Unrestricted Quantum Moduli Algebras.
I. The Case of Punctured Spheres

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Abstract. Let Σ be a finite type surface, and G a complex algebraic simple Lie group with Lie algebra $\mathfrak{g}$. The quantum moduli algebra of (Σ, G) is a quantization of the ring of functions of $X_G(\Sigma)$, the variety of G-characters of $\pi_1(\Sigma)$, introduced by Alekseev–Grosse–Schomerus and Buffenoir–Roche in the mid '90s. It can be realized as the invariant subalgebra of so-called graph algebras, which are $U_q(\mathfrak{g})$-module-algebras associated to graphs on Σ, where $U_q(\mathfrak{g})$ is the quantum group corresponding to G. We study the structure of the quantum moduli algebra in the case where Σ is a sphere with $n+1$ open disks removed, $n \geq 1$, using the graph algebra of the "daisy" graph on Σ to make computations easier. We provide new results that hold for arbitrary G and generic q, and develop the theory in the case where $q = \epsilon$, a primitive root of unity of odd order, and $G = \mathrm{SL}(2, \mathbb{C})$. In such a situation we introduce a Frobenius morphism that provides a natural identification of the center of the daisy graph algebra with a finite extension of the coordinate ring $\mathcal{O}(G^n)$. We extend the quantum coadjoint action of De-Concini–Kac–Procesi to the daisy graph algebra, and show that the associated Poisson structure on the center corresponds by the Frobenius morphism to the Fock–Rosly Poisson structure on $\mathcal{O}(G^n)$. We show that the set of fixed elements of the center under the quantum coadjoint action is a finite extension of $\mathbb{C}[X_G(\Sigma)]$ endowed with the Atiyah–Bott–Goldman Poisson structure. Finally, by using Wilson loop operators we identify the Kauffman bracket skein algebra $K_\zeta(\Sigma)$ at $\zeta := \mathrm{i}\epsilon^{1/2}$ with this quantum moduli algebra specialized at $q = \epsilon$. This allows us to recast in the quantum moduli setup some recent results of Bonahon–Wong and Frohman–Kania-Bartoszyńska–Lê on $K_\zeta(\Sigma)$.

Key words: quantum groups; invariant theory; character varieties; skein algebras

2020 Mathematics Subject Classification: 16R30; 17B37; 20G42; 57M27; 57R56; 81R50

1 Introduction

Let Σ be an oriented surface of finite type, and G a complex algebraic simple Lie group with Lie algebra $\mathfrak{g}$. In this paper we begin our investigation of the quantum moduli algebra defined by quantum lattice gauge field theory (qLGFT) on Σ with gauge algebra $U_\epsilon(\mathfrak{g})$, where $U_\epsilon(\mathfrak{g})$ is the adjoint unrestricted quantum group $U_\epsilon(\mathfrak{g})$ at a primitive root of unity ϵ. For technical simplicity we focus in this paper on the case where Σ has genus 0 and $n \geq 2$ boundary components, we assume that ϵ has odd order, and we prove our main results in the case of $G = \mathrm{SL}(2, \mathbb{C})$. Their formulation for arbitrary G has qualitatively the same form.

Our main motivation comes from quantum topology. We aim at showing that the quantum moduli algebras make a very efficient and unifying setting by which quantum invariant theory for manifolds equipped with G-characters can be studied. As an example, in this paper we will verify this postulate on the Kauffman bracket skein algebra $K_\zeta(\Sigma)$, where $\zeta = \mathrm{i}\epsilon^{1/2}$ is a primitive root of unity of order $4l$, with $l \geq 3$ odd. Namely, we will recast some recent results of Bonahon–Wong [17, 18] and Frohman–Kania-Bartoszyńska–Lê [45, 46] on $K_\zeta(\Sigma)$ in the setup of quantum moduli algebras, where they follow from our general results applied to the case of $\mathfrak{g} = \mathfrak{sl}(2)$.

Our approach consists genuinely of doing geometric invariant theory for quantum groups. In order to present our results, let us recall a few facts about qLGFTs.

The qLGFTs were introduced in the mid '90s by Alekseev–Grosse–Schomerus [1, 2, 3, 6] and Buffenoir–Roche [19, 20], who used as gauge algebras the quantum groups $H = U_q(\mathfrak{g})$ with q generic, or semi-simplifications thereof when q is a root of unity. Assuming that Σ has non-empty boundary, which simplifies this presentation and is the case studied in this paper, the qLGFT on Σ with gauge algebra H associates a H-module-algebra $\mathcal{L}_\Gamma(H)$, called graph algebra, to any ribbon graph Γ embedded in Σ and onto which Σ deformation retracts. The H-invariant subalgebra $\mathcal{L}_\Gamma(H)^H$ is independent up to isomorphism of the choice of Γ, and so is canonically associated to Σ. Abusing of notations we call $\mathcal{L}_\Gamma(H)^H$ "the" quantum moduli algebra of the qLGFT.

In these papers $\mathcal{L}_\Gamma(H)$ was defined by the method of "combinatorial quantization", which yields presentations by generators and relations given in matrix form. Assuming Γ has one vertex and no edge contractible in Σ, these presentations make $\mathcal{L}_\Gamma(H)$ a natural deformation quantization of the ring of regular functions $\mathcal{O}(G^E)$ endowed with the Fock–Rosly Poisson structure [4, 5, 43], where E is the number of edges (loops) of Γ. Hence $\mathcal{L}_\Gamma(H)^H$ is a deformation quantization of $\mathcal{O}(G^E)^G \subset \mathcal{O}(G^E)$, the ring of regular functions invariant under the coadjoint action of G, i.e., the ring of regular functions on the variety $X_G(\Sigma)$ of characters of representations $\pi_1(\Sigma) \to G$, endowed with the Atiyah–Bott–Goldman Poisson structure.

Bullock–Frohman–Kania-Bartoszyńska provided in [22] a coordinate free construction of the qLGFTs that works for any ribbon Hopf gauge algebra H, based on the Reshetikhin–Turaev functor. They also related the Wilson loop elements of $\mathcal{L}_\Gamma(H)^H$, introduced in [20] and associated to the isotopy classes of framed oriented links in $\Sigma \times [0, 1]$, to the Kauffman bracket skein algebra of Σ. In particular, by adapting their approach to $H = U_q(\mathfrak{sl}(2))$ (which is strictly speaking ribbon in a certain completion), they showed in [23] that the construction of Wilson loop elements yields an isomorphism from the skein algebra $K_q(\Sigma)$ defined over $\mathbb{C}(q)$ to the quantum moduli algebra for $H = U_q(\mathfrak{sl}(2))$. This isomorphism, that we call Wilson loop map, explained in a very natural way the emergence in qLGFTs of the Witten–Reshetikhin–Turaev mapping class group representations and of the Jones polynomial of links, already discovered in [6, 20]. We refer to [70] for a comprehensive account of the axiomatic and algebraic structures of qLGFTs, and to [14] for their re-appearance in the context of factorization homology.

With the exception of the work of Frolov [47], until recently the qLGFTs for (non semi-simplified) quantum groups at roots of unity were not studied, certainly because of their apparent complicated definitions. Although the combinatorial quantization approach is the less intrinsic, it has the advantage of providing computationally transparent connections with representation theory. The more topological approach of [22] makes some invariance statements obvious, but encapsulates part of representation theory in a diagrammatic calculus, leaving many aspects rather implicit.

A major progress has been achieved recently by Faitg in [38, 39, 40, 41]. He defined the qL-GFTs for arbitrary finite-dimensional ribbon factorizable Hopf gauge algebras H, not necessarily semisimple, using combinatorial quantization. He showed that the mapping class group representations associated to such qLGFTs coincide with those of Lyubashenko–Majid [67, 68], which were originally defined by categorical means, and provided explicit and ready-to-use formulas for Dehn twists. In particular this includes (with little adaptation) the restricted quantum group for $\mathfrak{sl}(2)$ at a primitive root of unity of even order. In this case, he generalized the Wilson loop map, obtained new non semisimple representations of $K_q(\Sigma)$, and established new relationships with the stated skein algebras. He showed also that the corresponding qLGFT mapping class group representations coincide in genus $g = 1$ with those derived from logarithmic conformal field theory in [42].

In the present paper we make a further step in the root of unity case; we note that another approach is being developed in [48]. We consider the qLGFTs on Σ with gauge algebra an

unrestricted (adjoint) quantum group $U_q(\mathfrak{g})$; when $q = \epsilon$ is a root of unity we mainly focus on the case $\mathfrak{g} = \mathfrak{sl}(2)$. Also for technical simplicity we assume ϵ is a primitive root of unity of odd order l; the case of even order can be treated similarly. As mentioned above, for simplicity also we focus on the case where Σ is a sphere with $n+1$ open disks removed, $n \geq 1$ (sometimes we say for short that Σ is punctured), but qualitatively similar results hold for surfaces of non zero genus.

First we construct the graph algebra $\mathcal{L}_{0,n} := \mathcal{L}_{0,n}(\mathfrak{g})$ associated to the "daisy graph" in Σ, made of one vertex, one loop encircling each deleted disk, and one "cilium" at the vertex, which provides an ordering of the loops by using the orientation of Σ. The embedding in Σ gives the daisy graph a structure of ribbon graph, as shown in the picture below:
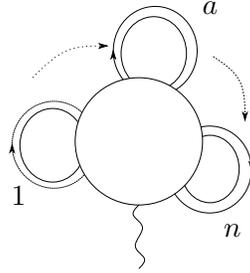
Figure 1.1. The daisy graph.

The graph algebra $\mathcal{L}_{0,n}$ is a module-algebra over the quantum group $U_q := U_q(\mathfrak{g})$ with ground ring $\mathbb{C}(q)$. We define $\mathcal{L}_{0,n}$ by means of combinatorial quantization based on U_q, that we reformulate also in terms of twists of module-algebras and braided tensor product. More precisely, U_q is not a ribbon Hopf algebra, but a suitable extension of the category of finite-dimensional U_q-modules is ribbon. So, to make sense of the construction of $\mathcal{L}_{0,n}$ we replace U_q by a categorical completion $\mathbb{U}_q$.

In particular, $\mathcal{L}_{0,1} = \mathcal{O}_q$ as a U_q-module, where $\mathcal{O}_q$ is the restricted dual of U_q endowed with the right coadjoint action of U_q, and the algebra structure of $\mathcal{L}_{0,1}$ is compatible with that action. Eventually, we find that a map due to Alekseev [1] yields an equivariant embedding of U_q-module-algebras

$$\Phi_n \colon \ \mathcal{L}_{0,n} \to \tilde{U}_q^{\otimes n},$$

where $\tilde{U}_q$ is the simply-connected quantum group associated to $\mathfrak{g}$, and $\mathcal{L}_{0,n}$ and $\tilde{U}_q^{\otimes n}$ are endowed with a coadjoint and an adjoint action of U_q respectively. In the case $n = 1$, Φ_1 coincides with a celebrated isomorphism of U_q-modules $\mathcal{O}_q \to \tilde{U}_q^{\mathrm{lf}}$, where $\tilde{U}_q^{\mathrm{lf}}$ is the subalgebra of locally finite elements of $\tilde{U}_q$, which was first introduced by Drinfeld and Reshetikhin–Semenov-Tian-Shansky [73], and further studied by Caldero, Joseph–Letzter and others (see, e.g., [13, 24, 53]).

For the purpose of defining specializations at $q = \epsilon$, we then introduce an integral form $\mathcal{L}_{0,n}^A$ of $\mathcal{L}_{0,n}$, that is, an A-algebra satisfying $\mathcal{L}_{0,n} = \mathcal{L}_{0,n}^A \otimes_A \mathbb{C}(q)$, where $A = \mathbb{C}[q, q^{-1}]$. It is a module-algebra over the *unrestricted* integral form U_A of U_q, as defined by De Concini–Kac–Procesi [27, 30]. The construction of $\mathcal{L}_{0,n}^A$ is based on Lusztig's [66] *restricted* integral form U_A^{res} of U_q and some integrality properties of the R-matrix on U_A^{res}-modules. The Alekseev map yields an equivariant embedding of U_A-module-algebras $\Phi_n \colon \mathcal{L}_{0,n}^A \to \tilde{U}_A^{\otimes n}$, and the iterated coproduct $\Delta^{(n-1)}$ of a (natural) integral form $\mathcal{O}_A$ of $\mathcal{O}_q$ defines a morphism of algebras $\Delta^{(n-1)} \colon \mathcal{L}_{0,1}^A \to \mathcal{L}_{0,n}^A$. We show (see Proposition 6.11 and Theorem 6.27 for details):

Theorem 1.1. *The invariant subalgebra* $\mathcal{L}_{0,n}^{U_q}$ *does not have non trivial zero divisors, and its center is a polynomial algebra, generated by* $\Delta^{(n-1)}(\mathcal{Z}(\mathcal{L}_{0,1}))$ *and* $\mathcal{Z}(\mathcal{L}_{0,n})$.

In the case $\mathfrak{g} = \mathfrak{sl}(2)$ this result also follows from the isomorphism $\mathcal{W}$ of Theorem 1.3 below, and the fact, proved in [16, 72], that the skein algebras satisfy the corresponding properties. In [72] it is also shown that the skein algebras are finitely generated and Noetherian.

The constructions and results above are developed from Section 2 to Section 6, for any of the quantum groups $U_q = U_q(\mathfrak{g})$. As they form the basis of all results that follow we give full details, though part of this material has already been considered in some ways in the litterature. Along the text and especially in Section 5 we consider in detail the case of $\mathfrak{g} = \mathfrak{sl}(2)$.

Section 7 is devoted to the center of the specializations of $\mathcal{L}_{0,n}^A$ at roots of unity. For every $\epsilon \in \mathbb{C}^\times$, set $U_\epsilon := U_A \otimes_A \mathbb{C}_\epsilon$ and

$$\mathcal{L}_{0,n}^\epsilon = \mathcal{L}_{0,n}^A \otimes_A \mathbb{C}_\epsilon,$$

where $\mathbb{C}_\epsilon = \mathbb{C}$ as a vector space, and as an A-module, q is evaluated as ϵ on $\mathbb{C}_\epsilon$. The Alekseev map affords an embedding of U_ϵ-modules $\Phi_n\colon \mathcal{L}_{0,n}^\epsilon \to \tilde{U}_\epsilon^{\otimes n}$. We study this map when ϵ is a primitive root of unity of odd order l.

De Concini–Kac–Procesi [27, 28, 30] showed that the center $\mathcal{Z}(\tilde{U}_\epsilon)$ can be identified with the coordinate ring of a Poisson–Lie group G^* (dual to G endowed with the standard Poisson–Lie structure), and that certain Hamiltonian vector fields on $\mathrm{Spec}\left(\mathcal{Z}(\tilde{U}_\epsilon)\right) \cong G^*$ can be integrated to define an *infinite dimensional* group $\mathcal{G}_{\mathrm{DCK}}$ acting by *analytic* automorphisms on $\mathrm{Spec}\left(\mathcal{Z}(\tilde{U}_\epsilon)\right)$, and by automorphisms on a suitable completion of U_ϵ. The orbits of this action, called *quantum coadjoint action*, lift the orbits of the conjugation action via the natural covering map $G^* \to G^0$, where G^0 is the big cell of G. They used this action to obtain a series of fundamental results on the simple U_ϵ-modules.

It is not hard to make the quantum coadjoint action explicit for $U_\epsilon(\mathfrak{sl}(2))$. Therefore, starting from Section 7 we restrict to the case of $\mathfrak{g} = \mathfrak{sl}(2)$, thus omitting $\mathfrak{g}$ from the notations (denoting, e.g., $\mathcal{L}_{0,n}^\epsilon(\mathfrak{sl}(2))$ by $\mathcal{L}_{0,n}^\epsilon$), and we put $G = \mathrm{SL}(2, \mathbb{C})$. In the case of $\mathfrak{g} = \mathfrak{sl}(2)$ the Alekseev map affords an isomorphism

$$\Phi_n\colon\; {}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon \to U_\epsilon^{\otimes n},$$

where ${}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon$ is the specialization at $q = \epsilon$ of a localization of $\mathcal{L}_{0,n}^A(\mathfrak{sl}(2))$ introduced in Section 6.

After proving preliminary results on the center $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ in Section 7.1, we extend the quantum coadjoint action by means of the Alekseev map, to get in Section 7.2 an action of a *finite dimensional* Lie group $\mathcal{G}$ on the fraction ring of $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ (hence a *partial action* on $\mathrm{Spec}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ by rational transformations) and on a suitable completion of $\mathcal{L}_{0,n}^\epsilon$. Then we study the invariant subalgebra $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$. The groups $\mathcal{G}$ and $\mathcal{G}_{\mathrm{DCK}}$ are different. In fact, it is necessary to adapt the constructions of De Concini–Kac–Procesi in the case $n > 1$ (see Remark 7.15). We note that the problem of extending the quantum coadjoint action to graph algebras has already been considered by Frolov in [47].

We can summarize the main results of Section 7 as follows (see Corollary 7.20 for a precise and more complete statement). Denote by $X_G(\Sigma)$ the variety of characters of representations $\pi_1(\Sigma) \to G$. We define an l-fold branched covering space $\tilde{G}$ of G, and an l^n-fold branched covering space $\tilde{X}'_G(\Sigma)$ of $X_G(\Sigma)$, related by a natural branched covering identification map $\tilde{\mathfrak{c}}\colon \tilde{X}'_G(\Sigma) \to \tilde{G}^n$. Then we prove:

Theorem 1.2. *The center $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ is naturally endowed with a Poisson bracket $\{\,,\,\}_{\mathrm{QCA}}$ inherited from the algebra structure of $\mathcal{L}_{0,n}^\epsilon$, so that there is an isomorphism of Poisson algebras*

$$\widetilde{\mathrm{Fr}}\colon\; \left(\mathcal{O}(\tilde{G}^n), \{\,,\,\}_{\mathrm{FR}}\right) \to \left(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon), \{\,,\,\}_{\mathrm{QCA}}\right),$$

where $\{\,,\,\}_{\mathrm{FR}}$ is the trivial extension to $\mathcal{O}(\tilde{G}^n)$ of the Fock–Rosly Poisson bracket on $\mathcal{O}(G^n)$. Moreover, $\widetilde{\mathrm{Fr}}$ yields an isomorphism of Poisson algebras

$$\widetilde{\mathrm{Fr}} \circ \tilde{\mathfrak{c}}^{-1*}\colon\; \left(\mathcal{O}(\tilde{X}'_G(\Sigma)), \{\,,\,\}_{\mathrm{Gold}}\right) \to \left(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}, \{\,,\,\}_{\mathrm{QCA}}\right),$$

where $\{\, ,\, \}_{\mathrm{Gold}}$ is the trivial extension to $\mathcal{O}\big(\tilde{X}'_G(\Sigma)\big)$ of the Atiyah–Bott–Goldman Poisson bracket on $\mathcal{O}(X_G(\Sigma))$.

The isomorphism $\widetilde{\mathrm{Fr}}$ provides a precise formulation of what means combinatorial quantization of the $\mathrm{Ad}(G)$-module-algebra $\mathcal{O}(G^n)$ at roots of unity. It maps the elements generating $\mathcal{O}(\tilde{G}^n)$ as an extension of $\mathcal{O}(G^n)$ to analogs of Casimir elements in $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$. On the subalgebra $\mathcal{O}(G^n)$, it is given by a *Frobenius map* Fr, analogous in the case $n = 1$ to the one defined for $\mathcal{O}_\epsilon = \mathrm{SL}_\epsilon(2)$ by Parshall–Wang in [71], though more complicated, see Definitions 7.1 and 7.3. It satisfies the remarkable identity (and a similar one for $n > 1$, see Proposition 7.6)

$$\mathrm{Fr}\big(\mathrm{Tr}\,\big(\underline{\overset{V_2}{M}}\big)\big) = T_l\big(\mathrm{qTr}\,\big(\overset{V_2}{M}\big)\big) = \mathrm{qTr}\,\big(T_l(V_2)M\big),$$

where $\underline{\overset{V_2}{M}}$ is the matrix of coordinate functions of G in its fundamental representation, $\overset{V_2}{M}$ is the matrix of generators of $\mathcal{L}^\epsilon_{0,1}$ in the fundamental representation V_2 of U_ϵ, T_l is the l-th Chebyshev polynomial of the first type (suitably normalized), $T_l(V_2)$ the corresponding virtual representation in the Grothendieck ring of U_q-modules, and qTr and Tr are the quantum trace and classical trace of 2×2 matrices respectively. This identity shows how Fr relates invariant functions on G to $\mathcal{G}$-invariant central elements. The appearance of the l-th Chebyshev polynomial T_l in this context relies on the fact that it generates the defining relation of $\mathcal{Z}(U_\epsilon)$, between the Casimir element Ω and the generators E^l, F^l, $K^{\pm l}$ of the "small" center $\mathcal{Z}_0(U_\epsilon) \subset \mathcal{Z}(U_\epsilon)$.

In the context of the quantum function algebra $\mathrm{SL}_\epsilon(2)$, identities similar to the above one and its extension to $n > 1$ have been obtained by Bonahon in [15].

In Section 8 we develop a topological (i.e., skein theoretic) formulation of some of the previous results. In Section 8.1 we give two definitions of a *Wilson loop functor* $\mathbb{W}$, defined on a category of ribbon oriented graphs in $\Sigma \times [0, 1]$ colored by U^{res}_A-modules, extending the Wilson loop map of [20] and defined for any $\mathfrak{g}$. One of these formulations uses the Reshetikhin–Turaev functor, and is close to the one of [22].

In Section 8.2 we consider the restriction W of $\mathbb{W}$ to *closed* colored ribbon oriented graphs. The image of W is $\big(\mathcal{L}^A_{0,n}\big)^{U_A} \otimes_A \mathbb{C}\big[q^{1/D}, q^{-1/D}\big]$, where we recall that $\big(\mathcal{L}^A_{0,n}\big)^{U_A}$ is the A-algebra of U_A-invariant elements of $\mathcal{L}^A_{0,n}$, and $A = \mathbb{C}\big[q, q^{-1}\big]$. Moreover, we prove the following result, which is an integral version (i.e., over the ring A) of a combination of [22, Theorem 10] and [23, Theorem 1]. Denote by $K_\zeta(\Sigma)$ the Kauffman bracket skein algebra of Σ, defined over the ring $\mathbb{C}\big[\zeta, \zeta^{-1}\big]$. We have (see Theorem 8.6, and Remark 8.7(3) for a statement over $\mathbb{Z}\big[\zeta, \zeta^{-1}\big]$):

Theorem 1.3. *When* $\mathfrak{g} = \mathfrak{sl}(2)$, *the linear map defined by* $\mathcal{W}(L) = \mathrm{i}^{\mathrm{lk}(L)}W(L)$ *on ribbon oriented links* L *colored by the fundamental representation, where* $\mathrm{lk}(L)$ *is the linking number of* L, *descends to an isomorphism of algebras* (*where* $\zeta := \mathrm{i}q^{1/2}$):

$$\mathcal{W}\colon\; K_\zeta(\Sigma) \to \big(\mathcal{L}^A_{0,n}\big)^{U_A} \otimes_A \mathbb{C}\big[\zeta, \zeta^{-1}\big].$$

By using the image by $\mathcal{W}$ of the multicurve basis of $K_\zeta(\Sigma)$ we prove in Theorem 8.8 the following result, about the specialization $\big(\mathcal{L}^A_{0,n}\big)^{U_A}_\epsilon = \big(\mathcal{L}^A_{0,n}\big)^{U_A} \otimes_A \mathbb{C}_\epsilon$.

Theorem 1.4. *The algebra* $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})^{\mathcal{G}}$ *is contained in* $\big(\mathcal{L}^A_{0,n}\big)^{U_A}_\epsilon$. *Moreover, the bracket* $\{\, ,\, \}_{\mathrm{QCA}}$ *extends to an action by derivations of* $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})^{\mathcal{G}}$ *on* $\big(\mathcal{L}^A_{0,n}\big)^{U_A}_\epsilon$.

Theorems 1.1, 1.2, 1.3 and 1.4 imply:

Corollary 1.5.

(1) *The skein algebra* $K_\zeta(\Sigma)$ *does not have non trivial zero divisors, and its center is the polynomial ring generated by the skein classes of the boundary components of* Σ.

(2) *When ζ is specialized to a root of unity ϵ' of order $4l$, with $l \geq 3$ odd, the center of $K_{\epsilon'}(\Sigma)$
     contains a subalgebra isomorphic to $\mathcal{O}(X_G(\Sigma))$, endowed with the image of the Poisson
     bracket $\{\,,\,\}_{\mathrm{Gold}}$, which extends to an action by derivations on $K_{\epsilon'}(\Sigma)$.*

Details are given in Section 9. As already mentioned above, (1) has been proved in [16]
and [72]. The claim (2) belongs to a corpus of results proved in [17, 18] and [45, 46], for any
finite type surface and root of unity ϵ'. One interest of our method is to be intrinsically algebro-
geometric, and valid for any complex simple Lie algebra $\mathfrak{g}$. We note that the Frobenius map
Fr discussed after Theorem 1.2 provides an explicit, geometric realization of the threading map
Ch: $K_{\epsilon'^{l^2}}(\Sigma) \to \mathcal{Z}_{\epsilon'}(\Sigma)$ of Bonahon–Wong, see [17] and also [46] (note that $\epsilon'^{l^2} \in \{\pm 1, \pm \mathrm{i}\}$).

In [11] and works in preparation we study the structure of the algebra $\mathcal{L}_{0,n}^{\epsilon}$ and its subalgebras
$(\mathcal{L}_{0,n}^{\epsilon})^{U_\epsilon}$ and $(\mathcal{L}_{0,n}^{A})_{\epsilon}^{U_A}$, their representations at roots of unity, and we extend the results of this
paper to arbitrary finite type surfaces.

We note that the quantum coadjoint action implies remarkable properties of the intertwiners
of quantum moduli algebras. In the $\mathfrak{sl}(2)$ case these properties should eventually recast the
quantum hyperbolic field theories (see [8, 9] and the references therein), and therefore quan-
tum Teichmüller theory (by the results of [10]) within the theory of quantum moduli algebras.
In another direction, integrating the action by derivations of $\mathcal{Z}(\mathcal{L}_{0,n}^{\epsilon})^{\mathcal{G}}$ should provide interes-
ting information on the space of finite-dimensional $(\mathcal{L}_{0,n}^{A})_{\epsilon}^{U_A}$-modules. There should be no major
difficulties in generalizing these results to the higher rank case.

2   Categorical completions

We recall here the notion of categorical completion of a Hopf algebra, which is connected to the
theory of multiplier Hopf algebras, see [79]. We need it because our main object of interest, the
algebra $\mathcal{L}_{0,n}(\mathfrak{g})$, is built from the Hopf algebra $U_q(\mathfrak{g})$ and its braided structure which exists only
in some completion. The categorical one is suited to algebras defined over $\mathbb{C}(q)$.

Let k be a field (in the sequel it will always be $\mathbb{C}$ or $\mathbb{C}(q)$), and U a k-associative algebra (not
necessarily with unit). We denote by μ the multiplication map of U, and by Mod_U the category
of left U-modules.

Let $F_U \colon \mathrm{Mod}_U \to \mathrm{Vect}$ be the forgetful functor from the category Mod_U to the category of
k-vector spaces. Denote by $\mathcal{U}$ the set of natural transformations from F_U to F_U. An element
of $\mathcal{U}$ is a collection $(a_X)_{X \in \mathrm{Mod}_U}$, where $a_X \in \mathrm{End}_k(X)$ satisfies $F_U(f) \circ a_X = a_Y \circ F_U(f)$ for any
objects X, Y in Mod_U and any arrow $f \in \mathrm{Hom}_U(X, Y)$. The direct product $\prod_{X \in \mathrm{Mod}_U} \mathrm{End}_k(X)$
is canonically endowed with a structure of unital k-algebra, and $\mathcal{U}$ is a unital subalgebra of
$\prod_{X \in \mathrm{Mod}_U} \mathrm{End}_k(X)$. The multiplication map is given in each factor by the composition map
$\mu_X \colon \mathrm{End}_k(X) \otimes \mathrm{End}_k(X) \to \mathrm{End}_k(X), u \otimes v \mapsto u \circ v$. The map

$$\iota_U \colon \quad U \to \mathcal{U}, \qquad a \mapsto (a_X)_{X \in \mathrm{Mod}_U},$$

where a_X is the endomorphism defined by the action of $a \in U$ on X, is a morphism of algebra.
When U has a unit 1, ι_U is an isomorphism with inverse

$$\iota_U^{-1} \colon \quad \mathcal{U} \to U, \qquad (a_X)_{X \in \mathrm{Mod}_U} \mapsto a_U(1),$$

where U is endowed with its structure of left-regular representation.

We will "enlarge" $\mathcal{U}$ by considering only k-finite dimensional U-modules; we stress that, in this
situation, the map corresponding to ι_U is not necessarily surjective nor injective (see below for
the case of $U = U_q(\mathfrak{g})$). Let FinVect and FinMod_U be respectively the full subcategories of Vect
and Mod_U whose objects consist of the finite dimensional k-vector spaces and U-modules. Let
$\mathrm{FinF}_U \colon \mathrm{FinMod}_U \to \mathrm{FinVect}$ be the forgetful functor, and $\hat{U}$ the set of natural transformations

from FinF_U to FinF_U. As $\mathcal{U}$ above, $\hat{U}$ is a unital subalgebra of $\prod_{X\in\mathrm{FinMod}_U}\mathrm{End}_k(X)$, and the map (keeping the same notation)

$$\iota_U\colon \quad U \to \hat{U}, \qquad a \mapsto (a_X)_{X\in\mathrm{FinMod}_U}$$

is a morphism of algebra. We will call $\hat{U}$ the FinMod_U-*categorical completion* of U.

We will often use special elements in $\hat{U}$ defined by series of elements of U. Consider a sequence of elements x_j of U, and assume that for any object X of FinMod_U, the set of indices j such that the endomorphism $(x_j)_X$ is not zero is finite. Then we can define an element $\sum_j \iota_U(x_j)$ of $\hat{U}$ by

$$\left(\sum_j \iota_U(x_j) \right)_X = \sum_j (x_j)_X. \tag{2.1}$$

When ι_U is injective this element is also denoted by $\sum_j x_j$.

Let U, V be $k-$algebras, and $\hat{U}$, $\hat{V}$ their FinMod-categorical completions. We define the categorical completed tensor product of $\hat{U}$ and $\hat{V}$, denoted by $\hat{U}\hat{\otimes}\hat{V}$, as the space of natural transformations from $F_{U,V}$ to $F_{U,V}$, where $F_{U,V}\colon \mathrm{FinMod}_U \times \mathrm{FinMod}_V \to \mathrm{FinVect}, (X,Y) \mapsto X\otimes_k Y$. An element of $\hat{U}\hat{\otimes}\hat{V}$ is a collection of linear maps $a_{X,Y} \in \mathrm{End}_k(X\otimes Y)$, for $X \in \mathrm{FinMod}_U$ and $Y \in \mathrm{FinMod}_V$, such that for any arrows $f \in \mathrm{Hom}_U(X,X')$ and $g \in \mathrm{Hom}_V(Y,Y')$ one has $(f \otimes g) \circ a_{X,Y} = a_{X',Y'} \circ (f \otimes g)$ (identifying $F_{U,V}(f,g)$ with the linear map $f \otimes g$). Again the componentwise composition map endows $\hat{U}\hat{\otimes}\hat{V}$ with a structure of associative algebra, and the map $\iota_{U,V}\colon U \otimes V \to \hat{U}\hat{\otimes}\hat{V}, u \otimes v \mapsto (u_X \otimes v_Y)_{X,Y}$ is a morphism of algebra. If $\mu\colon U \otimes U \to U$ is the product of U, we define $\hat{\mu}\colon \hat{U}\hat{\otimes}\hat{U} \to \hat{U}$ by

$$(\hat{\mu}(a))_X = \mu_X(a_{X,X})$$

for all $a \in \hat{U}\hat{\otimes}\hat{U}$ and objects X of FinMod_U. This construction is generalized straightforwardly to n-tuples of algebras $U_1,\ldots,U_n$. In particular, if $U_1 = \cdots = U_n = U$ we put

$$\hat{U}^{\hat{\otimes}n} := \hat{U}_1\hat{\otimes}\cdots\hat{\otimes}\hat{U}_n.$$

Adapting Sweedler's coproduct notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, we find convenient to write a sum $T = \sum_j u_j\hat{\otimes}v_j$ in $\hat{U}\hat{\otimes}\hat{V}$ as

$$T = \sum_{(T)} T_{(1)}\hat{\otimes}T_{(2)}.$$

Assume now that U is a k-bialgebra, with coproduct $\Delta\colon U \to U \otimes U$ and counit $\epsilon\colon U \to k$. Denote by X_0 the U-module structure on k defined by the counit ϵ. Set

$$\hat{\Delta}\colon \quad \hat{U} \to \hat{U}\hat{\otimes}\hat{U}, \qquad (a_Z)_{Z\in\mathrm{FinMod}_U} \mapsto (a_{X,Y})_{X,Y\in\mathrm{FinMod}_U},$$

where $a_{X,Y} := a_{X\otimes Y}$, and

$$\hat{\epsilon}\colon \quad \hat{U} \to k, \qquad a \mapsto a_{X_0}.$$

These maps are morphisms of algebras, and satisfy

$$\hat{\Delta} \circ \iota_U = \iota_{U,U} \circ \Delta, \qquad (\hat{\epsilon}\hat{\otimes}\mathrm{id})(\hat{\Delta}(a)) = (\mathrm{id}\hat{\otimes}\hat{\epsilon})(\hat{\Delta}(a)) = \mathrm{id}, \qquad (\hat{\Delta}\hat{\otimes}\mathrm{id})\hat{\Delta} = (\mathrm{id}\hat{\otimes}\hat{\Delta})\hat{\Delta}.$$

We will still call $\hat{U}$ a k-bialgebra although the tensor product is $\hat{\otimes}$ and not the algebraic one. In particular, given $a = (a_Z)_{Z\in\mathrm{FinMod}_U} \in \hat{U}$ we have

$$a_{(X,Y),Z} := \left((\hat{\Delta}\hat{\otimes}\mathrm{id})\hat{\Delta}(a)\right)_{X,Y,Z} = \left((\mathrm{id}\hat{\otimes}\hat{\Delta})\hat{\Delta}(a)\right)_{X,Y,Z} =: a_{X,(Y,Z)}.$$

We say that U is *quasicocommutative in its categorical completion* if there exists an invertible element $R \in \hat{U} \hat{\otimes} \hat{U}$ such that

$$\hat{\Delta}^{\mathrm{cop}}(a) = R\hat{\Delta}(a)R^{-1}$$

for all $a \in \hat{U}$, where $\hat{\Delta}^{\mathrm{cop}} := \sigma \circ \hat{\Delta}$, and $\sigma(a\hat{\otimes}b) = b\hat{\otimes}a$ for every $a, b \in \hat{U}$. In particular, this implies

$$\iota_{U\otimes U}\big(\Delta^{\mathrm{cop}}(a)\big) = \iota_{U\otimes U}(\sigma(\Delta(a))) = R\iota_{U\otimes U}(\Delta(a))R^{-1}$$

for all $a \in U$.

Assume that U is quasicocommutative in its categorical completion. We say it is quasitriangular, or braided, if moreover we have

$$\big(\hat{\Delta}\hat{\otimes}\mathrm{id}\big)(R) = R_{13}R_{23}, \qquad \big(\mathrm{id}\hat{\otimes}\hat{\Delta}\big)(R) = R_{13}R_{12}. \tag{2.2}$$

Then, we call R a universal R-matrix. We use the following notations:

$$R^+ := R, \qquad R^- := \sigma(R)^{-1}, \qquad R' := \sigma(R).$$

Finally assume that U is a Hopf algebra. For X a U-module, denote by X^* the dual module (with the standard action $a \cdot \lambda = \lambda \circ S(a)$, for all $a \in U$ and $\lambda \in X^*$). If X is finite dimensional, let $\psi_X \colon X^{**} \to X$ be the canonical isomorphism of vector spaces. We define an antimorphism of algebras $\hat{S} \colon \hat{U} \to \hat{U}$ by

$$\hat{S}(a)_X = \psi_X \circ {}^t(a_{X^*}) \circ \psi_X^{-1}$$

for all $X \in \mathrm{FinMod}_U$. It satisfies $\hat{\mu}\big(\hat{S}\hat{\otimes}\mathrm{id}\big)\hat{\Delta}(a) = \hat{\mu}\big(\mathrm{id}\hat{\otimes}\hat{S}\big)\hat{\Delta}(a) = \hat{\eta}\hat{\epsilon}$, where $\eta \colon k \to U$, $\lambda \mapsto \lambda 1_U$, and $\hat{\eta} \colon k \to \hat{U}$, $\lambda \mapsto \lambda 1_{\hat{U}}$, are the unit maps of U and $\hat{U}$ respectively. Hence $\hat{U}$ is a Hopf algebra in the generalized sense where the tensor product $\hat{\otimes}$ is used in place of the algebraic one.

When U is a Hopf algebra braided in its categorical completion, we say it is *ribbon in its categorical completion* if there exists $\theta \in \hat{U}$ such that θ is central, $\hat{\Delta}(\theta) = (R'R)^{-1}(\theta \otimes \theta)$, $\epsilon(\theta) = 1$ and $\hat{S}(\theta) = \theta$.

For the sake of notational simplicity, from now on we will omit the "^" symbols from the structure morphisms of the categorical completions $\hat{U}$ under consideration, thus denoting $\hat{\Delta}$ by Δ, $\hat{\otimes}$ by $\otimes$, and so on, like for U.

3   The case of $U_q(\mathfrak{g})$

3.1   Notations

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra of rank m. Denote by (a_{ij}) the Cartan matrix of $\mathfrak{g}$, and by d_i the unique coprime positive integers such that the matrix $(d_i a_{ij})$ is symmetric. Fix a Cartan subalgebra $\mathfrak{h}$ and a basis of simple roots $\alpha_i \in \mathfrak{h}_{\mathbb{R}}^*$. Denote by $(\,,\,)$ the unique inner product on $\mathfrak{h}_{\mathbb{R}}^*$ such that $d_i a_{ij} = (\alpha_i, \alpha_j)$. The root lattice Q is the $\mathbb{Z}$-lattice in $\mathfrak{h}_{\mathbb{R}}^*$ defined by $Q = \sum_{i=1}^m \mathbb{Z}\alpha_i$. The weight lattice P is the $\mathbb{Z}$-lattice formed by all $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ such that

$$\langle \lambda, \alpha_i \rangle := 2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}$$

for every $i = 1, \ldots, m$. So $P = \sum_{i=1}^m \mathbb{Z}\omega_i$, where the ω_i are the fundamental weights corresponding to the simple roots α_i, satisfying $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$. Put $Q_+ := \sum_{i=1}^m \mathbb{Z}_{\geq 0}\alpha_i$ and

$P_+ := \sum_{i=1}^{m} \mathbb{Z}_{\geq 0} \omega_i$, the cone of dominant integral weights. Denote by ρ half the sum of the positive roots, by N the number of positive roots, and by D the smallest positive integer such that $D(\lambda, \mu) \in \mathbb{Z}$ for every $\lambda, \mu \in P$. Note that $(\lambda, \alpha_i) \in \mathbb{Z}$ for every $\lambda \in P$, $\alpha \in Q$, and D is also the smallest positive integer such that $DP \subset Q$.

