) \in W$ and f is continuous in (s_0, y), it follows that there are two open sets U_y and V_y such that $(s_0, y) \in U_y \times V_y$ and $f((U_y \times V_y) \cap G^{(2)}) \subset W$. Since $\{V_y\}_{y \in K \cap G^{d(s_0)}}$ is an open cover of the compact set $K \cap G^{d(s_0)}$, it follows that there are $y_1, y_2, ..., y_n \in K$ such that $\underset{i=1,n}{\cup} V_{y_i} \supset K \cap G^{d(s_0)}$. Let us denote $U = \underset{i=1,n}{\cap} U_{y_i}$ and $V = \underset{i=1,n}{\cup} V_{y_i}$. Then U is an open neighborhood of s_0 and V is an open neighborhood of $K \cap G^{d(s_0)}$. Since K is compact and V open, it follows that $K \setminus V$ is compact and therefore $r(K \setminus V)$ is compact. Thus $r(K \setminus V)$ is compact, $r(K \setminus V) \subset A$ and A Hausdorff. Consequently, $r(K \setminus V)$ is relatively closed in A or equivalently $A \setminus r(K \setminus V)$ is an open subset of $G^{(0)}$. Furthermore $U_0 = U \cap G_{A \setminus r(K \setminus V)}$ is an open neighborhood of s_0 and $f\left((U_0 \times K) \cap G^{(2)}\right) \subset W$. Hence $s_0 \in U_0 \subset L$. □

Similarly, the following lemma can be proved:

Lemma 7. *Let G be a topological groupoid, let K be a compact set such that $d(K)$ has a Hausdorff neighborhood in $G^{(0)}$, let S be a topological space, let W be an open subset of S and let $f : G^{(2)} \to S$ be a function which is continuous in every point of $(K \times U) \cap G^{(2)}$, where U is an open set in G. Then*

$$L = \left\{y \in U : f(s, y) \in W \text{ for all } s \in K \cap G_{r(y)}\right\}$$

is an open set in G.

Proposition 8. *Let G be a locally compact groupoid and E be a Banach space. If a function $h : G \to E$ vanishes outside a compact set K and is continuous on an open neighborhood U of K, then it is left and right uniformly continuous on fibres.*

Proof. For each $x \in K$ let $A_{r(x)}$ be a Hausdorff open neighborhood of $r(x)$ and $K_x \subset U$ be a compact neighborhood of x such that $r(K_x) \subset A_{r(x)}$. Let U_x be the interior of K_x and let K'_x be a compact neighborhood of x contained in U_x. Thus

$$x \in K'_x \subset U_x \subset K_x \subset U.$$

Since K is compact there is finite family $\{x_i\}_i$ such that $\left\{K'_{x_i}\right\}_i$ is a cover of K. Let us denote

$$K'_i = K'_{x_i},\ K_i = K_{x_i},\ U_i = U'_{x_i}.$$

By Lemma 6, for each i

$$W_i = \left\{s \in G : sy \in U \text{ for all } \ y \in K_i \cap G^{d(s)}\right\}$$

is an open subset of G. Let us notice that $G^{(0)} \subset W_i$ and $W_i K_i \subset U$. Furthermore, also by Lemma 6,

$$W'_i = \left\{s \in W_i : sy \in U_i \text{ for all } \ y \in K'_i \cap G^{d(s)}\right\}$$

is an open subset of G. Moreover $G^{(0)} \subset W_i' \subset W_i$ and $W_i'K_i' \subset U_i \subset K_i$. Replacing W_i' (respectively, W_i) $W_i' \cap W_i'^{-1}$ (respectively, $W_i \cap W_i^{-1}$) we may assume that W_i' and W_i are two symmetric open neighborhoods of $G^{(0)}$ such that $W_i' \subset W_i$ and

$$K_i' \subset W_i'K_i' \subset K_i \subset W_iK_i \subset U.$$

Furthermore $W_0 = \bigcap_i W_i'$ is a symmetric open neighborhood of $G^{(0)}$ and for each i

$$K_i' \subset W_0K_i' \subset K_i \subset W_0K_i \subset U.$$

If we set

$$f : G^{(2)} \to E,\ f(s, y) = h(sy) - h(y)$$

then f is a continuous function in every point of $(W_0 \times K_i) \cap G^{(0)}$ for all i. Let $\varepsilon > 0$. By Lemma 6,

$$L_i = \left\{ s \in W_0 : |h(sy) - h(y)| < \varepsilon \text{ for all } \ y \in K_i \cap G^{d(s)} \right\}$$

is an open subset of G. Let us note that $G^{(0)} \subset L_i$ and consequently, $L = \bigcap_i L_i$ is an open symmetric neighborhood of $G^{(0)}$. Let $s \in L$ and $y \in G^{d(s)}$. Then either $y \in \bigcup_i K_i$, either $y \in G^{d(s)} \setminus \bigcup_i K_i$. In the first case, there is i such that $y \in K_i \cap G^{d(s)}$ and consequently, $|h(sy) - h(y)| < \varepsilon$. In the second case y as well as sy does not belong to K, otherwise

$$y \in s^{-1}K \subset L^{-1}K \subset W_0K \subset W_0 \bigcup_i K_i' \subset \bigcup_i W_0K_i' \subset \bigcup_i K_i.$$

Thus in this case $|h(sy) - h(y)| = 0$. Hence h is right uniformly continuous on fibres.