Let $q^{1/D}$ be a new variable, and set $q = \left(q^{1/D}\right)^D$, $q_i = q^{d_i}$. The quantum group $U_q = U_q(\mathfrak{g})$ is the algebra over $k = \mathbb{C}(q)$ with generators E_i, F_i, K_i, K_i^{-1}, $1 \leq i \leq m$, and defining relations (see, e.g., [25, Chapter 9.1]):

$$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$
$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,$$
$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{3.1}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0 \qquad \text{if} \quad i \neq j, \tag{3.2}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0 \qquad \text{if} \quad i \neq j. \tag{3.3}$$

Here we put for p, k non-negative integers with $0 \leq k \leq p$,

$$[0]_q! = 1, \qquad [p]_q! = [1]_q[2]_q \cdots [p]_q, \qquad [p]_q = \frac{q^p - q^{-p}}{q - q^{-1}}, \qquad \begin{bmatrix} p \\ k \end{bmatrix}_q = \frac{[p]_q!}{[p-k]_q![k]_q!}.$$

The algebra U_q is a Hopf algebra with the coproduct Δ, antipode S, and counit ε defined by

$$\Delta\left(K_i^{\pm 1}\right) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \qquad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta(F_i) = K_i^{-1} \otimes F_i + F_i \otimes 1,$$
$$S(E_i) = -E_i K_i^{-1}, \qquad S(F_i) = -K_i F_i, \qquad S\left(K_i^{\pm 1}\right) = K_i^{\mp 1},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \varepsilon(K_i) = 1.$$

In the case of $\mathfrak{g} = \mathfrak{sl}(2)$ we simply write $E = E_1$, $F = F_1$ and $K^{\pm 1} = K_1^{\pm 1}$ the generators of $U_q(\mathfrak{sl}(2))$. For every $\alpha \in Q$, $\alpha = \sum_{i=1}^{m} m_i \alpha_i$, we put

$$K_\alpha = \prod_{i=1}^{m} K_i^{m_i}.$$

The element $\ell = K_{2\rho}$ is group like and satisfies $S^2(x) = \ell x \ell^{-1}$, $x \in U_q$. Hence U_q is a *pivotal* Hopf algebra, with the pivotal element ℓ.

We will also consider the *simply-connected version* $\tilde{U}_q$ of U_q. It is obtained by adjoining invertible elements ℓ_i, $1 \leq i \leq m$, such that

$$K_i = \prod_{j=1}^{m} \ell_j^{a_{ji}}, \qquad \ell_i E_j \ell_i^{-1} = q_i^{\delta_{i,j}} E_j, \qquad \ell_i F_j \ell_i^{-1} = q_i^{-\delta_{i,j}} F_j, \qquad \Delta\left(\ell_i^{\pm 1}\right) = \ell_i^{\pm 1} \otimes \ell_i^{\pm 1}.$$

The expression of the pivotal element is

$$\ell = \prod_{j=1}^{m} \ell_j^2.$$

For instance, $\tilde{U}_q(\mathfrak{sl}(2))$ is obtained from $U_q(\mathfrak{sl}(2))$ by adjoining a square root of K. We denote $K^{\frac{1}{2}} = \ell_1$.

We fix a reduced expression $s_{i_1} \cdots s_{i_N}$ of the longest element of the Weyl group of $\mathfrak{g}$. It induces a total ordering of the positive roots,

$$\beta_1 = \alpha_{i_1}, \qquad \beta_2 = s_{i_1}(\alpha_{i_2}), \qquad \ldots, \qquad \beta_N = s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N}).$$

The root vectors of $\tilde{U}_q$ with respect to such an ordering are defined by

$$E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}), \qquad F_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}),$$

where T_i is Lusztig's algebra automorphism of $\tilde{U}_q$ associated to the simple root α_i [65, 66], see also [25, Chapter 8]). Let us just recall here that the monomials $F_{\beta_1}^{r_1} \cdots F_{\beta_N}^{r_N} K_\lambda E_{\beta_N}^{t_N} \cdots E_{\beta_1}^{t_1}$ ($r_i, t_i \in \mathbb{N}$, $\lambda \in P$) form a basis of $\tilde{U}_q$.

For every positive root β, we will denote $q_\beta = q^{(\beta,\beta)/2}$ and

$$\bar{E}_\beta = \left(q_\beta - q_\beta^{-1}\right)E_\beta, \qquad \bar{F}_\beta = \left(q_\beta - q_\beta^{-1}\right)F_\beta.$$

3.2 Categorical completions

By a theorem of Harish-Chandra, the map $\iota_U \colon U \to \hat{U}$ of Section 2 is injective when U is $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$ [31, Theorem 2.5.7]. It is also injective when $k = \mathbb{C}(q)$ and $U = U_q$ [51, Lemma 7.1.9].

In the case of $k = \mathbb{C}(q)$, only specific categorical completions of U_q are ribbon. We will use the one, that we denote by $\mathbb{U}_q$, defined as follows.

Recall that every finite dimensional U_q-module X is completely reducible (see, e.g., [25, Theorem 10.1.7]). If X is irreducible, it is a highest weight module parametrized by a dominant integral weight, say $\lambda \in P_+$, and signs $\sigma_i = \pm 1$, $i \in \{1, \ldots, m\}$. Then $X = \oplus_{\epsilon'} X_{\epsilon'}$, where the sum ranges over the tuples $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_m)$ such that $\epsilon'^{-1}_i \epsilon_i = q^{(\alpha, \alpha_i)}$ for some $\alpha \in Q_+$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$ and $\epsilon_i = \sigma_i q^{(\lambda, \alpha_i)}$, and $X_{\epsilon'} = \{x \in X \mid K_i \cdot x = \epsilon'_i x, i = 1, \ldots, m\}$ is the weight space of X of weight ϵ'. We say that a U_q-module X has *type* 1 if it is finite dimensional and the signs σ_i of all its irreducible components are equal to 1. Equivalently, the generators K_i are diagonalizable on X, with eigenvalues in $q_i^{\mathbb{Z}}$. The category $\mathcal{C}$ with objects the U_q-modules of type 1 is a semisimple tensor category (where by semisimple we mean that any object is completely reducible). We will systematically denote by V_λ the type 1 simple U_q-module parametrized by $\lambda \in P_+$.

The algebra $\tilde{U}_q$ is defined over $\mathbb{C}(q)$, and therefore also over $\mathbb{C}(q^{1/D})$. The finite dimensional $\tilde{U}_q$-modules over $\mathbb{C}(q^{1/D})$ are completely reducible. Similarly as for U_q, one defines the $\tilde{U}_q$-modules of type 1 as the finite dimensional $\tilde{U}_q$-modules over $\mathbb{C}(q^{1/D})$ on which the ℓ_i are diagonalizable with eigenvalues in $q_i^{(1/D)\mathbb{Z}}$. We denote by $\tilde{\mathcal{C}}$ the category of $\tilde{U}_q$-modules of type 1. Also, we denote by $\mathcal{C} \otimes \mathbb{C}(q^{1/D})$ the category obtained from $\mathcal{C}$ by extending coefficients of objects and morphisms to $\mathbb{C}(q^{1/D})$. The restriction functor $\tilde{\mathcal{C}} \to \mathcal{C} \otimes \mathbb{C}(q^{1/D})$ is an equivalence of categories.

This can be made explicit in the following way. As in Section 2, we can define the $\mathcal{C}$-categorical completion $\mathbb{U}'_q = \mathbb{U}_q(\mathfrak{g})'$ of U_q as the Hopf algebra of natural transformations from $F_{\mathcal{C}}$ to $F_{\mathcal{C}}$, where $F_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{FinVect}$ is the forgetful functor. Set

$$\mathbb{U}_q = \mathbb{U}'_q \otimes_k \mathbb{C}(q^{1/D}).$$

Extending the coefficient ring of $\mathcal{C}$ from $\mathbb{C}(q)$ to $\mathbb{C}(q^{1/D})$ allows one to embed $\tilde{U}_q$ in $\mathbb{U}_q$. Let us explain this. The type 1 simple U_q-module V_λ can be considered as a q-deformation of the finite dimensional simple $\mathfrak{g}$-module of highest weight λ. Therefore, the generators $H_i \in \mathfrak{g}$ such that $\alpha_i(H_j) = a_{ji}$ define elements of $\mathbb{U}'_q$: H_i acts on a weight space $X_{\epsilon'}$ of weight $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_m)$,

where $\epsilon_i' = q^{(\mu,\alpha_i)}$ and $\mu \in P$, by multiplication by (μ, α_i). Passing to the coefficient ring $\mathbb{C}(q^{1/D})$, one can define an element $q^{H_i/D} \in \mathbb{U}_q$, acting on $X_{\epsilon'} \otimes \mathbb{C}(q^{1/D})$ by multiplication by $q^{(\mu,\alpha_i)/D}$. Similarly, recalling that $(\mu, \omega_i) \in (1/D)\mathbb{Z}$, we can define the action of the generator $\ell_i \in \tilde{U}_q$ on $X_{\epsilon'} \otimes \mathbb{C}(q^{1/D})$ as the multiplication by $q^{(\mu,\omega_i)}$. This provides the claimed embedding $\tilde{U}_q \subset \mathbb{U}_q$.

Extending the coefficient ring of $\mathcal{C}$ from $\mathbb{C}(q)$ to $\mathbb{C}(q^{1/D})$ allows also to make sure that $\mathcal{C} \otimes \mathbb{C}(q^{1/D})$ is a braided and ribbon category, and that $\mathbb{U}_q$ is a braided and ribbon Hopf algebra. To see this, denote by $B \in M_m(\mathbb{Q})$ the matrix with entries $B_{ij} := d_j^{-1} a_{ij}$. By the same arguments as above one can define

$$\Theta := q^{\sum_{i,j=1}^m (B^{-1})_{ij} H_i \otimes H_j} \in \mathbb{U}_q^{\otimes 2} \tag{3.4}$$

as the operator acting on $X_\epsilon \otimes X_{\epsilon'} \otimes \mathbb{C}(q^{1/D})$ as the multiplication by $q^{(\mu,\nu)} \in q^{(1/D)\mathbb{Z}}$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$, $\epsilon' = (\epsilon_1', \ldots, \epsilon_m')$, $\epsilon_i = q^{(\mu,\alpha_i)}$, $\epsilon_i' = q^{(\nu,\alpha_i)}$ and $\mu, \nu \in P$. Recall that the Drinfeld universal R-matrix of the quantized universal enveloping algebra $U_h(\mathfrak{g})$, defined, e.g., in [25, Chapter 8.3.C] acts on tensor products $X \otimes Y$ of U_q-modules X and Y of type 1, where $q = e^h$, and that it can be written in the form

$$R := \Theta \check{R},$$

where Θ is defined in (3.4) and $\check{R}$, called the *quasi R-matrix*, is

$$\check{R} = \sum_{t_1,\ldots,t_N=0}^{\infty} \prod_{r=1}^N q_{\beta_r}^{\frac{1}{2} t_r(t_r+1)} \frac{\left(1 - q_{\beta_r}^{-2}\right)^{t_r}}{[t_r]_{q_{\beta_r}}!} (E_{\beta_r})^{t_r} \otimes (F_{\beta_r})^{t_r}. \tag{3.5}$$

Note that all terms in the above sum are zero but a finite number of them, since E_{β_r}, F_{β_r} act nilpotently on objects of $\mathcal{C}$, so from (2.1) one derives immediately that

$$R = (R_{X,Y}) \in \mathbb{U}_q^{\otimes 2}.$$

Observe that $R_{X,\cdot} \in \text{End}(X) \otimes \tilde{U}_q$. Indeed, $\check{R}_{X,\cdot} \in \text{End}(X) \otimes U_q$. Moreover, take weight spaces X_ϵ, $X_{\epsilon'}$ as in (3.4), with X_ϵ a weight subspace of X, and with basis vectors $\{e_i\}$ of X_ϵ, and dual basis $\{e^i\}$. Then, setting $\lambda = \sum_{t=1}^m k_t \omega_t$, we have

$$\left(\langle e^j \mid \cdot \mid e_i \rangle \otimes \text{id}\right)(\Theta) = \delta_{i,j} \prod_{t=1}^m \ell_t^{k_t} \in \tilde{U}_q. \tag{3.6}$$

For instance, in the case of $\mathfrak{g} = \mathfrak{sl}(2)$, we have $D = 2$ and $\Theta = q^{H \otimes H/2}$. Hence, identifying elements of U_q with their images by $\iota_{U_q} : U_q \to \mathbb{U}_q$ we can write

$$R = q^{H \otimes H/2} \sum_{n=0}^{+\infty} \frac{\left(q - q^{-1}\right)^n}{[n]_q!} q^{n(n-1)/2} E^n \otimes F^n. \tag{3.7}$$

One checks similarly that the ribbon element of $U_h(\mathfrak{g})$ induces an element of $\mathbb{U}_q$. In conclusion, U_q is naturally a braided and ribbon Hopf algebra in the categorical completion $\mathbb{U}_q$.

3.3 Integral forms and specializations

Let $A = \mathbb{C}[q, q^{-1}]$. The (De Concini–Kac) *unrestricted integral form* U_A is the A-subalgebra of U_q generated by the elements (see, e.g., [25, Chapter 9.2])

$$E_i, \ F_i, \ K_i^{\pm 1}, \ \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \qquad \text{for} \quad 1 \le i \le m.$$

We will also consider the smallest subalgebra $U'_A \subset U_A$ invariant under the Lusztig automorphisms T_i and containing the elements $(q_i - q_i^{-1})E_i$, $(q_i - q_i^{-1})F_i$ and $K_i^{\pm 1}$, for $1 \leq i \leq m$ (which is the unrestricted integral form considered by De Concini–Procesi [30]).

The (Lusztig) *restricted integral form* U_A^{res} is the A-subalgebra of U_q generated by the elements (see, e.g., [25, Chapter 9.3])

$$\frac{E_i^r}{[r]_{q_i}!}, \quad \frac{F_i^r}{[r]_{q_i}!}, \quad K_i^{\pm 1} \qquad \text{for any} \quad i, r \quad \text{with} \quad 1 \leq i \leq m, r \geq 1.$$

Note that $U_A \subset U_A^{\mathrm{res}}$. Both U_A and U_A^{res} have structures of Hopf A-algebras inherited from U_q, and they satisfy

$$U_A \otimes_A \mathbb{C}(q) = U_q, \qquad U_A^{\mathrm{res}} \otimes_A \mathbb{C}(q) = U_q.$$

Moreover they are free A-modules, with certain basis of PBW type, described, e.g., in [25, Chapter 9]. One defines $\tilde{U}_A$ in a similar way, replacing the generators $K_i^{\pm 1}$ by $\ell_i^{\pm 1}$.

We say a U_A^{res}-module has type 1 if it is a free A-module of finite rank with a basis made of eigenvectors for the elements K_i, with eigenvalues in $q^{\mathbb{Z}}$. Denote by $\mathcal{C}_A$ the category with objects the U_A^{res}-modules of type 1. It is a tensor category, which is not semisimple.

Remark 3.1. The fact that $\mathcal{C}_A$ is not semi simple can be seen on the following elementary example. Let V_2 be the two-dimensional type 1 simple $U_q(\mathfrak{sl}(2))$-module, with basis vectors v_+, v_- such that $K.v_+ = qv_+$, $E.v_+ = 0$, $v_- = F.v_+$. Define $v' := v_- \otimes v_+ - qv_+ \otimes v_-$ and $v'' := v_- \otimes v_+ + q^{-1}v_+ \otimes v_-$. Then $v_+ \otimes v_+$ generates a $U_A^{\mathrm{res}}(\mathfrak{sl}(2))$-module M of type 1 and rank 3, namely $M = Av_+ \otimes v_+ \oplus Av' \oplus Av_- \otimes v_-$, and v'' generates a $U_A^{\mathrm{res}}(\mathfrak{sl}(2))$-module N of type 1 and rank 1. The A-module $M \oplus N$ is strictly contained in the $U_A^{\mathrm{res}}(\mathfrak{sl}(2))$-module $V_2 \otimes V_2$, which is clearly of type 1, and it is not a direct summand thereof, for $(M \oplus N) \otimes \mathbb{C}(q) = V_2 \otimes V_2$. So $V_2 \otimes V_2$ provides an example of $U_A^{\mathrm{res}}(\mathfrak{sl}(2))$-module of type 1 which is not completely reducible.

We denote by $\mathcal{C}_A \otimes \mathbb{C}[q^{1/D}, q^{-1/D}]$ the category obtained by extending coefficients of objects and morphisms to $\mathbb{C}[q^{1/D}, q^{-1/D}]$. We have seen above that $\mathcal{C} \otimes \mathbb{C}(q^{1/D})$ is a ribbon category; let us now explain why this ribbon structure descends to $\mathcal{C}_A \otimes \mathbb{C}[q^{1/D}, q^{-1/D}]$. Define the categorical completion $\mathbb{U}_A^{\mathrm{res}}$ of U_A^{res} similarly as $\mathbb{U}_q$ above, that is

$$\mathbb{U}_A^{\mathrm{res}} = \mathbb{U}_A^{\mathrm{res}\prime} \otimes_A \mathbb{C}[q^{1/D}, q^{-1/D}],$$

where $\mathbb{U}_A^{\mathrm{res}\prime}$ is the Hopf algebra of natural transformations $F_{\mathcal{C}_A} \to F_{\mathcal{C}_A}$, and $F_{\mathcal{C}_A} : \mathcal{C}_A \to \mathrm{FinMod}_A$ is the forgetful functor (FinMod_A being the category of finite dimensional A-modules).

We will use the following property of the R-matrix. Recall that V_λ is the type 1 simple U_q-module parametrized by $\lambda \in P_+$. By a result of Lusztig [64, Proposition 4.2], V_λ contains an A-submodule $_A V_\lambda$, which is a U_A^{res}-module of type 1 such that $_A V_\lambda \otimes \mathbb{C}(q) = V_\lambda$ (hence $_A V_\lambda$ is a full A-sublattice). Moreover $_A V_\lambda$ is endowed with a *canonical* basis $\mathcal{B}_\lambda$ (the *Kashiwara–Lusztig* basis, see, e.g., [25, Chapter 14]).

Theorem 3.2 (integrality property)**.**

1. *For any U_A^{res}-module X, Y of type 1, the quasi-R-matrix $\check{R}_{X,Y}$ is an automorphism of the A-module $X \otimes Y$, and $\Theta_{X,Y}$, whence $R_{X,Y}$, is an automorphism of $X \otimes Y \otimes \mathbb{C}[q^{1/D}, q^{-1/D}]$.*

2. *Denote by $\mathcal{B}_{\lambda\mu}$ the basis of $V_\lambda \otimes V_\mu$ formed by the vectors $x \otimes y$, with $x \in \mathcal{B}_\lambda$, $y \in \mathcal{B}_\mu$. For every $\lambda, \mu \in P_+$, the matrix entries in the basis $\mathcal{B}_{\lambda\mu}$ of the endomorphisms $R_{V_\lambda, V_\mu}^{\pm 1} \in \mathrm{End}_{\mathbb{C}(q^{1/D})}(V_\lambda \otimes V_\mu)$ belong to $q^{\pm(\lambda,\mu)}\mathbb{C}[q, q^{-1}]$.*

3. *For every U_A^{res}-module X of type 1 we have $R_{X,\cdot} \in \mathrm{End}_A(X) \otimes \tilde{U}_A$.*

Proof. The claims (1) and (2) are classical results of Lusztig [66, Proposition 24.1.4 and Corollary 24.1.6]; see also [61]. They follow also from the formulas (3.4) and (3.5). The claim (3) is a consequence of the fact that $(E_{\beta_r})^{t_r}/[t_r]_{q_{\beta_r}}!$ acts on the A-lattice X, and finally that $\Theta_{X,\cdot} \in \mathrm{End}_A(X) \otimes \tilde{U}_A$, proved via (3.6) above. This concludes the proof. ∎

By Theorem 3.2(1) the braiding R of $\mathcal{C} \otimes \mathbb{C}(q^{1/D})$ yields a braiding on $\mathcal{C}_A \otimes \mathbb{C}[q^{1/D}, q^{-1/D}]$, and from this one derives easily that the same is true of the ribbon transformation. As a consequence, $\mathbb{U}_A^{\mathrm{res}}$ is a braided and ribbon Hopf algebra.

If $\epsilon \in \mathbb{C}^\times$, the *unrestricted specialisation* of U_q at $q = \epsilon$ is the $\mathbb{C}$-Hopf algebra

$$U_\epsilon = U_A \otimes_A \mathbb{C}_\epsilon, \tag{3.8}$$

where $\mathbb{C}_\epsilon = \mathbb{C}$ as a vector space, and as an A-module, q acts on $\mathbb{C}_\epsilon$ by ϵ. One defines $\tilde{U}_\epsilon$ in a similar way. The map $\iota_U \colon U \to \hat{U}$ of Section 2 is injective when $U = U_\epsilon$ or $\tilde{U}_\epsilon$ and ϵ is not a root of unity [50, Proposition 5.11].

When ϵ is not a root of unity, the finite dimensional U_ϵ-modules are completely reducible (see, e.g., [25, Theorem 10.1.14]). As above one can define the category $\mathcal{C}_\epsilon$ of U_ϵ-modules of type 1. The categorical $\mathcal{C}_\epsilon$-completion of U_ϵ is a braided and ribbon Hopf algebra. Note that, in the case of $\mathfrak{g} = \mathfrak{sl}(2)$, again when ϵ is not a root of unity, a similar construction of categorical completion of $U_\epsilon(\mathfrak{sl}(2))$ has been done in [44].

When ϵ is a root of unity, the category of finite dimensional U_ϵ-modules is not semisimple and not braided. When $\mathfrak{g} = \mathfrak{sl}(2)$ a classification of the simple modules is known [27, 75]. We will use it extensively in a sequel to this paper.

3.4 Quantum coordinate algebra

Let U be a Hopf algebra over a field k, X an object of FinMod_U, and π_X the representation of U associated to X. For every $v \in X$ and $w \in X^*$ we denote by ${}_X\phi_v^w \colon U \to k$ the linear form defined by

$${}_X\phi_v^w(a) = w(\pi_X(a)v), \qquad a \in U.$$

The linear form ${}_X\phi_v^w$ is called a *matrix coefficient* of π_X. Denote by U° the *restricted dual* of U, that is, the subspace of U^* generated by the matrix coefficients of the finite dimensional representations of U. It is naturally endowed with a structure of Hopf algebra, such that the bilinear form $\langle \cdot, \cdot \rangle \colon U^\circ \times U \to k$ is a Hopf pairing (see, e.g., [25, Chapter 4]).

When $U = U_q$ and $k = \mathbb{C}(q)$, we denote by $\mathcal{O}_q$ the Hopf subalgebra of U_q° generated by the matrix coefficients of the representations associated to the objects of $\mathcal{C}$, i.e., the U_q-modules of type 1. Equivalently one can define $\mathcal{O}_q$ as the set of $\mathbb{C}(q)$-linear maps

$$f \colon U_q \to \mathbb{C}(q)$$

such that $\mathrm{Ker}(f)$ contains a two sided ideal $I \subset U_q$ of finite codimension, and there is an $r \in \mathbb{N}$ such that $\prod_{s=-r}^r (K_i - q_i^s) \in I$ for every i. We denote by $\star$ the product on $\mathcal{O}_q$, induced from U_q°. Since the U_q-modules of type 1 are completely reducible, and the simple ones are the highest weight modules V_λ, $\lambda \in P_+$, the set of matrix coefficients ${}_{V_\lambda}\phi_{e_i}^{e^j}$, where $\{e_i\}$ is any basis of V_λ and $\{e^j\}$ the dual basis, provide a basis of $\mathcal{O}_q$ over $\mathbb{C}(q)$. The $\mathbb{C}(q)$-algebra $\mathcal{O}_q$ is also finitely generated by a finite number of elements, e.g., the matrix coefficients of the modules V_{ω_i} associated to the fundamental weights ω_i, $i = 1, \ldots, m$ (see, e.g., [25, Proposition 10.1.16]); when $\mathfrak{g}$ is of type A or C, the set of matrix coefficients of the natural representation V_{ω_1} already generates $\mathcal{O}_q$ (see [80] in the classical setup).

Because the morphism $\iota_{U_q}\colon U_q \to \mathbb{U}_q$ is injective (see Section 3.2), the Hopf pairing $\langle\cdot,\cdot\rangle\colon \mathcal{O}_q \times U_q \to \mathbb{C}(q)$ is non degenerate. By extending the coefficient ring from $\mathbb{C}(q)$ to $\mathbb{C}(q^{1/D})$, we can uniquely extend it to a bilinear pairing

$$\langle\cdot,\cdot\rangle\colon \ \big(\mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})\big) \times \mathbb{U}_q \to \mathbb{C}(q^{1/D})$$

such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{O}_q \otimes U_q & \xrightarrow{\ \langle\cdot,\cdot\rangle\ } & \mathbb{C}(q) \\
{\scriptstyle \mathrm{id}\otimes\iota_{U_q}}\big\downarrow & & \big\downarrow \\
\big(\mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})\big) \otimes \mathbb{U}_q & \xrightarrow{\ \langle\cdot,\cdot\rangle\ } & \mathbb{C}(q^{1/D}).
\end{array}
$$

This pairing is defined by

$$\langle {}_Y\phi_v^w, (a_X)\rangle = w(a_Y v)$$

for every $(a_X) \in \mathbb{U}_q$ and ${}_Y\phi_v^w \in \mathcal{O}_q$. It is non degenerate. From now on, we will denote $\mathcal{O}_q(q^{1/D}) := \mathcal{O}_q \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})$, and therefore the pairing above as $\langle\cdot,\cdot\rangle\colon \mathcal{O}_q(q^{1/D}) \times \mathbb{U}_q \to \mathbb{C}(q^{1/D})$.

3.5 Integral quantum coordinate algebra

Finally, we will use the integral form $\mathcal{O}_A$ of $\mathcal{O}_q$ introduced in [65], and further studied in [66] and [29] (see [29, Remark 4.1] for the equivalence of the two definitions). One defines $\mathcal{O}_A$ as the A-module generated by the set of matrix coefficients of the U_A^{res}-modules of type 1. Clearly, it is contained in $\mathcal{O}_q$; in fact, $\mathcal{O}_A$ is the subset of $\mathcal{O}_q$ formed by the linear maps $f\colon U_q \to \mathbb{C}(q)$ such that $f(U_A^{\mathrm{res}}) \subset \mathbb{C}[q,q^{-1}]$ (see [65, Section 7]). Equivalently one can define $\mathcal{O}_A$ as the set of A-linear maps

$$f\colon \ U_A^{\mathrm{res}} \to \mathbb{C}[q,q^{-1}]$$

such that $\mathrm{Ker}(f)$ contains a cofinite two sided ideal $I \subset U_A^{\mathrm{res}}$ and there is an $r \in \mathbb{N}$ such that $\prod_{s=-r}^r \big(K_i - q_i^s\big) \in I$ for every i (I cofinite meaning that there exists a free A-module M of finite rank such that $I \oplus M = U_A^{\mathrm{res}}$).

The Hopf algebra structure of $\mathcal{O}_q$ descends to a structure of Hopf A-algebra on $\mathcal{O}_A$. It is a full A-lattice in $\mathcal{O}_q$, that is, we have $\mathcal{O}_A \otimes_A \mathbb{C}(q) = \mathcal{O}_q$. The Hopf pairing $\langle\cdot,\cdot\rangle\colon \mathcal{O}_q \times U_q \to \mathbb{C}(q)$ restricts to a pairing of Hopf A-algebras $\mathcal{O}_A \otimes_A U_A^{\mathrm{res}} \to A$. Because $U_A^{\mathrm{res}} \otimes_A \mathbb{C}(q) = U_q$ and $\mathcal{O}_A \otimes_A \mathbb{C}(q) = \mathcal{O}_q$, this pairing is also non degenerate. We have described generators of $\mathcal{O}_q$ in Section 3.4. The A-algebra $\mathcal{O}_A$ is generated by the matrix coefficients of the lattices ${}_A V_\lambda$, $\lambda \in P_+$, introduced before Theorem 3.2 (see [65, Section 7.1]).

In [66], Lusztig proved that $\mathcal{O}_A$ is a free module over A (in fact over $\mathbb{Z}[q,q^{-1}]$). He provided a basis of $\mathcal{O}_A$ by considering a multiplier Hopf $\mathbb{C}(q)$-algebra $\dot{\mathbf{U}}$, containing a multiplier Hopf A-algebra $\dot{\mathbf{U}}_A$ as a full lattice, such that the so-called *unital* $\dot{\mathbf{U}}$-modules of finite dimension are exactly the U_q-modules of type 1, and the unital $\dot{\mathbf{U}}_A$-modules of finite rank are exactly the U_A^{res}-modules of type 1. The A-algebra $\dot{\mathbf{U}}_A$ is a free A-module, it has a canonical basis $\dot{\mathbf{B}}$, and the dual basis is a basis of $\mathcal{O}_A$.

When $\mathfrak{g} = \mathfrak{sl}(2)$, $\mathcal{O}_A$ is generated by the matrix coefficients a, b, c, d of the 2-dimensional type 1 simple U_q-module V_2 in the basis $\{v_-, v_+\}$ introduced before Theorem 3.2, and an A-basis is formed by the monomials $a^{\star r} \star b^{\star s} \star d^{\star t}$ and $a^{\star r} \star c^{\star u} \star d^{\star t}$, where $r, s, t \in \mathbb{N}$, $u > 0$ and $a^{\star r}$ is the product of a's r times, etc. (see [29, Lemma 1.3]).

4   The loop algebra $\mathcal{L}_{0,1}(\mathfrak{g})$

The algebra $\mathcal{L}_{0,1} = \mathcal{L}_{0,1}(\mathfrak{g})$ first appeared in relation with the reflection equation (see [60] and the references therein), and as the *braided group* associated to $\mathcal{O}_q$, which is the algebra of automorphisms of the ribbon category $\mathcal{C} \otimes \mathbb{C}(q^{1/D})$ of U_q-modules of type 1 (see [69, Examples 7.4.1 and 9.4.10]). In this paper we use a third definition of $\mathcal{L}_{0,1}$, by means of a twist of the U_q-bimodule $\mathcal{O}_q$ for the left and right coregular actions, following [33] (but using right modules instead of left ones). One merit of this definition is to be intrinsic, not given by generators and relations. That it is recovers the braided group of $\mathcal{O}_q$ is shown in [32, Section 5.2]. The twist uses R-matrices, whence the need of categorical completions. The reflection equation will be recovered in Proposition 4.6.

Denote the left and right coregular actions of U_q on $\mathcal{O}_q$ by

$$x \triangleright \alpha := \sum_{(\alpha)} \alpha_{(1)} \langle \alpha_{(2)}, x \rangle, \qquad \alpha \triangleleft x := \sum_{(\alpha)} \langle \alpha_{(1)}, x \rangle \alpha_{(2)}$$

for all $x \in U_q$ and $\alpha \in \mathcal{O}_q$, where $\langle\,,\,\rangle \colon \mathcal{O}_q \otimes U_q \to \mathbb{C}(q)$ is the duality pairing defined in Section 3.4, and $\Delta(\alpha) = \sum_{(\alpha)} \alpha_{(1)} \otimes \alpha_{(2)}$ (Sweedler's coproduct notation). The actions $\triangleright$ and $\triangleleft$ commute, and $\mathcal{O}_q$ is a U_q-*module algebra* for both actions, i.e., for all $x \in U_q$ and $\alpha, \beta \in \mathcal{O}_q$ we have

$$x \triangleright (\alpha \star \beta) = \sum_{(x)} (x_{(1)} \triangleright \alpha) \star (x_{(2)} \triangleright \beta), \qquad (\alpha \star \beta) \triangleleft x = \sum_{(x)} (\alpha \triangleleft x_{(1)}) \star (\beta \triangleleft x_{(2)}).$$

All this extends to define a structure of $\mathbb{U}_q$-module algebra on $\mathcal{O}_q(q^{1/D})$. Denote by $\mathbb{U}_q^{\mathrm{cop}}$ the Hopf algebra with the same algebra structure as $\mathbb{U}_q$ but the opposite coproduct Δ^{cop} and the antipode S^{-1}. Consider the Hopf algebra (equipped with the standard Hopf algebra structure on tensor product)

$$\mathbb{D}_q = \mathbb{U}_q \otimes \mathbb{U}_q^{\mathrm{cop}}.$$

It has a right action on $\mathcal{O}_q(q^{1/D})$ defined for pure tensors by

$$\alpha \cdot (x \otimes y) := S(y) \triangleright \alpha \triangleleft x$$

for every $x \in \mathbb{U}_q$, $y \in \mathbb{U}_q^{\mathrm{cop}}$ and $\alpha \in \mathcal{O}_q(q^{1/D})$, where $\otimes$ denotes the algebraic tensor product.

Recall the universal R-matrix $R \in \mathbb{U}_q \otimes \mathbb{U}_q$. Let

$$F := (R')_{23}(R')_{24} \in \left(\mathbb{U}_q \otimes \mathbb{U}_q^{\mathrm{cop}} \right)^{\otimes 2},$$

where $(R')_{kl} = \mathfrak{i}_{kl}(R')$, and $\mathfrak{i}_{kl} \colon \mathbb{U}_q \otimes \mathbb{U}_q \to \left(\mathbb{U}_q \otimes \mathbb{U}_q^{\mathrm{cop}} \right)^{\otimes 2}$ identifies the subalgebra $\mathbb{U}_q \otimes 1$ (resp. $1 \otimes \mathbb{U}_q$) of $\mathbb{U}_q \otimes \mathbb{U}_q$ with the k-th (resp. l-th) tensorand of $\left(\mathbb{U}_q \otimes \mathbb{U}_q^{\mathrm{cop}} \right)^{\otimes 2}$. The tensor F is a *twist* of $\mathbb{U}_q \otimes \mathbb{U}_q^{\mathrm{cop}}$: by definition, this means that F is invertible and satisfies

$$(\varepsilon_{\mathbb{D}_q} \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon_{\mathbb{D}_q})(F) = 1 \tag{4.1}$$

and

$$F_{12}(\Delta_{\mathbb{D}_q} \otimes \mathrm{id})(F) = F_{23}(\mathrm{id} \otimes \Delta_{\mathbb{D}_q})(F). \tag{4.2}$$

Put $u := \sum_{(F)} F_{(1)} S(F_{(2)})$. Denote by $S_{\mathbb{D}_q}$ the antipode of $\mathbb{D}_q$ and set

$$\Delta_{\mathbb{D}_q}^F(x) := F\Delta_{\mathbb{D}_q}(x)F^{-1}, \qquad S_{\mathbb{D}_q}^F(x) := uS_{\mathbb{D}_q}(x)u^{-1}, \qquad x \in \mathbb{D}_q. \tag{4.3}$$

The maps $\Delta_{\mathbb{D}_q}^F$, $S_{\mathbb{D}_q}^F$ and the counit of $\mathbb{D}_q$ define the comultiplication, the antipode and the counit of a new structure of Hopf algebra on the algebra $\mathbb{D}_q$. It is called *the twist of $\mathbb{D}_q$ by F*. Denote it by $\mathbb{A}_q$. Since $\mathbb{A}_q$ and $\mathbb{D}_q$ coincide as algebras, the right action of $\mathbb{D}_q$ on $\mathcal{O}_q(q^{1/D})$ is also a right action of $\mathbb{A}_q$. Define a new product on $\mathcal{O}_q(q^{1/D})$ by

$$\alpha\beta = \sum_{(F)} (\alpha \cdot F_{(1)}) \star (\beta \cdot F_{(2)}). \tag{4.4}$$

Because of (4.1) and (4.2), this product defines on $\mathcal{O}_q(q^{1/D})$ a structure of associative unital algebra. Explicitly, by writing $R = \sum_{(R)} R_{(1)} \otimes R_{(2)} = \sum_{(R)} R_{(1')} \otimes R_{(2')}$ we have

$$F = \sum_{(R),(R)} 1 \otimes R_{(2)}R_{(2')} \otimes R_{(1)} \otimes R_{(1')}.$$

Then, by using $(S \otimes S)(R) = R$ we get

$$\alpha\beta = \sum_{(R),(R)} (R_{(2')}S(R_{(2)}) \rhd \alpha) \star (R_{(1')} \rhd \beta \lhd R_{(1)}) \tag{4.5}$$

and conversely

$$\alpha \star \beta = \sum_{(R),(R)} (R_{(2)}R_{(2')} \rhd \alpha)(S(R_{(1')}) \rhd \beta \lhd R_{(1)}). \tag{4.6}$$

Note that, by the expression of R and the fact that the generators E_i, F_i of U_q act nilpotently on finite dimensional U_q-modules, there is only a finite number of non zero terms in the last sum. Therefore the expression (4.4) is well defined. It is easy to check that the right $\mathbb{A}_q$-module $\mathcal{O}_q(q^{1/D})$ endowed with the product (4.4) is a right $\mathbb{A}_q$-module algebra. We denote it $\mathcal{L}_{0,1}(q^{1/D})$, and call it the *twist* by F of the $\mathbb{D}_q$-module algebra $\mathcal{O}_q(q^{1/D})$.

We claim that the product (4.4) restricted to $\mathcal{O}_q \subset \mathcal{O}_q(q^{1/D})$ is defined over the subfield $\mathbb{C}(q)$. Recall that $\mathcal{O}_q$ is generated by the matrix coefficients $_{V_\lambda}\phi_{e_i}^{e^j}$, where $\lambda \in P_+$ and $\{e_i\}$, $\{e^i\}$ are dual basis of weight vectors of V_λ. By equation (4.5), and using $(\mathrm{id} \otimes S)(R) = (\mathrm{id} \otimes \ell)R^{-1}(\mathrm{id} \otimes \ell^{-1})$, we have

$$\alpha\beta = \sum_{(R),(R^{-1})} \left(R_{(2')}\ell(R^{-1})_{(2)}\ell^{-1} \rhd \alpha\right) \star \left(R_{(1')} \rhd \beta \lhd (R^{-1})_{(1)}\right). \tag{4.7}$$

Therefore, if $\alpha = {}_{V_\lambda}\phi_{e_i}^{e^j}$ and $\beta = {}_{V_\mu}\phi_{f_m}^{f^n}$, we obtain

$$_{V_\lambda}\phi_{e_i}^{e^j} \, {}_{V_\mu}\phi_{f_m}^{f^n} = \sum_{(R),(R^{-1}),j',n',n''} \left(\pi_{V_\lambda}\left(R_{(2')}\ell(R^{-1})_{(2)}\ell^{-1}\right)\right)_{e_i}^{e^{j'}}$$
$$\times \left(\pi_{V_\mu}(R_{(1')})\right)_{f_m}^{f^{n'}} \left(\pi_{V_\mu}(R^{-1})_{(1)}\right)_{f_{n''}}^{f^n} \, {}_{V_\lambda}\phi_{e_{j'}}^{e^j} \star {}_{V_\mu}\phi_{f_{n'}}^{f^{n''}}. \tag{4.8}$$

Recall that the matrix entries of the endomorphisms $R_{V_\lambda,V_\mu}^{\pm 1} \in \mathrm{End}_{\mathbb{C}(q^{1/D})}(V_\lambda \otimes V_\mu)$ belong to $q^{\pm(\lambda,\mu)}\mathbb{C}(q)$ (see Theorem 3.2(2) for the stronger integral statement). Then we see that the factors $q^{+(\lambda,\mu)}$ cancel the factors $q^{-(\lambda,\mu)}$ in the last expression. Noting moreover that the matrix entries of ℓ belong to A, we finally obtain that the coefficients in the sum belong to $\mathbb{C}(q)$.

Proposition 4.1. *The subspace $\mathcal{O}_q \subset \mathcal{O}_q(q^{1/D})$ endowed with the product (4.4) is a right module $\mathbb{C}(q)$-algebra over $U_q \otimes U_q^{\mathrm{cop}} \subset \mathbb{A}_q$. We denote it $\mathcal{L}_{0,1}(\mathfrak{g})$, or simply $\mathcal{L}_{0,1}$.*

The following result is due to Majid [69] (see [33] for a simpler proof):

Proposition 4.2. *The coproduct* $\Delta\colon \mathbb{U}_q \to \mathbb{U}_q \otimes \mathbb{U}_q$ *yields a morphism of Hopf algebras* $\Delta\colon \mathbb{U}_q \to \mathbb{A}_q$.

By using the morphism of Hopf algebras $U_q \hookrightarrow \mathbb{U}_q \xrightarrow{\Delta} \mathbb{A}_q$ we can pull-back the action of $\mathbb{A}_q$ on $\mathcal{L}_{0,1}$, which thus becomes a right U_q-module algebra. It is easily seen that the action of U_q is the right coadjoint action, defined by

$$\mathrm{coad}^r(x)(\alpha) = \sum_{(x)} S(x_{(2)}) \triangleright \alpha \triangleleft x_{(1)}, \qquad \forall x \in U_q, \quad \forall \alpha \in \mathcal{L}_{0,1}.$$

Next we are going to recall a fundamental relation between $\mathcal{L}_{0,1}$ and $\tilde{U}_q$. Recall that U_q is a right U_q-module algebra for the right adjoint action, defined by

$$\mathrm{ad}^r(y)(x) = \sum_{(y)} S(y_{(1)}) x y_{(2)}, \qquad \forall x, y \in U_q.$$

This action extends to actions on the simply-connected quantum group $\tilde{U}_q$ and $\mathbb{U}_q$, and thus defines on them structures of right U_q-module algebras. Denote by $\mathcal{Z}(\tilde{U}_q)$ the center of $\tilde{U}_q$. We have

$$\mathcal{Z}(\tilde{U}_q) = \left\{ z \in \tilde{U}_q, \forall x \in U_q, \mathrm{ad}^r(x)(z) = \epsilon(x)z \right\}. \tag{4.9}$$

The set of *locally finite* elements of $\tilde{U}_q$ is defined by

$$\tilde{U}_q^{\mathrm{lf}} := \left\{ x \in \tilde{U}_q \mid \mathrm{rk}_{\mathbb{C}(q)}(\mathrm{ad}^r(U_q)(x)) < \infty \right\}.$$

It is a U_q-module subalgebra of $\tilde{U}_q$. Finally, the set of coad^r-invariant elements of $\mathcal{L}_{0,1}$ is defined by

$$\mathcal{L}_{0,1}^{U_q} := \left\{ \alpha \in \mathcal{L}_{0,1} \mid \forall y \in U_q, \mathrm{coad}^r(y)(\alpha) = \varepsilon(y)\alpha \right\}.$$

Since $\mathcal{L}_{0,1}$ is a U_q-module algebra, $\mathcal{L}_{0,1}^{U_q}$ is a subalgebra of $\mathcal{L}_{0,1}$.

Theorem 4.3. *Define* $\Phi_1\colon \mathcal{L}_{0,1} \to \mathbb{U}_q$, $\Phi_1(\alpha) = (\alpha \otimes \mathrm{id})(RR')$. *We have:*

1. Φ_1 *is a morphism of algebra, equivariant and injective.*
2. *The image of* Φ_1 *is* $\tilde{U}_q^{\mathrm{lf}}$.
3. Φ_1 *induces an isomorphism from* $\mathcal{L}_{0,1}^{U_q}$ *to* $\mathcal{Z}(\tilde{U}_q)$.

We call Φ_1 the *RSD map*, after Reshetikhin and Semenov-Tian-Shansky [73] and Drinfeld [34, 35], who considered it first.

A proof of Theorem 4.3 can be found in [13, Theorem 3]. To make a complete correspondence with that statement, note that (3.6) implies that Φ_1 takes values in $\tilde{U}_q$. The difficult parts of Theorem 4.3 are the injectivity of Φ_1 and the claim (2). Note that the third claim follows from the first two and (4.9).

An alternative proof that Φ_1 is an equivariant morphism is given in [33, Proposition 4.7], based on the construction of a left U_q^{op}-comodule structure on $\mathcal{L}_{0,1}$. The equivariance of Φ_1 also follows from a simple computation shown in Proposition 6.6 below.

Note that because the elements of the restricted dual are necessarily coad^r finite, the equivariance of Φ_1 implies that their images by Φ_1 are necessarily ad^r-finite.

Remark 4.4. We give a simple self-contained proof of Theorem 4.3 in the $\mathfrak{sl}(2)$ case in Proposition 5.4.

Remark 4.5. It is a result of [52] that the $\mathrm{ad}^r(U_q)$-module $\tilde{U}_q^{\mathrm{lf}}$ is generated by the elements $\ell_{-\lambda}$, $\lambda \in 2P_+$, where for every weight $\mu = \sum_{i=1}^m n_i \omega_i$ we set $\ell_\mu = \prod_{i=1}^m \ell_i^{n_i}$. Moreover, there is an Ore subset S of $\tilde{U}_q^{\mathrm{lf}}$ such that $\tilde{U}_q$ is a free module of finite rank over the skew fraction ring $S^{-1}\tilde{U}_q^{\mathrm{lf}}$ (see [53]). The set S is the Abelian group generated by the elements $\ell_{-\lambda}$, where $\lambda \in 2P_+ \cap Q_+$.

Finally, we provide a definition of $\mathcal{L}_{0,1}$ by generators and relations. Though well-known, we include a proof for completeness. As $\mathcal{L}_{0,1}$ and $\mathcal{O}_q$ are isomorphic as linear spaces, the matrix coefficients $_V\phi_{e_j}^{e^i}$ generate $\mathcal{L}_{0,1}$. Let V be an object of the category $\mathcal{C}$, (e_i) a basis of V, (e^i) the dual basis, and E_i^j the corresponding basis of $\mathrm{End}(V)$, defined by $E_i^j(e_k) = \delta_{j,k}e_i$. Define

$$\overset{V}{M} = \sum_{i,j} E_i^j \otimes {_V\phi_{e_j}^{e^i}} \in \mathrm{End}(V) \otimes \mathcal{L}_{0,1}. \tag{4.10}$$

This expression defines $\overset{V}{M}$ independently of the choice of basis (e_i), since we have the following naturality property: if V, W are objects of $\mathcal{C}$, and $f: V \to W$ a morphism of U_q-modules, then

$$\overset{W}{M}(f \otimes \mathrm{id}) = (f \otimes \mathrm{id})\overset{V}{M}. \tag{4.11}$$

Given two objects V and W of $\mathcal{C}$ we write

$$\overset{V}{M}_1 = \sum_{i,j} E_i^j \otimes \mathrm{id} \otimes {_V\phi_{e_j}^{e^i}} \in \mathrm{End}(V) \otimes \mathrm{End}(W) \otimes \mathcal{L}_{0,1}$$

and similarly $\overset{W}{M}_2 = \sum_{i,j} \mathrm{id} \otimes E_i^j \otimes {_W\phi_{e_j}^{e^i}}$. We view $\overset{V \otimes W}{M}$ as an element of $\mathrm{End}(V) \otimes \mathrm{End}(W)$ by using the standard isomorphism

$$\mathrm{End}(V) \otimes \mathrm{End}(W) \cong \mathrm{End}(V \otimes W).$$

Finally we view $R_{V,W}, R'_{V,W} \in \mathrm{End}(V) \otimes \mathrm{End}(W)$ as elements of $\mathrm{End}(V) \otimes \mathrm{End}(W) \otimes 1 \subset \mathrm{End}(V) \otimes \mathrm{End}(W) \otimes \mathcal{L}_{0,1}$.

Proposition 4.6. *The following fusion relation holds true in* $\mathrm{End}(V) \otimes \mathrm{End}(W) \otimes \mathcal{L}_{0,1}$:

$$\overset{V \otimes W}{M} = \overset{V}{M}_1 \, R'_{V,W} \, \overset{W}{M}_2 \, (R'_{V,W})^{-1}, \tag{4.12}$$

where the product of $\mathcal{L}_{0,1}$ *is used to multiply the matrix elements of* $\overset{V}{M}$ *and* $\overset{W}{M}$*. It implies the reflection equation:*

$$R_{V,W} \, \overset{V}{M}_1 \, R'_{V,W} \, \overset{W}{M}_2 = \overset{W}{M}_2 \, R_{V,W} \, \overset{V}{M}_1 \, R'_{V,W}. \tag{4.13}$$

Conversely, the naturality relations (4.11) *and the fusion relations* (4.12), *for every objects V and W of $\mathcal{C}$, are a defining set of relations for* $\mathcal{L}_{0,1}$. *That is, $\mathcal{L}_{0,1}$ can be viewed as the quotient of the algebra freely generated over $k = \mathbb{C}(q)$ by the matrix coefficients $_V\phi_{e_j}^{e^i}$, for all objects V of the category $\mathcal{C}$, by the ideal generated by the relations* (4.11) *and* (4.12).

Proof. Let us write $R_{V,W} = \sum_{(R)} R_{(1)} \otimes R_{(2)} = \sum_{(R)} R_{(1')} \otimes R_{(2')}$. The product of $\mathcal{O}_q$ is defined by

$$_{(V \otimes W)}\phi_{e_j \otimes e_l}^{e^i \otimes e^k} = {_V\phi_{e_j}^{e^i}} \star {_W\phi_{e_l}^{e^k}}.$$

Then the relation (4.12) is equivalent to

$$\sum_{i,j,k,l} E_i^j \otimes E_k^l \otimes {}_V\phi_{e_j}^{e^i} \star {}_W\phi_{e_l}^{e^k} = \sum_{(R),(R),i,j,k,l} E_i^j R_{(2)}R_{(2')} \otimes R_{(1)} E_k^l S(R_{(1')}) \otimes {}_V\phi_{e_j}^{e^i} {}_W\phi_{e_l}^{e^k}.$$

The isomorphism $V \otimes V^* \to \mathrm{End}(V)$, $v \otimes f \mapsto (w \mapsto f(w)v)$, maps $R_{(1)}e_k \otimes S(R_{(1')})^* e^l$ to $R_{(1)} E_k^l S(R_{(1')})$ and $e_i \otimes (R_{(2)}R_{(2')})^* e^j$ to $E_i^j R_{(2)}R_{(2')}$. Hence the above relation can be written as

$$\sum_{i,j,k,l} E_i^j \otimes E_k^l \otimes {}_V\phi_{e_j}^{e^i} \star {}_W\phi_{e_l}^{e^k} = \sum_{(R),(R),i,j,k,l} E_i^j \otimes E_k^l \otimes \left({}_V\phi_{R_{(2)}R_{(2')}e_j}^{e^i}\right)\left({}_V\phi_{S(R_{(1')})e_l}^{R_{(1)}^* e^k}\right).$$

Now we have

$${}_V\phi_{S(R_{(1')})e_l}^{R_{(1)}^* e^k} = S(R_{(1')}) \triangleright {}_V\phi_{e_l}^{e^k} \triangleleft R_{(1)}, \qquad {}_V\phi_{R_{(2)}R_{(2')}e_j}^{e^i} = R_{(2)}R_{(2')} \triangleright {}_W\phi_{e_j}^{e^i},$$

where we use the coregular actions $\triangleright$, $\triangleleft$ and we denote now by $R_{(1)}, R_{(1')}, R_{(2)}, R_{(2')} \in \mathbb{U}_q$ the components of the universal R-matrix, instead of $R_{V,W}$. Identifying the matrix coefficients in i, j, k, l we recover the relation (4.6). Hence it is equivalent to the fusion relation, which thus provides a defining set of relations for $\mathcal{L}_{0,1}$.

Finally, note that by (4.11) we have

$$\sigma_{V,W} R_{V,W} \overset{V \otimes W}{M} = \overset{W \otimes V}{M} \sigma_{V,W} R_{V,W},$$

where $\sigma_{V,W} \colon V \otimes W \to W \otimes V$ is the flip map. Then

$$\sigma_{V,W} R_{V,W} \overset{V}{M_1} R'_{V,W} \overset{W}{M_2} (R'_{V,W})^{-1} = \overset{W}{M_1} R'_{W,V} \overset{V}{M_2} (R'_{W,V})^{-1} \sigma_{V,W} R_{V,W}$$
$$= \sigma_{V,W} \overset{W}{M_2} R_{V,W} \overset{V}{M_1},$$

which implies the reflection equation. ∎

Remark 4.7. As usual, denote by a_V or $\pi_V(a) \in \mathrm{End}(V)$ the component of an element $a = (a_V) \in \mathbb{U}_q$ associated to the object V of $\mathcal{C}$. The right coadjoint action of U_q on $\mathcal{L}_{0,1}$ can be written in matrix form as (see, e.g., the proof of (6.21) below for a similar computation)

$$\mathrm{coad}^r(y)(\overset{V}{M}) = \sum_{(y)} \left((y_{(1)V} \otimes \mathrm{id}) \overset{V}{M} (S(y_{(2)})_V \otimes \mathrm{id})\right). \tag{4.14}$$

Let $\lambda \in P_+$, V_λ the type 1 simple U_q-module of highest weight λ, and π_{V_λ} the associated representation. Denote by $\mathrm{Tr}_{V_\lambda} \colon \mathrm{End}(V_\lambda) \to k$ the trace on $\mathrm{End}(V_\lambda)$. Put

$$\mathrm{qTr}_{V_\lambda}(\overset{V_\lambda}{M}) := (\mathrm{Tr}_{V_\lambda} \otimes \mathrm{id})\left((\pi_{V_\lambda}(\ell) \otimes \mathrm{id}) \overset{V_\lambda}{M}\right) \in \mathcal{L}_{0,1}.$$

Proposition 4.8.

1. *The elements* $\mathrm{qTr}_{V_\lambda}(\overset{V_\lambda}{M})$, $\lambda \in P_+$, *form a basis of* $\mathcal{L}_{0,1}^{U_q}$.

2. *The elements* $(\mathrm{Tr}_{V_\lambda} \otimes \mathrm{id})\left((\pi_{V_\lambda} \otimes \mathrm{id})((\ell \otimes 1)(RR'))\right)$, $\lambda \in P_+$, *form a basis of* $\mathcal{Z}(\tilde{U}_q)$.

Proof. The first part is an immediate consequence of Proposition 6.22 below (namely, it is the case $n = 1$, so that $a_\lambda(x) \in \mathrm{End}_{U_q}(V_\lambda)$ is a scalar for every $\lambda \in P_+$). The second part is a consequence of the first and Theorem 4.3. ∎

Remark 4.9. Let V_{ω_1} be the fundamental representation of $\tilde{U}_q$. The center $\mathcal{Z}(\tilde{U}_q)$ contains the elements (see [12, 74])

$$(\mathrm{Tr}_{V_{\omega_1}} \otimes \mathrm{id})\big((\pi_{V_{\omega_1}} \otimes \mathrm{id})\big((\ell \otimes \mathrm{id})\big((RR')^k\big)\big)\big), \qquad k \in \{1, \ldots, m\}.$$

These elements generate $\mathcal{Z}(\tilde{U}_q)$ when $\mathfrak{g}$ is of type A_m or C_m; see [12] for a more precise description in the other cases. Using Theorem 4.3 and

$$(\mathrm{id} \otimes \Phi_1)\big(\overset{V}{M}\big) = (\pi_V \otimes \mathrm{id})(RR')$$

we deduce that the elements

$$_k\omega := (\mathrm{Tr}_{V_{\omega_1}} \otimes \mathrm{id})\big((\pi_{V_{\omega_1}}(\ell) \otimes \mathrm{id}) \overset{V_{\omega_1}}{M}{}^k\big), \qquad k \in \{1, \ldots, m\},$$

belong to and generate $\mathcal{L}_{0,1}^{U_q}$ when $\mathfrak{g}$ is of type A_m or C_m. In particular, for $\mathfrak{g} = \mathfrak{sl}(2)$ we have $\mathcal{L}_{0,1}^{U_q} = \mathbb{C}(q)[\omega]$, with

$$\omega = \mathrm{qTr}_{V_2}\big(\overset{V_2}{M}\big) = \mathrm{Tr}_{V_2}\big(K_{V_2} \overset{V_2}{M}\big), \qquad K_{V_2} := \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \tag{4.15}$$

where V_2 is the 2-dimensional type 1 simple $U_q(\mathfrak{sl}(2))$-module, and as usual K_{V_2} is the endomorphism of V_2 given by the action of K.

Next we define the integral form $\mathcal{L}_{0,1}^A$ of $\mathcal{L}_{0,1}$. Recall that $\mathcal{O}_A \subset \mathcal{O}_q$ is a Hopf A-subalgebra, and that it is a free A-module, and a full A-sublattice of $\mathcal{O}_q$.

Proposition 4.10. *The space $\mathcal{O}_A$ endowed with the product* (4.4) *is an A-algebra. We denote it $\mathcal{L}_{0,1}^A$. Moreover, $\mathcal{L}_{0,1}^A$ is the A-algebra generated by the matrix coefficients $_X\phi_{e_j}^{e^i}$, for all objects X of the category $\mathcal{C}_A$, with defining relations* (4.11) *and* (4.12)*.*

Note that, since $\mathcal{L}_{0,1}^A$ coincides with $\mathcal{O}_A$ as an A-module, it is a free A-module and we have $\mathcal{L}_{0,1}^A \otimes_A \mathbb{C}(q) = \mathcal{L}_{0,1}$.

Proof. We have to check that products of elements of an A-basis of $\mathcal{O}_A$ can be expressed as linear combinations in this basis with coefficients in A. Denote by $\{\alpha_i\}$ a basis of $\mathcal{O}_A$. By Proposition 4.1 we have

$$\alpha_i \alpha_j = \sum_k m_{ij}^k \alpha_k, \tag{4.16}$$

where each $m_{ij}^k \in \mathbb{C}(q)$. The basis elements α_i are linear combinations over A of matrix coefficients of U_A^{res}-modules of type 1. Let X and Y be two such modules, and $\{e_i\}, \{e^i\}$ and $\{f_n\}, \{f^n\}$ dual basis of X and Y respectively. As in (4.8) we have

$$_X\phi_{e_i}^{e^j}{}_Y\phi_{f_m}^{f^n} = \sum_{(R),(R^{-1}),j',n',n''} \big(\pi_X\big(R_{(2')}\ell(R^{-1})_{(2)}\ell^{-1}\big)\big)_{e_i}^{e^{j'}} \big(\pi_Y\big(R_{(1')}\big)\big)_{f_m}^{f^{n'}}$$

$$\times \big(\pi_Y(R^{-1})_{(1)}\big)_{f_{n''}}^{f^n}{}_X\phi_{e_{j'}}^{e^j} \star {}_Y\phi_{f_{n'}}^{f^{n''}}.$$

The elements $_X\phi_{e_{j'}}^{e^j} \star {}_Y\phi_{f_{n'}}^{f^{n''}}$ being in $\mathcal{O}_A$, they can expressed as linear combinations over A of the basis elements α_i. Also, by Theorem 3.2(1) the coefficients in this sum belong to $\mathbb{C}[q^{1/D}, q^{-1/D}]$. Therefore each coefficient m_{ij}^k in (4.16) belongs to $\mathbb{C}[q^{1/D}, q^{-1/D}]$. Since $\mathbb{C}(q) \cap \mathbb{C}[q^{1/D}, q^{-1/D}] = \mathbb{C}[q, q^{-1}]$, this proves the first claim. The arguments of Proposition 4.6 apply as well to $\mathcal{L}_{0,1}^A$, which implies the second claim. ∎

Lemma 4.11. *The action* coad^r *yields on* $\mathcal{L}_{0,1}^A$ *a structure of right U_A^{res}-module algebra, whence of U_A-module algebra. Moreover, $\Phi_1\big(\mathcal{L}_{0,1}^A\big)$ is a full A-sublattice of $\tilde{U}_q^{\mathrm{lf}}$, that we denote by $\tilde{U}_A^{\mathrm{lf}}$.*

Proof. The first claim is clear, for the action coad^r endows the A-module $\mathcal{L}_{0,1}^A$ with a structure of right U_A^{res}-module algebra, dual by the (non degenerate) pairing $\mathcal{O}_A \otimes_A U_A^{\mathrm{res}} \to A$ to the structure defined by ad^r on U_A^{res}. The inclusion $U_A \subset U_A^{\mathrm{res}}$ yields the second claim. The third follows from Theorem 4.3 and the fact that $\mathcal{L}_{0,1}^A$ is a full A-sublattice of $\mathcal{L}_{0,1}$. ∎

5 The example of $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$

In this section we provide a presentation by generators and relations of $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$, and using it we give an elementary proof of Theorem 4.3 in this case.

Let V_r be the r-dimensional type 1 simple $U_q(\mathfrak{sl}(2))$-module. Put on V_2 the basis vectors v_+, v_- such that $K.v_+ = qv_+$, $E.v_+ = 0$, $v_- = F.v_+$, and define in this basis

$$\overset{V_2}{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{End}(V_2) \otimes \mathcal{L}_{0,1}(\mathfrak{sl}(2)). \tag{5.1}$$

Lemma 5.1. *The algebra $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$ is generated by the matrix elements a, b, c, d of $\overset{V_2}{M}$. These satisfy the relations*

$$\begin{aligned} ad &= da, & ab - ba &= -\big(1 - q^{-2}\big)bd, \\ db &= q^2 bd, & cb - bc &= \big(1 - q^{-2}\big)\big(da - d^2\big), \\ cd &= q^2 dc, & ac - ca &= \big(1 - q^{-2}\big)dc, \end{aligned} \tag{5.2}$$

as well as $ad - q^2 bc = 1$. Moreover $\omega = qa + q^{-1}d$ is central.

Note that ω coincides with the element (4.15).

Proof. The family of matrix elements of $\overset{V}{M}$ for all type 1 $U_q(\mathfrak{sl}(2))$-modules V spans $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$ over $\mathbb{C}(q)$, since as a vector space it is the same as $\mathcal{O}_q(\mathfrak{sl}(2))$. Any finite dimensional $U_q(\mathfrak{sl}(2))$-module is completely reducible, and any simple one is a direct summand of some tensor power of V_2. Hence the fusion relation (4.12) implies that $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$ is generated by the matrix elements a, b, c, d. The relations (5.2) follow easily from the reflection equation (4.13) associated to V_2, using the expression

$$R_{V_2,V_2} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

and they imply that $qa + q^{-1}d$ is central. Because $V_2 \otimes V_2$ admits the trivial representations as a subrepresentation, there exist non zero intertwiners

$$\phi \colon V_2 \otimes V_2 \to \mathbb{C}(q), \qquad \psi \colon \mathbb{C}(q) \to V_2 \otimes V_2.$$

As a consequence $\phi \circ \overset{V_2 \otimes V_2}{M} \circ \psi$ is proportional to the unit element of $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$. Using the fusion relation, an easy computation provides the additional relation $ad - q^2 bc = 1$. ∎

Consider the RSD map $\Phi_1 \colon \mathcal{L}_{0,1}(\mathfrak{sl}(2)) \to \mathbb{U}_q(\mathfrak{sl}(2))$, $\alpha \mapsto (\alpha \otimes \mathrm{id})(RR')$. A straightforward computation using the expression (3.7) shows that

$$\Phi_1(a) = K + q^{-1}(q - q^{-1})^2 FE, \qquad \Phi_1(b) = q^{-1}(q - q^{-1})F,$$
$$\Phi_1(c) = (q - q^{-1})K^{-1}E, \qquad\qquad \Phi_1(d) = K^{-1}. \tag{5.3}$$

Therefore $\mathrm{Im}(\Phi_1)$ is contained in $U_q(\mathfrak{sl}(2))$. The image of the central element ω is

$$\Omega = \Phi_1(\omega) = qK + q^{-1}K^{-1} + (q - q^{-1})^2 FE, \tag{5.4}$$

which is $(q - q^{-1})^2$ times the standard Casimir element of $U_q(\mathfrak{sl}(2))$.

Next we show that the relations (5.2) and $ad - q^2 bc = 1$ yield a presentation of $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$. Let $\tilde{\mathcal{L}}_{0,1}$ be the algebra generated by elements $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ satisfying all these relations. Denote by

$$j \colon \ \tilde{\mathcal{L}}_{0,1} \to \mathcal{L}_{0,1}(\mathfrak{sl}(2))$$

the unique morphism of algebra sending $\tilde{x}$ to x for $x \in \{a, b, c, d\}$.

Proposition 5.2. *The monomials $\tilde{a}^\alpha \tilde{b}^\beta \tilde{c}^\gamma$ and $\tilde{d}^\delta \tilde{b}^\beta \tilde{c}^\gamma$, where $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ and $\alpha \geq 1$, form a basis of $\tilde{\mathcal{L}}_{0,1}$ over $\mathbb{C}(q)$. Moreover, j is an isomorphism, Φ_1 is injective, and the center of $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$ is the polynomial algebra $\mathbb{C}(q)[\omega]$.*

Proof. By inspection of the relations of Lemma 5.1, it is easily seen that the given monomials form a generating set of $\tilde{\mathcal{L}}_{0,1}$. As for linear independence, consider the Verma $U_q(\mathfrak{sl}(2))$-module M_x, $x \in \mathbb{C}^\times$, with basis vectors v_n, $n \in \mathbb{N}$, and action

$$Ev_0 = 0, \qquad Kv_n = xq^{-2n}v_n, \qquad Fv_n = v_{n+1}.$$

Then

$$Ev_n = [n]_q \frac{xq^{1-n} - x^{-1}q^{n-1}}{q - q^{-1}} v_{n-1}, \qquad \Omega v_n = (qx + q^{-1}x^{-1})v_n,$$

where Ω is as above. By the formulas (5.3) we have

$$\Phi_1(a^\alpha b^\beta c^\gamma) = q^{-\alpha - \beta}(q - q^{-1})^{\beta + \gamma}(\Omega - q^{-1}K^{-1})^\alpha F^\beta (K^{-1}E)^\gamma$$

and

$$\Phi_1(d^\delta b^\beta c^\gamma) = q^{-\beta}(q - q^{-1})^{\beta + \gamma}K^{-\delta}F^\beta (K^{-1}E)^\gamma.$$

Consider a linear relation with coefficients $A_{\alpha\beta\gamma}, B_{\delta\beta'\gamma'} \in \mathbb{C}(q)$:

$$\sum_{\alpha,\beta,\gamma} A_{\alpha\beta\gamma} \tilde{a}^\alpha \tilde{b}^\beta \tilde{c}^\gamma + \sum_{\delta,\beta',\gamma'} B_{\delta\beta'\gamma'} \tilde{d}^\delta \tilde{b}^{\beta'} \tilde{c}^{\gamma'} = 0.$$

Applying $\Phi_1 \circ j$ we get (keeping the same names for the resulting coefficients in $\mathbb{C}(q)$):

$$\sum_{\alpha,\beta,\gamma,\gamma'} \left(A_{\alpha\beta\gamma}(\Omega - q^{-1}K^{-1})^\alpha F^\beta (K^{-1}E)^\gamma + B_{\delta\beta'\gamma'}K^{-\delta}F^{\beta'}(K^{-1}E)^{\gamma'} \right) = 0. \tag{5.5}$$

By acting on the highest weight vector v_0 all terms on the left hand side vanish, but those with $\gamma = \gamma' = 0$. Hence

$$\sum_{\alpha,\beta} A_{\alpha\beta 0}(\Omega - q^{-1}K^{-1})^\alpha F^\beta v_0 + \sum_{\delta\beta'} B_{\delta\beta'0}K^{-\delta}F^{\beta'}v_0 = 0$$

implying for each β the relation

$$\sum_\alpha A_{\alpha\beta0}\big(qx + q^{-1}x^{-1}(1 - q^{2\beta})\big)^\alpha + \sum_\delta B_{\delta\beta0}x^{-\delta}q^{2\beta\delta} = 0.$$

This is a Laurent polynomial in x. Since $\delta \geq 1$, the highest degree term in x has vanishing coefficient, $A_{\alpha\beta0} = 0$, and hence $B_{\delta\beta0} = 0$. So (5.5) has no terms with $\gamma = \gamma' = 0$. Then, by acting on v_1 it results again that (5.5) has no term with $\gamma = \gamma' = 1$. Iterating this argument, an obvious recurrence implies that all the coefficients vanish, $A_{\alpha\beta\gamma} = B_{\delta\beta'\gamma'} = 0$, which implies the linear independence of the monomials $\tilde{a}^\alpha \tilde{b}^\beta \tilde{c}^\gamma$ and $\tilde{d}^\delta \tilde{b}^\beta \tilde{c}^\gamma$ and therefore proves that they form a basis of $\tilde{\mathcal{L}}_{0,1}$.

As a by-product we see that $\Phi_1 \circ j$ is injective, and therefore j is injective too. It is also surjective because the monomials $a^\alpha b^\beta c^\gamma$ and $d^\delta b^\beta c^\gamma$ form a generating family of $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$. It follows that Φ_1 is injective as well.

Finally, let z be in the center of $\mathcal{L}_{0,1}$. Then $\Phi_1(z)$ commutes with $\Phi_1(\mathcal{L}_{0,1}(\mathfrak{sl}(2)))$, and hence with K, E, F. Thus it belongs to the center of $U_q(\mathfrak{sl}(2))$, which is $\mathbb{C}(q)[\Omega]$. Therefore $\Phi_1(z) = P(\Omega) = P(\Phi_1(\omega)) = \Phi_1(P(\omega))$. The result follows from the injectivity of Φ_1. ∎

Remark 5.3. A consequence of this proposition is that the algebras $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$ and $U_q(\mathfrak{sl}(2))$ are not isomorphic. Indeed the former has the family of one dimensional representations ρ_{xy} ($x \in \mathbb{C}$, $y \in \mathbb{C}^\times$), defined by

$$\rho_{xy}(d) = 0, \qquad \rho_{xy}(a) = x, \qquad \rho_{xy}(b) = -q^{-1}y, \qquad \rho_{xy}(c) = q^{-1}y^{-1}.$$

Then $K \notin \mathrm{Im}(\Phi_1)$, for otherwise $K = \Phi_1(\lambda)$, and injectivity of Φ_1 and $K^{-1} = \Phi_1(d)$ would imply $\lambda d = d\lambda = 1$. But this is not possible since $\rho_{xy}(d) = 0$. In fact the family $\{\rho_{xy}\}_{x,y}$ cannot be obtained by pull-back of representations of $U_q(\mathfrak{sl}(2))$ and the category of finite dimensional modules of $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$ is not semisimple (see [21, Proposition 9]).

From the above results we can now derive an easy proof of the particular case of Theorem 4.3 for $\mathfrak{g} = \mathfrak{sl}(2)$. Note that $\tilde{U}_q^{\mathrm{lf}}(\mathfrak{sl}(2)) = U_q^{\mathrm{lf}}(\mathfrak{sl}(2))$ by the result of [52] recalled in Remark 4.5. Also, the inclusion $U_q^{\mathrm{lf}}(\mathfrak{sl}(2)) \subset U_q(\mathfrak{sl}(2))$ is strict, for $K \notin U_q^{\mathrm{lf}}(\mathfrak{sl}(2))$.

Proposition 5.4. *The RSD map yields an isomorphism of U_q-module algebras*

$$\Phi_1 \colon \ \mathcal{L}_{0,1}(\mathfrak{sl}(2)) \to U_q^{\mathrm{lf}}(\mathfrak{sl}(2)).$$

Proof. That Φ_1 is an equivariant morphism follows from the same arguments as for the Alekseev map (see Theorem 6.6). Injectivity was shown in Proposition 5.2. We prove that $\mathrm{Im}(\Phi_1) = U_q^{\mathrm{lf}}(\mathfrak{sl}(2))$ by following closely the arguments of Section 3.11 of Joseph–Letzter [52]. For every integer $m \geq 0$ we have

$$\mathrm{ad}^r(K)\big((EK^{-1})^m\big) = q^{-2m}(EK^{-1})^m, \qquad \mathrm{ad}^r(E)\big((EK^{-1})^m\big) = 0. \tag{5.6}$$

Moreover,

$$\mathrm{ad}^r(F)\big(EK^{-1}\big) = \frac{q}{q - q^{-1}}\big(\Omega - (q + q^{-1})K^{-1}\big), \qquad \mathrm{ad}^r\big(F^2\big)\big(EK^{-1}\big) = -\big(q + q^{-1}\big)F,$$

$$\mathrm{ad}^r(F^3)\big(EK^{-1}\big) = 0, \qquad \mathrm{ad}^r(E)\big(K^{-1}\big) = EK^{-1}\big(q^{-2} - 1\big),$$

$$\mathrm{ad}^r(F)\big(K^{-1}\big) = F\big(1 - q^{-2}\big), \qquad \mathrm{ad}^r(E)(F) = -\mathrm{ad}^r(F)\big(EK^{-1}\big), \qquad \mathrm{ad}^r(F)(F) = 0.$$

These relations imply that EK^{-1}, K^{-1} and F belong to $U_q^{\mathrm{lf}}(\mathfrak{sl}(2))$. Because Ω is invariant under the action ad^r, the formulas (5.3) imply that $\Phi_1(a)$, $\Phi_1(b)$, $\Phi_1(c)$ and $\Phi_1(d)$ belong to $U_q^{\mathrm{lf}}(\mathfrak{sl}(2))$. Therefore $\mathrm{Im}(\Phi_1) \subset U_q^{\mathrm{lf}}(\mathfrak{sl}(2))$.

These above relations imply also that $\Phi_1(c)$ is a highest weight vector for the action ad^r, generating a copy of V_3, the 3-dimensional simple $U_q(\mathfrak{sl}(2))$-module of type 1.

Let us show by induction that $\mathrm{ad}^r(U_q(\mathfrak{sl}(2)))\big(\Phi_1(c)^m\big) = V_{2m+1}$, for an arbitrary positive integer m. In view of (5.6), it remains to prove that $\mathrm{ad}^r\left(F^k\right)\big((EK^{-1})^m\big) \neq 0$ for all integers $k \leq 2m$, and $\mathrm{ad}^r\left(F^k\right)\big((EK^{-1})^m\big) = 0$ for $k \geq 2m+1$. This holds true for $m = 1$.

If this is true for a given m, then using the formula of $\Delta(F)$, the q-binomial identity, and the fact that $U_q(\mathfrak{sl}(2))$ is an ad^r-module algebra, we get

$$\mathrm{ad}^r\left(F^k\right)\big((EK^{-1})^{m+1}\big) = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_{q^{-2}} \mathrm{ad}^r\left(K^{-i}F^{k-i}\right)\big((EK^{-1})^m\big)\,\mathrm{ad}^r\left(F^i\right)\big(EK^{-1}\big)$$

$$= \sum_{i=0}^{2} \begin{bmatrix} k \\ i \end{bmatrix}_{q^{-2}} \mathrm{ad}^r\left(K^{-i}F^{k-i}\right)\big((EK^{-1})^m\big)\,\mathrm{ad}^r\left(F^i\right)\big(EK^{-1}\big),$$

since $\mathrm{ad}^r\left(F^i\right)\big(EK^{-1}\big) = 0$ if and only if $i \geq 3$. By the induction hypothesis, for $k \geq 2m+3$ each of the three terms of the sum vanishes. For $k = 2m+2$, only the term for $i = 2$ is non zero, which is equal to

$$\begin{bmatrix} k \\ 2 \end{bmatrix}_{q^{-2}} \mathrm{ad}^r\left(K^{-2}F^{2m}\right)\big((EK^{-1})^m\big)\big(-(q+q^{-1})F\big).$$

By induction this term is non zero. Therefore $\mathrm{ad}^r(F^k)\big((EK^{-1})^m\big) \neq 0$ for all integers $k \leq 2m+2$. This proves our claim.

Finally the multiplication map $(\oplus_{m \geq 0} V_{2m+1}) \otimes \mathbb{C}(q)[\Omega] \longrightarrow U_q^{\mathrm{lf}}(\mathfrak{sl}(2))$ is easily shown to be an isomorphism, as in [52, Section 3.11]. Since $\Omega = \Phi_1(\omega)$, and V_{2m+1} is generated by $\Phi_1(c^{2m})$, we deduce the inclusion $U_q^{\mathrm{lf}}(\mathfrak{sl}(2)) \subset \mathrm{Im}(\Phi_1)$. ∎

Remark 5.5. If we denote $\mathbb{H} = \oplus_{m \geq 0} V_{2m+1}$, the isomorphism of modules

$$\mathbb{H} \otimes \mathbb{C}(q)[\Omega] \longrightarrow U_q^{\mathrm{lf}}(\mathfrak{sl}(2))$$

is an explicit example of the theorem of separation of variables of [13, 53] in the case of $\mathfrak{sl}(2)$ (this case was first described in [52]). As shown in these works, the multiplicity $[\mathbb{H}, V_k]$ for $k \geq 1$ is the dimension (0 or 1) of the set of zero weight vectors of V_k.

Let us make explicit the result of [53] mentionned in Remark 4.5. By the relations (5.2) and Proposition 5.2, it is immediate that for all $x \in \mathcal{L}_{0,1}(\mathfrak{sl}(2))$ there exist elements $y, y' \in \mathcal{L}_{0,1}(\mathfrak{sl}(2))$ such that $dy = xd$ and $y'd = dx$. The element d is regular because $\Phi_1(d)$ is invertible and Φ_1 is injective, so $\{d^n\}_{n \in \mathbb{N}}$ is a *left and right multiplicative Ore set* in $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$. The localization of $\mathcal{L}_{0,1}(\mathfrak{sl}(2))$ over $\{d^n\}_{n \in \mathbb{N}}$ is well-defined (see [49, Theorem 1.9 and Corollary 6.4]). Let us denote it by $_{\mathrm{loc}}\mathcal{L}_{0,1}(\mathfrak{sl}(2))$.

Proposition 5.6. *The map* $\Phi_1 \colon {}_{\mathrm{loc}}\mathcal{L}_{0,1}(\mathfrak{sl}(2)) \to U_q(\mathfrak{sl}(2))$ *defined by* $\Phi_1\big(d^{-1}\big) = K$ *and the formulas* (5.3) *is an isomorphism of* $U_q(\mathfrak{sl}(2))$-*module algebras.*

Proof. Since $\Phi_1(d)$ is invertible, Φ_1 extends to a morphism $_{\mathrm{loc}}\mathcal{L}_{0,1}(\mathfrak{sl}(2)) \to U_q(\mathfrak{sl}(2))$ uniquely (see [49, Proposition 6.3]). It is an isomorphism, because the monomials $d^\delta b^\beta c^\gamma$ with $\beta, \gamma \in \mathbb{N}$ and $\delta \in \mathbb{Z}$ make a basis of $_{\mathrm{loc}}\mathcal{L}_{0,1}(\mathfrak{sl}(2))$, and they are sent by Φ_1 to a PBW basis of $U_q(\mathfrak{sl}(2))$. ∎

Finally consider the integral form $\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))$.

Lemma 5.7. *The following holds:*

1. *The A-algebra $\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))$ is generated by a, b, c, d with the defining relations* (5.2).

2. *The localization of $\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))$ over the set $\{d^n\}_{n\in\mathbb{N}}$, that we denote by $_{\mathrm{loc}}\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))$, is generated by a, b, c, $d^{\pm 1}$ with the defining relations (5.2).*

3. *The RSD map yields embeddings of U_A-module algebras $\Phi_1 \colon \mathcal{L}_{0,1}^A(\mathfrak{sl}(2)) \to U_A^{\mathrm{lf}}(\mathfrak{sl}(2))$ and $\Phi_1 \colon {}_{\mathrm{loc}}\mathcal{L}_{0,1}^A(\mathfrak{sl}(2)) \to U_A(\mathfrak{sl}(2))$, and $\Phi_1\big({}_{\mathrm{loc}}\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))\big) = U_A'(\mathfrak{sl}(2))$ is the A-subalgebra $U_A'(\mathfrak{sl}(2)) \subset U_A(\mathfrak{sl}(2))$, generated by $\big(q - q^{-1}\big)E$, $\big(q - q^{-1}\big)F$ and $K^{\pm 1}$.*

Proof. (1) It is shown in [29], Proposition 1.3, that $\mathcal{O}_A(\mathfrak{sl}(2)) \subset \mathcal{O}_q(\mathfrak{sl}(2))$ is the A-subalgebra generated by a, b, c, d. Since $\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))$ coincides with $\mathcal{O}_A(\mathfrak{sl}(2))$ as an A-module, any element of $\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))$ is a linear combination over A of monomials in a, b, c, d with respect to the product $\star$ of $\mathcal{O}_A(\mathfrak{sl}(2))$. By the relation (4.6) and Proposition 4.10, any such monomial is a linear combination over A of monomials in a, b, c, d with respect to the product (4.4) of $\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))$. This implies the first claim. The second follows from the fact that the relations (5.2) are defined over A. The assertion (2) is an immediate consequence of (1), and (3) follows from Lemma 4.11. We use the formulas (5.3) to determine $\Phi_1\big({}_{\mathrm{loc}}\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))\big)$. ∎

6 The "daisy" graph algebra $\mathcal{L}_{0,n}(\mathfrak{g})$

6.1 Definition and first properties

We define $\mathcal{L}_{0,n}$ as a *twisted product* of n copies of the $U_q \otimes U_q^{\mathrm{cop}}$-module algebra $\mathcal{L}_{0,1}$ (see Proposition 4.1), following [32] (but using right modules instead of left ones). Since the twist uses R-matrices, as for $\mathcal{L}_{0,1}$ the construction uses as well categorical completions. Eventually we will see that $\mathcal{L}_{0,n}$ is also the braided tensor product of n copies of the U_q-module algebra $\mathcal{L}_{0,1}$ (see, e.g., [69] for the notion of braided tensor product).