Similarly, h is left uniformly continuous on fibres. $\square$

Proposition 9. *Let G be a locally compact groupoid and K_1 and K_2 be two compact subsets of G such that K_1 is contained in the interior of K_2. Then there is a symmetric open neighborhood W of $G^{(0)}$ such that*

$$WK_1 \subset K_2 \text{ and } K_1W \subset K_2.$$

Proof. Let let U be the interior of K_2. For each $x \in K_1$ let $A_{r(x)}$ be a Hausdorff open neighborhood of $r(x)$ and $K_x \subset U$ be a compact neighborhood of x such that $r(K_x) \subset A_{r(x)}$. Since K_1 is compact, there is finite family $\{x_i\}_i$ such that $\{K_{x_i}\}_i$ is a cover of K. For each i, let us write K_i for K_{x_i}. By Lemma 6, for each i

$$W_i = \left\{ s \in G : sy \in U \text{ for all } \ y \in K_i \cap G^{d(s)} \right\}$$

is an open subset of G. It is easy to see that $G^{(0)} \subset W_i$ and $W_i K_i \subset U$. If we denote by $W_1 = \bigcap\limits_i W_i$, we obtain an open neighborhood of $G^{(0)}$ having the property that

$$W_1 K_1 \subset W_1 \bigcup_i K_i \subset \bigcup_i W_1 K_i \subset U \subset K_2.$$

Similarly, we can find an open neighborhood W_2 of $G^{(0)}$ such that $K_1 W_2 \subset K_2$. If we take $W = (W_1 \cap W_2) \cap (W_1 \cap W_2)^{-1}$, then $WK_1 \subset K_2$, $K_1 W \subset K_2$ and W is a symmetric open neighborhood of $G^{(0)}$. □

Remark 10. *Let K be a compact subset of $G^{(0)}$ and let W be a neighborhood of $G^{(0)}$. If $K_0 \subset G$ is a compact neighborhood of K, then according Proposition 9 there is a symmetric open neighborhood W_0 of $G^{(0)}$ such that*

$$W_0 K \subset K_0 \text{ and } KW_0 \subset K_0.$$

Without loss of generality, we may assume $W_0 \subset W$. The compact set K can be covered with a finite family of open subsets $\{U_i\}$ of G such that each U_i is the interior of a compact set K_i with the property that K_i has a Hausdorff neighborhood $V_i \subset W_0$. Let $K_1 = \bigcup\limits_i K_i$. Let φ_i be a nonnegative continuous compactly supported function defined on V_i such that $\varphi_i(u) = 1$ for all $u \in K_i$. Let us also denote by φ_i the extension of φ_i by 0 outside V_i. Then $\varphi = \sum_i \varphi_i$ is left and right uniformly continuous on fibres. Moreover if $K_\varphi = \bigcup\limits_i supp(\varphi_i)$, then K_φ is a compact set and the function φ vanishes outside $K_\varphi \subset W_0$. Since $\varphi(x) \geq 1$ for all $x \in K_1 \subset K_\varphi$, if follows that if $u \in K$, then u belongs to the interior of $\{x : \varphi(x) \geq 1\} \cap G^u$ (respectively, $\{x : \varphi(x) \geq 1\} \cap G_u$). Thus for each compact subset K of $G^{(0)}$ and each neighborhood W of $G^{(0)}$, there is a symmetric open neighborhood $W_0 \subset W$ of $G^{(0)}$ and a nonnegative function φ defined on G such that

1. *$W_0 K$ and KW_0 are contained in a compact subset of G;*
2. *φ is a left and right uniformly continuous on fibres which vanishes outside a compact set contained in W_0;*
3. *For all $u \in K$, the interior of the set $\{x : \varphi(x) \geq 1\} \cap G^u$ (respectively, of the set $\{x : \varphi(x) \geq 1\} \cap G_u$) is nonempty.*

3 Versions of the reduced groupoid C^*-algebra

The construction of the full/reduced C^*-algebras of a groupoid (introduced in [5]) extends the well-known case of a group. In the case of a locally compact Hausdorff groupoid G the space $C_c(G)$ of continuous functions with compact support is made into a $*$-algebra and endowed with the full/reduced C^*-norm. The full/reduced C^*-algebra of G is the completion of $C_c(G)$. If G is a not necessarily Hausdorff, locally compact groupoid, then as pointed out by A. Connes [3], one has to modify the choice of $C_c(G)$ (because $C_c(G)$ it is too small to capture the topological or differential structure of G). Usually $C_c(G)$ is replaced with the space $\mathcal{C}_c(G)$ of complex valued functions on G spanned by functions f which vanishes outside a compact set K contained in an open Hausdorff subset U of G and being continuous on U. Since in a non-Hausdorff space a compact set may not be closed, the functions in $\mathcal{C}_c(G)$ are not necessarily continuous on G. In the Hausdorff case $C_c(G)$ and $\mathcal{C}_c(G)$ coincide.