We need to recall a few notions. Consider Hopf algebras A and B, and a *bicharacter $F \in B \otimes A$*. By definition, F is an invertible tensor and satisfies

$$(\Delta_B \otimes \mathrm{id}_A)(F) = F_{23}F_{13}, \qquad (\mathrm{id}_B \otimes \Delta_A)(F) = F_{12}F_{13}. \tag{6.1}$$

Viewing F as an element of $(1 \otimes B) \otimes (A \otimes 1) \subset (A \otimes B)^{\otimes 2}$, it is readily checked that

$$F_{12}(\Delta_{A\otimes B} \otimes \mathrm{id})(F) = F_{23}(\mathrm{id} \otimes \Delta_{A\otimes B})(F)$$

and

$$(\varepsilon_{A\otimes B} \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon_{A\otimes B})(F) = 1.$$

Therefore F is a twist of $A \otimes B$, endowed with the standard Hopf algebra product structure (see (4.3) for the similar operation applied to $\mathbb{D}_q$). Denote by $A \otimes^F B$ the resulting structure of Hopf algebra. Given a right A-module algebra M, and a right B-module algebra N, one defines the twisted tensor product $M \otimes^F N$ as the space $M \otimes N$ endowed with the product

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') = \sum_{(F)} \alpha(\alpha' \cdot F_{(2)}) \otimes (\beta \cdot F_{(1)})\beta'$$

for every $\alpha, \beta \in M$, $\alpha', \beta' \in N$. This product gives $M \otimes^F N$ a structure of associative and unital right module algebra over $A \otimes^F B$, containing $M \otimes^F 1$ and $1 \otimes^F N$ as A- and B-module algebras respectively.

Remark 6.1. In the case where $A = B$ is quasi-triangular with R-matrix R, and $F = R'$, this construction gives the braided tensor product of A-module algebras.

These constructions generalizes straightforwardly to the categorical completions we consider. Taking $A = B = \mathbb{U}_q$, the identities (2.2) for the universal R-matrix imply that R' is a bicharacter of $\mathbb{U}_q \otimes \mathbb{U}_q$, and that we have (compare with (4.2))

$$\mathbb{A}_q = \mathbb{U}_q \otimes^{R'} \mathbb{U}_q.$$

We are going to iterate the above twist constructions. To this aim, observe that given homomorphisms of Hopf algebras $f_A \colon \mathbb{U}_q \to A$, $f_B \colon \mathbb{U}_q \to B$, the element $(f_B \otimes f_A)(R')$ is a bicharacter of $A \otimes B$, and by Proposition 4.2 the map

$$f_A \odot f_B := (f_A \otimes f_B) \circ \Delta \colon \ \mathbb{U}_q \to A \otimes^{(f_B \otimes f_A)(R')} B$$

is a morphism of Hopf algebras. Because of (6.1), taking $H := A = B$ the operation $\odot$ endows the set of all morphisms of Hopf algebras $\mathbb{U}_q \to H$ with a structure of associative monoid. One checks easily that it has the counit as neutral element. Clearly, $\odot$ is natural in the sense that $(g_A \circ f_A) \odot (g_B \circ f_B) = (g_A \otimes g_B) \circ (f_A \odot f_B)$ for any morphisms of Hopf algebras $g_A \colon A \to A'$, $g_B \colon B \to B'$.

In particular, by using the family of morphisms $\mathrm{id}_{\mathbb{U}_q}^{\odot n}$, $n \in \mathbb{N}$, we define Hopf algebras $\mathbb{U}_q^{\odot n}$ as follows:

$$\mathbb{U}_q^{\odot 0} = \mathbb{C}(q^{1/D}), \qquad \mathbb{U}_q^{\odot 1} = \mathbb{U}_q$$

and for $n \geq 1$,

$$\mathbb{U}_q^{\odot(n+1)} := \mathbb{U}_q^{\odot n} \otimes^{J_n} \mathbb{U}_q, \qquad \text{where} \quad J_n := \big(\mathrm{id}_{\mathbb{U}_q} \otimes \mathrm{id}_{\mathbb{U}_q}^{\odot n} \big)(R').$$

Therefore, an immediate induction shows that $\mathbb{U}_q^{\odot n}$ is the twist of $\mathbb{U}_q^{\otimes n}$ by $\prod_{k=n-1}^{1} J_k$. Because of (6.1), for every $k, l \in \mathbb{N}$ we have an equality of Hopf algebras

$$\mathbb{U}_q^{\odot(k+l)} = \mathbb{U}_q^{\odot k} \odot \mathbb{U}_q^{\odot l}. \tag{6.2}$$

For instance, since $\mathbb{A}_q = \mathbb{U}_q^{\odot 2}$ and $\mathrm{id}_{\mathbb{U}_q}^{\odot 2} = \Delta$, the Hopf algebra $\mathbb{A}_q^{\odot 2} = \mathbb{U}_q^{\odot 4}$ is the twist of $\mathbb{A}_q \otimes \mathbb{A}_q$ by

$$F_2 := \big(\mathrm{id}_{\mathbb{U}_q}^{\odot 2} \odot \mathrm{id}_{\mathbb{U}_q}^{\odot 2} \big)(R') = (\Delta \otimes \Delta)(R') = R'_{23} R'_{13} R'_{24} R'_{14} \in \mathbb{A}_q \otimes \mathbb{A}_q. \tag{6.3}$$

We are mainly concerned with the Hopf algebras $\mathbb{A}_q^{\odot n} = \mathbb{U}_q^{\odot 2n}$. Denote by F_n the twist from $\mathbb{A}_q^{\otimes n}$ to $\mathbb{A}_q^{\odot n}$.

We define $\mathcal{L}_{0,n}(q^{1/D})$ as the twist by F_n of the right $\mathbb{A}_q^{\otimes n}$-module algebra $\mathcal{L}_{0,1}(q^{1/D})^{\otimes n}$ (endowed with the componentwise action and product). By construction $\mathcal{L}_{0,n}(q^{1/D})$ is a right $\mathbb{A}_q^{\odot n}$-module algebra, and it coincides with $\mathcal{L}_{0,1}(q^{1/D})^{\otimes n}$ as a $\mathbb{A}_q^{\odot n}$-module.

The above definition of $\mathcal{L}_{0,n}(q^{1/D})$ by induction can be formulated by means of exchange relations between its factors $\mathcal{L}_{0,1}$. Let us explain this. For every k, l the right $\mathbb{A}_q^{\odot n}$-module algebra $\mathcal{L}_{0,k}(q^{1/D}) \odot \mathcal{L}_{0,l}(q^{1/D})$ is the twist by $F(l,k) = (\Delta^{\odot l} \otimes \Delta^{\odot k})(R')$ of the $\mathbb{A}_q^{\odot k} \otimes \mathbb{A}_q^{\odot l}$-module algebra $\mathcal{L}_{0,k}(q^{1/D}) \otimes \mathcal{L}_{0,l}(q^{1/D})$. By associativity of $\odot$ (see (6.2)) we have

$$\mathcal{L}_{0,k}(q^{1/D}) \odot \mathcal{L}_{0,l}(q^{1/D}) = \mathcal{L}_{0,k+l}(q^{1/D}).$$

Under this factorisation, by setting $k + l = n$, for every $\alpha, \alpha' \in \mathcal{L}_{0,k}(q^{1/D})$ and $\beta, \beta' \in \mathcal{L}_{0,l}(q^{1/D})$ the product of $\mathcal{L}_{0,n}(q^{1/D})$ takes the form

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') = \sum_{(F(l,k))} \alpha(\alpha' \cdot F(l,k)_{(2)}) \otimes (\beta \cdot F(l,k)_{(1)})\beta', \tag{6.4}$$

where the products $\alpha(\alpha' \cdot F(l,k)_{(2)})$ and $(\beta \cdot F(l,k)_{(1)})\beta'$ are taken in $\mathcal{L}_{0,k}(q^{1/D})$ and $\mathcal{L}_{0,l}(q^{1/D})$ respectively. Moreover, for every $1 \leq a \leq n$ the map $i_a \colon \mathcal{L}_{0,1}(q^{1/D}) \to \mathcal{L}_{0,n}(q^{1/D})$, identifying $\mathcal{L}_{0,1}(q^{1/D})$ with the a-th factor of $\mathcal{L}_{0,1}(q^{1/D})^{\otimes n}$ by putting 1's elsewhere $\big(1$ being the unit of $\mathcal{L}_{0,1}(q^{1/D})\big)$, is an embedding of module algebras. We will use the notations

$$\mathcal{L}_{0,n}(q^{1/D})^{(a)} := \mathrm{Im}(i_a), \qquad (\alpha)^{(a)} := i_a(\alpha).$$

Taking $(\alpha)^{(a)} \in \mathcal{L}_{0,n}(q^{1/D})^{(a)}$ and $(\beta)^{(b)} \in \mathcal{L}_{0,n}(q^{1/D})^{(b)}$ with $a < b$, and $F = F(1,1) \in \mathbb{A}_q \otimes \mathbb{A}_q$ as in (6.3), we deduce from (6.4) that $\big($all products being taken in $\mathcal{L}_{0,n}(q^{1/D})\big)$

$$(\beta)^{(b)}(\alpha)^{(a)} = \sum_{(F)} \big(\alpha \cdot F_{(2)}\big)^{(a)}\big(\beta \cdot F_{(1)}\big)^{(b)}. \tag{6.5}$$

We have:

Proposition 6.2. *The subspace $\mathcal{L}_{0,1}^{\otimes n} \subset \mathcal{L}_{0,n}(q^{1/D})$ endowed with the product defined by (4.4) on each component and by the exchange relation (6.5) between distinct components is a right module $\mathbb{C}(q)$-algebra over $\big(U_q \otimes U_q^{\mathrm{cop}}\big)^{\odot n} \subset \mathbb{A}_q^{\odot n}$. We denote it $\mathcal{L}_{0,n}$. Moreover, $\mathcal{L}_{0,n}$ is a U_q-module algebra, namely the braided tensor product of n copies of the U_q-module algebra $\mathcal{L}_{0,1}$, with U_q-action, that we still denote by coad^r, given by*

$$\mathrm{coad}^r(y)\big(\alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)}\big) = \sum_{(y)} \mathrm{coad}^r(y_{(1)})\big(\alpha^{(1)}\big) \otimes \cdots \otimes \mathrm{coad}^r(y_{(n)})\big(\alpha^{(n)}\big) \tag{6.6}$$

for all $y \in U_q$ and $\alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)} \in \mathcal{L}_{0,n}$.

Proof. We have to show that the product (6.5) restricted to $\mathcal{L}_{0,1}^{\otimes n} \subset \mathcal{L}_{0,n}(q^{1/D})$ is defined over the subfield $\mathbb{C}(q)$. Indeed, the injection $i_a \colon \mathcal{L}_{0,1} \to \mathcal{L}_{0,n}(q^{1/D})$ being a morphism of algebras, the product of elements $\alpha, \beta \in i_a(\mathcal{L}_{0,1})$ is defined over $\mathbb{C}(q)$ (see Proposition 4.1). Therefore it is enough to check that the exchange relation (6.5) is defined over $\mathbb{C}(q)$ as well. We proceed as in (4.7) and (4.8). For matrix coefficients $\big({}_{V_\lambda}\phi_{e_j}^{e^i}\big)^{(a)}$ and $\big({}_{V_\mu}\phi_{e_l}^{e^k}\big)^{(b)}$, where $a < b$, (6.5) can be written as

$$\big({}_{V_\mu}\phi_{e_l}^{e^k}\big)^{(b)}\big({}_{V_\lambda}\phi_{e_j}^{e^i}\big)^{(a)}$$

$$= \sum_{(R^1)\cdots(R^4)} \Big(\big({}_{V_\lambda}\phi_{e_j}^{e^i}\big)^{(a)} \cdot \big(R_{(1)}^1 R_{(1)}^2 \otimes R_{(1)}^3 R_{(1)}^4\big)\Big)\Big(\big({}_{V_\mu}\phi_{e_l}^{e^k}\big)^{(b)} \cdot \big(R_{(2)}^2 R_{(2)}^4 \otimes R_{(2)}^1 R_{(2)}^3\big)\Big)$$

$$= \sum_{(R^1)\cdots(R^4)} \big(S\big(R_{(1)}^3 R_{(1)}^4\big) \triangleright \big({}_{V_\lambda}\phi_{e_j}^{e^i}\big)^{(a)} \triangleleft R_{(1)}^1 R_{(1)}^2\big)\big(S\big(R_{(2)}^1 R_{(2)}^3\big) \triangleright \big({}_{V_\mu}\phi_{e_l}^{e^k}\big)^{(b)} \triangleleft R_{(2)}^2 R_{(2)}^4\big)$$

$$= \sum_{(R^1)\cdots(R^4),i',j',k',l'} \pi_{V_\lambda}\big(R_{(1)}^1 R_{(1)}^2\big)_{e_{i'}}^{e^i} \, \pi_{V_\lambda}\big(S\big(R_{(1)}^3 R_{(1)}^4\big)\big)_{e_j}^{e^{j'}}$$
$$\times \pi_{V_\mu}\big(R_{(2)}^2 R_{(2)}^4\big)_{e_{k'}}^{e^k} \pi_{V_\mu}\big(S\big(R_{(2)}^1 R_{(2)}^3\big)\big)_{e_l}^{e^{l'}} \big({}_{V_\lambda}\phi_{e_{j'}}^{e^{i'}}\big)^{(a)} \big({}_{V_\mu}\phi_{e_{l'}}^{e^{k'}}\big)^{(b)}$$

$$= \sum_{(R^1)\cdots(R^4),i',j',k',l'} \pi_{V_\lambda}\big(R_{(1)}^{1\,-1} R_{(1)}^2\big)_{e_{i'}}^{e^i} \, \pi_{V_\lambda}\big(R_{(1)}^{4\,-1} R_{(1)}^3\big)_{e_j}^{e^{j'}}$$
$$\times \pi_{V_\mu}\big(R_{(2)}^2 R_{(2)}^{4\,-1}\big)_{e_{k'}}^{e^k} \pi_{V_\mu}\big(R_{(2)}^3 \ell R_{(2)}^{1\,-1} \ell^{-1}\big)_{e_l}^{e^{l'}} \big({}_{V_\lambda}\phi_{e_{j'}}^{e^{i'}}\big)^{(a)} \big({}_{V_\mu}\phi_{e_{l'}}^{e^{k'}}\big)^{(b)},$$

where we denote by $R^1, \ldots, R^4$ the four R-matrices in (6.3), $R^i = \sum_{(R)} R_{(1)}^i \otimes R_{(2)}^i$ as usual, and ℓ is the pivotal element. Since the matrix entries of $R_{V_\lambda, V_\mu}^{\pm 1} \in \mathrm{End}_{\mathbb{C}(q^{1/D})}(V_\lambda \otimes V_\mu)$ belong to $q^{\pm(\lambda,\mu)}\mathbb{C}(q)$ (see Theorem 3.2(2) for the stronger integral statement), the factors $q^{+(\lambda,\mu)}$ cancel

the factors $q^{-(\lambda,\mu)}$ in the last expression. The matrix entries of ℓ belong to A, so we finally obtain that the coefficients in the sum belong to $\mathbb{C}(q)$.

Finally, as for the action of U_q on $\mathcal{L}_{0,n}$, consider the structure of U_q-module algebra on $\mathcal{L}_{0,1}$ obtained by pulling-back the action of $\left(U_q \otimes U_q^{\mathrm{cop}}\right)$ via Δ (see the comment after Proposition 4.2). It follows from Remark 6.1 that $\mathcal{L}_{0,n} = \mathcal{L}_{0,1}^{\odot n}$ is the braided tensor of n copies of $\mathcal{L}_{0,1}$. The restriction of the action of U_q on each factor $\mathcal{L}_{0,1}$ being coad^r, the formula (6.6) is a consequence of the properties of actions of module algebras. ∎

The commutation relation (6.5) yields a presentation of $\mathcal{L}_{0,n}$ by generators and relations. This presentation is well-known (see [2, 19, 32, 70]), but for completeness we prefer to give a proof. Let V be an object of the category $\mathcal{C}$. Similarly to (4.10), define

$$\overset{V}{M}{}^{(a)} = \sum_{i,j} E_i^j \otimes \left(_V\phi_{e_j}^{e^i}\right)^{(a)} = \sum_{i,j} E_i^j \otimes \left(1^{\otimes(a-1)} \otimes (_V\phi_{e_j}^{e^i}) \otimes 1^{\otimes(n-a)}\right) \in \mathrm{End}(V) \otimes \mathcal{L}_{0,n}.$$

Note that the matrix coefficients of the set of matrices $\overset{V}{M}{}^{(a)}$, for every object V of $\mathcal{C}$ and $1 \leq a \leq n$, generate the algebra $\mathcal{L}_{0,n}$.

Proposition 6.3. *For every $a < b$ the matrices $\overset{V}{M}{}^{(a)}$ satisfy the fusion equation (4.12), the naturality relations (4.11), and the following* exchange relation *(in $\mathrm{End}(V) \otimes \mathrm{End}(W) \otimes \mathcal{L}_{0,n}$)*

$$R_{V,W} \; \overset{V}{M}_1{}^{(a)} R_{V,W}^{-1} \; \overset{W}{M}_2{}^{(b)} = \overset{W}{M}_2{}^{(b)} R_{V,W} \; \overset{V}{M}_1{}^{(a)} R_{V,W}^{-1}. \tag{6.7}$$

Moreover, all these relations determine the product of $\mathcal{L}_{0,n}$. Hence the algebra $\mathcal{L}_{0,n}$ can be viewed as the quotient of the algebra freely generated over $\mathbb{C}(q)$ by the matrix coefficients $\left(_V\phi_{e_j}^{e^i}\right)^{(a)}$, for all objects V of $\mathcal{C}$ and every $1 \leq a \leq n$, by the ideal generated by the relations (4.11), (4.12) and (6.7).

Proof. The fusion and naturality relations follow from the fact that $\mathfrak{i}_a \colon \mathcal{L}_{0,1} \to \mathcal{L}_{0,n}$ is a morphism of algebras. The matrix coefficients $\left(_V\phi_{e_j}^{e^i}\right)^{(a)}$ generate the subalgebras $\mathcal{L}_{0,n}^{(a)}$, whence $\mathcal{L}_{0,n}$ too. Conversely, by Proposition 4.6 the fusion and naturality relations determine the product of $\mathcal{L}_{0,n}^{(a)}$. Hence it is enough to show that the exchange relations (6.7) and the commutation relation (6.5) are equivalent. Let us write $R_{V,W} = \sum_{(R)} R_{(1)} \otimes R_{(2)} = \sum_{(R)} R_{(1')} \otimes R_{(2')}$. Then (6.7) is equivalent to

$$\sum_{(R),(R),i,j,k,l} R_{(1)} E_i^j S\left(R_{(1')}\right) \otimes R_{(2)} R_{(2')} E_k^l \otimes \left(_V\phi_{e_j}^{e^i}\right)^{(a)} \left(_W\phi_{e_l}^{e^k}\right)^{(b)}$$
$$= \sum_{(R),(R),i,j,k,l} R_{(1)} E_i^j S\left(R_{(1')}\right) \otimes E_k^l R_{(2)} R_{(2')} \otimes \left(_W\phi_{e_l}^{e^k}\right)^{(b)} \left(_V\phi_{e_j}^{e^i}\right)^{(a)}.$$

The isomorphism $V \otimes V^* \to \mathrm{End}(V)$, $v \otimes f \mapsto (w \mapsto f(w)v)$, maps $R_{(1)}e_i \otimes S\left(R_{(1')}\right)^* e^j$ to $R_{(1)} E_i^j S\left(R_{(1')}\right)$, $R_{(2)} R_{(2')} e_k \otimes e^l$ to $R_{(2)} R_{(2')} E_k^l$, and $e_k \otimes (R_{(2)} R_{(2')})^* e^l$ to $E_k^l R_{(2)} R_{(2')}$. Hence the above relation can be written as

$$\sum_{(R),(R),i,j,k,l} E_i^j \otimes E_k^l \otimes \left(_V\phi_{S(R_{(1')})e_j}^{R_{(1)}^* e^i}\right)^{(a)} \left(_W\phi_{e_l}^{(R_{(2)}R_{(2')})^* e^k}\right)^{(b)}$$
$$= \sum_{(R),(R),i,j,k,l} E_i^j \otimes E_k^l \otimes \left(_W\phi_{R_{(2)}R_{(2')}e_l}^{e^k}\right)^{(b)} \left(_V\phi_{S(R_{(1')})e_j}^{R_{(1)}^* e^i}\right)^{(a)}.$$

Now we have

$$
\begin{aligned}
{}_V\phi^{R^*_{(1)}e^i}_{S(R_{(1')})e_j} &= S\big(R_{(1')}\big) \triangleright {}_V\phi^{e^i}_{e_j} \triangleleft R_{(1)} = {}_V\phi^{e^i}_{e_j} \cdot \big(R_{(1)} \otimes R_{(1')}\big), \\
{}_W\phi^{(R_{(2)}R_{(2')})^*e^k}_{e_l} &= {}_W\phi^{e^k}_{e_l} \triangleleft R_{(2)}R_{(2')} = {}_W\phi^{e^k}_{e_l} \cdot \big(R_{(2)}R_{(2')} \otimes 1\big), \\
{}_W\phi^{e^k}_{R_{(2)}R_{(2')}e_l} &= R_{(2)}R_{(2')} \triangleright {}_W\phi^{e^k}_{e_l} = {}_W\phi^{e^k}_{e_l} \cdot \big(1 \otimes S^{-1}\big(R_{(2)}R_{(2')}\big)\big),
\end{aligned}
$$

where we use the coregular actions $\triangleright, \triangleleft$ and on the right hand sides we denote by $R_{(1)}, R_{(1')}$, $R_{(2)}, R_{(2')} \in \mathbb{U}_q$ the components of the universal R-matrix (instead of $R_{V,W}$). Using that $(S \otimes S)(R) = R$, and denoting by m the product map of $\mathcal{L}_{0,n}$, the above relation eventually becomes

$$
\sum_{i,j,k,l} E^j_i \otimes E^l_k \otimes m\big(\big(\big({}_V\phi^{e^i}_{e_j}\big)^{(a)} \otimes \big({}_W\phi^{e^k}_{e_l}\big)^{(b)}\big) \cdot (R_{13}R_{23})\big)
$$
$$
= \sum_{i,j,k,l} E^j_i \otimes E^l_k \otimes m\big(\big(\big({}_W\phi^{e^k}_{e_l}\big)^{(b)} \otimes \big({}_V\phi^{e^i}_{e_j}\big)^{(a)}\big) \cdot (R^-_{24}R^-_{23})\big).
$$

Identifying terms we get the commutation relation

$$
\big({}_W\phi^{e^k}_{e_l}\big)^{(b)}\big({}_V\phi^{e^i}_{e_j}\big)^{(a)} = m\big(\big(\big({}_V\phi^{e^i}_{e_j}\big)^{(a)} \otimes \big({}_W\phi^{e^k}_{e_l}\big)^{(b)}\big) \cdot (R_{14}R_{24}R_{13}R_{23})\big)
$$
$$
= \sum_{(F)} \big(\big({}_V\phi^{e^i}_{e_j}\big)^{(a)} \cdot F_{(2)}\big)\big(\big({}_W\phi^{e^k}_{e_l}\big)^{(b)} \cdot F_{(1)}\big), \tag{6.8}
$$

where $F = R'_{23}R'_{13}R'_{24}R'_{14} = \sigma_{12,34}(R_{14}R_{24}R_{13}R_{23})$. This is the same as (6.5), so it is equivalent to the exchange relation. ∎

Let us denote $\overset{V}{M}_{ij} := {}_V\phi^{e^j}_{e_i}$. The following proposition gives a formulation of the action coad^r in (6.6) on these matrix coefficients.

Proposition 6.4. *For all $1 \le a \le n$ we have:*

$$
\mathrm{coad}^r(y)\big(\overset{V}{M}_{ij}{}^{(a)}\big) = \sum_{(y),k,l} \pi_V(y_{(1)})_{ik}\, \overset{V}{M}_{kl}{}^{(a)} \pi_V\big(S(y_{(2)})_{lj}\big) \tag{6.9}
$$

also written as

$$
\mathrm{coad}^r(y)\big(\overset{V}{M}{}^{(a)}\big) = \sum_{(y)} \big(y_{(1)V} \otimes \mathrm{id}\big) \overset{V}{M}{}^{(a)}\big(S(y_{(2)})_V \otimes \mathrm{id}\big).
$$

Proof. The formula (6.9) extends uniquely to any product of matrices $\overset{V}{M}{}^{(a)}$ by the defining property of actions of module algebras. That it is equivalent to (6.6) is immediate from the definition of the coregular actions. ∎

Remark 6.5. The fact that $\mathcal{L}_{0,n}$ is a right U_q-module algebra (see Proposition 6.2) follows also by verifying that the ideal generated by its defining relations (4.11), (4.12) and (6.7) of Proposition 6.3 is stable under coad^r, using the formula (6.9). In the original papers on qLGFTs, these defining relations were imposed in order that this U_q-module algebra structure holds.

6.2 The Alekseev map

We have defined $\mathcal{L}_{0,n}$ as a twist of $\mathcal{L}_{0,1}^{\otimes n}$ in Proposition 6.2, and obtained a presentation by generators and relations in Proposition 6.3. Although this presentation seems complicated, the Alekseev map, defined below, identifies $\mathcal{L}_{0,n}$ as a module subalgebra of $\tilde{U}_q^{\otimes n}$.

Let V be a U_q-module of type 1. For every $1 \le a \le n$ set

$$R_{Va} = (\pi_V \otimes \mathsf{i}_a)(R) \in \mathrm{End}(V) \otimes \tilde{U}_q^{\otimes n},$$

where as usual $R \in \mathbb{U}_q \otimes \mathbb{U}_q$ is the universal R-matrix, $\pi_V \colon \mathbb{U}_q \to \mathrm{End}(V)$ the canonical projection, and $\mathsf{i}_a \colon \tilde{U}_q \to \tilde{U}_q^{\otimes n}$ the map defined by $\mathsf{i}_a(x) = 1^{\otimes(a-1)} \otimes x \otimes 1^{\otimes(n-a)}$. That R_{Va} is a matrix with entries in $\tilde{U}_q^{\otimes n}$ follows from (3.6). Consider the linear map

$$\Phi_n \colon \quad \mathrm{Vect}_{\mathbb{C}(q)} \left\{ \left({}_V\phi_{e_j}^{e^i} \right)^{(a)} \,\middle|\, V \in \mathrm{Ob}(\mathcal{C}),\ 1 \le a \le n,\ 1 \le i, j \le \dim(V) \right\} \longrightarrow \tilde{U}_q^{\otimes n}$$

defined by

$$(\mathrm{id} \otimes \Phi_n)\big(\overset{V}{M}{}^{(n)} \big) = R_{Vn} R'_{Vn},$$

$$(\mathrm{id} \otimes \Phi_n)\big(\overset{V}{M}{}^{(a)} \big) = (R_{Vn} \cdots R_{Va+1}) R_{Va} R'_{Va} (R_{Vn} \cdots R_{Va+1})^{-1}, \qquad 1 \le a < n. \qquad (6.10)$$

Since $\overset{V}{M}{}^{(a)} = \sum_{i,j} E_i^j \otimes \left({}_V\phi_{e_j}^{e^i} \right)^{(a)}$, we have $(\mathrm{id} \otimes \Phi_n)\big(\overset{V}{M}{}^{(a)} \big) \in \mathrm{End}(V) \otimes \tilde{U}_q^{\otimes n}$. When $n = 1$, Φ_n coincides with the RSD map $\Phi_1 \colon \mathcal{L}_{0,1} \to \tilde{U}_q$. We call Φ_n the *Alekseev map*. It was first introduced by Alekseev in [1].

We can represent $(\mathrm{id} \otimes \Phi_n)\big(\overset{V}{M}{}^{(a)} \big)$ by the oriented colored braid shown in the figure below, where we use the standard graphical encoding of invariant operators of ribbon categories (see [78]). In this figure, the vertical strand with label a carries the a-th factor of $\tilde{U}_q^{\otimes n}$, and the one with label V carries $\mathrm{End}(V)$. Positive crossings carry the operator $\check{R} = \sigma \circ R$, where $\sigma(x \otimes y) = y \otimes x$ as usual, and negative crossings carry the operator $\check{R}^{-1}$.
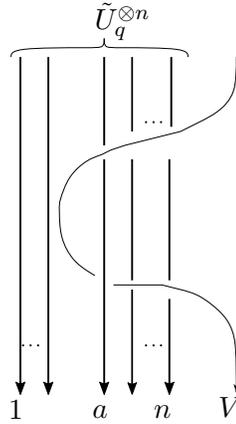
Figure 6.1. The colored braid representing $(\mathrm{id} \otimes \Phi_n)\big(\overset{V}{M}{}^{(a)} \big)$.

Let us endow $\tilde{U}_q^{\otimes n}$ with the following action of U_q

$$\mathrm{ad}^r(y)(x) = \sum_{(y)} \Delta^{(n)}(S(y_{(1)})) x \Delta^{(n)}(y_{(2)}) \qquad (6.11)$$

for all $y \in U_q$, $x \in \tilde{U}_q^{\otimes n}$. Then $\tilde{U}_q^{\otimes n}$ becomes a right U_q-module algebra.

The next result is due to Alekseev [1].

Theorem 6.6. *The Alekseev map yields an embedding of module algebras $\Phi_n\colon \mathcal{L}_{0,n} \to \tilde{U}_q^{\otimes n}$. Moreover it satisfies*

$$(\mathrm{id} \otimes \Phi_n)\big(\overset{V}{M}{}^{(1)} \cdots \overset{V}{M}{}^{(n)}\big) = \big(\pi_V \otimes \Delta^{(n-1)}\big)(RR'). \tag{6.12}$$

Proof. Let us extend Φ_n (by keeping the same notation) in the natural way to the algebra freely generated by the matrix coefficients $\big({}_V\phi_{e_j}^{e^i}\big)^{(a)}$.

By Proposition 6.3, Φ_n induces a well-defined algebra morphism $\mathcal{L}_{0,n} \to \tilde{U}_q^{\otimes n}$ if it preserves the fusion and exchange relations. Using the graphical encoding recalled in Figure 6.1, this is shown in the next two figures in the case $n = 2$, which generalizes immediately to any n. The symbol $\doteq$ means equality up to isotopy. Similarly, the relation (6.12) is proved by the third figure.
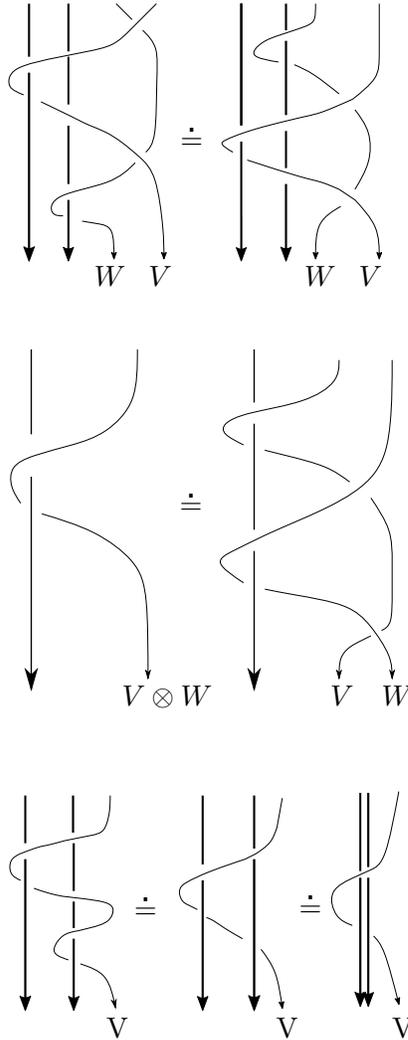
Figure 6.2. The colored braid identities representing the exchange relation (6.7) (top), the fusion relation (4.12) (middle), and the relation (6.12) (bottom).

The linear map defined by $\overset{V}{M}{}^{(a)} \mapsto R_{Va}R'_{Va}$ on the $\mathbb{C}(q)$-vector space spanned by all the matrix coefficients $\big({}_V\phi_{e_j}^{e^i}\big)^{(a)}$, for every $1 \le a \le n$ and object V of $\mathcal{C}$, is injective by Theorem 4.3. It differs from Φ_n by a linear isomorphism (induced on the a-th tensorand by conjugating with $R_{Vn} \cdots R_{Va+1}$), so Φ_n is injective.

Finally, let us show that Φ_n is a morphism of U_q-modules. Since both multiplications in $\mathcal{L}_{0,n}$ and $\tilde{U}_q^{\otimes n}$ commute with the respective actions of U_q, it is enough to check this on generators. For those given by the matrix coefficients we then have to show that

$$(\mathrm{id} \otimes \Phi_n)\big(\mathrm{coad}^r(y)\big(\overset{V}{M}{}^{(a)}\big)\big) = (\mathrm{id} \otimes \mathrm{ad}^r(y))\big((\mathrm{id} \otimes \Phi_n)\big(\overset{V}{M}{}^{(a)}\big)\big) \tag{6.13}$$

for every object V in $\mathcal{C}$, $1 \le a \le n$ and $y \in U_q$. In the case $a = n$, Φ_n has the same expression as the RSD map. Hence (6.13) follows from the statement of equivariance in Proposition 4.3. This can also be proved easily:

$$
\begin{aligned}
(\mathrm{id} \otimes \mathrm{ad}^r&(y))\big((\mathrm{id} \otimes \Phi_n)\big(\overset{V}{M}{}^{(n)}\big)\big) \\
&= (\mathrm{id} \otimes \mathrm{ad}^r(y))(R_{Vn}R'_{Vn}) \\
&= \sum_{(RR'),(y)} \pi_V\big((RR')_{(1)}\big)S(y_{(1)})(RR')_{(2)}y_{(2)} \\
&= \sum_{(RR'),(y)} \pi_V\big(y_{(1)}S(y_{(2)})\big)\pi_V\big((RR')_{(1)}\big)S(y_{(3)})(RR')_{(2)}y_{(4)} \\
&= \sum_{(RR'),(y)} \pi_V(y_{(1)})\pi_V\big((RR')_{(1)}\big)\pi_V\big(S(y_{(2)})\big)(RR')_{(2)}S(y_{(3)})y_{(4)} \\
&= \sum_{(RR'),(y)} \pi_V(y_{(1)})\pi_V\big((RR')_{(1)}\big)\pi_V\big(S(y_{(2)})\big)(RR')_{(2)} \\
&= (\mathrm{id} \otimes \Phi_n)\big(\mathrm{coad}^r(y)\big(\overset{V}{M}{}^{(n)}\big)\big),
\end{aligned}
$$

where we have used $[RR', (S \otimes S)(\Delta(y))] = 0$.

More generally, by writing the actions explicitly, that result implies as well the relation

$$
\begin{aligned}
\sum_{(y)} \big(&\big(\pi_V(y_{(1)}) \otimes 1\big)R_{Va}R'_{Va}\big(\pi_V(S(y_{(2)})) \otimes 1\big)\big) \\
&= \sum_{(y)} \big(1 \otimes \mathsf{i}_a\big(S(y_{(1)})\big)\big)R_{Va}R'_{Va}\big(1 \otimes \mathsf{i}_a(y_{(2)})\big). \tag{6.14}
\end{aligned}
$$

Now set

$$R^{(a)} = R_{Vn} \cdots R_{Va+1}$$

and denote by $\mathsf{i}_{an} \colon \tilde{U}_q^{\otimes n-a} \to 1^{\otimes a} \otimes \tilde{U}_q^{\otimes n-a} \subset \tilde{U}_q^{\otimes n}$ the identification map with the final $n - a$ tensorands. Then

$$
\begin{aligned}
(1 \otimes \Phi_n)\big(\mathrm{coad}^r(y)\big(\overset{V}{M}{}^{(a)}\big)\big) &= \sum_{(y)} \big(\pi_V(y_{(1)}) \otimes 1\big)R^{(a)}R_{Va}R'_{Va}R^{(a)-1}\big(\pi_V(S(y_{(2)})) \otimes 1\big) \\
&= \sum_{(y)} \big(\pi_V(y_{(1)}) \otimes 1\big)R^{(a)}R_{Va}R'_{Va} \\
&\quad \times \big(1 \otimes \mathsf{i}_{an}\big(\Delta^{(n-a-1)}\big(\underbrace{S(y_{(2)})y_{(3)}}_{=\varepsilon(y_{(2)})}\big)\big)\big)R^{(a)-1}\big(\pi_V(S(y_{(4)})) \otimes 1\big) \\
&= \sum_{(y)} \big(\pi_V(y_{(1)}) \otimes 1\big)R^{(a)}\big(1 \otimes \mathsf{i}_{an}\big(\Delta^{(n-a-1)}\big(S(y_{(2)})\big)\big)\big) \\
&\quad \times R_{Va}R'_{Va}\big(1 \otimes \mathsf{i}_{an}\big(\Delta^{(n-a-1)}(y_{(3)})\big)\big)R^{(a)-1}\big(\pi_V\big(S(y_{(4)})\big) \otimes 1\big).
\end{aligned}
$$

Since $R\Delta = \Delta' R$ in $\mathbb{U}_q^{\otimes 2}$, we have $\sum_{(y)} R^{-1}(y_{(2)} \otimes y_{(1)}) = \sum_{(y)}(y_{(1)} \otimes y_{(2)})R^{-1}$ and then

$$\sum_{(y)} \big(1 \otimes S(y_{(1)})\big)R(y_{(2)} \otimes 1) = \sum_{(y)}(y_{(1)} \otimes 1)R\big(1 \otimes S(y_{(2)})\big),$$

$$\sum_{(y)} \big(S(y_{(1)}) \otimes 1\big)R^{-1}(1 \otimes y_{(2)}) = \sum_{(y)}(1 \otimes y_{(1)})R^{-1}\big(S(y_{(2)}) \otimes 1\big)$$

by applying $\mathrm{id} \otimes S$, and using $(\mathrm{id} \otimes S)(R^{-1}) = R$. We deduce recursively

$$\sum_{(y)} \big(\pi_V(y_{(1)}) \otimes 1\big)R^{(a)}\big(1 \otimes i_{an}\big(\Delta^{(n-a-1)}\big(S(y_{(2)})\big)\big)\big)$$

$$= \big(1 \otimes i_{an}\big(\Delta^{(n-a-1)}\big(S(y_{(1)})\big)\big)\big)R^{(a)}\big(\pi_V(y_{(2)}) \otimes 1\big)$$

and

$$\sum_{(y)} \big(1 \otimes i_{an}\big(\Delta^{(n-a-1)}(y_{(1)})\big)\big)R^{(a)-1}\big(\pi_V\big(S(y_{(2)})\big) \otimes 1\big)$$

$$= \big(\pi_V\big(S(y_{(1)})\big) \otimes 1\big)R^{(a)-1}\big(1 \otimes i_{an}\big(\Delta^{(n-a-1)}(y_{(2)})\big)\big).$$

So

$$(1 \otimes \Phi_n)\big(\mathrm{coad}^r(y)\big(\overset{V}{M}{}^{(a)}\big)\big)$$

$$= \sum_{(y)} \big(1 \otimes i_{an}\big(\Delta^{(n-a-1)}\big(S(y_{(1)})\big)\big)\big)R^{(a)}\big(\pi_V(y_{(2)}) \otimes 1\big)$$

$$\times R_{Va}R'_{Va}\big(\pi_V\big(S(y_{(3)})\big) \otimes 1\big)R^{(a)-1}\big(1 \otimes i_{an}\big(\Delta^{(n-a-1)}(y_{(4)})\big)\big)$$

$$= \sum_{(y)} \big(1 \otimes i_{an}\big(\Delta^{(n-a-1)}\big(S(y_{(1)})\big)\big)\big)R^{(a)}\big(1 \otimes i_a\big(S(y_{(2)})\big)\big)$$

$$\times R_{Va}R'_{Va}\big(1 \otimes i_a(y_{(3)})\big)R^{(a)-1}\big(1 \otimes i_{an}\big(\Delta^{(n-a-1)}(y_{(4)})\big)\big)$$

$$= \sum_{(y)} \big(1 \otimes i_{an}\big(\Delta^{(n-a-1)}\big(S(y_{(1)})\big)\big)\big)\big(1 \otimes i_a\big(S(y_{(2)})\big)\big)R^{(a)}$$

$$\times R_{Va}R'_{Va}R^{(a)-1}\big(1 \otimes i_a(y_{(3)})\big)\big(1 \otimes i_{an}\big(\Delta^{(n-a-1)}(y_{(4)})\big)\big)$$

$$= \sum_{(y)} \big(1 \otimes i_{a-1n}\big(\Delta^{(n-a)}\big(S(y_{(1)})\big)\big)\big)R^{(a)}R_{Va}R'_{Va}R^{(a)-1}\big(1 \otimes i_{a-1n}\big(\Delta^{(n-a)}(y_{(2)})\big)\big)$$

$$= \sum_{(y)} \big(1 \otimes \Delta^{(n-1)}\big(S(y_{(1)})\big)\big)R^{(a)}R_{Va}R'_{Va}R^{(a)-1}\big(1 \otimes \Delta^{(n-1)}(y_{(2)})\big),$$

where we use (6.14) in the second equality, and the others follow from trivial commutations between elements lying on different tensorands, and the property $(S \otimes S)\Delta' = \Delta S$. This proves (6.13). The relation (6.12) has been proved with a picture which encapsulates the following simple computation:

$$(\mathrm{id} \otimes \Phi_n)\big(\overset{V}{M}{}^{(1)} \cdots \overset{V}{M}{}^{(n)}\big) = \prod_{a=1}^{n} \big((R_{Vn} \cdots R_{Va+1})R_{Va}R'_{Va}(R_{Vn} \cdots R_{Va+1})^{-1}\big)$$

$$= R_{Vn} \cdots R_{V1}R'_{V1} \cdots R'_{Vn} = \big(\pi_V \otimes \Delta^{(n-1)}\big)(RR').$$

This concludes the proof. ∎

Denote by $\big(\tilde{U}_q^{\otimes n}\big)^{\mathrm{lf}}$ the set of locally finite elements of $\tilde{U}_q^{\otimes n}$ with respect to the action (6.11). Consider the subalgebras $(\mathcal{L}_{0,n})^{U_q}$ and $\big(\tilde{U}_q^{\otimes n}\big)^{U_q}$ of invariant elements of the U_q-module algebras $\mathcal{L}_{0,n}$ and $\tilde{U}_q^{\otimes n}$ respectively. We have:

Proposition 6.7. *The Alekseev map sends $\mathcal{L}_{0,n}$ isomorphically onto $\big(\tilde{U}_q^{\otimes n}\big)^{\mathrm{lf}}$, and its restriction to invariant elements affords an isomorphism $\Phi_n\colon \mathcal{L}_{0,n}^{U_q} \to \big(\tilde{U}_q^{\otimes n}\big)^{U_q}$.*

Proof. Consider the first claim. By (6.9) the action coad^r on $\mathcal{L}_{0,n}$ preserves the spaces of matrix coefficients of each object V of $\mathcal{C}$. Since it gives $\mathcal{L}_{0,n}$ a structure of module algebra, it follows that coad^r is locally finite. The equivariance of Φ_n then implies that its image is contained in $\big(\tilde{U}_q^{\otimes n}\big)^{\mathrm{lf}}$. As Φ_n is injective, it remains to prove that it maps $\mathcal{L}_{0,n}$ surjectively onto $\big(\tilde{U}_q^{\otimes n}\big)^{\mathrm{lf}}$. Let us denote by ad_n^r the action (6.11). We can consider another action of U_q on $\tilde{U}_q^{\otimes n}$ defined as the n-fold tensor product of the adjoint action of U_q on $\tilde{U}_q^{\otimes n}$, which we denote $(\mathrm{ad}^r)^{\otimes n}$. It is thus defined as follows:

$$(\mathrm{ad}^r)^{\otimes n}(y)\big(a^{(1)} \otimes \cdots \otimes a^{(n)}\big) = \sum_{(y)} \mathrm{ad}^r(y_{(1)})\big(a^{(1)}\big) \otimes \cdots \otimes \mathrm{ad}^r(y_{(n)})\big(a^{(n)}\big).$$

We first show that these two actions on $\tilde{U}_q^{\otimes n}$ define isomorphic modules. For simplicity of notations let us consider the case $n = 2$. Consider the map

$$\psi\colon \quad \mathbb{U}_q^{\otimes 2} \to \mathbb{U}_q^{\otimes 2}, \qquad a \otimes b \mapsto R_{12}^{-1}(a \otimes 1)R_{12}(1 \otimes b).$$

We claim that ψ intertwines $(\mathrm{ad}^r)^{\otimes 2}$ and ad_2^r. This follows from a small variation of the previous proof. Indeed we have

$$\begin{aligned}
\mathrm{ad}_2^r(y)(\psi(a \otimes b)) &= \sum_{(y),(R),(R^{-1})} S(y_{(2)})(R^{-1})_{(1)}aR_{(1)}y_{(3)} \otimes S(y_{(1)})(R^{-1})_{(2)}R_{(2)}by_{(4)} \\
&= \sum_{(y),(R),(R^{-1})} (R^{-1})_{(1)}S(y_{(1)})aR_{(1)}y_{(3)} \otimes (R^{-1})_{(2)}S(y_{(2)})R_{(2)}by_{(4)} \\
&= \sum_{(y),(R),(R^{-1})} (R^{-1})_{(1)}S(y_{(1)})ay_{(2)}R_{(1)} \otimes (R^{-1})_{(2)}R_{(2)}S(y_{(3)})by_{(4)} \\
&= \psi((\mathrm{ad}^r)^{\otimes 2}(a \otimes b))).
\end{aligned}$$

It is easy to show that ψ is an isomorphism, it descends to a map $\psi\colon \big(\tilde{U}_q^{\mathrm{lf}}\big)^{\otimes 2} \to \big(\tilde{U}_q^{\otimes 2}\big)^{\mathrm{lf}}$, and that $\psi(a \otimes b) = \sum_{(R)} \mathrm{ad}^r(R_{(1)})(a) \otimes R_{(2)}b$. Now, let $z \in \big(\tilde{U}_q^{\otimes 2}\big)^{\mathrm{lf}}$. Then $\psi^{-1}(z)$ is locally finite for the action $(\mathrm{ad}^r)^{\otimes 2}$. By the main theorem of [58], the set of locally finite elements of $\tilde{U}_q^{\otimes n}$ for $(\mathrm{ad}^r)^{\otimes n}$ is equal to $\big(\tilde{U}_q^{\mathrm{lf}}\big)^{\otimes n}$. Therefore $\psi^{-1}(z) \in \tilde{U}_q^{\mathrm{lf}} \otimes \tilde{U}_q^{\mathrm{lf}}$, and from the surjectivity of the map Φ_1 onto $\tilde{U}_q^{\mathrm{lf}}$, we deduce that $z = \psi \circ \Phi_1^{\otimes 2}(t)$ for some $t \in \mathcal{L}_{0,1}^{\otimes 2}$. But because of the identity $R_{12}^{-1}R_{01}R_{01}'R_{12} = R_{02}R_{01}R_{01}'R_{02}^{-1}$, which is a consequence of the Yang–Baxter equation, we obtain that $z = \psi \circ \Phi_1^{\otimes 2}(t) = \Phi_2(t)$, where we have identified the two vector spaces $\mathcal{L}_{0,2}$ and $\mathcal{L}_{0,1}^{\otimes 2}$. This shows Φ_2 is a surjection onto $\big(\tilde{U}_q^{\otimes 2}\big)^{\mathrm{lf}}$. The generalisation to any n is straightforward.

By the previous theorem we have an inclusion $\Phi_n\big(\mathcal{L}_{0,n}^{U_q}\big) \subset \big(\tilde{U}_q^{\otimes n}\big)^{U_q}$. The above argument applies in particular to ad_n^r-invariant elements $z \in \big(\tilde{U}_q^{\otimes n}\big)^{U_q}$. Together with the injectivity of Φ_n it implies that this inclusion is an equality. Therefore the second claim follows. ∎

Remark 6.8. This last result is a generalisation to $n \geq 1$ of Theorem 4.3(3).

6.3 Integral form

Let $\{\alpha_i\}_{i \in I}$ be a basis of the free A-module $\mathcal{L}^A_{0,1}$. Put $(\alpha_i)^{(a)} := \mathsf{i}_a(\alpha_i)$, where as usual $\mathsf{i}_a \colon \mathcal{L}^A_{0,1} \to (\mathcal{L}^A_{0,1})^{\otimes n}$ is the inclusion map, $1 \le a \le n$. A basis of the free A-module $(\mathcal{L}^A_{0,1})^{\otimes n}$ is given by the elements $\alpha_{i_1 \cdots i_n} = (\alpha_{i_1})^{(1)} \otimes \cdots \otimes (\alpha_{i_n})^{(n)}$, the products in $\mathcal{L}^A_{0,n}$ of the elements $(\alpha_{i_1})^{(1)}, \ldots, (\alpha_{i_n})^{(n)}$ (in this order from 1 to n), indexed by the tuples $(i_1, \ldots, i_n) \in I^n$. For every basis elements $\alpha_{i_1 \cdots i_n}$, $\alpha_{j_1 \cdots j_n}$ we have

$$\alpha_{i_1 \cdots i_n} \alpha_{j_1 \cdots j_n} = \sum_{k_1 \cdots k_n} m^{k_1 \cdots k_n}_{i_1 \cdots i_n, j_1 \cdots j_n} \alpha_{k_1 \cdots k_n}$$

for some coefficients $m^{k_1 \cdots k_n}_{i_1 \cdots i_n, j_1 \cdots j_n} \in \mathbb{C}(q)$. By Proposition 4.10 we know that $(\alpha_i)^{(a)}(\alpha_j)^{(a)}$ is a linear combination over A of basis elements $(\alpha_k)^{(a)}$, for every i, j and a. Therefore, it remains to show that any product $(\alpha_i)^{(b)}(\alpha_j)^{(a)}$, for $a < b$, is a linear combination over A of basis elements $(\alpha_k)^{(a)}(\alpha_l)^{(b)}$. Any $(\alpha_k)^{(a)}$ is a linear combination over A of matrix cooefficients of U^{res}_A-modules of type 1. Let X and Y be two such modules. By (6.8) the exchange relation between matrix coefficients $\left({}_X\phi^{e^i}_{e_j}\right)^{(a)}$ and $\left({}_Y\phi^{e^k}_{e_l}\right)^{(b)}$, for $a < b$, can be written as

$$\left({}_Y\phi^{e^k}_{e_l}\right)^{(b)}\left({}_X\phi^{e^i}_{e_j}\right)^{(a)} = \sum_{(F)} \left(\left({}_X\phi^{e^i}_{e_j}\right)^{(a)} \cdot F_{(2)}\right)\left(\left({}_Y\phi^{e^k}_{e_l}\right)^{(b)} \cdot F_{(1)}\right),$$

where $F = R'_{23}R'_{13}R'_{24}R'_{14}$. By using Theorem 3.2 as in the proof of Proposition 4.10, we see that the sum is a linear combination over $\mathbb{C}[q^{1/D}, q^{-1/D}]$ of basis elements $(\alpha_k)^{(a)}(\alpha_l)^{(b)}$. Therefore each coefficient $m^{k_1 \cdots k_n}_{i_1 \cdots i_n, j_1 \cdots j_n} \in \mathbb{C}(q) \cap \mathbb{C}[q^{1/D}, q^{-1/D}] = \mathbb{C}[q, q^{-1}]$.

This proves the first claim of the following statement. The second follows from the arguments of Proposition 6.3, which apply as well to $\mathcal{L}^A_{0,n}$.

Proposition 6.9. *The A-submodule $(\mathcal{L}^A_{0,1})^{\otimes n}$ of $\mathcal{L}_{0,n}$ is an A-subalgebra. We denote it $\mathcal{L}^A_{0,n}$. Moreover, $\mathcal{L}^A_{0,n}$ is the A-algebra generated by the matrix coefficients $\left({}_X\phi^{e^i}_{e_j}\right)^{(a)}$, for all objects X of the category $\mathcal{C}_A$ and $1 \le a \le n$, with defining relations given by the naturality relations (4.11), the fusion relations (4.12), and the exchange relations (6.7).*

Note that, by the properties of $\mathcal{L}^A_{0,1}$, $\mathcal{L}^A_{0,n}$ is a free A-module and we have $\mathcal{L}^A_{0,n} \otimes_A \mathbb{C}(q) = \mathcal{L}_{0,n}$.

Lemma 6.10. *The action coad^r on $\mathcal{L}_{0,n}$ yields on $\mathcal{L}^A_{0,n}$ a structure of right U_A-module algebra, and the Alekseev map restricts to an embedding of U_A-module algebras $\Phi_n \colon \mathcal{L}^A_{0,n} \to \tilde{U}^{\otimes n}_A$.*

Proof. The first claim follows immediately from the case of $\mathcal{L}^A_{0,1}$ (see Lemma 4.11), since $\mathcal{L}^A_{0,n} = (\mathcal{L}^A_{0,1})^{\otimes n}$ as an A-module. For the second claim, one simply note that Theorem 3.2 implies that $R_{Xn} \cdots R_{Xa+1} \in \mathrm{End}_A(X) \otimes \tilde{U}^{\otimes n}_A$, for every U^{res}_A-module X of type 1, and use that $\mathrm{id} \otimes \Phi_n$ is defined on generators of $\mathrm{End}_A(X) \otimes \mathcal{L}^A_{0,n}$ as $\mathrm{id} \otimes \Phi^{\otimes n}_1$ followed by conjugations with matrices of the form $R_{Xn} \cdots R_{Xa+1}$. $\blacksquare$

Proposition 6.11. *$\mathcal{L}_{0,n}$, and therefore its subalgebras $\mathcal{L}^A_{0,n}$ and $(\mathcal{L}^A_{0,n})^{U_A}$, does not have non trivial zero divisors.*

Proof. Because of the injectivity of Φ_n it is sufficient to show that $\tilde{U}^{\otimes n}_q$ and $\tilde{U}^{\otimes n}_A$ have no non trivial zero divisors. We note that $U_A(\mathfrak{g})^{\otimes n} = U_A(\mathfrak{g}^{\oplus n})$. Then the result for $U^{\otimes n}_A$ is a consequence of [27, Corollary 1.8] applied to $\mathfrak{g}^{\oplus n}$. Note that in that paper it is assumed that the Cartan matrix is indecomposable but their method, which consists in proving that an associated graded algebra is quasipolynomial, does not use this assumption. It applies as well for $\tilde{U}^{\otimes n}_A$, $\tilde{U}^{\otimes n}_q$. $\blacksquare$

In the case of $\mathfrak{g} = \mathfrak{sl}(2)$ we can restrict the target:

Proposition 6.12. $\Phi_n\colon \mathcal{L}_{0,n}(\mathfrak{sl}(2)) \to \tilde{U}_q(\mathfrak{sl}(2))^{\otimes n}$ *takes values in* $U_q(\mathfrak{sl}(2))^{\otimes n}$, *and yields an embedding of* U_A-*module algebras* $\Phi_n\colon \mathcal{L}^A_{0,n}(\mathfrak{sl}(2)) \to U_A(\mathfrak{sl}(2))^{\otimes n}$.

Proof. The second claim follows from the first and Lemma 6.10. Consider the first claim. For every $1 \le a \le n$ and object V of $\mathcal{C}$ we have $R_{Va}R'_{Va} \in \mathrm{End}(V) \otimes U^{\mathrm{lf}}_q(\mathfrak{sl}(2))^{\otimes n}$ by Proposition 5.4. We have to show that conjugating by $R_{Vn} \cdots R_{Va+1} \in \mathrm{End}(V) \otimes \tilde{U}_q(\mathfrak{sl}(2))^{\otimes n}$ maps $\mathrm{End}(V) \otimes U^{\mathrm{lf}}_q(\mathfrak{sl}(2))^{\otimes n}$ to $\mathrm{End}(V) \otimes U_q(\mathfrak{sl}(2))^{\otimes n}$. It is enough to prove it for $V = V_2$, and for the conjugation by R_{Va+1} only (the general case follows from this one by an easy induction). Recall the expression of R in (3.7). Using that E, F act nilpotently on V_2 with order 2, and $R^{-1} = (S \otimes \mathrm{id})(R)$, we get

$$R_{Va+1} = (\pi_{V_2} \otimes i_{a+1})(R) = \begin{pmatrix} 1 \otimes q^{H/2} & (q - q^{-1})1 \otimes q^{H/2}F \\ 0 & 1 \otimes q^{-H/2} \end{pmatrix}, \tag{6.15}$$

$$R^{-1}_{Va+1} = (\pi_{V_2} \otimes i_{a+1})(R^{-1}) = \begin{pmatrix} 1 \otimes q^{-H/2} & -q(q - q^{-1})1 \otimes q^{H/2}F \\ 0 & 1 \otimes q^{H/2} \end{pmatrix}, \tag{6.16}$$

where $q^{H/2} \in \tilde{U}_q(\mathfrak{sl}(2))$ is defined in Section 3, and for each matrix entry we write only the components in the a-th and $a+1$-th tensorands of $\tilde{U}_q(\mathfrak{sl}(2))^{\otimes n}$ (the others being 1's). With this convention, let

$$N := \begin{pmatrix} u \otimes 1 & v \otimes 1 \\ w \otimes 1 & x \otimes 1 \end{pmatrix} \in \mathrm{End}(V) \otimes \tilde{U}_q(\mathfrak{sl}(2))^{\otimes n}. \tag{6.17}$$

Then

$$R_{Va+1} N R^{-1}_{Va+1} = \begin{pmatrix} u \otimes 1 & -(q^2 - 1)u \otimes KF - q^2(q - q^{-1})w \otimes KF^2 \\ + q^{-1}(q - q^{-1})w \otimes F & +v \otimes K + q^{-1}(q - q^{-1})x \otimes KF \\ w \otimes K^{-1} & -q(q - q^{-1})w \otimes F + x \otimes 1 \end{pmatrix}.$$

This matrix has entries in $U_q(\mathfrak{sl}(2))^{\otimes n}$ if the matrix entries of N belong to $U_q(\mathfrak{sl}(2))^{\otimes n}$. Conjugating recursively $R_{Va}R'_{Va}$ with R_{Vi}, for i from $a+1$ to n, by the same computation we deduce that Φ_n takes values in $U_q(\mathfrak{sl}(2))^{\otimes n}$. ∎

6.4   Localization when $\mathfrak{g} = \mathfrak{sl}(2)$

We are now going to define a localization of $\mathcal{L}_{0,n}(\mathfrak{sl}(2))$, which will satisfy a generalization of Proposition 5.6. We need the following lemma. For every $u \in U_q(\mathfrak{sl}(2))$, denote by $u^{(i)}$ the element of $U_q(\mathfrak{sl}(2))^{\otimes n}$ with u in the i-th tensorand and 1's elsewhere. Analogously to (5.1), for every $1 \le i \le n$, put

$$\overset{V}{M}{}^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} \in \mathrm{End}(V) \otimes \mathcal{L}_{0,n}(\mathfrak{sl}(2))^{(i)}, \tag{6.18}$$

where $V = V_2$.

Lemma 6.13. *For every* $1 \le i \le n$, $\Phi^{\otimes n}_1\big(\mathcal{L}^{(i)}_{0,n}(\mathfrak{sl}(2))\big)$ *is contained in the subalgebra of* $U_q(\mathfrak{sl}(2))^{\otimes n}$ *generated by* $\mathrm{Im}(\Phi_n)$ *and the elements* $K^{(i+1)}, \ldots, K^{(n)}$.

Proof. The case $i = n$ is clear, as $\Phi^{\otimes n}_1\big(\mathcal{L}^{(n)}_{0,n}(\mathfrak{sl}(2))\big) = \Phi_n\big(\mathcal{L}^{(n)}_{0,n}(\mathfrak{sl}(2))\big)$ by definition. We argue by decreasing induction on $i \in \{1, \ldots, n\}$. Take

$$\tilde{M} = (\mathrm{id} \otimes \Phi^{\otimes n}_1)\big(\overset{V}{M}{}^{(n-1)}\big)$$

in (6.17). The entries of $\tilde{M}$ generate the algebra $\Phi_1^{\otimes n}\big(\mathcal{L}_{0,n}^{(n-1)}(\mathfrak{sl}(2))\big)$. By the formula of $R_{Vn}\tilde{M}R_{0n}^{-1}$ we have $\Phi_n\big(c^{(n-1)}\big) = w \otimes K^{-1}$. Hence $w \otimes 1 = \big(w \otimes K^{-1}\big)(1 \otimes K)$ belongs to the algebra by $\Phi_n(\mathcal{L}_{0,n}(\mathfrak{sl}(2)))$ and $K^{(n)}$. Since $q^{-1}\big(q - q^{-1}\big)(1 \otimes F) = \Phi_1^{\otimes n}\big(b^{(n)}\big) = \Phi_n\big(b^{(n)}\big)$, the same is true of $-q\big(q - q^{-1}\big)w \otimes F$, and hence eventually also of $x \otimes 1$, $u \otimes 1$ and $v \otimes 1$ by using again the formula of $R_{Vn}\tilde{M}R_{Vn}^{-1}$. This proves the statement for $i = n - 1$. Inducting on i, using the matrix $\tilde{M}$ of generators of $\mathcal{L}_{0,n}^{(i)}(\mathfrak{sl}(2))$, the same reasoning proves the result for all values of i. ■

Lemma 6.14. *Define elements $\xi^{(i)} \in \mathcal{L}_{0,n}(\mathfrak{sl}(2))$, $i = 1, \ldots, n$ by*

$$\xi^{(i)} = \big(\overset{V}{M}{}^{(i)} \cdots \overset{V}{M}{}^{(n)}\big)_{22},$$

where 22 *denotes the lower right matrix element. The elements $\xi^{(i)}$ are commuting and satisfy:*

$$\Phi_n\big(\xi^{(i)}\big) = \big(K^{-1}\big)^{(i)} \cdots \big(K^{-1}\big)^{(n)}.$$

Proof. As in (6.12) we have

$$(\mathrm{id} \otimes \Phi_n)\big(\overset{V}{M}{}^{(i)} \cdots \overset{V}{M}{}^{(n)}\big) = R_{Vn} \cdots R_{Vi}R'_{Vi} \cdots R_{Vn} = \big(\pi_V \otimes 1^{\otimes(i-1)} \otimes \Delta^{(n-i)}\big)(RR').$$

The lower right matrix element of $(\pi_V \otimes \mathrm{id})(RR')$ is equal to K^{-1}. As a result, by applying $\Delta^{(n-i)}$ we obtain

$$\Phi_n\big(\big(\overset{V}{M}{}^{(i)} \cdots \overset{V}{M}{}^{(n)}\big)_{22}\big) = 1^{\otimes(i-1)} \otimes \Delta^{(n-i)}\big(K^{-1}\big) = \big(K^{-1}\big)^{(i)} \cdots \big(K^{-1}\big)^{(n)}.$$

By injectivity of Φ_n this proves that the elements $\xi^{(i)}$ are commuting. ■

The elements $\xi^{(i)}$ commute, and $\big\{\big(\xi^{(1)k}\big)\big\}_{k \in \mathbb{N}}$ is an Ore set of $\mathcal{L}_{0,n}(\mathfrak{sl}(2))$. In fact, it is easy to see that for every $i = 1, \ldots, n$ the element $\xi^{(i)}$ is an Ore element of $\mathcal{L}_{0,n}^{(i \leq)}(\mathfrak{sl}(2))$, where $\mathcal{L}_{0,n}^{(i \leq)}(\mathfrak{sl}(2))$ is the subalgebra of $\mathcal{L}_{0,n}(\mathfrak{sl}(2))$ generated by the subalgebras $\mathcal{L}_{0,n}^{(a)}(\mathfrak{sl}(2))$, $a \geq i$. Indeed, because $\mathcal{L}_{0,n}$ has no non trivial zero divisors, $\xi^{(i)}$ is a regular element. The set $\big\{\big(\xi^{(i)k}\big)\big\}_{k \in \mathbb{N}}$ is multiplicatively closed, and it is an Ore subset of $\mathcal{L}_{0,n}^{(i \leq)}(\mathfrak{sl}(2))$, since for all $x \in \mathcal{L}_{0,n}^{(i \leq)}(\mathfrak{sl}(2))$ there exist elements $y, y' \in \mathcal{L}_{0,n}^{(i \leq)}(\mathfrak{sl}(2))$ such that $x\xi^{(i)} = \xi^{(i)}y$ and $\xi^{(i)}x = y'\xi^{(i)}$. This is shown as follows: $\Phi_n(x)\Phi_n\big(\xi^{(i)}\big) = \Phi_n(x)\big(K^{-1}\big)^{(i)} \cdots \big(K^{-1}\big)^{(n)} = \big(K^{-1}\big)^{(i)} \cdots \big(K^{-1}\big)^{(n)} \mathrm{ad}^r(K)(\Phi_n(x))$, where we have used the fact that $\Phi_n(x) \in 1^{\otimes i-1} \otimes U_q^{\otimes(n-i+1)}$. But $\mathrm{ad}^r(K)(\Phi_n(x)) = \Phi_1(\mathrm{coad}^r(K)(x))$, and therefore the Ore conditions are satisfied with $y = \mathrm{coad}^r(K)(x)$.

This argument does not permit to conclude that the multiplicative sets $\big\{\big(\xi^{(i)k}\big)\big\}_{k \in \mathbb{N}}$ are Ore sets of $\mathcal{L}_{0,n}(\mathfrak{sl}(2))$. We therefore cannot localize with respect to the elements $\xi^{(i)}$ as easily as for $\xi^{(1)}$. We proceed in a different way.

We first explain the case $n = 2$. The element $\xi^{(1)} = \big(\overset{V}{M}{}^{(1)} \overset{V}{M}{}^{(2)}\big)_{22} = d^{(1)}d^{(2)} + c^{(1)}b^{(2)}$ is a regular Ore element of $\mathcal{L}_{0,2}$, so we can define the localisation $\mathcal{L}_{0,2}\big[\xi^{(1)-1}\big]$. We want to define the inverse of the element $\xi^{(2)}$, and a new algebra $\mathcal{L}_{0,2}\big[\xi^{(1)-1}\big]\big[\xi^{(2)-1}\big]$ such that $\mathcal{L}_{0,2}\big[\xi^{(1)-1}\big] \subset \mathcal{L}_{0,2}\big[\xi^{(1)-1}\big]\big[\xi^{(2)-1}\big]$ and Φ_2 extends naturally to an homomorphism of algebras

$$\Phi_2\colon \ \mathcal{L}_{0,2}\big[\xi^{(1)-1}\big]\big[\xi^{(2)-1}\big] \to U_q(\mathfrak{sl}(2))^{\otimes 2}.$$

This can be done by writing explicitly the exchange relations between $\overset{V}{M}{}^{(1)}$ and $\overset{V}{M}{}^{(2)}$ involving $d^{(2)} = \xi^{(2)}$:

$$c^{(1)}d^{(2)} = d^{(2)}c^{(1)},$$

$$d^{(2)}a^{(1)} = a^{(1)}d^{(2)} + \left(1 - q^{-2}\right)c^{(1)}b^{(2)},$$
$$d^{(2)}d^{(1)} = d^{(1)}d^{(2)} + \left(1 - q^2\right)c^{(1)}b^{(2)},$$
$$d^{(2)}b^{(1)} = b^{(1)}d^{(2)} + \left(1 - q^2\right)\left(a^{(1)} - d^{(1)}\right)b^{(2)}.$$

We define $\mathcal{L}_{0,2}\big[\xi^{(1)-1}\big]\big[\xi^{(2)-1}\big]$ to be the algebra generated by the elements $a^{(1)}$, $b^{(1)}$, $c^{(1)}$, $d^{(1)}$, $a^{(2)}$, $b^{(2)}$, $c^{(2)}$, $d^{(2)}$, $\xi^{(1)-1}$, $\xi^{(2)-1}$, where $a^{(1)},\ldots,d^{(2)}$ satisfy the exchange relation (5.2) and (6.7), $\xi^{(1)-1}$ is the inverse in the sense of Ore of the element $\xi^{(1)} = d^{(1)}d^{(2)} + c^{(1)}b^{(2)}$ and the following relations for $\xi^{(2)-1}$, the inverse of $\xi^{(2)} = d^{(2)}$, hold true:

$$\xi^{(2)}\xi^{(2)-1} = \xi^{(2)-1}\xi^{(2)} = 1,$$
$$\xi^{(2)-1}c^{(1)} = c^{(1)}\xi^{(2)-1},$$
$$\xi^{(2)-1}a^{(1)} = a^{(1)}\xi^{(2)-1} - q^{-2}\left(1 - q^{-2}\right)c^{(1)}b^{(2)}\xi^{(2)-2},$$
$$\xi^{(2)-1}d^{(1)} = d^{(1)}\xi^{(2)-1} - q^{-2}\left(1 - q^2\right)c^{(1)}b^{(2)}\xi^{(2)-2},$$
$$\xi^{(2)-1}b^{(1)} = b^{(1)}\xi^{(2)-1} + q^{-1}\left(1 - q^{-2}\right)\left(a^{(1)} - d^{(1)}\right)b^{(2)}\xi^{(2)-2}$$
$$\qquad\qquad + q^{-5}\left(1 - q^2\right)\left(1 - q^{-4}\right)c^{(1)}\left(b^{(2)}\right)^2\xi^{(2)-3}.$$

The last relations are chosen in order that $\mathcal{L}_{0,2}\big[\xi^{(1)-1}\big] \subset \mathcal{L}_{0,2}\big[\xi^{(1)-1}\big]\big[\xi^{(2)-1}\big]$ as an algebra and are a direct consequence of the exchange relations (6.7) and the invertibility of $\xi^{(2)}$. As a result Φ_2 extends to a morphism of algebra $\Phi_2\colon \mathcal{L}_{0,2}\big[\xi^{(1)-1}\big]\big[\xi^{(2)-1}\big] \to U_q(\mathfrak{sl}(2))^{\otimes 2}$. This morphism of algebra will be shown to be an isomorphism in Proposition 6.16.

The construction of the localisation for $\mathcal{L}_{0,n}(\mathfrak{sl}(2))$, $n > 2$, is defined by the same procedure. We want to define a localisation $\mathcal{L}_{0,n}(\mathfrak{sl}(2))\big[\xi^{(n)-1},\ldots,\xi^{(1)-1}\big]$. The set of elements $\big\{\xi^{(n)k}\big\}_k$ being an Ore subset of $\mathcal{L}_{0,n}(\mathfrak{sl}(2))$, we can define the standard localisation $\mathcal{L}_{0,n}(\mathfrak{sl}(2))\big[\xi^{(n)-1}\big]$. Note that the matrix $\overset{V}{M}{}^{(2)}\cdots\overset{V}{M}{}^{(n)}$ has the same reflection equation with $\overset{V}{M}{}^{(1)}$ as $\overset{V}{M}{}^{(2)}$. Therefore we can define $\mathcal{L}_{0,n}(\mathfrak{sl}(2))\big[\xi^{(n)-1}\big]\big[\xi^{(n-1)-1}\big]$ by the same method as when $n = 2$, and so on.

Definition 6.15. By iterating the preceding construction we define:

$$_{\mathrm{loc}}\mathcal{L}_{0,n}(\mathfrak{sl}(2)) = \mathcal{L}_{0,n}(\mathfrak{sl}(2))\big[\xi^{(n)-1}\big]\big[\xi^{(n-1)-1}\big]\cdots\big[\xi^{(1)-1}\big].$$

In the sequel it will be convenient to define invertible elements $\delta^{(i)} \in {}_{\mathrm{loc}}\mathcal{L}_{0,n}(\mathfrak{sl}(2))$, $i = 1,\ldots,n$, satisfying $\xi^{(i)} = \delta^{(i)}\cdots\delta^{(n)}$, i.e., $\delta^{(i)} = \xi^{(i)}\xi^{(i+1)-1}$.

The elements $\delta^{(i)}$, $i = 1,\ldots,n$, are invertible, commute and satisfy $\Phi_n\big(\delta^{(i)}\big) = \left(K^{-1}\right)^{(i)}$.

We can define the localization $_{\mathrm{loc}}\mathcal{L}_{0,n}^A(\mathfrak{sl}(2))$ of the integral form $\mathcal{L}_{0,n}^A(\mathfrak{sl}(2))$ in the very same way (recall Lemma 5.7 for the case $n = 1$). Let

$$\Phi_n\colon \quad _{\mathrm{loc}}\mathcal{L}_{0,n}(\mathfrak{sl}(2)) \to U_q(\mathfrak{sl}(2))^{\otimes n}$$

be the unique morphism of module algebras extending the Alekseev map. Recall the subalgebra $U'_A(\mathfrak{sl}(2))$ of $U_A(\mathfrak{sl}(2))$, defined in Lemma 5.7.

Proposition 6.16. *The extended Alekseev map* $\Phi_n\colon {}_{\mathrm{loc}}\mathcal{L}_{0,n}(\mathfrak{sl}(2)) \to U_q(\mathfrak{sl}(2))^{\otimes n}$ *is an isomorphism of U_q-module algebras, and it restricts to an isomorphism of U_A-module algebras* $\Phi_n\colon {}_{\mathrm{loc}}\mathcal{L}_{0,n}^A(\mathfrak{sl}(2)) \to U'_A(\mathfrak{sl}(2))^{\otimes n}$.

Proof. All claims are clear by the previous results, except surjectivity. For the first claim, it follows from Lemma 6.13, since the algebra generated by $\Phi_1^{\otimes n}\big(\mathcal{L}_{0,n}^{(i)}(\mathfrak{sl}(2))\big)$ and $K^{(i)}$ is $U_q(\mathfrak{sl}(2))^{(i)}$ (the i-th tensorand). For the second claim, it follows from Lemma 5.7(3). ∎

Remark 6.17. This proposition (first claim) justifies the localization of $\mathcal{L}_{0,n}(\mathfrak{sl}(2))$ by the elements $\xi^{(1)},\ldots,\xi^{(n)}$. The result of localization, if any, by the elements $d^{(1)},\ldots,d^{(n)}$ is not clear to us; note that they do not commute.

6.5 Invariant elements

We go back to the general situation of $\mathcal{L}_{0,n}^A = \mathcal{L}_{0,n}^A(\mathfrak{g})$ for an arbitrary $\mathfrak{g}$. We need the following fact, proved by Majid [69] for $\mathcal{O}_q$. For completeness we recall the main ideas of the proof given in [32], which extends immediately to $\mathcal{O}_A$.

Proposition 6.18. *The iterated coproduct* $\Delta^{(n-1)}\colon \mathcal{O}_A \to \mathcal{O}_A^{\otimes n}$, *considered as a linear map* $\mathcal{L}_{0,1}^A \to \mathcal{L}_{0,n}^A$, *is an algebra morphism and satisfies the following commutative diagram:*

$$
\begin{array}{ccc}
\mathcal{L}_{0,1}^A & \xrightarrow{\Delta^{(n-1)}} & \mathcal{L}_{0,n}^A \\
{\scriptstyle \Phi_1}\downarrow & & \downarrow{\scriptstyle \Phi_n} \\
\tilde{U}_A & \xrightarrow{\Delta^{(n-1)}} & \tilde{U}_A^{\otimes n}.
\end{array}
$$

Proof. The main point of the proof is that, for every U_A^{res}-module X of type 1, we have

$$
\big(\mathrm{id} \otimes \Delta^{(n-1)}\big)\big(\overset{X}{M}\big) = \overset{X}{M}{}^{(1)} \cdots \overset{X}{M}{}^{(n)}. \tag{6.19}
$$

This follows by a direct application of the definitions. But Proposition 4.6 implies that the map $\Delta^{(n-1)}\colon \mathcal{L}_{0,1}^A \to \mathcal{L}_{0,n}^A$ is an algebra morphism if and only if the matrix on the left hand side of (6.19) satisfies the fusion relation. This is easily shown to be true for the right hand side by using the exchange relation (6.7) recursively. This proves the proposition. The commutativity diagram is a reformulation of (6.12). ∎

Consider now the algebras of invariant elements,

$$
\mathcal{L}_{0,n}^{U_q} := \big\{ \alpha \in \mathcal{L}_{0,n} \mid \forall y \in U_q,\ \mathrm{coad}^r(y)(\alpha) = \varepsilon(y)\alpha \big\},
$$
$$
\big(\mathcal{L}_{0,n}^A\big)^{U_A} := \big\{ \alpha \in \mathcal{L}_{0,n}^A \mid \forall y \in U_A,\ \mathrm{coad}^r(y)(\alpha) = \varepsilon(y)\alpha \big\}.
$$

Note that $\big(\mathcal{L}_{0,n}^A\big)^{U_A} = \mathcal{L}_{0,n}^{U_q} \cap \mathcal{L}_{0,n}^A$, and therefore $\mathcal{L}_{0,n}^{U_q} = \big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes_A \mathbb{C}(q)$. Since $\mathcal{L}_{0,n}^A$ is a free A-module, and A is a principal ideal domain, the A-submodule $\big(\mathcal{L}_{0,n}^A\big)^{U_A}$ is free.

Proposition 6.19. *The algebra* $\big(\mathcal{L}_{0,n}^A\big)^{U_A}$ *is the centralizer of* $\Delta^{(n-1)}\big(\mathcal{L}_{0,1}^A\big)$ *in* $\mathcal{L}_{0,n}^A$. *As a corollary we have* $\mathcal{Z}\big(\mathcal{L}_{0,n}^A\big)^{U_A} = \mathcal{Z}\big(\mathcal{L}_{0,n}^A\big)$.