For developing an algebraic theory of functions on a locally compact groupoid (more precisely, to define convolution that gives the algebra structure on $\mathcal{C}_c(G)$), one needs an analogue of Haar measure on locally compact groups. This analogue is a system of measures, called Haar system, subject to suitable invariance and smoothness conditions called respectively "left invariance" and "continuity". Unlike the case of locally compact group, Haar system on groupoid need not exists, and if it does, it will not usually be unique. However, for instance, on locally compact Hausdorff second countable groupoids one can construct systems of measures that satisfy "left invariance" condition ([1]). But the continuity assumption has topological consequences for the groupoid. It entails that the range map (and hence the domain map) is open ([9, Proposition I. 4]). On the other hand A. K. Seda proved that the "continuity" condition is crucial in construction of the groupoid C^*-algebra ([6]). More precisely, Seda proved that in the Hausdorff case the "continuity" assumption is a necessary condition for the invariance of $C_c(G)$ under the convolution product.

The purpose of this section is to propose various versions of reduced C^*-algebras associated to a locally compact groupoid endowed with a system of measures that satisfies the "left invariance" condition (but not necessary the "continuity" condition). If the continuity of system of measures hold, then the usual reduced C^*-algebra is obtained. For the case a locally compact Hausdorff groupoid with the property that the range map is locally injective (semi étale in the sense of [7]), we recover the C^*-algebras introduced in [7].

Notation 11. *Let G be a locally compact groupoid. Let us denote by $\mathcal{UF}_{cb}(G)$ the space of bounded functions which vanish outside a compact set and which are left and right uniformly continuous on fibres.*

Definition 12. *Let G be a locally compact groupoid with Hausdorff fibres (i.e. G^u is Hausdorff for every $u \in G^{(0)}$). A* left pre-Haar system *on G is a family of positive Radon measures, $\{\nu^u,\ u \in G^{(0)}\}$, with the following properties:*

1. *The support of ν^u is G^u for all $u \in G^{(0)}$;*
2. *$\int f(y)\, d\nu^{r(x)}(y) = \int f(xy)\, d\nu^{d(x)}(y)$ for all $x \in G$ and all $f : G \to \mathbb{C}$ having the property that $f|G^u$ is a compactly supported continuous function.*

For all u, we denote by ν_u the Radon measure on G_u defined by

$$\nu_u(f) = \int f\left(x^{-1}\right) d\nu^u(x),$$

where $f : G_u \to \mathbb{C}$ is a compactly supported continuous function.

Definition 13. *Let G be a locally compact groupoid with Hausdorff fibres. The pre-Haar system $\left\{\nu^u,\ u \in G^{(0)}\right\}$ on G is said to be* bounded on compact sets *if*

$$\sup\left\{\nu^u(K),\ u \in G^{(0)}\right\} < \infty$$

for each compact set K in G.

Lemma 14. *Let G be a locally compact groupoid with Hausdorff fibres. Let us assume that G is endowed with a left pre-Haar system $\left\{\nu^u,\ u \in G^{(0)}\right\}$ bounded on compact sets. Then for all $f \in \mathcal{UF}_{cb}(G)$ the maps*

$$\begin{aligned} x &\mapsto \int f(y)\, d\nu^{r(x)}(y) \quad [: G \to \mathbb{C}] \\ x &\mapsto \int f(y)\, d\nu^{d(x)}(y) \quad [: G \to \mathbb{C}] \end{aligned}$$

are left and right uniformly continuous on fibres.

Proof. Let K be a compact set with the property that f vanishes outside K and let K_0 be a compact neighborhood of K. By Lemma 9, there is a symmetric open neighborhood W of $G^{(0)}$ such that

$$WK \subset K_0 \text{ and } KW \subset K_0.$$

Let $\varepsilon > 0$. Since f is left uniformly continuous on fibres there is an open neighborhood W_ε of $G^{(0)}$, such that

$$|f(xy) - f(y)| < \varepsilon \text{ for all } x \in W \text{ and } y \in G^{d(x)}$$

Without loss of generality, we may assume that $W_\varepsilon \subset W$ and that W_ε is symmetric. Then

$$K \subset W_\varepsilon K \subset K_0$$

Furthermore, we have

$$|f(xy) - f(y)| < \varepsilon 1_{WK}(y) \le \varepsilon 1_{K_0}(y) \text{ for all } x \in W_\varepsilon \text{ and } y \in G^{d(x)},$$

where 1_{WK}, respectively 1_{K_0} denotes the indicator function of WK, respectively of K_0.

$$\int |f(xy) - f(y)|\, d\nu^{d(x)}(y) < \int \varepsilon 1_{K_0}(y)\, d\nu^{d(x)}(y) \text{ for all } x \in W_\varepsilon$$

$$\begin{aligned}
\left|\int f(xy)\,d\nu^{d(x)}(y) - \int f(y)\,d\nu^{d(x)}(y)\right| &< \varepsilon\nu^{d(x)}(K_0) \text{ for all } x \in W_\varepsilon \\
\left|\int f(y)\,d\nu^{r(x)}(y) - \int f(y)\,d\nu^{d(x)}(y)\right| &< \varepsilon\nu^{d(x)}(K_0) \text{ for all } x \in W_\varepsilon \\
\left|\int f(y)\,d\nu^{d(z)}(y) - \int f(y)\,d\nu^{d(zx)}(y)\right| &< \varepsilon\nu^{d(x)}(K_0) \text{ for all } x \in W_\varepsilon,\ z \in G_{r(x)}
\end{aligned}$$

Thus $x \mapsto \int f(y)\,d\nu^{d(x)}(y)$ is right (and obviously, left) uniformly continuous on fibres. Similarly $x \mapsto \int f(y)\,d\nu^{r(x)}(y)$ is left and right uniformly continuous on fibres. □

Notation 15. *Let G be a locally compact groupoid with Hausdorff fibres and let $\nu = \{\nu^u,\ u \in G^{(0)}\}$ be a pre-Haar system bounded on compact sets. For each function $f : G \to \mathbb{C}$ with the property that $f|G^u$ is ν^u-integrable or/and $f|G_u$ is ν_u-integrable for all u, let us denote*

$$\begin{aligned}
\|f\|_{I,r} &= \sup\left\{\int |f(y)|\,d\nu^u(y),\ u \in G^{(0)}\right\},\ \mathcal{I}_r(G) = \left\{f :\ \|f\|_{I,r} < \infty\right\} \\
\|f\|_{I,d} &= \sup\left\{\int \left|f\left(y^{-1}\right)\right|d\nu^u(y),\ u \in G^{(0)}\right\},\ \mathcal{I}_d(G) = \left\{f :\ \|f\|_{I,d} < \infty\right\} \\
\|f\|_I &= \max\left\{\|f\|_{I,r},\ \|f\|_{I,d}\right\},\ \mathcal{I}(G) = \{f :\ \|f\|_I < \infty\}.
\end{aligned}$$

Let $f : G \to \mathbb{C}$ be a function that vanishes outside a compact set. Obviously, if f and has the property that for all u, $f|G^u$ and $f|G_u$ are continuous, then f belongs to $\mathcal{I}(G)$. If for all u, $f|G^u$ is continuous (respectively, for all u, $f|G_u$ is continuous), then $f \in \mathcal{I}_r(G)$ (respectively, $\mathcal{I}_d(G)$). In particular $\mathcal{UF}_{cb}(G) \subset \mathcal{I}(G)$.

For $f : G \to \mathbb{C}$, the *involution* is defined by

$$f^*(x) = \overline{f\left(x^{-1}\right)},\ x \in G.$$

For $f, g : G \to \mathbb{C}$ the *convolution* is defined by:

$$\begin{aligned}
f * g(x) &= \int f(xy)\,g\left(y^{-1}\right)d\nu^{d(x)}(y) \\
&= \int f(y)\,g\left(y^{-1}x\right)d\nu^{r(x)}(y),\ x \in G.
\end{aligned}$$

provided the integrals have sense (obviously, $f * g(x) \in \mathbb{C}$ is correctly, defined if f and g have the property that for all u, $f|G^u$and $g|G_u$ are continuous and compactly supported). It is not difficult to check that if $f, g \in \mathcal{I}_r(G)$ (respectively, $f, g \in \mathcal{I}_d(G)$), then $\|f * g\|_{I,r} \le \|f\|_{I,r}\|g\|_{I,r}$ (respectively, $\|f * g\|_{I,d} \le \|f\|_{I,d}\|g\|_{I,d}$).