Proof. Clearing denominators it is enough to give the details for $\mathcal{L}_{0,n}^{U_q}$. First note that an element z of $\tilde{U}_q^{\otimes n}$ is invariant under the right adjoint action of U_q if and only if it commutes with $\Delta^{(n-1)}(x)$ for every $x \in U_q$. Indeed, we have

$$
z\Delta^{(n-1)}(x) = \sum_{(x)} z\varepsilon(x_{(1)})\Delta^{(n-1)}(x_{(2)}) = \sum_{(x)} \varepsilon(x_{(1)})z\Delta^{(n-1)}(x_{(2)})
$$
$$
= \sum_{(x)} \Delta^{(n-1)}(x_{(1)})\Delta^{(n-1)}(S(x_{(2)}))z\Delta^{(n-1)}(x_{(3)}) = \sum_{(x)} \Delta^{(n-1)}(x_{(1)})\,\mathrm{ad}^r(x_{(2)})(z).
$$

Hence $z\Delta^{(n-1)}(x) = \sum_{(x)} \Delta^{(n-1)}(x_1)\varepsilon(x_{(2)})z = \Delta^{(n-1)}(x)z$ if z is an invariant element. Conversely, if z commutes with $\Delta^{(n-1)}(x)$ for every $x \in U_q$, then

$$
\mathrm{ad}^r(x)(z) = \sum_{(x)} \Delta^{(n-1)}(S(x_{(1)}))\Delta^{(n-1)}(x_{(2)})z = \Delta^{(n-1)}(\varepsilon(x))z = \varepsilon(x)z.
$$

This proves our claim. Now, let $\alpha \in \mathcal{L}_{0,n}^{U_q}$. Then $\Phi_n(\alpha)$ is an $\mathrm{ad}^r(U_q)$-invariant element of $\tilde{U}_q^{\otimes n}$, by Theorem 6.6. By the claim above, and the fact that commuting with elements of $\tilde{U}_q$ or U_q is the same, $\Phi_n(\alpha)$ commutes with the matrix coefficients of $(\pi_V \otimes \Delta^{(n-1)})(RR')$, for every object V of $\mathcal{C}$. By the injectivity of Φ_n and the relation (6.12), α commutes with the matrix coefficients of (6.19) for all objects V of $\mathcal{C}$. These generate the algebra $\Delta^{(n-1)}(\mathcal{L}_{0,1})$, so $\mathcal{L}_{0,n}^{U_q}$ lies in the centralizer of the latter.

Conversely, the same reasoning shows that if α lies in the centralizer of $\Delta^{(n-1)}(\mathcal{L}_{0,1})$, then $\Phi_n(\alpha)$ commutes with the matrix coefficients of $(\pi_V \otimes \Delta^{(n-1)})(RR')$, for every object V of $\mathcal{C}$. From Theorem 4.3 we deduce that $\Phi_n(\alpha)$ lies in the centralizer of $\Delta^{(n-1)}(\tilde{U}_q^{\mathrm{lf}})$. This is the same as the centralizer of $\Delta^{(n-1)}(\tilde{U}_q)$ by Remark 4.5(1), so as above we deduce that $\Phi_n(\alpha)$ is an invariant element of $\tilde{U}_q^{\otimes n}$, and by injectivity and equivariance of Φ_n that α is an invariant element of $\mathcal{L}_{0,n}$.

The corollary is immediate, because a central element of $\mathcal{L}_{0,n}$ necessarily commutes with $\Delta^{(n-1)}(\mathcal{L}_{0,1})$, so it is invariant. ∎

We now give an explicit basis of the algebra of invariant elements $\mathcal{L}_{0,n}^{U_q}$. Let n be an integer greater than 1. For every $2 \leq k \leq n$ we denote by $S(k) \in \mathbb{U}_q^{\otimes n}$ the element defined by

$$S(k) = \mathrm{id}^{\otimes(k-2)} \otimes (\mathrm{id} \otimes \Delta^{(n-k)})(R).$$

Let $\lambda_1, \ldots, \lambda_n \in P_+$, and $V_{\lambda_1}, \ldots, V_{\lambda_n}$ the type 1 simple U_q-modules of highest weights $\lambda_1, \ldots, \lambda_n$ respectively. Put $[\lambda] = (\lambda_1, \ldots, \lambda_n)$, and consider the U_q-module

$$V_{[\lambda]} = \bigotimes_{j=1}^{n} V_{\lambda_j}.$$

Define $\overset{[\lambda]}{\mathbb{M}} \in \mathrm{End}(V_{[\lambda]}) \otimes \mathcal{L}_{0,n}$ by $\overset{[\lambda]}{\mathbb{M}} = \overset{V_\lambda}{M}$ if $n = 1$, and if $n \geq 2$ by

$$\overset{[\lambda]}{\mathbb{M}} = \overset{V_{\lambda_1}}{M}{}_1^{(1)} \prod_{k=2}^{n} \left(S(k)_{V_{\lambda_1}, \ldots, V_{\lambda_n}}^{-1} \overset{V_{\lambda_k}}{M}{}_k^{(k)} \right) \prod_{k=n}^{2} S(k)_{V_{\lambda_1}, \ldots, V_{\lambda_n}}. \tag{6.20}$$

For example for $n = 2$ we have

$$\overset{[\lambda]}{\mathbb{M}} = \overset{V_{\lambda_1}}{M}{}_1^{(1)} R_{12}^{-1} \overset{V_{\lambda_2}}{M}{}_2^{(2)} R_{12}$$

and for $n = 3$,

$$\overset{[\lambda]}{\mathbb{M}} = \overset{V_{\lambda_1}}{M}{}_1^{(1)} (\mathrm{id} \otimes \Delta)(R^{-1}) \overset{V_{\lambda_2}}{M}{}_2^{(2)} (R^{-1})_{23} \overset{V_{\lambda_3}}{M}{}_3{}^{(3)} R_{23} (\mathrm{id} \otimes \Delta)(R).$$

The reason for considering the matrices $\overset{[\lambda]}{\mathbb{M}}$ comes from the two following propositions. The first generalizes the identity (4.14).

Proposition 6.20. *The right coadjoint action of U_q on the matrix elements of $\overset{[\lambda]}{\mathbb{M}}$ can be written in matrix form as*

$$\mathrm{coad}^r(y)\left(\overset{[\lambda]}{\mathbb{M}}\right) = \sum_{(y)} \left((\pi_{V_{[\lambda]}}(y_{(1)}) \otimes \mathrm{id}) \overset{[\lambda]}{\mathbb{M}} (\pi_{V_{[\lambda]}}(S(y_{(2)})) \otimes \mathrm{id}) \right). \tag{6.21}$$

Proof. We show it for $n = 2$, since the general proof follows from it by an easy induction on n. We have

$$\operatorname{coad}^r(y)\big(\overset{[\lambda]}{\mathbb{M}}\big) = \operatorname{coad}^r(y)\big(\overset{V_{\lambda_1}}{M}{}_1^{(1)} R_{12}^{-1}\, \overset{V_{\lambda_2}}{M}{}_2^{(2)} R_{12}\big)$$

$$= \sum_{(y)} (y_{(1)})_1\, \overset{V_{\lambda_1}}{M}{}_1^{(1)} S(y_{(2)})_1 R_{12}^{-1}(y_{(3)})_2\, \overset{V_{\lambda_2}}{M}{}_2^{(2)} S(y_{(4)})_2 R_{12}$$

$$= \sum_{(y)} (y_{(1)})_1\, \overset{V_{\lambda_1}}{M}{}_1^{(1)} (y_{(2)})_2 R_{12}^{-1}(S(y_{(3)}))_1\, \overset{V_{\lambda_2}}{M}{}_2^{(2)} S(y_{(4)})_2 R_{12}$$

$$= \sum_{(y)} (y_{(1)} \otimes y_{(2)})\, \overset{V_{\lambda_1}}{M}{}_1^{(1)} R_{12}^{-1}\, \overset{V_{\lambda_2}}{M}{}_2^{(2)} (S(y_{(3)}) \otimes S(y_{(4)})) R_{12}$$

$$= \sum_{(y)} (y_{(1)} \otimes y_{(2)})\, \overset{V_{\lambda_1}}{M}{}_1^{(1)} R_{12}^{-1}\, \overset{V_{\lambda_2}}{M}{}_2^{(2)} R_{12}\big(S(y_{(4)}) \otimes S(y_{(3)})\big)$$

$$= \sum_{(y)} \big((\pi_{V_{[\lambda]}}(y_{(1)}) \otimes \operatorname{id})\, \overset{[\lambda]}{\mathbb{M}}\, (\pi_{V_{[\lambda]}}(S(y_{(2)})) \otimes \operatorname{id})\big).$$

All equalities are clear but the third and fifth, which follow from $(\operatorname{id}\otimes S^{-1})(R) = R^{-1}$, $R\Delta R^{-1} = \Delta^{\operatorname{cop}}$, $\Delta \circ S = (S \otimes S) \circ \Delta$, and the fact that S is an algebra antimorphism. The result follows. ∎

Denote by $\big(\overset{\lambda}{e}_i\big)$ a basis of V_λ, by $\big(\overset{\lambda}{e}{}^i\big)$ the dual basis, and put $\overset{[\lambda]}{e}_{[i]} := \overset{\lambda_1}{e}_{i_1} \otimes \cdots \otimes \overset{\lambda_n}{e}_{i_n}$. We thus get a basis $\big(\overset{[\lambda][i]}{e}\big)$ of $V_{[\lambda]}$. Denote by $\big(\overset{[\lambda][i]}{e}\big)$ the dual basis.

Proposition 6.21. *The elements* $\big(\overset{[\lambda][j]}{e} \otimes \operatorname{id}\big)\, \overset{[\lambda]}{\mathbb{M}}\, \big(\overset{[\lambda]}{e}_{[i]} \otimes \operatorname{id}\big)$ *of* $\mathcal{L}_{0,n}$, *where* $[\lambda] \in P_+^n$, $[i]$ *labels the basis of* $V_{[\lambda]}$, *and* $[j]$ *the dual basis, form a basis of* $\mathcal{L}_{0,n}$. *Equivalently, for every* $x \in \mathcal{L}_{0,n}$ *there exists a unique family of endomorphisms* $a_{[\lambda]}(x) \in \operatorname{End}_{\mathbb{C}(q)}\big(V_{[\lambda]}\big)$, $[\lambda] \in P_+^n$, *which are zero except possibly for a finite number of terms, such that*

$$x = \sum_{[\lambda]} \big(\operatorname{Tr}_{V_{[\lambda]}} \otimes \operatorname{id}\big)\big((\pi_{V_\lambda}(\ell)a_{[\lambda]}(x) \otimes \operatorname{id})\, \overset{[\lambda]}{\mathbb{M}}\big) = \sum_{[\lambda]} \operatorname{qTr}_{V_{[\lambda]}} \big((a_{[\lambda]}(x) \otimes \operatorname{id})\, \overset{[\lambda]}{\mathbb{M}}\big), \qquad (6.22)$$

where as usual ℓ *is the pivotal element and* $\pi_{V_\lambda}(\ell)$ *is the endomorphism of* $V_{[\lambda]}$ *given by the action of* ℓ.

Note that $\pi_{V_\lambda}(\ell)$ is introduced in (6.22) in order to simplify the statement of the next proposition.

Proof. We show the first claim of the proposition for $n = 2$; the general proof is similar. In this case it is sufficient to prove that the set of matrix elements of the tensors

$$\overset{V_{\lambda_1}}{M}{}_1^{(1)} R_{12}^{-1}\, \overset{V_{\lambda_2}}{M}{}_2^{(2)} = \sum_{(R)} (R_{(2)})_2\, \overset{V_{\lambda_1}}{M}{}_1^{(1)}\, \overset{V_{\lambda_2}}{M}{}_2^{(2)} S(R_{(1)})_1, \qquad [\lambda] \in P_+^n,$$

is a basis of $\mathcal{L}_{0,2}$. Note that we use $R^{-1} = (S \otimes \operatorname{id})(R) = \sum_{(R)} S(R_{(1)}) \otimes R_{(2)}$ in the above equality. Let $T = (S \otimes S^{-1})(R^{-1})$, that we denote as usual by $T = \sum_{(T)} T_{(1)} \otimes T_{(2)}$. From the identity $1 \otimes 1 = (S \otimes \operatorname{id})(R^{-1}R) = \sum_{(R),(T)} S(R_{(1)})T_{(1')} \otimes S(T_{(2')})R_{(2)}$, we obtain

$$\overset{V_{\lambda_1}}{M}{}_1^{(1)}\, \overset{V_{\lambda_2}}{M}{}_2^{(2)} = \sum_{(T)} S(T_{(2)})_2\, \overset{V_{\lambda_1}}{M}{}_1^{(1)} R_{12}^{-1}\, \overset{V_{\lambda_2}}{M}{}_2^{(2)} (T_{(1)})_1.$$

The conclusion follows from this, since the set of matrix elements of $\overset{V_{\lambda_1}}{M}{}_1^{(1)} \overset{V_{\lambda_2}}{M}{}_2^{(2)}$, $[\lambda] \in P_+^2$, is a basis of $\mathcal{L}_{0,2}$. The second claim is a trivial consequence of the first one. ∎

By combining the two previous propositions we obtain the following characterisation of the invariants elements.

Proposition 6.22. *An element $x \in \mathcal{L}_{0,n}$ belongs to $\mathcal{L}_{0,n}^{U_q}$ if and only if for every $[\lambda] \in P_+^n$ we have $a_{[\lambda]}(x) \in \mathrm{End}_{U_q}\big(V_{[\lambda]}\big)$. Therefore, the elements*

$$v_{[\lambda]}\big(a_{[\lambda]}^{(k)}\big) := \mathrm{qTr}_{V_{[\lambda]}}\big(a_{[\lambda]}^{(k)} \overset{[\lambda]}{\mathbb{M}}\big), \tag{6.23}$$

where $\big\{a_{[\lambda]}^{(k)}\big\}_k$ is any basis of $\mathrm{End}_{U_q}\big(V_{[\lambda]}\big)$, make a basis of $\mathcal{L}_{0,n}^{U_q}$.

Proof. We have $x \in \mathcal{L}_{0,n}^{U_q}$ if and only if $\mathrm{coad}^r(y)(x) = \epsilon(y)x$ for every $y \in U_q$, or equivalently for every $y \in U_q$. From Proposition 6.20, the basis provided by Proposition 6.21, and the fact that $S^2(y) = \ell y \ell^{-1}$, this is equivalent to $\sum_{(y)} S^{-1}(y_{(2)})_{V_{[\lambda]}} a_{[\lambda]}(x)(y_{(1)})_{V_{[\lambda]}} = \epsilon(y)a_{[\lambda]}(x)$ for every $[\lambda] \in P_+^n$, which is also equivalent to $y_{V_{[\lambda]}} a_{[\lambda]}(x) = a_{[\lambda]}(x)y_{V_{[\lambda]}}$. ∎

Finally, let us consider the integral form $\big(\mathcal{L}_{0,n}^A\big)^{U_A}$. Matrices $\overset{[X]}{\mathbb{M}}$ can be defined by the formula (6.20), replacing $V_{\lambda_1}, \ldots, V_{\lambda_n}$ with arbitrary U_A^{res}-modules $X_1, \ldots, X_n$ of type 1. Clearly these matrices still satisfy the equivariance property of Proposition 6.20, and the arguments of Proposition 6.21 imply that their matrix elements form a generating family of the A-module $\mathcal{L}_{0,n}^A$. When the U_A^{res}-modules X_i span the set of full A-sublattices $_A V_{\lambda_i}$ of the U_q-modules V_{λ_i}, such elements form a free family. One can still define invariant elements $v_X(a_X) \in \big(\mathcal{L}_{0,n}^A\big)^{U_A}$ by

$$v_X(a_X) := \mathrm{qTr}_X\big(a_X \overset{X}{\mathbb{M}}\big),$$

where $X := X_1 \otimes \cdots \otimes X_n$ for arbitrary U_A^{res}-modules $X_1, \ldots, X_n$ of type 1, and $a_X \in \mathrm{End}_{U_A^{\mathrm{res}}}(X)$. The A-module $\mathrm{End}_{U_A^{\mathrm{res}}}(X)$ is free, as it is a submodule of the free A-module $\mathrm{End}_{U_A^{\mathrm{res}}}(X)$ and A is a principal ideal domain. Then, taking $X_i := {}_A V_{\lambda_i}$ and $a_X^{(k)}$ basis elements of $\mathrm{End}_{U_A^{\mathrm{res}}}(X)$, the elements $v_X\big(a_X^{(k)}\big)$ form a free family over A, and by Proposition 6.22 a basis of $\mathcal{L}_{0,n}^{U_q}$.

In the case of $\mathfrak{g} = \mathfrak{sl}(2)$, let us state the following result. It is a direct consequence of Theorem 8.6 and the fact that the skein algebra $K_\zeta(\Sigma)$ is finitely generated and Noetherian, which is proved in [72, Theorem 3], by topological means. The method relies on the fact that $K_\zeta(\Sigma)$ has a natural filtration, and consists in proving that the associated graded algebra is finitely generated and Noetherian.

Theorem 6.23. *The algebra $\big(\mathcal{L}_{0,n}^A(\mathfrak{sl}(2))\big)^{U_A(\mathfrak{sl}(2))}$ is finitely generated and Noetherian.*

6.6 Centers

Next we turn to the center $\mathcal{Z}(\mathcal{L}_{0,n})$ of $\mathcal{L}_{0,n}$ and $\mathcal{Z}\big(\mathcal{L}_{0,n}^{U_q}\big)$ of $\mathcal{L}_{0,n}^{U_q}$.

Proposition 6.24. *The Alekseev map affords an isomorphism from $\mathcal{Z}(\mathcal{L}_{0,n})$ to $\mathcal{Z}\big(\tilde{U}_q^{\otimes n}\big)$. In particular this implies $\mathcal{Z}(\mathcal{L}_{0,n}) = \mathcal{Z}(\mathcal{L}_{0,1})^{\otimes n}$. Moreover, the elements*

$$_\lambda\omega^{(i)} := \mathrm{qTr}_{V_\lambda}\big(\overset{V_\lambda}{M}{}^{(i)}\big), \qquad \lambda \in P_+, i \in \{1, \ldots, n\},$$

belong to the center $\mathcal{Z}(\mathcal{L}_{0,n})$ and the family of elements $\prod_{i=1}^n {}_{\lambda_i}\omega^{(i)}$, where $\lambda_1, \ldots, \lambda_n \in P_+$, form a basis of $\mathcal{Z}(\mathcal{L}_{0,n})$. The elements $_\lambda\omega^{(i)}$ belong to $\mathcal{L}_{0,n}^A$, and they form a free family.

Proof. We have $\Phi_n(\mathcal{Z}(\mathcal{L}_{0,n})) = \mathcal{Z}(\Phi_n(\mathcal{L}_{0,n}))$ by Theorem 6.6. We claim that if $x \in \tilde{U}_q^{\otimes n}$ commutes with $\Phi_n(\mathcal{L}_{0,n})$, it is central. Indeed, x commutes with $\Phi_n\big(\mathcal{L}_{0,n}^{(n)}\big) = 1^{\otimes(n-1)} \otimes \tilde{U}_q^{\mathrm{lf}}$, and hence with $1^{\otimes(n-1)} \otimes \tilde{U}_q$, and the matrix coefficients of $R_{Vn}^{\pm1}$ by Remark 4.5. (For instance, when $\mathfrak{g} = \mathfrak{sl}(2)$, $K = q^H$ and $q^{H/2}$ commute and are diagonalizable on objects of $\mathcal{C}$, and since x commutes with $1^{\otimes(n-1)} \otimes K$, it commutes with $1^{\otimes(n-1)} \otimes q^{H/2}$ too, and the conclusion follows from (6.15) and (6.16)). Using that x commutes also with $\Phi_n\big(\mathcal{L}_{0,n}^{(n-1)}\big)$, and hence with the matrix coefficients of $R_{Vn}R_{Vn-1}R'_{Vn-1}R_{Vn}^{-1}$, we deduce that it commutes with the matrix coefficients of $R_{Vn-1}R'_{Vn-1}$, and hence with every element of $1^{\otimes(n-2)} \otimes \tilde{U}_q \otimes 1$. Continuing in this way recursively, we get that $x \in \mathcal{Z}\big(\tilde{U}_q^{\otimes n}\big)$, which proves our claim. It implies that $\Phi_n(\mathcal{Z}(\mathcal{L}_{0,n})) \subset \mathcal{Z}\big(\tilde{U}_q^{\otimes n}\big)$.

Let us prove the converse inclusion. Put $R^{(a)} := R_{Vn} \cdots R_{Va+1}$. For every $1 \leq a \leq n$ we have

$$
\begin{aligned}
\Phi_n\big(\mathrm{qTr}_{V_\lambda}\big(\overset{V_\lambda}{M}{}^{(a)}\big)\big) &= (\mathrm{Tr}_{V_\lambda} \otimes\mathrm{id})\big(\big(\pi_{V_\lambda}(\ell) \otimes \mathrm{id}\big) R^{(a)} R_{Va} R'_{Va} R^{(a)-1}\big) \\
&= (\mathrm{Tr}_{V_\lambda} \otimes\mathrm{id})\big(\big(\pi_{V_\lambda}(\ell) \otimes \mathrm{id}\big) R_{Va} R'_{Va}\big) \\
&= \mathfrak{i}_a\big(\Phi_1\big(\mathrm{qTr}_{V_\lambda}\big(\overset{V_\lambda}{M}\big)\big)\big),
\end{aligned}
\tag{6.24}
$$

where Φ_1 in the last equality is the RSD map, and the second equality follows from $R^{-1} = (S \otimes \mathrm{id})(R)$ and $S^2(x) = \ell x \ell^{-1}$ for every $x \in \mathbb{U}_q$. By Theorem 4.3 and Proposition 4.8, the family of these elements, when λ spans P_+, forms a basis of the center of the a-th tensorand of $\tilde{U}_q^{\otimes n}$. Therefore $\Phi_n(\mathcal{Z}(\mathcal{L}_{0,n})) = \mathcal{Z}\big(\tilde{U}_q^{\otimes n}\big)$. By injectivity of Φ_n this proves the first claim. The computation (6.24) concludes the description of $\mathcal{Z}(\mathcal{L}_{0,n})$. The last claim is clear. ∎

In particular, recall from (6.18) that for $\mathfrak{g} = \mathfrak{sl}(2)$ and every $1 \leq i \leq n$ we put

$$
\overset{V_2}{M}{}^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} \in \mathrm{End}(V_2) \otimes \mathcal{L}_{0,n}(\mathfrak{sl}(2))^{(i)}.
$$

Set

$$
\omega^{(i)} = qa^{(i)} + q^{-1}d^{(i)} = \mathrm{qTr}_{V_2}\big(\overset{V_2}{M}{}^{(i)}\big).
$$

By (6.24) we have

$$
\Phi_n\big(\omega^{(i)}\big) = \Omega^{(i)},
$$

that is, $(q - q^{-1})^2$ times the Casimir element of the i-th tensorand of $U_q(\mathfrak{sl}(2))^{\otimes n}$, and then $\mathcal{Z}(U_q(\mathfrak{sl}(2))^{\otimes n}) = \mathbb{C}(q)\big[\Omega^{(1)}, \ldots, \Omega^{(n)}\big]$. The proposition implies

$$
\mathcal{Z}(\mathcal{L}_{0,n}(\mathfrak{sl}(2))) = \mathbb{C}(q)\big[\omega^{(1)}, \ldots, \omega^{(n)}\big].
$$

More generally, by the same arguments and the results recalled in Remark 4.9, there is an analogous description of $\mathcal{Z}(\mathcal{L}_{0,n}(\mathfrak{g}))$ for an arbitrary finite dimensional complex simple Lie algebra $\mathfrak{g}$ of type A, B, C or D.

Remark 6.25. When $\mathfrak{g} = \mathfrak{sl}(2)$, by using the A-basis of the A-module $\mathcal{L}_{0,1}^A(\mathfrak{sl}(2)) = \mathcal{O}_A(\mathfrak{sl}(2))$ given in [29, Proposition 1.3], the arguments of Proposition 5.2 show that $\mathcal{Z}\big(\mathcal{L}_{0,1}^A(\mathfrak{sl}(2))\big) = A[\omega]$. Then it is easy to see as above that the family of elements $\prod_{i=1}^n \lambda_i \omega^{(i)}$ form an A-basis of $\mathcal{Z}\big(\mathcal{L}_{0,n}^A(\mathfrak{sl}(2))\big)$, and that $\mathcal{Z}\big(\mathcal{L}_{0,n}^A(\mathfrak{sl}(2))\big) = A\big[\omega^{(1)}, \ldots, \omega^{(n)}\big]$.

For every $\lambda \in P_+$ denote

$$\lambda\eta = \mathrm{qTr}_{V_\lambda}\big(\overset{V_\lambda}{M}{}^{(1)} \cdots \overset{V_\lambda}{M}{}^{(n)}\big).$$

In the specific case of $\mathfrak{g} = \mathfrak{sl}(2)$, let us put

$$\eta := \mathrm{qTr}_{V_2}\big(\overset{V_2}{M}{}^{(1)} \cdots \overset{V_2}{M}{}^{(n)}\big). \tag{6.25}$$

Lemma 6.26. $\mathcal{Z}\big(\mathcal{L}_{0,n}^{U_q}\big)$ *contains the commutative algebra generated over $\mathbb{C}(q)$ by the elements* $\lambda_1\omega^{(1)}, \ldots, \lambda_n\omega^{(n)}$ *and* $\lambda\eta$ *for all* $\lambda, \lambda_1, \ldots, \lambda_n \in P_+$.

Proof. Clearly $\mathcal{Z}\big(\mathcal{L}_{0,n}^{U_q}\big)$ contains $\mathcal{Z}(\mathcal{L}_{0,n})^{U_q}$, whence the elements $\lambda_1\omega^{(1)}, \ldots, \lambda_n\omega^{(n)}$ for every $\lambda_1, \ldots, \lambda_n \in P_+$, by Propositions 6.24 and 6.19. Moreover, by this result $\mathcal{L}_{0,n}^{U_q}$ is the centralizer of $\Delta^{(n-1)}(\mathcal{L}_{0,1})$ in $\mathcal{L}_{0,n}$. Therefore $\mathcal{Z}\big(\mathcal{L}_{0,n}^{U_q}\big) \supset \Delta^{(n-1)}(\mathcal{Z}(\mathcal{L}_{0,1}))$. By Proposition 4.8 and the relation (6.19), the elements $\lambda\eta$, $\lambda \in P_+$, form a basis of $\Delta^{(n-1)}(\mathcal{Z}(\mathcal{L}_{0,1}))$. The result follows. ∎

Note that the $\mathbb{C}(q)$-algebra generated by the elements $\lambda_1\omega^{(1)}, \ldots, \lambda_n\omega^{(n)}$ and $\lambda\eta$ for all weights $\lambda, \lambda_1, \ldots, \lambda_n \in P_+$ is the polynomial algebra generated by these elements when $\lambda, \lambda_1, \ldots, \lambda_n$ are fundamental weights. We have even a much better result:

Theorem 6.27. *We have an isomorphism*

$$\mathcal{Z}\big(\mathcal{L}_{0,n}^{U_q}\big) \cong \Delta^{(n-1)}(\mathcal{Z}(\mathcal{L}_{0,1})) \otimes_{\mathbb{C}(q)} \mathcal{Z}(\mathcal{L}_{0,1})^{\otimes n}.$$

Therefore $\mathcal{Z}\big(\mathcal{L}_{0,n}^{U_q}\big)$ is the polynomial algebra generated over $\mathbb{C}(q)$ by the elements $\lambda_1\omega^{(1)}, \ldots,$ $\lambda_n\omega^{(n)}$ *and* $\lambda\eta$ *for all fundamental weights $\lambda, \lambda_1, \ldots, \lambda_n$. In particular, when $\mathfrak{g} = \mathfrak{sl}(2)$ we have*

$$\mathcal{Z}\big(\mathcal{L}_{0,n}(\mathfrak{sl}(2))^{U_q(\mathfrak{sl}(2))}\big) = \mathbb{C}(q)\big[\omega^{(1)}, \ldots, \omega^{(n)}, \eta\big]$$

and

$$\mathcal{Z}\big(\mathcal{L}_{0,n}^A(\mathfrak{sl}(2))^{U_A(\mathfrak{sl}(2))}\big) = A\big[\omega^{(1)}, \ldots, \omega^{(n)}, \eta\big].$$

This is a consequence of the following lemmas, which are interesting by themselves. Denote by $U_h := U_h(\mathfrak{g})$ the Hopf algebra over $\mathbb{C}[[h]]$ generated topologically (in the h-adic sense) by E_i, F_i, H_i, where $i = 1, \ldots, m$, satisfying the relations $[H_i, H_j] = 0$, $[H_i, E_j] = a_{ij}E_j$, $[H_i, F_j] = -a_{ij}F_j$ and (3.1)–(3.3), where K_i is replaced by $q_i^{H_i}$ and q_i by $e^{d_i h}$. By the same formula as (6.11) with U_h in place of U_q, U_h acts on $U_h^{\otimes n}$ by the right adjoint action. Let $\mathcal{Z}(U_h)$ be the center of U_h.

Lemma 6.28. *The h-adic completion of $\Delta^{(n-1)}(\mathcal{Z}(U_h)) \otimes_{\mathbb{C}[[h]]} \mathcal{Z}(U_h)^{\otimes n}$, considered as a subalgebra of $U_h^{\otimes n}$, is the center of $\big(U_h^{\otimes n}\big)^{U_h}$.*

Proof. Denote by $U = U(\mathfrak{g})$ the envelopping algebra of $\mathfrak{g}$, and by Δ_0 its canonical coproduct. We have $U_h/hU_h = U$. As shown in [76, Lemma 3.10], a direct application of Theorem 10.1 of [57] proves the result for $h = 0$, i.e., for U. Because $\mathfrak{g}$ is finite dimensional and semisimple, Drinfeld's results in [34, 35] show that there exists an isomorphism of algebras $\phi: U_h \to U[[h]]$, equal to the identity on the quotient U_h/hU_h, and there exists an invertible element $J \in U^{\otimes 2}[[h]]$ such that $\forall x \in U_h, (\phi \otimes \phi)(\Delta(x)) = J\Delta_0(\phi(x))J^{-1}$. By using J one can easily define an invertible element $J_n \in U^{\otimes n}[[h]]$ such that $\forall x \in U_h$, $(\phi^{\otimes n})(\Delta^{(n-1)}(x)) = J_n\Delta_0^{(n-1)}(\phi(x))J_n^{-1}$. As a result the map $\psi: \big(U_h^{\otimes n}\big)^{U_h} \to (U^{\otimes n})^U[[h]]$, $\psi(x) = J_n^{-1}(\phi^{\otimes n}(x))J_n$, is an isomorphism of algebras. We obviously have $\phi(\mathcal{Z}(U_h)) = \mathcal{Z}(U)[[h]]$. The lemma follows after having checked that $\psi^{-1}\big(\mathcal{Z}(U)^{\otimes n}[[h]]\big) = \mathcal{Z}(U_h)^{\otimes n}$ and $\psi^{-1}\big(\Delta_0^{(n-1)}(\mathcal{Z}(U)[[h]])\big) = \Delta^{(n-1)}(\mathcal{Z}(U_h))$. ∎

Lemma 6.29. *The center of* $(U_q^{\otimes n})^{U_q}$ *is* $\mathcal{Z}\big((U_q^{\otimes n})^{U_q}\big) = \Delta^{(n-1)}(\mathcal{Z}(U_q)) \otimes_{\mathbb{C}(q)} \mathcal{Z}(U_q)^{\otimes n}$.

Proof. By Lemma 6.26 it only remains to prove the inclusion "$\subset$". Let $\varphi \colon U_A \otimes_A \mathbb{C}[[h]] \to U_h$ be the morphism of algebras defined by $\varphi(K_i) = e^{hd_iH_i}$, $\varphi(E_i) = E_i$, $\varphi(F_i) = F_i$. Let $z \in \mathcal{Z}\big((U_q^{\otimes n})^{U_q}\big)$. Up to multiplication by an element of A we can assume that $z \in \mathcal{Z}\big((U_A^{\otimes n})^{U_A}\big)$. Because $\varphi^{\otimes n}(z)$ commutes with $\varphi^{\otimes n}\big((U_A^{\otimes n})^{U_A} \otimes_A \mathbb{C}[[h]]\big)$, it commutes with its h-adic completion. Consider the family of elements $v_{[\lambda]}(a_{[\lambda]})$ defined by (6.23) with ${}_A V_{[\lambda]} := {}_A V_{\lambda_1} \otimes \cdots \otimes {}_A V_{\lambda_1}$ and $a_{[\lambda]}$ in $\mathrm{End}_{U_A^{\mathrm{res}}}\big({}_A V_{\lambda_1} \otimes \cdots \otimes {}_A V_{\lambda_n}\big)$. By Proposition 6.22 (see the comments therefater) they form a $\mathbb{C}(q)$-basis of $\mathcal{L}_{0,n}^{U_q}$. Proposition 6.7 then implies that the elements $\Phi_n(v_{[\lambda]}\big(a_{[\lambda]}^{(k)}\big)$ form a basis of $(U_q^{\otimes n})^{U_q}$. These results still hold by working with U_h and the topological version of $\mathcal{L}_{0,n}$ defined over $\mathbb{C}[[h]]$, so the elements $\varphi^{\otimes n}\big(\Phi_n\big(v_{[\lambda]}\big(a_{[\lambda]}^{(k)}\big)\big)\big)$ form a $\mathbb{C}[[h]]$-topological basis of $(U_h^{\otimes n})^{U_h}$. Therefore $\varphi^{\otimes n}(z)$ centralizes $(U_h^{\otimes n})^{U_h}$, whence $\varphi^{\otimes n}(z) \in \Delta^{(n-1)}(\mathcal{Z}(U_h)) \otimes_{\mathbb{C}[[h]]} \mathcal{Z}(U_h)^{\otimes n}$ by Lemma 6.28. This proves $z \in \Delta^{(n-1)}(\mathcal{Z}(U_q)) \otimes_{\mathbb{C}(q)} \mathcal{Z}(U_q)^{\otimes n}$. ∎

Proof of Theorem 6.27. We have an isomorphism $\mathcal{L}_{0,n}^{U_q} \cong \big(\tilde{U}_q^{\otimes n}\big)^{U_q}$ by Proposition 6.7, and $\mathcal{Z}\big((\tilde{U}_q^{\otimes n})^{U_q}\big) = \Delta^{(n-1)}\big(\mathcal{Z}(\tilde{U}_q)\big) \otimes_{\mathbb{C}(q)} \mathcal{Z}(\tilde{U}_q)^{\otimes n}$ by the last lemma applied to $\tilde{U}_q$ instead of U_q. But $\Phi_n^{-1}\big(\mathcal{Z}(\tilde{U}_q)^{\otimes n}\big) = \mathcal{Z}(\mathcal{L}_{0,1})^{\otimes n}$ by Proposition 6.24, which is the vector space generated by ${}_{\lambda_1}\omega^{(1)}, \ldots, {}_{\lambda_n}\omega^{(n)}$ for $\lambda_1, \ldots, \lambda_n \in P_+$ (by Proposition 4.8), and $\Phi_n^{-1}\big(\Delta^{(n-1)}\big(\mathcal{Z}(\tilde{U}_q)\big)\big)$ is the vector space generated by ${}_{\lambda}\eta$ for all $\lambda \in P_+$ (by Proposition 4.8 and the relation (6.19)). This proves all claims but the last one, which in turn follows from $\mathcal{Z}\big((\mathcal{L}_{0,n}^A)^{U_A}\big) = \mathcal{Z}\big(\mathcal{L}_{0,n}^{U_q}\big) \cap \mathcal{L}_{0,n}^A$, and the fact that $\omega^{(1)}, \ldots, \omega^{(n)}, \eta \in \mathcal{L}_{0,n}^A$. ∎

6.7 Specializations

Let $\epsilon \in \mathbb{C}^\times$. We defined the unrestricted specialization $U_\epsilon := U_A \otimes_A \mathbb{C}_\epsilon$ of U_q in (3.8). Recall that $\mathbb{C}_\epsilon$ is the A-module $\mathbb{C}$, where q acts by multiplication by ϵ. Similarly, the unrestricted specialisation of $\mathcal{L}_{0,n}^A$ at ϵ is the U_ϵ-module algebra

$$\mathcal{L}_{0,n}^\epsilon = \mathcal{L}_{0,n}^A \otimes_A \mathbb{C}_\epsilon.$$

We need to consider the specialization at $q = \epsilon$ of the Alekseev map. By Lemma 6.10 and Proposition 6.12 we know that $\Phi_n \colon \mathcal{L}_{0,n}^A \to \tilde{U}_A^{\otimes n}$ is an embedding of U_A-module algebras, and in the case $\mathfrak{g} = \mathfrak{sl}(2)$ it maps into $U_A(\mathfrak{sl}(2))^{\otimes n}$. Moreover, by Proposition 6.16 the latter embedding extends to an isomorphism $\Phi_n \colon {}_{\mathrm{loc}}\mathcal{L}_{0,n}^A(\mathfrak{sl}(2)) \to U_A'(\mathfrak{sl}(2))^{\otimes n}$.

When $q = \epsilon$ the formulas (5.3) show that $\Phi_1 \colon \mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}(2)) \to U_\epsilon(\mathfrak{sl}(2))$ is an embedding, and $\Phi_1 \colon {}_{\mathrm{loc}}\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}(2)) \to U_\epsilon(\mathfrak{sl}(2))$ an isomorphism. For an arbitrary $\mathfrak{g}$, $\Phi_1 \colon \mathcal{L}_{0,1}^\epsilon \to \tilde{U}_\epsilon$ is also an embedding; this follows from the following facts. In [29, Sections 4 and 6], De Concini–Lyubashenko introduced an embedding of algebras $\mu'' \colon \mathcal{O}_A \to \tilde{U}_A(\mathfrak{b}_-) \otimes \tilde{U}_A(\mathfrak{b}_+)$ (where $\tilde{U}_A(\mathfrak{b}_\pm)$ is the subalgebra of $\tilde{U}_A$ associated to the Borel subalgebra $\mathfrak{b}_\pm$ of $\mathfrak{g}$), and they proved that it affords an embedding $\mu''_\epsilon \colon \mathcal{O}_\epsilon \to \tilde{U}_\epsilon(\mathfrak{b}_-) \otimes \tilde{U}_\epsilon(\mathfrak{b}_+)$. We have $\Phi_1 = m \circ (\mathrm{id} \otimes S^{-1}) \circ \mu''$. Moreover, $\mathrm{Im}(\mu'')$ is contained in the subalgebra of $\tilde{U}_\epsilon(\mathfrak{b}_-) \otimes \tilde{U}_\epsilon(\mathfrak{b}_+)$ with basis elements

$$\bar{F}_{\beta_1}^{n_1} \cdots \bar{F}_{\beta_N}^{n_N} K_{n_1\beta_1 + \cdots + n_N\beta_N} K_\lambda \otimes K_{-\lambda} K_{-p_1\beta_1 - \cdots - p_N\beta_N} \bar{E}_{\beta_1}^{p_1} \cdots \bar{E}_{\beta_N}^{p_N},$$

where $\lambda \in P$ and $n_1, \ldots, n_N, p_1, \ldots, p_N \in \mathbb{N}$. The map $m \circ (\mathrm{id} \otimes S^{-1})$ sends this basis to a free family of $\tilde{U}_\epsilon$. As a result $\Phi_1 \colon \mathcal{L}_{0,1}^\epsilon \to \tilde{U}_\epsilon$ is injective. Since Φ_n differs from $\Phi_1^{\otimes n}$ by a linear isomorphism (see (6.10)), it follows that

$$\Phi_n \colon \ \mathcal{L}_{0,n}^\epsilon \to \tilde{U}_\epsilon^{\otimes n}$$

is an embedding of U_ϵ-module algebras.

Moreover, in the $\mathfrak{sl}(2)$ case, by Proposition 6.16 and the fact that $\Phi_1\colon {}_{\mathrm{loc}}\mathcal{L}_{0,1}^\epsilon(\mathfrak{sl}(2)) \to U_\epsilon(\mathfrak{sl}(2))$ is an isomorphism, it follows that

$$\Phi_n\colon\ {}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon(\mathfrak{sl}(2)) \to U_\epsilon(\mathfrak{sl}(2))^{\otimes n} \tag{6.26}$$

is an isomorphism. Note that, by Lemma 4.11, when $n = 1$ we have an isomorphism

$$\Phi_1\colon\ \mathcal{L}_{0,1}^\epsilon \to \tilde{U}_A^{\mathrm{lf}} \otimes_A \mathbb{C}_\epsilon.$$

It is important to note that taking the specialization at a root of unity $q = \epsilon$ and taking locally finite elements are non commuting operations. For instance $\tilde{U}_\epsilon$ is a free module of finite rank over its center (see, e.g., [25, Section 9.2]). Hence it coincides with its subalgebra of locally finite elements $(\tilde{U}_\epsilon)^{\mathrm{lf}}$. On another hand, $\tilde{U}_A^{\mathrm{lf}} \otimes_A \mathbb{C}_\epsilon$ is strictly contained in $\tilde{U}_\epsilon$; for instance it does not contain the elements K_i.