Let $f_1 : G \to \mathbb{C}$ be a function which is left uniformly continuous on fibres and let $g_1 \in \mathcal{I}_d(G)$. Then there is a neighborhood W_ε of $G^{(0)}$ such that:

$$|f_1(x) - f_1(y)| < \varepsilon \text{ for all } (x, y) \in G_d \times_d G,\ xy^{-1} \in W_\varepsilon.$$

For all $(x, y) \in G_d \times_d G$, we have

$$\begin{aligned}
& |f_1 * g_1(x) - f_1 * g_1(y)| = \\
= & \left| \int f_1(xz) g_1\left(z^{-1}\right) d\nu^{d(x)}(z) - \int f_1(yz) g_1\left(z^{-1}\right) d\nu^{d(y)}(z) \right| \\
\leq & \int |f_1(xz) - f_1(yz)| \left|g_1\left(z^{-1}\right)\right| d\nu^{d(y)}(z) \\
\leq & \varepsilon \|g_1\|_{I,d},
\end{aligned}$$

and therefore $f_1 * g_1$ is left uniformly continuous on fibres. Consequently, for all u, $f_1 * g_1|G_u$ is continuous.

Let $f_2 : G \to \mathbb{C}$ be a function which is right uniformly continuous on fibres and let $g_2 \in\in \mathcal{I}_r(G)$. Then there is a neighborhood W_ε of $G^{(0)}$ such that:

$$|f_2(x) - f_2(y)| < \varepsilon \text{ for all } (x, y) \in G_r \times_r G,\ x^{-1}y \in W_\varepsilon.$$

For all $(x, y) \in G_r \times_r G$, we have

$$\begin{aligned}
& |g_2 * f_2(x) - g_2 * f_2(y)| = \\
= & \left| \int g_2(z) f_2\left(z^{-1}x\right) d\nu^{r(x)}(z) - \int g_2(z) f_2\left(z^{-1}y\right) d\nu^{r(y)}(z) \right| \\
\leq & \int |g(z_2)| \left|f_2\left(z^{-1}x\right) - f_2\left(z^{-1}y\right)\right| d\nu^{r(y)}(z) \\
\leq & \varepsilon \|g_2\|_{I,r}
\end{aligned}$$

and therefore $g_2 * f$ is right uniformly continuous on fibres. Consequently, for all u, $g_2 * f_2|G^u$ is continuous.

Theorem 16. *Let G be a locally compact groupoid with Hausdorff fibres and let $\nu = \left\{\nu^u,\ u \in G^{(0)}\right\}$ be a left pre-Haar system bounded on compact sets. Then $\mathcal{UF}_{cb}(G)$ is closed under involution and convolution, and consequently it is a $*$-algebra.*

Let us fix $u \in G^{(0)}$ and $f \in \mathcal{UF}_{cb}$. Let $\xi : G_u \to \mathbb{C}$ be such that $\xi \in L^1(G_u, \nu_u) \cap L^2(G_u, \nu_u)$. Let us extend ξ with zero outside G_u and let us denote the extension by $\tilde{\xi} : G \to \mathbb{C}$. Obviously, $\tilde{\xi} \in \mathcal{I}_d(G)$ ($\left\|\tilde{\xi}\right\|_{I,d} = \|\xi\|_{L^1(G_u,\nu_u)}$) and therefore $f * \tilde{\xi}$ is left uniformly continuous on fibres. Thus $f * \tilde{\xi}|G_u$ is continuous. If $\eta \in L^2(G_u, \nu_u)$, then

$$\begin{aligned}
& \left|\int f * \tilde{\xi}(x)\, \overline{\eta(x)} d\nu_u(x)\right| \\
= & \left|\int \left(\int f(xy)\, \xi\left(y^{-1}\right) d\nu^{d(x)}(y)\right) \overline{\eta(x)} d\nu_u(x)\right| \\
\leq & \int \left(\int |f(xy)| \left|\xi\left(y^{-1}\right)\right| d\nu^{d(x)}(y)\right) \left|\overline{\eta(x)}\right| d\nu_u(x) \\
\leq & \int \left(\int |f(xy)| \left|\xi\left(y^{-1}\right)\right| |\eta(x)|\, d\nu^{u}(y)\right) d\nu_u(x) \\
\leq & \int\int \left|f\left(xy^{-1}\right)\right| |\xi(y)|\, |\eta(x)|\, d\nu_u(y)\, d\nu_u(x) \\
\leq & \|f\|_I \|\xi\|_{L^2(G_u,\nu_u)} \|\eta\|_{L^2(G_u,\nu_u)}
\end{aligned}$$

and consequently, $\|f * \xi\|_{L^2(G_u,\nu_u)} \leq \|f\|_I \|\xi\|_{L^2(G_u,\nu_u)}$. Hence the operator $Ind_u(f)$ defined by $Ind_u(f)\xi = f * \xi$ on $L^1(G_u, \nu_u) \cap L^2(G_u, \nu_u)$ can be extended to a bounded operator on $L^2(G_u, \nu_u)$. Thus we obtain a non-degenerate representation ($*$-homomorphism): $f \mapsto Ind_u(f)$ $\left[: \mathcal{UF}_{cb} \to \mathcal{B}\left(L^2(G_u, \nu_u)\right)\right]$ with the property that

$$\|Ind_u(f)\| \leq \|f\|_I .$$

Indeed, it is not difficult to check that

$$\begin{aligned}
Ind_u(f_1 * f_2) &= Ind_u(f_1)\, Ind_u(f_2)\, ,\ f_1, f_2 \in \mathcal{UF}_{cb}(G) \\
Ind_u(f^*) &= Ind_u(f)^*\, ,\ f \in \mathcal{UF}_{cb}(G)\, .
\end{aligned}$$

Let us prove that $\left\{Ind_u(f)\xi : f \in \mathcal{UF}_{cb}(G)\, ,\ \xi \in L^2(G_u, \nu_u)\right\}$dense in $L^2(G_u, \nu_u)$ ($f \mapsto Ind_u(f)$ non-degenerate). Using a similar argument as in [8, Lemma 4.5, p. 569] we can prove that

$$C_c(G_u) \subset \{f|G_u, f \in \mathcal{CF}_c(G)\} \subset \{f|G_u, f \in \mathcal{UF}_{cb}(G)\} \subset C_c(G_u)\, .$$

Thus any function $g \in C_c(G_u)$ can be obtain as restriction to G_u of a function $f \in \mathcal{UF}_{cb}(G)$.

Let $f \in \mathcal{UF}_{cb}(G)$ and K_f be a compact subset of G such that f vanishes outside K_f . Let $K \subset G^{(0)}$ be a compact set and let W be a neighborhood of $G^{(0)}$. Then there is a symmetric open neighborhood $W_0 \subset W$ of $G^{(0)}$ and a nonnegative function $\varphi_{K,W}$ defined on G (Remark 10) such that

1. $\varphi_{K,W}$ is a left and right uniformly continuous on fibres which vanishes outside a compact set contained in W_0;

2. For all $u \in K$, the interior of the set $\{x : \varphi_{K,W}(x) \geq 1\} \cap G^u$ (respectively, of the set $\{x : \varphi_{K,W}(x) \geq 1\} \cap G_u$) is nonempty.

Let $\psi_{K,W} : G^{(0)} \to \mathbb{R}$ be a function defined by $\psi_{K,W}(u) = \frac{1}{\int \varphi_{K,W}(x) d\nu^u(x)}$ if $u \in K$ and $\psi(u) = 0$ otherwise. Let $e_{K,W} : G \to \mathbb{R}$ be a function defined by $e_{K,W}(x) = \psi_{K,W}(d(x)) \varphi_{K,W}(x^{-1})$, $x \in G$. It is easy to see that for all $x \in d^{-1}(K)$,

$$\begin{aligned}\int e_{K,W}\left(y^{-1}\right) d\nu^{d(x)}(y) &= \int \psi_{K,W}(r(y)) \varphi_{K,W}(y) d\nu^{d(x)}(y) \\ &= \psi_{K,W}(d(x)) \int \varphi_{K,W}(y) d\nu^{d(x)}(y) = 1.\end{aligned}$$

Moreover for all $u \in G^{(0)}$, $e_{K,W}|G_u = \psi_{K,W}(u) \varphi_{K,W} \circ inv|G_u \in C_c(G_u) \subset L^2(G_u, \nu_u)$ (where $inv(x) = x^{-1}$).

Let us assume that $K \subset G^{(0)}$ is a compact neighborhood of $d(K_f)$. Then

$$f(x) \int e_{K,W}\left(y^{-1}\right) d\nu^{d(x)}(y) = f(x) \text{ for all } x \in G.$$

Let $\varepsilon > 0$ and let us suppose that W is contained in a neighborhood of $G^{(0)}$ with the property that

$$|f(x) - f(y)| < \varepsilon \text{ for all } (x, y) \in G_r \times_r G,\ x^{-1}y \in W.$$

Thus for K and W with the above properties, taking into account that $e_{K,W}$ vanishes outside a compact set contained in $W_0 = W_0^{-1} \subset W$, we obtain:

$$\begin{aligned} & |Ind_u(f) e_{K,W}(x) - f(x)| \\ = & \left| \int f(y) e_{K,W}\left(y^{-1}x\right) d\nu^{r(x)}(y) - f(x) \right| \\ = & \left| \int f(xy) e_{K,W}\left(y^{-1}\right) d\nu^{d(x)}(y) - f(x) \int e_{K,W}\left(y^{-1}\right) d\nu^{d(x)}(y) \right| \\ \leq & \int |f(xy) - f(x)| e_{K,W}\left(y^{-1}\right) d\nu^{d(x)}(y) \\ < & \ \varepsilon \text{ for all } x \in G_u \end{aligned}$$

Hence if $\tilde{K}_f$ is a compact set such that $Ind_u(f) e_{K,W} - f$ vanishes outside $\tilde{K}_f$, then

$$\|Ind_u(f) e_{K,W} - f\|_{L^2(G_u, \nu_u)} < \varepsilon \nu_u\left(\tilde{K}_f^{-1}\right).$$

Consequently, the linear span of

$$\left\{Ind_u(f)\xi :\ f \in C_c(G),\ \xi \in L^2(G_u, \nu_u)\right\}$$

is dense in $L^2(G_u, \nu_u)$.