The algebra $\tilde{U}_q^{\mathrm{lf}}$ is very complicated [51], we know neither generators nor basis for arbitrary $\mathfrak{g}$. This hurdle prevents us to give a precise description of $\tilde{U}_A^{\mathrm{lf}}$, $\tilde{U}_A^{\mathrm{lf}} \otimes_A \mathbb{C}_\epsilon$, and $\big(\tilde{U}_A^{\otimes n}\big)^{\mathrm{lf}} \otimes_A \mathbb{C}_\epsilon$ for $n > 1$.

Finally, recall that $\big(\mathcal{L}_{0,n}^A\big)^{U_A}$ is the centralizer of $\Delta^{(n-1)}\big(\mathcal{L}_{0,1}^A\big)$, see Proposition 6.19. By the same arguments the algebra $\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}$ of U_ϵ-invariant elements of $\mathcal{L}_{0,n}^\epsilon$ is the centralizer of $\Delta^{(n-1)}(\mathcal{L}_{0,1}^\epsilon)$. Therefore we have an inclusion $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) \subset \big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}$, and since clearly $\big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes_A \mathbb{C}_\epsilon$ is a subset of $\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}$, multiplication defines a morphism of algebras

$$\big(\big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes_A \mathbb{C}_\epsilon\big) \otimes \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) \to \big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}.$$

In the rest of this paper we will simplify notations by setting

$$\big(\mathcal{L}_{0,n}^A\big)_\epsilon^{U_A} := \big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes_A \mathbb{C}_\epsilon. \tag{6.27}$$

The arguments of Proposition 6.11 apply as well to the specialization $q = \epsilon$ (using that $\tilde{U}_\epsilon$ has no non trivial zero divisors), so we have:

Proposition 6.30. *The algebra $\mathcal{L}_{0,n}^\epsilon$ does not have non trivial zero divisors, and therefore the subalgebras $\big(\mathcal{L}_{0,n}^\epsilon\big)^{U_\epsilon}$ and $\big(\mathcal{L}_{0,n}^A\big)_\epsilon^{U_A}$ too.*

7 Center of $\mathcal{L}_{0,n}^\epsilon(\mathfrak{sl}(2))$ and quantum coadjoint action

7.1 Center of $\mathcal{L}_{0,n}^\epsilon(\mathfrak{sl}(2))$

From now on $\mathfrak{g} = \mathfrak{sl}(2)$ and ϵ is a primitive l-th root of unity. We assume that $l \geq 3$ and l is odd. We make this latter choice to simplify the exposition; the case of l even can be treated in a similar way, as all our constructions below rely on the description of the center of $U_\epsilon(\mathfrak{sl}(2))$, which is done for all primitive roots of unity ϵ in [27, Section 3]. We omit $\mathfrak{g}$ from the notations of the various algebras, and denote $U_\epsilon(\mathfrak{sl}(2))$ by U_ϵ, etc.

By the relations (5.3) and (5.4), $\Phi_1(\mathcal{L}_{0,1}^\epsilon) = \tilde{U}_A^{\mathrm{lf}} \otimes_A \mathbb{C}_\epsilon = U_A^{\mathrm{lf}} \otimes_A \mathbb{C}_\epsilon$ is the algebra generated over $\mathbb{C}$ by Ω, EK^{-1}, F and K^{-1}. Adding the generator K gives U_ϵ. By results of [27], the center $\mathcal{Z}(U_\epsilon)$ of U_ϵ is the $\mathbb{C}$-algebra generated by E^l, F^l, $K^{\pm l}$ and Ω satisfying the relation

$$\prod_{j=1}^l (\Omega - c_j) = \big(\epsilon - \epsilon^{-1}\big)^{2l} E^l F^l + K^l + K^{-l} - 2, \tag{7.1}$$

where $c_j = \epsilon^j + \epsilon^{-j}$. Let $\mathcal{Z}_0(U_\epsilon)$ be the subalgebra of $\mathcal{Z}(U_\epsilon)$ generated by E^l, F^l and $K^{\pm l}$. It is a sub-Hopf algebra of U_ϵ, with

$$
\begin{aligned}
&\Delta\big(K^{\pm l}\big) = K^{\pm l} \otimes K^{\pm l}, \qquad \Delta\big(E^l\big) = E^l \otimes K^l + 1 \otimes E^l, \qquad \Delta\big(F^l\big) = K^{-l} \otimes F^l + F^l \otimes 1, \\
&S\big(E^l\big) = -E^l K^{-l}, \qquad S\big(F^l\big) = -K^l F^l, \qquad S\big(K^{\pm l}\big) = K^{\mp l}, \\
&\varepsilon\big(E^l\big) = \varepsilon\big(F^l\big) = 0, \qquad \varepsilon\big(K^l\big) = 1.
\end{aligned}
$$

Consider the sequence of polynomials T_k, $k \in \mathbb{N}$, defined recursively by

$$
T_0(x) = 2, \qquad T_1(x) = x, \qquad T_k(x) = x T_{k-1}(x) - T_{k-2}(x) \qquad \text{for} \quad k \geq 2. \tag{7.2}
$$

Note that $T_k(x)/2$ is the k-th Chebyshev polynomial of the first type in the variable $x/2$. One has $T_k\big(u + u^{-1}\big) = u^k + u^{-k}$, from which one derives easily that $T_l(x) - 2 = \prod_{j=1}^{l}(x - c_j)$. Therefore, the relation (7.1) can be written as:

$$
T_l(\Omega) = \big(\epsilon - \epsilon^{-1}\big)^{2l} E^l F^l + K^l + K^{-l}. \tag{7.3}
$$

Since l is odd, it is also equivalent to

$$
\prod_{j=1}^{l}(\Omega + c_j) = \big(\epsilon - \epsilon^{-1}\big)^{2l} E^l F^l + K^l + K^{-l} + 2. \tag{7.4}
$$

By (5.3) we have

$$
\begin{aligned}
&\Phi_1(\omega) = \Omega, \qquad \Phi_1\big(b^l\big) = \big(\epsilon - \epsilon^{-1}\big)^l F^l, \\
&\Phi_1\big(c^l\big) = \big(\epsilon - \epsilon^{-1}\big)^l (EK^{-1})^l, \qquad \Phi_1\big(d^l\big) = K^{-l}. \tag{7.5}
\end{aligned}
$$

Hence

$$
T_l(\omega) = b^l c^l d^{-l} + d^{-l} + d^l. \tag{7.6}
$$

Using that Φ_1 is equivariant, injective, and surjective when extended to the localization, we deduce

$$
\begin{aligned}
&\mathcal{Z}\big({}_{\mathrm{loc}}\mathcal{L}_{0,1}^\epsilon\big) = \mathbb{C}\big[\omega, b^l, c^l, d^{\pm l}\big]/\mathcal{I}, \\
&\mathcal{Z}\big(\mathcal{L}_{0,1}^\epsilon\big) = \mathbb{C}\big[\omega, b^l, c^l, d^l\big]/\mathcal{I},
\end{aligned}
$$

where $\mathcal{I}$ is the ideal of $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$ generated by $\big(T_l(\omega) - d^l\big)d^l - b^l c^l - 1$. By the presentation of $\mathcal{L}_{0,1}^A$ in Lemma 5.7, we have

$$
\mathcal{Z}\big(\mathcal{L}_{0,1}^\epsilon\big) = \big(\mathcal{L}_{0,1}^\epsilon\big)^{U_\epsilon}.
$$

Alternatively, we have $\big(U_A^{\mathrm{lf}} \otimes_A \mathbb{C}_\epsilon\big)^{U_\epsilon} = \mathcal{Z}\big(U_A^{\mathrm{lf}} \otimes_A \mathbb{C}_\epsilon\big)$ by the arguments of Corollary 6.19 for $n = 1$. Therefore $\mathcal{Z}(\mathcal{L}_{0,1}^\epsilon) = \Phi_1^{-1}\big(\big(U_A^{\mathrm{lf}} \otimes_A \mathbb{C}_\epsilon\big)^{U_\epsilon}\big) = \big(\mathcal{L}_{0,1}^\epsilon\big)^{U_\epsilon}$.

We can now define a notion of quantum Frobenius homomorphism for $\mathcal{L}_{0,1}^\epsilon$, similar in spirit to the one defined for $\mathcal{O}_\epsilon$ in [71], which is a map $\mathcal{O}_1 \to \mathcal{Z}(\mathcal{O}_\epsilon)$. Consider the specialization $\mathcal{L}_{0,1}^1$ of $\mathcal{L}_{0,1}^A$ at $q = 1$. We have $\mathcal{L}_{0,1}^1 = \mathcal{O}_1 = \mathcal{O}(G)$ as commutative algebras (with $G = \mathrm{SL}(2,\mathbb{C})$). Denote by $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$ the images of the generators a, b, c, d of $\mathcal{L}_{0,1}^A$ under the specialization map $\mathcal{L}_{0,1}^A \to \mathcal{L}_{0,1}^1$. They satisfy $\underline{ad} - \underline{bc} = 1$. Let us define $Q_l \in \mathbb{C}[X,Y]$ by $Q_l(X,Y) = T_l(\epsilon X + \epsilon^{-1} Y) - X^l - Y^l$. Recall that a and d commute. So we put:

Definition 7.1. The Frobenius map $\mathrm{Fr}\colon \mathcal{L}_{0,1}^1 \to \mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$ is the homomorphism of algebras given by

$$\mathrm{Fr}(\underline{a}) = a^l + Q_l(a,d) = T_l(\omega) - d^l, \qquad \mathrm{Fr}(\underline{b}) = b^l, \qquad \mathrm{Fr}(\underline{c}) = c^l, \qquad \mathrm{Fr}(\underline{d}) = d^l.$$

We shall denote

$$\overset{V_2}{\underline{M}} = \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \qquad \text{and} \qquad \mathrm{Fr}\, \overset{V_2}{\underline{M}} = \begin{pmatrix} a^l + Q_l(a,d) & b^l \\ c^l & d^l \end{pmatrix}.$$

Note that:

- $\det\big(\mathrm{Fr}\,\overset{V_2}{\underline{M}}\big) - 1 = (T_l(\omega) - d^l)d^l - b^l c^l - 1$, i.e., the generator of the ideal $\mathcal{I}$.
- $T_l\big(\mathrm{qTr}\,\big(\overset{V_2}{M}\big)\big) = \mathrm{Tr}\,\big(\mathrm{Fr}\,\overset{V_2}{\underline{M}}\big)$.

The notions above can be developed similarly for every $\mathcal{L}_{0,n}^\epsilon$, $n \geq 1$. First, recall the additional generators $\delta^{(i)-1}$ of the localization $_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon$ introduced in Definition 6.15.

Proposition 7.2. *We have*

$$\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon) = \mathbb{C}\big[\omega^{(i)}, b^{(i)l}, c^{(i)l}, d^{(i)l}, i = 1, \ldots, n\big]/\big(\mathcal{I}^{(i)}, i = 1, \ldots, n\big),$$
$$\mathcal{Z}(_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon) = \mathbb{C}\big[\omega^{(i)}, b^{(i)l}, c^{(i)l}, d^{(i)l}, \big(\delta^{(i)l}\big)^{-1}, i = 1, \ldots, n\big]/\big(\mathcal{I}^{(i)}, i = 1, \ldots, n\big),$$

where $\mathcal{I}^{(i)}$ is the ideal generated by the element $\big(T_l\big(\omega^{(i)}\big) - d^{(i)l}\big)d^{(i)l} - b^{(i)l}c^{(i)l} - 1$.

Proof. First we prove that $\Phi_n\big(b^{(i)l}\big), \Phi_n\big(c^{(i)l}\big), \Phi_n\big(d^{(i)l}\big) \in \mathcal{Z}(U_\epsilon^{\otimes n})$ for every $i = 1, \ldots, n$. By injectivity of Φ_n it will follow that $b^{(i)l}, c^{(i)l}, d^{(i)l} \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$. By (7.5) the claim is true for $i = n$. Let $1 \leq i \leq n-1$, and denote $R^{(i)} := R_{V_2 n} \cdots R_{V_2 i+1} \in \mathrm{End}(V_2) \otimes \tilde{U}_q^{\otimes n}$, with q an indeterminate, as in the previous sections. Define $r_{11}^{(i)}, r_{12}^{(i)}, r_{21}^{(i)}, r_{22}^{(i)} \in \tilde{U}_q^{\otimes n}$ by

$$R^{(i)} = \begin{pmatrix} r_{11}^{(i)} & r_{12}^{(i)} \\ r_{21}^{(i)} & r_{22}^{(i)} \end{pmatrix}$$

and $m_{11}^{(i)}, m_{12}^{(i)}, m_{21}^{(i)}, m_{22}^{(i)} \in 1^{\otimes(i-1)} \otimes U_q^{\mathrm{lf}} \otimes 1^{\otimes(n-i)}$ by

$$\mathrm{id}_{V_2} \otimes \big(1^{\otimes(i-1)} \otimes \Phi_1 \otimes 1^{\otimes(n-i)}\big) \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} = \begin{pmatrix} m_{11}^{(i)} & m_{12}^{(i)} \\ m_{21}^{(i)} & m_{22}^{(i)} \end{pmatrix},$$

where as in (6.18) we put

$$\overset{V_2}{M}{}^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} \in \mathrm{End}(V_2) \otimes \mathcal{L}_{0,n}^{(i)}.$$

By Proposition 5.4, $m_{11}^{(i)}, m_{12}^{(i)}, m_{21}^{(i)}, m_{22}^{(i)}$ generate the subalgebra $1^{\otimes(i-1)} \otimes U_q^{\mathrm{lf}} \otimes 1^{\otimes(n-i)}$ of $U_q^{\otimes n}$. These elements satisfy the relations (5.2), with $m_{11}^{(i)}, m_{12}^{(i)}, m_{21}^{(i)}, m_{22}^{(i)}$ replacing a, b, c, d respectively, and they commute with $r_{11}^{(i)}, r_{12}^{(i)}, r_{21}^{(i)}, r_{22}^{(i)}$, since the non trivial tensor components of the latters do not lie on the i-th tensorand of $U_q^{\otimes n}$. Denote $\mathfrak{i}_{in}\colon \tilde{U}_q^{\otimes(n-i)} \to \tilde{U}_q^{\otimes n}$ the identification map with the last $n-i$ tensorands. We have $R^{(i)} = \big(\pi_{V_2} \otimes \mathfrak{i}_{in} \circ \Delta^{(n-i-1)}\big)(R)$ by the relation (2.2). So the formulas (6.15) and (6.16) yield

$$r_{21}^{(i)} = 0, \qquad r_{22}^{(i)} = r_{11}^{(i)-1}, \qquad r_{12}^{(i)}r_{11}^{(i)} = qr_{11}^{(i)}r_{12}^{(i)}, \qquad r_{12}^{(i)} = r_{11}^{(i)}f_{12}^{(i)}$$

and

$$R^{(i)-1} = \begin{pmatrix} r_{11}^{(i)-1} & -qr_{12}^{(i)} \\ 0 & r_{11}^{(i)} \end{pmatrix},$$

where

$$r_{11}^{(i)} = \mathsf{i}_{in} \circ \Delta^{(n-i-1)}(q^{H/2}), \qquad f_{12}^{(i)} = \mathsf{i}_{in} \circ \Delta^{(n-i-1)}((q-q^{-1})F).$$

Therefore

$$R^{(i)}\big(\mathrm{id}_{V_2} \otimes \big(1^{\otimes(i-1)} \otimes \Phi_1 \otimes 1^{\otimes(n-i)}\big)\big) \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} R^{(i)-1}$$
$$= \begin{pmatrix} m_{11}^{(i)} + r_{12}^{(i)}r_{11}^{(i)-1}m_{21}^{(i)} & * \\ r_{11}^{(i)-2}m_{21}^{(i)} & -qr_{11}^{(i)-1}r_{12}^{(i)}m_{21}^{(i)} + m_{22}^{(i)} \end{pmatrix}.$$

Note also that, by definition,

$$\big(\mathrm{id}_{V_2} \otimes \Phi_n\big) \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} = R^{(i)}\big(\mathrm{id}_{V_2} \otimes \big(1^{\otimes(i-1)} \otimes \Phi_1 \otimes 1^{\otimes(n-i)}\big)\big) \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} R^{(i)-1}.$$

Hence

$$\Phi_n\big(c^{(i)}\big) = r_{11}^{(i)-2}m_{21}^{(i)},$$
$$\Phi_n\big(d^{(i)}\big) = -qr_{11}^{(i)-1}r_{12}^{(i)}m_{21}^{(i)} + m_{22}^{(i)}.$$

Let now take the specialization $q = \epsilon$ as above, a primitive l-th root of unity where $l \geq 3$ is odd. We have

$$\Phi_n\big(c^{(i)l}\big) = \big(r_{11}^{(i)-2}m_{21}^{(i)}\big)^l = r_{11}^{(i)-2l}m_{21}^{(i)l}, \tag{7.7}$$
$$\Phi_n\big(d^{(i)l}\big) = \big(-\epsilon r_{11}^{(i)-1}r_{12}^{(i)}m_{21}^{(i)} + m_{22}^{(i)}\big)^l = -r_{11}^{(i)-l}r_{12}^{(i)l}m_{21}^{(i)l} + m_{22}^{(i)l}, \tag{7.8}$$

by using $m_{21}^{(i)}m_{22}^{(i)} = \epsilon^2 m_{22}^{(i)}m_{21}^{(i)}$, the q-binomial formula (see, e.g., [54, Proposition IV.2.2]), and the vanishing at $q = \epsilon$ of the q-Gauss binomial coefficients $[l]_q/[k]_q[l-k]_q$, $0 < k < l$. Now

$$m_{21}^{(i)l} = \big(\epsilon - \epsilon^{-1}\big)^l 1^{\otimes(i-1)} \otimes \big(K^{-1}E\big)^l \otimes 1^{\otimes(n-i)}, \qquad r_{11}^{(i)-2l} = \mathsf{i}_{in} \circ \Delta^{(n-i-1)}\big(K^{-l}\big).$$

These are central elements of $U_\epsilon^{\otimes n}$, so $\Phi_n\big(c^{(i)l}\big)$ is central. As

$$r_{11}^{(i)-l}r_{12}^{(i)l} = f_{12}^{(i)l} = \mathsf{i}_{in} \circ \Delta^{(n-i-1)}\big((\epsilon - \epsilon^{-1})^l F^l\big)$$

and $\mathcal{Z}_0(U_\epsilon)$ is a Hopf algebra, $r_{11}^{(i)-l}r_{12}^{(i)l}$ is a central element of $U_\epsilon^{\otimes n}$. Again, $m_{21}^{(i)l}$ and $m_{22}^{(i)l}$ being central in $U_\epsilon^{\otimes n}$, $\Phi_n\big(d^{(i)l}\big)$ is central.

Finally, recalling that $\mathcal{L}_{0,n}^{(i)}$ is isomorphic to $\mathcal{L}_{0,1}$, by specializing q to ϵ we get

$$d^{(i)l}T_l\big(\omega^{(i)}\big) - d^{(i)2l} - 1 = b^{(i)l}c^{(i)l}. \tag{7.9}$$

We know that $\omega^{(i)} \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ and we just proved that $d^{(i)l}, c^{(i)l} \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$. Therefore, for every $x \in \mathcal{L}_{0,n}$ we get $b^{(i)l}c^{(i)l}x = xb^{(i)l}c^{(i)l} = b^{(i)l}xc^{(i)l}$, i.e., $\big(b^{(i)l}x - xb^{(i)l}\big)c^{(i)l} = 0$, $c^{(i)l}$ is not a zero divisor, we deduce $b^{(i)l}x - xb^{(i)l} = 0$. Hence $b^{(i)l}$ is central in $\mathcal{L}_{0,n}^\epsilon$.

A formula of $\Phi_n\big(b^{(i)l}\big)$ can be obtained as follows. Recall that (6.24) implies

$$\Phi_n\big(\omega^{(i)}\big) = \Omega^{(i)}. \tag{7.10}$$

This can also be checked by using the above formulas:

$$\Phi_n\big(\omega^{(i)}\big) = q\Phi_n\big(a^{(i)}\big) + q^{-1}\Phi_n\big(d^{(i)}\big) = q\big(m_{11}^{(i)} + q^{-1}f_{12}^{(i)}m_{21}^{(i)}\big) + q^{-1}\big(m_{22}^{(i)} - qf_{12}^{(i)}m_{21}^{(i)}\big)$$
$$= qm_{11}^{(i)} + q^{-1}m_{22}^{(i)}.$$

Then, by applying Φ_n to the relation (7.9) and using (7.7), (7.8) and (7.10), one finds

$$\Phi_n\big(b^{(i)l}\big) = -r_{11}^{(i)l}r_{12}^{(i)l}\big(T_l\big(\Omega^{(i)}\big) - 2m_{22}^{(i)l}\big) + r_{11}^{(i)2l}m_{12}^{(i)l} - r_{12}^{(i)2l}m_{21}^{(i)l}. \tag{7.11}$$

We can now achieve the proof. Note that $\mathbb{C}\big[\omega^{(1)},\ldots,\omega^{(n)}\big] = \mathcal{Z}\big(\mathcal{L}_{0,n}^A\big) \otimes_A \mathbb{C}_\epsilon$ by Remark 6.25. The inclusion $\mathcal{Z}\big(\mathcal{L}_{0,n}^A\big) \otimes_A \mathbb{C}_\epsilon \subset \mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)$ is clear, and the natural embedding $\mathcal{L}_{0,n}^\epsilon \to {}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon$ maps $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ into $\mathcal{Z}({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon)$. By the case $n=1$ the elements $\omega^{(i)}$, $b^{(i)l}$, $c^{(i)l}$, $d^{(i)l}$ generate $\mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon{}^{(i)}\big)$, with ideal of relations $\mathcal{I}^{(i)}$. The set of these ideals for $i = 1,\ldots,n$ provide all the relations in $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$, for there are no others in $\mathrm{Im}(\Phi_n)$ (as shows, e.g., Corollary 7.4 below). Therefore one has inclusions

$$\mathbb{C}\big[\omega^{(i)}, b^{(i)l}, c^{(i)l}, d^{(i)l}, i = 1,\ldots,n\big]/\big(\mathcal{I}^{(i)}, i = 1,\ldots,n\big) \subset \mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big) \subset \mathcal{Z}\big({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon\big).$$

The conclusion follows at once, since by their very definition the elements $\delta^{(i)\pm l}$ are central in ${}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon$, and Φ_n maps the algebra generated by them and the left-hand side isomorphically to $U_\epsilon^{\otimes n}$. $\blacksquare$

Analogously to the case $n = 1$, recalling that $\mathcal{L}_{0,1}^1 = \mathcal{O}(G)$ we put:

Definition 7.3. The Frobenius map $\mathrm{Fr}\colon \big(\mathcal{L}_{0,1}^1\big)^{\otimes n} \to \mathcal{Z}\big(\mathcal{L}_{0,n}^\epsilon\big)$ is the homomorphism of algebras given by

$$\mathrm{Fr}\big(\underline{a}^{(i)}\big) = a^{(i)l} + Q_l\big(a^{(i)}, d^{(i)}\big), \qquad \mathrm{Fr}\big(\underline{b}^{(i)}\big) = b^{(i)l}, \qquad \mathrm{Fr}\big(\underline{c}^{(i)}\big) = c^{(i)l}, \qquad \mathrm{Fr}\big(\underline{d}^{(i)}\big) = d^{(i)l}.$$

We shall denote

$$\underline{\overset{V_2}{M}}{}^{(i)} = \begin{pmatrix} \underline{a}^{(i)} & \underline{b}^{(i)} \\ \underline{c}^{(i)} & \underline{d}^{(i)} \end{pmatrix} \qquad \text{and} \qquad \mathrm{Fr}\,\underline{\overset{V_2}{M}}{}^{(i)} = \begin{pmatrix} a^{(i)l} + Q_l\big(a^{(i)}, d^{(i)}\big) & b^{(i)l} \\ c^{(i)l} & d^{(i)l} \end{pmatrix}, \tag{7.12}$$

where we recall that $a^{(i)l} + Q_l\big(a^{(i)}, d^{(i)}\big) = T_l\big(\omega^{(i)}\big) - d^{(i)l}$. We can express $\Phi_n \circ \big(\Phi_1^{\otimes n}\big)^{-1}$ on the center as follows. Set

$$\mathcal{R}^{(i)} = \begin{pmatrix} r_{11}^{(i)l} & r_{12}^{(i)l} \\ 0 & r_{11}^{(i)-l} \end{pmatrix},$$

$$\mathcal{M}^{(i)} = \big(1^{\otimes(i-1)} \otimes \Phi_1 \otimes 1^{\otimes(n-i)}\big)\big(\mathrm{Fr}\,\underline{\overset{V_2}{M}}{}^{(i)}\big) = \begin{pmatrix} T_l\big(\Omega^{(i)}\big) - m_{22}^{(i)l} & m_{12}^{(i)l} \\ m_{21}^{(i)l} & m_{22}^{(i)l} \end{pmatrix},$$

where we use the notations in the proof of Proposition 7.2. These matrices belong to $\mathrm{End}(V_2) \otimes \mathcal{Z}_0(\tilde{U}_\epsilon)^{\otimes n}$ and $\mathrm{End}(V_2) \otimes \mathcal{Z}_0(U_\epsilon)^{\otimes n}$ respectively. Here we note that $\mathcal{Z}(\tilde{U}_\epsilon)$ is generated by Ω, E^l, F^l and $K^{\pm\frac{l}{2}}$ satisfying the relation (7.1), and we define $\mathcal{Z}_0(\tilde{U}_\epsilon)$ as the subalgebra generated by E^l, F^l and $K^{\pm\frac{l}{2}}$.

By using (7.7), (7.8), (7.10) and (7.11) it is easy to check that:

Corollary 7.4. *The map* $\Phi_n \circ \left(\Phi_1^{\otimes n}\right)^{-1} \colon \mathcal{Z}_0(U_\epsilon)^{\otimes n} \to \mathcal{Z}_0(U_\epsilon)^{\otimes n}$ *is given by*

$$\left(\mathrm{id}_{V_2} \otimes \left(\Phi_n \circ (\Phi_1^{\otimes n})^{-1}\right)\right)\left(\mathcal{M}^{(i)}\right) = \mathcal{R}^{(i)}\mathcal{M}^{(i)}\mathcal{R}^{(i)-1}.$$

We will find useful later to have explicit formulas. Let us introduce the following generators of $\mathcal{Z}_0(U_\epsilon)$:

$$x = -\left(\epsilon - \epsilon^{-1}\right)^l E^l K^{-l}, \qquad y = \left(\epsilon - \epsilon^{-1}\right)^l F^l, \qquad z^{\pm 1} = K^{\pm l}. \tag{7.13}$$

Similarly, denote by $\mathcal{Z}_0(\tilde{U}_\epsilon) \subset \tilde{U}_\epsilon$ the subalgebra generated by x, y, z and

$$z'^{\pm 1} = K^{\pm \frac{l}{2}}.$$

For every $a \in \{x, y, z^{\pm 1}\}$ put $a^{(i)} = 1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(n-i)}$. We can view $\mathcal{Z}_0(U_\epsilon)^{\otimes n}$ as a polynomial algebra in the variables $x^{(i)}$, $y^{(i)}$, $\left(z^{\pm 1}\right)^{(i)}$, and $\mathcal{Z}_0(\tilde{U}_\epsilon)^{\otimes n}$ as a polynomial algebra in $x^{(i)}$, $y^{(i)}$, $\left(z'^{\pm 1}\right)^{(i)}$. Then

$$\mathcal{M}^{(i)} = \begin{pmatrix} T_l\left(\Omega^{(i)}\right) - (z^{-1})^{(i)} & y^{(i)} \\ -x^{(i)} & (z^{-1})^{(i)} \end{pmatrix} = \begin{pmatrix} z^{(i)}\left(1 - x^{(i)}y^{(i)}\right) & y^{(i)} \\ -x^{(i)} & (z^{-1})^{(i)} \end{pmatrix}. \tag{7.14}$$

Also,

$$r_{11}^{(i)l} = \left(z'^{-1}\right)^{(i+1)} \cdots \left(z'^{-1}\right)^{(n)}, \tag{7.15}$$

$$r_{12}^{(i)l} = \left(z'^{-1}\right)^{(i+1)} \cdots \left(z'^{-1}\right)^{(n)} \left(y^{(i+1)} + \sum_{j=1}^{n-i-1} \left(z^{-1}\right)^{(i+1)} \cdots \left(z^{-1}\right)^{(i+j)} y^{(i+j+1)}\right) \tag{7.16}$$

yields

$$\Phi_n\left(c^{(i)l}\right) = r_{11}^{(i)-2l} m_{21}^{(i)l} = -x^{(i)}\left(z^{-1}\right)^{(i+1)} \cdots \left(z^{-1}\right)^{(n)},$$

$$\begin{aligned}\Phi_n\left(d^{(i)l}\right) &= m_{22}^{(i)l} - m_{21}^{(i)l} r_{11}^{(i)-l} r_{12}^{(i)l} \\ &= \left(z^{-1}\right)^{(i)} + x^{(i)}\left(y^{(i+1)} + \sum_{j=1}^{n-i-1} \left(z^{-1}\right)^{(i+1)} \cdots \left(z^{-1}\right)^{(i+j)} y^{(i+j+1)}\right).\end{aligned}$$

One can readily express $\Phi_n\left(b^{(i)l}\right) = -r_{11}^{(i)l} r_{12}^{(i)l}\left(\left(T_l(\Omega^{(i)}) - 2m_{22}^{(i)l}\right) + r_{11}^{(i)2l} m_{12}^{(i)l} - r_{12}^{(i)2l} m_{21}^{(i)l}\right)$ as well as a polynomial in the variables $x^{(j)}$, $y^{(j)}$, $\left(z^{\pm 1}\right)^{(j)}$.

In [71], Parshall–Wang showed that the quantum Frobenius homomorphism $\mathcal{O}_1 \to \mathcal{Z}(\mathcal{O}_\epsilon)$ they defined is a morphism of coalgebra. As we now explain, our quantum Frobenius homomorphism $\mathrm{Fr} \colon \mathcal{L}_{0,1}^1 \to \mathcal{Z}(\mathcal{L}_{0,1}^\epsilon)$ satisfies a similar property.

Recall the algebra morphism $\Delta^{(n-1)} \colon \mathcal{L}_{0,1}^A \to \mathcal{L}_{0,n}^A$ (see Proposition 6.18). Denote again by $\Delta^{(n-1)} \colon \mathcal{L}_{0,1}^\epsilon \to \mathcal{L}_{0,n}^\epsilon$ its evaluation at ϵ. For every $1 \leq i_1 < \cdots < i_k \leq n$ we define $\iota_{i_1,\ldots,i_k} \colon \mathcal{L}_{0,k}^\epsilon \to \mathcal{L}_{0,n}^\epsilon$ to be the identification map of the j-th tensorand of $\mathcal{L}_{0,k}^\epsilon$ with the i_j-th tensorand of $\mathcal{L}_{0,n}^\epsilon$, $1 \leq j \leq k$.

Lemma 7.5. *For every* $1 \leq i_1 < \cdots < i_k \leq n$ *we have*

$$\iota_{i_1,\ldots,i_k} \circ \Delta^{(k-1)}\left(\mathrm{Fr}\,\overset{V_2}{\underline{M}}\right) = \mathrm{Fr}\,\overset{V_2}{\underline{M}}{}^{(i_1)} \cdots \mathrm{Fr}\,\overset{V_2}{\underline{M}}{}^{(i_k)}. \tag{7.17}$$

Proof. First we show (7.17) in the case $k = n = 2$, that is

$$\Delta\big(\mathrm{Fr}\,\underset{}{\overset{V_2}{\underline{M}}}\big) = \mathrm{Fr}\,\underset{}{\overset{V_2}{\underline{M}}}{}^{(1)}\,\mathrm{Fr}\,\underset{}{\overset{V_2}{\underline{M}}}{}^{(2)}. \tag{7.18}$$

The commutation relations of $\overset{V_2}{M}{}^{(1)}$ and $\overset{V_2}{M}{}^{(2)}$ being complicated, we cannot compute the matrix components directly (which is the way used in [71] for their quantum Frobenius homomorphism $\mathcal{O}_1 \to \mathcal{Z}(\mathcal{O}_\epsilon)$, where such computations reduce to the q-binomial identity). Instead, we first use the Alekseev map. Indeed, because Φ_2 is an algebra embedding, it is sufficient to show that (7.18) holds after having been composed with Φ_2. We have

$$\Phi_1\big(\mathrm{Fr}\,\overset{V_2}{\underline{M}}\big) = \begin{pmatrix} T_l(\Omega) - K^{-l} & \big(\epsilon - \epsilon^{-1}\big)^l F^l \\ \big(\epsilon - \epsilon^{-1}\big)^l (EK^{-1})^l & K^{-l} \end{pmatrix}.$$

Therefore (7.18) is a consequence of the following four equations in $U_\epsilon^{\otimes 2}$:

$$\Delta\big(K^{-l}\big) = \Phi_2\big(c^{(1)l}b^{(2)l} + d^{(1)l}d^{(2)l}\big),$$
$$\Delta\big(\big(\epsilon - \epsilon^{-1}\big)^l\big(EK^{-1}\big)^l\big) = \Phi_2\big(c^{(1)l}\big(a^{(2)l} + Q_l\big(a^{(2)}, d^{(2)}\big)\big) + d^{(1)l}c^{(2)l}\big),$$
$$\Delta\big(\big(\epsilon - \epsilon^{-1}\big)^l F^l\big) = \Phi_2\big(\big(a^{(1)l} + Q_l\big(a^{(1)}, d^{(1)}\big)\big)b^{(2)l} + b^{(1)l}d^{(2)l}\big),$$
$$\Delta\big(T_l(\Omega) - K^{-l}\big) = \Phi_2\big(\big(a^{(1)l} + Q_l\big(a^{(1)}, d^{(1)}\big)\big)\big(a^{(2)l} + Q_l\big(a^{(2)}, d^{(2)}\big)\big) + b^{(1)l}c^{(2)l}\big).$$

We listed them in order of complexity. The first equation, using the explicit expression of Φ_2 on components, is rewritten as

$$\Delta\big(K^{-l}\big) = \big(\epsilon - \epsilon^{-1}\big)^{2l}\big(K^{-1}E \otimes K^{-1}\big)^l\big(1 \otimes F^l\big)$$
$$+ \big(K^{-1} \otimes 1 - \epsilon\big(\epsilon - \epsilon^{-1}\big)^2 K^{-1}E \otimes F\big)^l\big(1 \otimes K^{-l}\big).$$

This relation holds thanks to the q-binomial identity. The other three relations, although more complicated, are shown similarly by a direct computation and using the expressions (7.7), (7.8), (7.10) and (7.11).

Because the relative commutations relations of $\overset{V_2}{M}{}^{(1)}, \ldots, \overset{V_2}{M}{}^{(k)}$ and $\overset{V_2}{M}{}^{(i_1)}, \ldots, \overset{V_2}{M}{}^{(i_k)}$ for general sequences $i_1 < \cdots < i_k$ are the same, the proof of (7.17) follows immediately from the case where $i_j = j$, for $1 \leq j \leq k$. This, in turn, follows from (7.18) by induction on k. This concludes the proof. $\blacksquare$

The following consequence of the Lemma will be a key tool in Section 9.

Proposition 7.6. *For every $1 \leq i_1 < \cdots < i_k \leq n$ we have*

$$T_l\big(\mathrm{qTr}\,\big(\overset{V_2}{M}{}^{(i_1)} \cdots \overset{V_2}{M}{}^{(i_k)}\big)\big) = \mathrm{Tr}\,\big(\mathrm{Fr}\,\underset{}{\overset{V_2}{\underline{M}}}{}^{(i_1)} \cdots \mathrm{Fr}\,\underset{}{\overset{V_2}{\underline{M}}}{}^{(i_k)}\big).$$

In particular, this element is central in $\mathcal{L}_{0,n}^\epsilon$.

Proof. Let $\iota_{i_1,\ldots,i_k}\colon \mathcal{L}_{0,k}^\epsilon \to \mathcal{L}_{0,n}^\epsilon$ be the identification map of the j-th tensorand of $\mathcal{L}_{0,k}^\epsilon$ with the i_j-th tensorand of $\mathcal{L}_{0,n}^\epsilon$, $1 \leq j \leq k$. We have

$$T_l\big(\mathrm{qTr}\,\big(\overset{V_2}{M}{}^{(i_1)} \cdots \overset{V_2}{M}{}^{(i_k)}\big)\big) = T_l\big(\mathrm{qTr}\,\big(\big(\mathrm{id} \otimes \big(\iota_{i_1,\ldots,i_k} \circ \Delta^{(k-1)}\big)\big)\big(\overset{V_2}{M}\big)\big)\big)$$

$$= T_l\big(\big(\iota_{i_1,\ldots,i_k} \circ \Delta^{(k-1)}\big)\big(\mathrm{qTr}\,\big(\overset{V_2}{M}\big)\big)\big)$$

$$= \big(\iota_{i_1,\ldots,i_k} \circ \Delta^{(k-1)}\big)\big(T_l\big(\mathrm{qTr}\,\big(\overset{V_2}{M}\big)\big)\big)$$

$$= \left(\iota_{i_1,\dots,i_k} \circ \Delta^{(k-1)} \right) \left(T_l(\omega) \right).$$

In the first equality we used the formula (6.19), the second and fourth equalities follow from definitions, and the third comes from the fact that $\Delta^{(k-1)} \colon \mathcal{L}_{0,1}^A \to \mathcal{L}_{0,k}^A$ is a homomorphism of algebras. By the identity (7.6) and the fact that $\mathcal{Z}_0(U_\epsilon) \cong \mathbb{C}\big[b^l, c^l, d^{\pm l}\big]$ is a Hopf subalgebra of U_ϵ, we have $\Delta^{(k-1)}(T_l(\omega)) \in \mathcal{Z}(\mathrm{loc}\mathcal{L}_{0,1}^\epsilon)^{\otimes k}$. This and Proposition 7.2 imply that the above element is central in $\mathcal{L}_{0,n}^\epsilon$. Moreover

$$
\begin{aligned}
\iota_{i_1,\dots,i_k} \circ \Delta^{(k-1)}(T_l(\omega)) &= \iota_{i_1,\dots,i_k} \circ \Delta^{(k-1)}\big(\mathrm{Tr}\,\big(\mathrm{Fr}\,\underline{\overset{V_2}{M}}\big)\big) \\
&= \mathrm{Tr}\,\Big(\big(\mathrm{id} \otimes \iota_{i_1,\dots,i_k} \circ \Delta^{(k-1)}\big)\big(\mathrm{Fr}\,\underline{\overset{V_2}{M}}\big)\Big) \\
&= \mathrm{Tr}\,\big(\mathrm{Fr}\,\underline{\overset{V_2}{M}}{}^{(i_1)} \cdots \mathrm{Fr}\,\underline{\overset{V_2}{M}}{}^{(i_k)}\big),
\end{aligned}
$$

where the first equality follows from the observations we made before Proposition 7.2, and the others from the previous lemma. $\blacksquare$

7.2 $\mathcal{G}$-invariant central elements and SL$(2,\mathbb{C})$-characters

We are going to relate $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ with the algebra of regular functions on the variety of SL$(2,\mathbb{C})$-characters of the sphere with $n+1$ punctures, endowed with the Atiyah–Bott–Goldman Poisson structure. This is achieved in Section 7.2.2. To this aim we recall a few preliminary results in the next section.

7.2.1 The quantum coadjoint action for U_ϵ

We refer to [27, 28, 30] for details about the material discussed in this section. It can be formulated for any of the quantum groups $U_\epsilon(\mathfrak{g})$, but we restrict to $U_\epsilon(\mathfrak{sl}(2))$ as we shall need Proposition 7.2 in Section 7.2.2.