Notation 17. *Let G be a locally compact groupoid with Hausdorff fibres and let $\nu = \left\{\nu^u,\ u \in G^{(0)}\right\}$ a left pre-Haar system bounded on compact sets. For each $f \in \mathcal{UF}_{cb}(G)$ let us define*

$$\|f\|_{red} = \sup_{u \notin G^{(0)}} \|Ind_u(f)\| \leq \|f\|_I$$

(the reduced norm *of f).*

If $\|Ind_u(f)\| = 0$ for all u, then $Ind_u(f)\, e_{K,W} = 0$ for all functions $e_{K,W}$ and all u. Thus for all $\varepsilon > 0$

$$\|f\|_{L^2(G_u,\nu_u)} = \|Ind_u(f)\, e_{K,W} - f\|_{L^2(G_u,\nu_u)} < \varepsilon \nu_u\left(\tilde{K}_f^{-1}\right),$$

and consequently, $\|f\|_{L^2(G_u,\nu_u)} = 0$. Since $f|G_u$ is continuous and the support of ν_u is G_u, it follows that $f(x) = 0$ for all $x \in G_u$. Therefore $\|\cdot\|_{red}$ is a norm. Taking into account that $Ind_u(f_1 * f_2) = Ind_u(f_1)\, Ind_u(f_2)$ and $Ind_u(f^*) = Ind_u(f)^*$, it is easy to see that in fact $\|\cdot\|_{red}$ is a C^*-norm.

Definition 18. *Let G be a locally compact groupoid with Hausdorff fibres and let $\nu = \left\{\nu^u,\ u \in G^{(0)}\right\}$ a left pre-Haar system bounded on compact sets. We define the C^*-algebra $\mathcal{C}^*_{red}(G,\nu)$ to be the completion of $\mathcal{UF}_{cb}(G)$) in the norm $\|\cdot\|_{red}$.*

Remark 19. *The C^*-algebra defined above might be too big. Therefore in order to capture various properties of the topological structure of G we may need to consider various subalgebras of $\mathcal{C}^*_{red}(G,\nu)$ such as:*

1. *the C^*-subalgebra of $\mathcal{C}^*_{red}(G,\nu)$ generated by the space of complex valued compactly supported continuous functions on G. This C^*-subalgebra may be $\{0\}$ - see for instance [4, Example 1.2/p. 51].*

2. *the C^*-subalgebra of $\mathcal{C}^*_{red}(G,\nu)$ generated by the space of complex valued functions on G spanned by functions f having the property that f vanishes outside a compact set K and is continuous on a neighborhood of K.*

3. *the C^*-subalgebra of $\mathcal{C}^*_{red}(G,\nu)$ generated by he space of complex valued functions on G spanned by functions f having the property that f vanishes outside a compact set K and is continuous on a* Hausdorff *neighborhood of K.*

In the Hausdorff case all these three C^*-subalgebras coincide. Moreover, if ν is continuous, in the sense that the map

$$u \mapsto \int f(x)\, d\nu^u(x) \quad \left[: G^{(0)} \to \mathbb{C}\right]$$

is continuous for all continuous compactly supported functions f, then all those three C^*-subalgebras coincide with the usual reduced C^*-algebra (used in [5]).

On the other hand, for the particular case of a locally compact Hausdorff groupoid with the property that the range map is locally injective (semi étale in the sense of [7]), $\mathcal{C}^*_{red}(G,\nu)$ (with v consisting in counting measures on the corresponding fibres) coincides with the C^*-algebra $B^*_r(G)$ introduced in [7] and all the three C^*-subalgebras proposed above with $C^*_r(G)$.

Alternatively, we may consider C^*-subalgebras of $\mathcal{C}^*_{red}(G,\nu)$ obtain as a closures of the $*$-subalgebras of $\mathcal{UF}_{cb}(G)$ defined in the following theorem.

Theorem 20. *([2]) Let G be a locally compact groupoid with Hausdorff fibres and let $\{\nu^u,\ u\in G^{(0)}\}$ be a pre-Haar system on G bounded on compact sets. Let $\mathcal{C}$ be a class of complex valued functions on $G^{(0)}$ closed to addition and scalar multiplication (for instance, the class of functions continuous, or the class of Borel functions, etc.). Let*

$$\mathcal{A}_c^{\mathcal{C}}=\left\{\begin{array}{r} f\in\mathcal{UF}_{cb}(G),\ u\mapsto\int f(y)\varphi(d(y))\,d\nu^u(y)\in\mathcal{C}\text{ for all }\varphi\in\mathcal{C}\text{ and}\\ u\mapsto\int\overline{f(y^{-1})}\varphi(d(y))\,d\nu^u(y)\in\mathcal{C}\text{ for all }\varphi\in\mathcal{C}\end{array}\right\}.$$

Then $\mathcal{A}_c^{\mathcal{C}}$ is a $*$-subalgebra of $\mathcal{UF}_{cb}(G)$.

Proof. G is a "locally abstractly compact" groupoid in the sense of [2] (for $\mathcal{K}$ being the family of compact subsets of G). □

Let us consider the following special cases in Theorem 20:

(i) $\mathcal{C}(\tau)=\{\varphi:G^0\to\mathbb{C},\ \varphi\text{ continuous with respect to }\tau\}$, where τ is a topology on $G^{(0)}$.

(ii) $\mathcal{C}(\mathcal{B})=\{\varphi:G^0\to\mathbb{C},\ \varphi\text{ measurable with respect to }\sigma\text{-algebra }\mathcal{B}\text{ on }G^{(0)}\}$.

Particular cases

Let us consider now two classes of groupoids: the groups and the sets (as co-trivial groupoids):

1. **Groups:** If G is locally compact Hausdorff group, then G (as a groupoid) admits an essentially unique (left) pre-Haar system $\{\lambda\}$ where λ is a Haar measure on G. Let $C_c(G)$ be space of complex valued continuous functions with compact support on G. The convolution of f, $g\in C_c(G)$ (G is seen as a groupoid) is given by:

$$f*g(x)=\int f(xy)\,g\left(y^{-1}\right)d\lambda(y)\ \text{ (usual convolution on } G)$$

and the involution by

$$f^*(x)=\overline{f\left(x^{-1}\right)}.$$

The involution defined above is slightly different from the usual involution on groups. In fact if Δ is the modular function of the Haar measure λ, then $f \mapsto \Delta^{1/2} f$ is $*$-isomorphism from $C_c(G)$ for G seen as a groupoid to $C_c(G)$ for G seen as a group.

Therefore
$$\mathcal{A}_c^{\mathcal{C}(\tau)} = \mathcal{A}_c^{\mathcal{C}(\mathcal{B})} = \mathcal{UF}_{cb}(G) = C_c(G)$$

and $C^*_{red}(G, \nu)$ (with $\nu = \{\lambda\}$), as well as all its C^*-subalgebras proposed above, are $*$-isomorphic with the usual reduced C^*-algebra of G.

2. **Sets:** If X is locally compact space. Then X seen as groupoid (under the operations $xx = x$, $x^{-1} = x$ for all $x \in X$) is a *locally compact groupoid.*

   The system of measures $\nu = \{\varepsilon_x,\ x \in X\}$ is a pre-Haar system on X ($\varepsilon_x(f) = f(x)$) bounded on the compact sets. The convolution of $f, g \in C_c(G)$ is given by:

$$f * g(x) = \int f(xy)\, g\left(y^{-1}\right) d\varepsilon_x(y) = f(xx)\, g\left(x^{-1}\right) = f(x)\, g(x)$$

   and the involution by
$$f^*(x) = \overline{f(x)}.$$

In this framework:

a. $\mathcal{UF}_{cb}(X)$ is the space of bounded functions which vanish outside a compact set. For all $f \in \mathcal{UF}_{cb}(X)$,

$$\|f\|_{red} = \|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Thus $C^*_{red}(X, \nu)$ is the algebra of bounded functions f on X which vanishes at infinity (in the sense that for each $\varepsilon > 0$, there is a compact set K such that $|f(x)| < \varepsilon$ for all $x \notin K$).

b. $\mathcal{A}_c^{\mathcal{C}(\tau)}$ is the space of τ-continuous bounded functions f which vanish outside a compact set.

c. $\mathcal{A}_c^{\mathcal{C}(\mathcal{B})}$ is the space of bounded functions f which vanish outside a compact set and which are measurable with respect to the σ-algebra $\mathcal{B}$.

When X is a locally compact Hausdorff space and τ is the topology of X, then $\mathcal{A}_c^{\mathcal{C}(\tau)} = C_c(X)$, the space of continuous functions with compact support on X and the C^*-subalgebra obtain as the closure of $\mathcal{A}_c^{\mathcal{C}(\tau)}$ is $C_0(X)$ (the algebra of complex valued continuous functions f on X which vanishes at infinity).

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Mădălina Buneci
University Constantin Brâncuşi,
Str. Geneva, Nr. 3, 210136 Târgu-Jiu, Romania.
e-mail: ada@utgjiu.ro
http://www.utgjiu.ro/math/mbuneci/