Consider the sets $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ and $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ of algebra homomorphisms from $\mathcal{Z}(U_\epsilon)$ and $\mathcal{Z}_0(U_\epsilon)$ to $\mathbb{C}$, respectively. They are affine algebraic sets. An element of $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ is called a *central character* of U_ϵ. The inclusion $\mathcal{Z}_0(U_\epsilon) \subset \mathcal{Z}(U_\epsilon)$ induces a regular (restriction) map

$$\tau \colon \ \mathrm{Spec}(\mathcal{Z}(U_\epsilon)) \longrightarrow \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon)). \tag{7.19}$$

Since $\mathcal{Z}_0(U_\epsilon)$ is a polynomial algebra, any $\chi \in \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ is entirely defined by its values $(x_\chi, y_\chi, z_\chi) \in \mathbb{C}^2 \times \mathbb{C}^*$ on the tuple (x, y, z) of generators of $\mathcal{Z}_0(U_\epsilon)$ defined in (7.13). By (7.1)–(7.4), any $\chi \in \mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ is entirely defined by its values $(x_\chi, y_\chi, z_\chi, \Omega_\chi) \in \mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{C}$ on the tuple (x, y, z, Ω) of generators of $\mathcal{Z}(U_\epsilon)$, solutions to one of the equivalent equations

$$\prod_{j=1}^{l}(\Omega_\chi \mp c_j) = -x_\chi y_\chi z_\chi + z_\chi + z_\chi^{-1} \mp 2,$$

where $c_j = \epsilon^j + \epsilon^{-j}$. Hence τ has degree l, and $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ is a branched covering of $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon)) = \mathbb{C}^2 \times \mathbb{C}^*$ of degree l, a hypersurface in $\mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{C}$ with quadratic singularities at the points $(0, 0, \pm 1, \pm c_j)$, $j = 1, \dots, (l-1)/2$.

Because $\mathcal{Z}_0(U_\epsilon)$ is a commutative Hopf algebra, $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ has a canonical group structure defined dually by

$$\chi_1 \chi_2(u) := (\chi_1 \otimes \chi_2)\Delta(u), \qquad \chi_1^{-1}(u) := \chi_1(S(u)), \qquad e(u) := \varepsilon(u) \tag{7.20}$$

for any $u \in \mathcal{Z}_0(U_\epsilon)$ and $\chi_1, \chi_2 \in \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$, where $e \in \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ is the identity element. In formulas:

$$x_{\chi_1\chi_2} = x_{\chi_1} + z_{\chi_1}^{-1} x_{\chi_2}, \qquad y_{\chi_1\chi_2} = y_{\chi_1} + y_{\chi_2} z_{\chi_1}^{-1}, \qquad z_{\chi_1\chi_2} = z_{\chi_1} z_{\chi_2},$$

$$x_{\chi^{-1}} = -z_\chi x_\chi, \qquad y_{\chi^{-1}} = -y_\chi z_\chi, \qquad z_{\chi^{-1}} = z_\chi^{-1},$$

$$x_e = 0, \qquad y_e = 0, \qquad z_e = 1.$$

This can be formulated as follows. Put $G = \mathrm{SL}(2,\mathbb{C})$, and let G^* be the group formed by the pairs of matrices

$$\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a^{-1} & 0 \\ c & a \end{pmatrix} \right) \in \mathrm{SL}(2,\mathbb{C})^{\mathrm{op}} \times \mathrm{SL}(2,\mathbb{C})^{\mathrm{op}},$$

where $\mathrm{SL}(2,\mathbb{C})^{\mathrm{op}}$ is $\mathrm{SL}(2,\mathbb{C})$ endowed with the opposite multiplication. Set

$$\psi \left(\begin{pmatrix} z' & z'y \\ 0 & z'^{-1} \end{pmatrix}, \begin{pmatrix} z'^{-1} & 0 \\ z'x & z' \end{pmatrix} \right) = (x, y, z'^2).$$

Identifying $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon)))$ with $\mathbb{C}^2 \times \mathbb{C}^*$ by mapping χ to (x_χ, y_χ, z_χ) defined as above, it is readily checked that this defines a surjective morphism of algebraic groups

$$\psi\colon\ G^* \to \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon)))$$

with kernel the subgroup generated by $-(I, I)$, where I is the 2-by-2 identity matrix. Put

$$\bar{G}^* = G^*/\{\pm(I, I)\}.$$

We will denote the quotient isomorphism by

$$\bar{\psi}\colon\ \bar{G}^* \to \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))).$$

Let us endow the ring of regular functions $\mathcal{O}(G)$ with the Sklyanin–Drinfeld Poisson bracket $\{\,,\,\}$, associated to the classical r-matrix

$$\mathfrak{r} = \frac{1}{4} H \otimes H + E \otimes F \in \mathfrak{g} \otimes \mathfrak{g}.$$

Recall that it can be given the following expression (see, e.g., [77], or [7, 25] for a setup close to ours). First note that it is entirely determined by its values on the matrix coefficients (coordinate functions) $l_{11}, l_{12}, l_{21}, l_{22}$ of the fundamental representation of G on $\mathbb{C}^2$. Put

$$L = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} = \sum_{r,s=1}^{2} E_r^s \otimes l_{rs} \in \mathrm{End}\left(\mathbb{C}^2\right) \otimes \mathcal{O}(G). \tag{7.21}$$

Denote by $\{ \overset{1}{L}, \overset{2}{L} \}$ and $\overset{1}{L}\overset{2}{L}$ the tensors in $\mathrm{End}\left(\mathbb{C}^2\right) \otimes \mathrm{End}\left(\mathbb{C}^2\right) \otimes \mathcal{O}(G)$ defined by

$$\{ \overset{1}{L}, \overset{2}{L} \} = \sum_{r,s,t,u=1}^{2} E_r^s \otimes E_t^u \otimes \{l_{rs}, l_{tu}\}, \qquad \overset{1}{L}\overset{2}{L} = \sum_{r,s,t,u=1}^{2} E_r^s \otimes E_t^u \otimes l_{rs} l_{tu}$$

and as usual put $\overset{1}{L} = \sum_{r,s=1}^{2} E_r^s \otimes 1 \otimes l_{rs}$, $\overset{2}{L} = \sum_{t,u=1}^{2} 1 \otimes E_t^u \otimes l_{tu}$. The r-matrix in the fundamental representation is

$$\mathfrak{r} = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{End}\left(\mathbb{C}^2\right) \otimes \mathrm{End}\left(\mathbb{C}^2\right),$$

which we identify with $\mathfrak{r} \otimes 1 \in \mathrm{End}\left(\mathbb{C}^2\right) \otimes \mathrm{End}\left(\mathbb{C}^2\right) \otimes \mathcal{O}(G)$. Then the bracket $\{\,,\,\}$ on $\mathcal{O}(G)$ is defined by the identity

$$\{ \overset{1}{L}, \overset{2}{L} \} = [\mathfrak{r}, \overset{1}{L}\overset{2}{L}] = \mathfrak{r}\,\overset{1}{L}\overset{2}{L} - \overset{1}{L}\overset{2}{L}\,\mathfrak{r}.$$

The group G with the bracket $\{\,,\,\}$ is a Poisson–Lie group. As such, it admits a dual Poisson–Lie group, which is precisely the group G^* defined as above. Its Poisson bracket $\{\,,\,\}_{G^*}$ is defined by

$$\{ \overset{1}{L}_\pm, \overset{2}{L}_\pm \}_{G^*} = [\mathfrak{r}, \overset{1}{L}_\pm \overset{2}{L}_\pm], \qquad \{ \overset{1}{L}_+, \overset{2}{L}_- \}_{G^*} = [\mathfrak{r}, \overset{1}{L}_+ \overset{2}{L}_-], \tag{7.22}$$

where (L_+, L_-) is the pair of matrices of coordinate functions on $G^* \subset \mathrm{SL}(2,\mathbb{C})^{\mathrm{op}} \times \mathrm{SL}(2,\mathbb{C})^{\mathrm{op}}$. We will denote by $\{\,,\,\}_{\bar{G}^*}$ the induced bracket on $\bar{G}^*$. The push-forward $\psi_*\{\,,\,\}_{G^*}$ is a Poisson bracket on $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ that can be defined directly via the quantum coadjoint action, that we now recall.

The specialization morphism $ev_\epsilon \colon U_A \otimes \mathbb{C}[q, q^{-1}] \to U_\epsilon$, $q \mapsto \epsilon$, is surjective, with kernel the subalgebra $(q^l - q^{-l})U_A \otimes \mathbb{C}[q, q^{-1}]$. Given an element $x \in U_\epsilon$, let us denote by $\tilde{x} \in ev_\epsilon^{-1}(x)$ any preimage of x. For every $a \in \mathcal{Z}(U_\epsilon)$, $u \in U_\epsilon$ we have $[\tilde{a}, \tilde{u}] = \tilde{a}\tilde{u} - \tilde{u}\tilde{a} \in (q^l - q^{-l})U_A \otimes \mathbb{C}[q, q^{-1}]$. So, let us put

$$D_a(u) = -\lim_{q \to \epsilon} \frac{[\tilde{a}, \tilde{u}]}{l(q^l - q^{-l})}, \qquad a \in \mathcal{Z}_0(U_\epsilon). \tag{7.23}$$

It is easy to check that the maps $D_a \colon U_\epsilon \to U_\epsilon$ are well-defined (they do not depend on the choices of $\tilde{a}$ and $\tilde{u}$), and that they are derivations of U_ϵ preserving $\mathcal{Z}_0(U_\epsilon)$ and $\mathcal{Z}(U_\epsilon)$. Hence they define algebraic vector fields on $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$. Since Ω is central in $U_q(\mathfrak{sl}(2))$, D_a is the zero map when $a \in \mathbb{C}[\Omega]$.

Let us introduce the elements e, f defined in [27] by

$$e = -xz = (\epsilon - \epsilon^{-1})^l E^l, \qquad f = -yz = -(\epsilon - \epsilon^{-1})^l F^l K^l. \tag{7.24}$$

Direct computations from the definition and the fact that $D_e\Omega = 0$ imply the formulas:

$$D_z(K) = 0, \qquad D_z(E) = -\frac{1}{l}zE, \qquad D_z(F) = \frac{1}{l}zF, \tag{7.25}$$

$$D_e(K) = \frac{1}{l}eK, \qquad D_e(E) = 0, D_e(F) = -\frac{1}{l}(\epsilon - \epsilon^{-1})^{l-1}[K;1]E^{l-1}, \tag{7.26}$$

$$D_y(K) = -\frac{1}{l}yK, \qquad D_y(F) = 0, D_y(E) = \frac{1}{l}(\epsilon - \epsilon^{-1})^{l-1}[K;-1]F^{l-1}, \tag{7.27}$$

where $[K;r] = \frac{K\epsilon^r - K^{-1}\epsilon^{-r}}{\epsilon - \epsilon^{-1}}$, $r \in \mathbb{Z}$.

A Poisson bracket $\{\,,\,\}_{\mathrm{QCA}}$ is defined on $\mathcal{Z}(U_\epsilon)$ by

$$\{a, b\}_{\mathrm{QCA}} = D_a(b).$$

With this Poisson bracket $\mathcal{Z}(U_\epsilon)$ is a Poisson algebra, $\mathcal{Z}_0(U_\epsilon)$ being a Poisson ideal. The Poisson structure on $\mathcal{Z}_0(U_\epsilon)$ is completely defined by the formulas

$$\{y, x\}_{\mathrm{QCA}} = -1 + xy + z^{-2}, \qquad \{z, x\}_{\mathrm{QCA}} = -zx, \qquad \{z, y\}_{\mathrm{QCA}} = yz \tag{7.28}$$

which can be straithforwardy checked using the definition. From them it is easy to show that the Poisson center of $\mathcal{Z}_0(U_\epsilon)$ is the commutative algebra generated by $-xyz + z + z^{-1}$.

Let us introduce the family of automorphisms of $U_q(\mathfrak{sl}(2))$ defined as follows. Let $r \in \mathbb{Z}$, and τ_r, T_r the automorphisms of $U_q(\mathfrak{sl}(2))$ defined by

$$\tau_r(K) = K, \qquad \tau_r(E) = K^r E, \qquad \tau_r(F) = FK^{-r},$$
$$T_r(K) = K^{-1}, \qquad T_r(E) = -FK^{-r}, \qquad T_r(F) = -K^r E.$$

Note that $T_r = \tau_r \circ T_0$ and T_1 is the braid group automorphism. We have

$$T_1(x) = y, \qquad T_1(y) = z^2 x.$$

These automorphisms of $U_q(\mathfrak{sl}(2))$ define automorphisms of the specialisation U_ϵ and we will keep the same notation for them. If T is any automorphism of the type T_r or τ_r, it is easy to show from the definition of D_a for $a \in \mathcal{Z}(U_\epsilon)$ that

$$D_{T(a)} \circ T = T \circ D_a.$$

Denote by G^0 the *big cell* of $G = \mathrm{SL}(2, \mathbb{C})$. It consists of the matrices with non vanishing lower right entry, and satisfies $G^0 = B_+ B_-$, where B_+ and B_- are the subgroups of G of upper and lower triangular matrices. We have an unramified 2-fold covering

$$\sigma : \quad \begin{array}{ccc} G^* & \longrightarrow & G^0, \\ (b_+, b_-) & \longmapsto & b_+ b_-^{-1}. \end{array}$$

It induces a diffeomorphism

$$\bar{\sigma} : \quad \bar{G}^* \to G^0.$$

Setting $z := z'^2$ and

$$\mathcal{M} = \sigma \left(\begin{pmatrix} z' & z'y \\ 0 & z'^{-1} \end{pmatrix}, \begin{pmatrix} z'^{-1} & 0 \\ z'x & z' \end{pmatrix} \right) \tag{7.29}$$

we have

$$\mathcal{M} = \begin{pmatrix} z - zxy & y \\ -x & z^{-1} \end{pmatrix}.$$

Consider the Poisson bracket $\{\,,\,\}_{\mathrm{FR}}$ on $\mathcal{O}(G)$ defined by

$$\left\{ \overset{1}{L}, \overset{2}{L} \right\}_{\mathrm{FR}} = \mathfrak{r}\, \overset{1}{L}\overset{2}{L} - \overset{1}{L}\overset{2}{L}\, \mathfrak{r}' + \overset{2}{L}\, \mathfrak{r}'\, \overset{1}{L} - \overset{1}{L}\, \mathfrak{r}\, \overset{2}{L},$$

where $\mathfrak{r}'$ is $\mathfrak{r}$ post-composed with the flip map $a \otimes b \mapsto b \otimes a$. The bracket $\{\,,\,\}_{\mathrm{FR}}$ has been introduced in [77] and generalized in the work of Fock–Rosly [43] (which explains our notation). Note that $\mathcal{O}(G^0)$ is the localization of the algebra $\mathcal{O}(G)$ with respect to the matrix coefficient l_{22} in (7.21). As a consequence, the Poisson bracket $\{,\}_{\mathrm{FR}}$ being quadratic, it can be extended to $\mathcal{O}(G^0)$.

The next result sums up the relationships between the brackets $\{\,,\,\}_{\mathrm{FR}}$, $\{\,,\,\}_{G^*}$, and $\{\,,\,\}_{\mathrm{QCA}}$:

Theorem 7.7.

1. *The map $\bar{\psi} : (\bar{G}^*, \{\,,\,\}_{\bar{G}^*}) \to (\mathrm{Spec}((\mathcal{Z}_0(U_\epsilon)), \{\,,\,\}_{\mathrm{QCA}})$ is an isomorphism of complex Poisson–Lie groups [28, 30].*

2. *The map $\bar{\sigma} : (\bar{G}^*, \{\,,\,\}_{\bar{G}^*}) \to (G^0, \{\,,\,\}_{\mathrm{FR}})$ is a diffeomorphism of complex Poisson manifolds [77].*

The first claim is proved in [28, 30] for the simply-connected quantum groups $\tilde{U}_\epsilon(\mathfrak{g})$, where $\mathfrak{g}$ is a complex finite dimensional simple Lie algebra (see Remark 7.9 below). We state it rather in the case of $U_\epsilon(\mathfrak{sl}(2))$, the adjoint quantum group for $\mathfrak{g} = \mathfrak{sl}(2)$, where it follows from straighforward computations using the formulas (7.22) and (7.28). The second statement follows as well from straighforward computations. Namely, by using the identities (7.22) and $\sigma^*(L) = L_+ L_-^{-1}$, and the fact that $\mathfrak{r} + \mathfrak{r}'$ is ad-invariant, one can identify the formulas of $\{\,,\,\}_{\mathrm{FR}}$ and

$$\sigma_*(\{\,,\,\}_{G^*})(L \otimes L) = \{\sigma^*(\overset{1}{L}), \sigma^*(\overset{2}{L})\}_{G^*}.$$

We leave the verifications to the reader.

The next statement summarizes the results of De Concini–Kac–Procesi on the quantum coadjoint action. Recall the elements $e = -xz, f = -yz$ of $\mathcal{Z}_0(U_\epsilon)$ (see (7.24)). Let us identify $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ with G^0 using the map $\bar{\sigma} \circ \bar{\psi}^{-1}$ of Theorem 7.7, and hence the derivations D_e, D_f, D_z of $\mathcal{Z}_0(U_\epsilon)$ with algebraic vector fields on G^0 (and hence on G). Denote by $\underline{H}, \underline{X}, \underline{Y}$ the left-invariant vector fields on G associated to the generators H, X, Y of $\mathfrak{sl}(2)$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Denote by $\hat{\mathcal{Z}}_0(U_\epsilon)$ the vector space of power series in the generators x, y, and $z^{\pm 1}$ of $\mathcal{Z}_0(U_\epsilon)$ whose sums converge when evaluated on any $\chi \in \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon)) \cong \mathbb{C}^2 \times \mathbb{C}^*$, $\chi = (x_\chi, y_\chi, z_\chi)$ (thus defining holomorphic functions). Set

$$\hat{U}_\epsilon = U_\epsilon \otimes_{\mathcal{Z}_0(U_\epsilon)} \hat{\mathcal{Z}}_0(U_\epsilon), \qquad \hat{\mathcal{Z}}(U_\epsilon) = \mathcal{Z}(U_\epsilon) \otimes_{\mathcal{Z}_0(U_\epsilon)} \hat{\mathcal{Z}}_0(U_\epsilon).$$

Theorem 7.8 ([28, 30])**.**

(a) *We have $D_z = z\underline{H}/2$, $D_e = -z\underline{Y}$, $D_f = z\underline{X}$.*

(b) *For every $t \in \mathbb{C}$ the power series $\exp(tD_e)$, $\exp(tD_f)$ converge to automorphisms of $\hat{U}_\epsilon$ preserving $\hat{\mathcal{Z}}_0(U_\epsilon)$ and $\hat{\mathcal{Z}}(U_\epsilon)$, and fixing $\mathbb{C}[\Omega]$.*

Denote by $\mathcal{G}_{\mathrm{DCK}}$ the subgroup of $\mathrm{Aut}(\hat{U}_\epsilon)$ generated by the 1-parameter groups $(\exp(tD_e))_{t \in \mathbb{C}}$ and $(\exp(tD_f))_{t \in \mathbb{C}}$. It acts dually by holomorphic transformations on $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ and $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ by defining $g.\chi$ on $u \in \hat{\mathcal{Z}}_0(U_\epsilon)$, for every $g \in \mathcal{G}$ and $\chi \in \mathrm{Spec}(\mathcal{Z}(U_\epsilon))$, by $u(g.\chi) = (g^{-1}.u)(\chi)$. Then:

(c) *The diffeomorphism $\bar{\sigma} \circ \bar{\psi}^{-1}\colon \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon)) \to G^0$ maps the action of $\mathcal{G}_{\mathrm{DCK}}$ on the tangent spaces of $(0, 0, \pm 1) \in \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ to the coadjoint action of G on $\mathfrak{sl}(2, \mathbb{C})^*$, the tangent spaces of $\pm I \in G^0$.*

(d) *For any conjugacy class Γ in $\mathrm{SL}(2, \mathbb{C})$, $(\bar{\psi} \circ \bar{\sigma}^{-1})(\Gamma \cap G^0)$ is a (non empty) $\mathcal{G}_{\mathrm{DCK}}$-orbit in $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$, and τ^{-1} of this orbit has l connected components, all of whose are $\mathcal{G}_{\mathrm{DCK}}$-orbits in $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$.*

(e) *An element $a \in \mathcal{Z}(U_\epsilon)$ is $\mathcal{G}_{\mathrm{DCK}}$-invariant if and only if $a \in \mathbb{C}[\Omega]$. Dually, the sets of fixed points of the action of $\mathcal{G}_{\mathrm{DCK}}$ on $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ and $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ are respectively $(0, 0, \pm 1) := (\bar{\psi} \circ \bar{\sigma}^{-1})(\{\pm I\})$ and*

$$\mathcal{D} := \tau^{-1}((0, 0, \pm 1)) = \{(0, 0, \pm 1, \pm c_j), j = 1, \ldots, (l-1)/2\}.$$

(f) *The $\mathcal{G}_{\mathrm{DCK}}$-orbits on $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ and $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ are the symplectic leaves of $\{\,,\,\}_{\mathrm{QCA}}$.*

Remark 7.9. In the statements of Theorems 7.7 and 7.8 we use our conventions, which differ from those in [28, 30] in the following ways:

(i) Theorem 7.7(1) for the simply connected quantum group $\tilde{U}_\epsilon(\mathfrak{sl}(2))$ instead of $U_\epsilon(\mathfrak{sl}(2))$, as in [28, 30], states an isomorphism of $(G^*, \{\,,\,\}_{G^*})$ with $(\mathrm{Spec}(\mathcal{Z}_0(\tilde{U}_\epsilon)), \{\,,\,\}_{\mathrm{QCA}})$; as we take opposite comultiplications on $\tilde{U}_\epsilon(\mathfrak{sl}(2))$, we get opposite multiplications of G^* (whence $\bar{G}^*$ for $U_\epsilon(\mathfrak{sl}(2))$).

(ii) Our derivations D_a differ by a sign, which we introduce in order to get the equality of $\bar{\sigma}_*\{\,,\,\}_{\bar{G}^*}$ with $\{\,,\,\}_{\mathrm{FR}}$ in Theorem 7.7(2).

(iii) In [27, 28, 30], the matrix $\mathcal{M}$ in (7.29) is different because they use the opposite coproduct on $\tilde{U}_\epsilon(\mathfrak{sl}(2))$; their braid group automorphism T is in our notation T_{-1}, which satisfies $T_{-1}(e) = f$.

It is easily checked that the identities (7.28) imply

$$D_z\mathcal{M} = \left[\frac{z}{2}\underline{H}, \mathcal{M}\right], \qquad D_x\mathcal{M} = \left[-z^{-1}\underline{X}, \mathcal{M}\right], \qquad D_y\mathcal{M} = \left[z^{-1}\underline{Y}, \mathcal{M}\right], \tag{7.30}$$

where $\mathcal{M}$ is defined by (7.29). Hence we get

$$D_z = \frac{z}{2}\underline{H}, \qquad D_x = -z^{-1}\underline{X}, \qquad D_y = z^{-1}\underline{Y}.$$

This is the content of Theorem 7.8(a) expressed in terms of the derivations D_z, D_x, D_y. In Theorem 7.8(b), U_ϵ is enlarged to $\hat{U}_\epsilon$ in order to define the automorphisms $\exp(tD_e)$, $\exp(tD_f)$ because the derivations D_e, D_f are not nilpotent.

For reasons that will be explained in Remark 7.15, we will need a result analogous to Theorem 7.8 but based on different derivations. This leads to substantial differences in the details, so we give all proofs.

Consider the following derivations on U_ϵ:

$$\mathcal{E} = zD_x, \qquad \mathcal{F} = -zD_y, \qquad \mathcal{H} = -2z^{-1}D_z. \tag{7.31}$$

Lemma 7.10. *The derivations $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{H}$ satisfy the following commutation relations, generating therefore the Lie algebra $\mathfrak{sl}(2)$:*

$$[\mathcal{H}, \mathcal{E}] = 2\mathcal{E}, \qquad [\mathcal{H}, \mathcal{F}] = -2\mathcal{F}, \qquad [\mathcal{E}, \mathcal{F}] = \mathcal{H}. \tag{7.32}$$

Proof. Straightforward computation using (7.28). ∎

For any $\mathbb{C}$-algebra A and endomorphism $\mathcal{V} \in \mathrm{End}(A)$ we can define

$$\exp(t\mathcal{V}): \quad A \quad \longrightarrow \quad A[[t]],$$
$$a \quad \longmapsto \quad \sum_{n\geq 0} \frac{t^n}{n!}\mathcal{V}^n(a).$$

If $\mathcal{V}$ is a derivation, $\exp(t\mathcal{V})$ is a morphism of algebras. It admits a unique extension (by continuity for the t-adic topology) $\exp(t\mathcal{V})\colon A[[t]] \to A[[t]]$ which is an automorphism with inverse $\exp(-t\mathcal{V})$.

We now give expressions of $\exp(t\mathcal{F})$, $\exp(t\mathcal{E})\colon U_\epsilon \to U_\epsilon[[t]]$. For every $\alpha \in \mathbb{C}$ consider the following elements of $\mathbb{C}[[t]]$:

$$(1+t)^\alpha = \sum_{n\geq 0} \frac{1}{n!} t^n \alpha(\alpha-1)\cdots(\alpha-n+1),$$

$$\psi_\alpha(t) = \sum_{n\geq 1} \frac{(-1)^n}{n!} t^{n-1}\alpha(\alpha-1)\cdots(\alpha-n+1) = \frac{(1-t)^\alpha - 1}{t}.$$

For any $a \in \mathcal{Z}_0(U_\epsilon)$, we can similarly define elements $(1+ta)^\alpha$, $\psi_\alpha(ta) \in \mathcal{Z}_0(U_\epsilon)[[t]]$.

Lemma 7.11. *The action of* $\exp(t\mathcal{F})$ *is given by:*

$$\exp(t\mathcal{F})(K) = (1 - tyz)^{-1/l}K, \qquad \exp(t\mathcal{F})\big(K^{-1}\big) = (1 - tyz)^{1/l}K^{-1},$$
$$\exp(t\mathcal{F})(F) = F,$$
$$\exp(t\mathcal{F})(E) = E - \big(\epsilon - \epsilon^{-1}\big)^{l-2}\big(K\epsilon^{-1}tz\psi_{-1/l}(tyz) + K^{-1}\epsilon tz\psi_{1/l}(tyz)\big)F^{l-1}.$$

Proof. From (7.27) we have

$$\mathcal{F}(K) = \frac{zy}{l}K, \qquad \mathcal{F}(F) = 0, \qquad \mathcal{F}(E) = -\big(\epsilon - \epsilon^{-1}\big)^{l-1}\frac{z}{l}[K, -1]F^{l-1}. \tag{7.33}$$

Iterating this, a straightforward computation proves the lemma. ∎

Lemma 7.12. *The action of* $\exp(t\mathcal{E})$ *is given by*

$$\exp(t\mathcal{E})(K) = (1 - txz)^{-1/l}K, \qquad \exp(t\mathcal{E})\big(K^{-1}\big) = (1 - txz)^{1/l}K^{-1},$$
$$\exp(t\mathcal{E})\big(K^{-1}E\big) = K^{-1}E,$$
$$\exp(t\mathcal{E})(FK) = FK + \big(\epsilon - \epsilon^{-1}\big)^{l-2}\big(K\epsilon tz\psi_{-1/l}(txz) + K^{-1}\epsilon^{-1}tz\psi_{1/l}(txz)\big)\big(K^{-1}E\big)^{l-1}.$$

Proof. This is a little trickier than the previous proof, because it is D_y which appears in (7.27) and $\mathcal{E}$ involves D_x. We therefore use the automorphism T_{-1} which is such that $T_{-1}(y) = -x$. As a result, by applying it to (7.33), we obtain

$$D_x\big(K^{-1}\big) = -\frac{x}{l}K^{-1}, \qquad D_x\big(K^{-1}E\big) = 0, \qquad D_x(FK) = \big(\epsilon - \epsilon^{-1}\big)^{l-1}\frac{1}{l}[K, 1]\big(K^{-1}E\big)^{l-1}$$

which imply

$$\mathcal{E}(K) = \frac{zx}{l}K, \qquad \mathcal{E}\big(K^{-1}E\big) = 0, \qquad \mathcal{E}(FK) = \big(\epsilon - \epsilon^{-1}\big)^{l-1}\frac{z}{l}[K, 1]\big(K^{-1}E\big)^{l-1}.$$

Iterating this, a straightforward computation proves the lemma. ∎

Because $\mathcal{E}$ and $\mathcal{F}$ leave invariant $\mathcal{Z}(U_\epsilon)$ and $\mathcal{Z}_0(U_\epsilon)$ they define maps $\exp(t\mathcal{E})$, $\exp(t\mathcal{F})$: $\mathcal{Z}(U_\epsilon) \to \mathcal{Z}(U_\epsilon)[[t]]$ and $\mathcal{Z}_0(U_\epsilon) \to \mathcal{Z}_0(U_\epsilon)[[t]]$.

Proposition 7.13. *We have*

$$\exp(t\mathcal{E})(x) = x, \qquad \exp(t\mathcal{F})(y) = y, \qquad \exp(t\mathcal{E})(\Omega) = \exp(t\mathcal{F})(\Omega) = \Omega,$$
$$\exp(t\mathcal{E})(z) = (1 - tzx)^{-1}z, \qquad \exp(t\mathcal{E})(y) = y + t\big(-xyz + z - z^{-1}\big) + t^2x,$$
$$\exp(t\mathcal{F})(z) = (1 - tzy)^{-1}z, \qquad \exp(t\mathcal{F})(x) = x + t\big(-xyz + z - z^{-1}\big) + t^2y.$$

Proof. Rather than using the explicit action of $\exp(t\mathcal{E})$ and $\exp(t\mathcal{F})$ on U_ϵ, we remark that from (7.30) we have

$$\exp(t\mathcal{E})(\mathcal{M}) = \exp(-tX)\mathcal{M}\exp(tX), \qquad \exp(t\mathcal{F})(\mathcal{M}) = \exp(-tY)\mathcal{M}\exp(tY). \tag{7.34}$$

The announced expressions follow by writing the matrix elements of these equations. ∎

Let $\mathcal{G}$ be the free product of $(\mathbb{C}, +)$ with itself; it can be equivalently seen as the group generated by elements ϕ_s, ψ_s, where $s \in \mathbb{C}$, with the relations $\phi_s\phi_{s'} = \phi_{s+s'}$, $\psi_s\psi_{s'} = \psi_{s+s'}$ for every $s, s' \in \mathbb{C}$.

We can define a partial action of $\mathcal{G}$ on $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ and on $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$, in the sense of [37]. Let $s \in \mathbb{C}$, and denote by $D(\phi_s)$ the set of $\chi \in \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ such that for all $u \in \mathcal{Z}_0(U_\epsilon)$

the series $(\exp(s\mathcal{E})(u))(\chi)$ is normally convergent in a small neighborhood of s. Equivalently $D(\phi_s) = \{\chi \in \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon)), |sx_\chi z_\chi| < 1\}$. We define an action of ϕ_s on $D(\phi_s)$ by

$$u(\phi_s.\chi) = (\exp(-s\mathcal{E})(u))(\chi). \tag{7.35}$$

Similarly, put $D(\psi_s) = \{\chi \in \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon)), |sy_\chi z_\chi| < 1\}$ and define an action of ψ_s on $D(\psi_s)$ by

$$u(\psi_s.\chi) = (\exp(-s\mathcal{F})(u))(\chi). \tag{7.36}$$

The domains $D(\phi_s)$ (resp. $D(\psi_s)$) cover $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ as s varies in $\mathbb{C}$. By results of Exel ([37] and [36, Examples 1 and 4]), the set formed by the local actions (7.35) (resp. (7.36)) on the domains $D(\phi_s)$ (resp. $D(\psi_s)$) defines a partial action of the one-parameter group $(\phi_s)_{s\in\mathbb{C}}$ (resp. $(\psi_s)_{s\in\mathbb{C}}$) on $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$, and combining the two we get a partial action of their free product $\mathcal{G}$ on $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$. Similarly we get a partial action of $\mathcal{G}$ on $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ by replacing $\mathcal{Z}_0(U_\epsilon)$ by $\mathcal{Z}(U_\epsilon)$.

We call them the *partial quantum coadjoint actions* of $\mathcal{G}$. Orbits (called *partial orbits* in [37]) are defined as for group actions: the $\mathcal{G}$-orbit of a point $\chi \in \mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ is the set of points $g.\chi$, for every possible $g \in \mathcal{G}$ such that $g.\chi$ is defined.

The next result states the analogs of Theorem 7.8(c)–(f) obtained by replacing $\mathcal{G}_{\mathrm{DCK}}$ by $\mathcal{G}$. In particular, it describes the $\mathcal{G}$-orbits.

Theorem 7.14.

(a) *The diffeomorphism $\bar{\sigma} \circ \bar{\psi}^{-1}$ maps the action of $\mathcal{G}$ on the tangent spaces of $(0, 0, \pm 1) \in \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ to the coadjoint action of G on $\mathfrak{sl}(2,\mathbb{C})^* = T_{\pm I}(G^0)$.*

(b) *For any conjugacy class Γ in G, $(\bar{\psi}\circ\bar{\sigma}^{-1})(\Gamma\cap G^0)$ is a (non empty) $\mathcal{G}$-orbit in $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$, and τ^{-1} of this orbit has l connected components, all of whose are $\mathcal{G}$-orbits in $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$.*

(c) *An element $a \in \mathcal{Z}(U_\epsilon)$ is $\mathcal{G}$-invariant if and only if $a \in \mathbb{C}[\Omega]$. Dually, the sets of fixed points of the partial $\mathcal{G}$-action on $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ and $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ are respectively $(0, 0, \pm 1) := (\bar{\psi} \circ \bar{\sigma}^{-1})(\{\pm I\})$ and*

$$\mathcal{D} := \tau^{-1}((0, 0, \pm 1)) = \{(0, 0, \pm 1, \pm c_j), j = 1, \ldots, (l-1)/2\}.$$

(d) *The $\mathcal{G}$-orbits on $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ and $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ are the symplectic leaves of $\{\,,\,\}_{\mathrm{QCA}}$.*

Proof. (a) Using $\bar{\sigma}^{-1*}\circ\bar{\psi}^*$ to identify $\mathcal{Z}_0(U_\epsilon)$ with $\mathcal{O}(G^0)$, this is a direct consequence of (7.34) and the fact that $\mathcal{M} = \pm I$ at the fixed point.

(b) That $\Gamma \cap G^0$ is non-empty and connected is classical. Then so is $(\bar{\psi} \circ \bar{\sigma}^{-1})(\Gamma \cap G^0)$. Take a point $\chi \in (\bar{\psi} \circ \bar{\sigma}^{-1})(\Gamma \cap G^0)$. Evaluating (7.34) at χ shows that $\mathcal{G} \cdot \chi$ is contained in $(\bar{\psi} \circ \bar{\sigma}^{-1})(\Gamma \cap G^0)$. We claim that $\mathcal{G} \cdot \chi$ is an open and closed subset, so by connectedness of $(\bar{\psi} \circ \bar{\sigma}^{-1})(\Gamma \cap G^0)$ it coincides with it. Indeed, it is an open subset because any point of $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ (whence of $\mathcal{G} \cdot \chi$) belongs to the domains $D(\phi_s)$ and $D(\psi_s)$ for s small enough. Since the one-parameter groups $(\phi_s)_{s\in\mathbb{C}}$ and $(\psi_s)_{s\in\mathbb{C}}$ are obtained by integrating the derivations $\mathcal{E}$ and $\mathcal{F}$, it follows from Lemma 7.10 that $\mathcal{G} \cdot \chi$ contains a neighborhood in $(\bar{\psi} \circ \bar{\sigma}^{-1})(\Gamma \cap G^0)$ of each of its points. By the same reason any limit point of $\mathcal{G}\cdot\chi$ in $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ has a neighborhood where the partial $\mathcal{G}$-action is defined and which intersects $\mathcal{G} \cdot \chi$. Therefore it belongs to $\mathcal{G} \cdot \chi$, which shows that $\mathcal{G} \cdot \chi$ is also closed in $(\bar{\psi} \circ \bar{\sigma}^{-1})(\Gamma \cap G^0)$. Because the covering map τ is unramified of degree l, the result for $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ follows at once.

(c) Any $\chi \in \mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ belongs to the domains $D(\phi_s)$, $D(\psi_s)$ for s small enough. Solving the equations $\phi_s(\chi) = \chi$ and $\psi_s(\chi) = \chi$ by using the formulas in Proposition 7.13 imposes $\chi = (0, 0, \pm 1)$. This gives the fixed point set of $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$; the result for $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ follows

immediately. As for the first claim, note that for any conjugacy class Γ of maximal dimension the set $\Gamma \cap G^0$ contains a diagonal matrix, and that the union of such sets forms a Zariski open and dense subset of G. Hence (b) above implies that a central invariant element is completely determined by its value at points $(0, 0, z_\chi, \Omega_\chi)$. Therefore it is non zero if and only if it belongs to $\mathbb{C}[\Omega]$.

(d) The groups $\mathcal{G}$ and $\mathcal{G}_{\mathrm{DCK}}$ are obtained by integrating the derivations $\mathcal{E}$, $\mathcal{F}$, and D_e, D_f respectively, and the Lie algebras generated by these two pairs of derivations have the same span at every point of $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))$ or $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$. Then the conclusion follows from Theorem 7.8(f). $\blacksquare$

Remark 7.15. A difficulty with the group $\mathcal{G}$ is that its elements act only on subsets of $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$: in the formulas (7.34), the parameter $t \in \mathbb{C}$ must be such that the lower right entries of the computed matrices are non zero. However it has various merits as compared to the group $\mathcal{G}_{\mathrm{DCK}}$ of Theorem 7.8:

(i) $\mathcal{G}$ is finite dimensional, associated to the Lie algebra $\mathfrak{sl}(2)$ (by Lemma 7.10), whereas $\mathcal{G}_{\mathrm{DCK}}$ is infinite-dimensional.

(ii) The partial action of $\mathcal{G}$ on $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ is by birational transformations (by Proposition 7.13), whereas $\mathcal{G}_{\mathrm{DCK}}$ is a subgroup of $\mathrm{Aut}(\hat{U}_\epsilon)$, acting on $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ by holomorphic (entire) transformations.

(iii) $\mathcal{G}$ can be generalized straightforwardly to $\mathcal{L}_{0,n}^\epsilon$, whereas $\mathcal{G}_{\mathrm{DCK}}$ does not. This is our main motivation for developing this construction. We do it in the next section.

Remark 7.16. We think useful to have in mind the following description of $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ (see [27]). Denote by $G//G$ the affine variety with coordinate ring $\mathcal{O}(G)^G$, the ring of regular functions on G invariant under the coadjoint action of G. We have an isomorphism

$$G//G \cong \mathbb{C}^* / (t \sim t^{-1}).$$

In fact, denote by T the torus of G formed by the diagonal matrices, and by $\mathcal{O}(T)$ its coordinate ring. Then $T \cong \mathbb{C}^*$ and $\mathcal{O}(T) \cong \mathbb{C}[t, t^{-1}]$, where t is the coordinate function of the upper left entry of elements of T. The Weyl group W of G acts on T by inversion, and

$$\mathcal{O}(G)^G \cong \mathcal{O}(T)^W = \mathbb{C}[t + t^{-1}].$$

Consider the maps

$$p\colon\ G \longrightarrow G//G, \qquad p_k\colon\ G//G \longrightarrow G//G,$$

where $p\colon G \to G//G$ is the quotient map, and p_k is induced by the k-th power map $g \mapsto g^k$, $g \in G$, $k \in \mathbb{Z}$. Note here that on coordinate functions we have $p_k(t + t^{-1}) = t^k + t^{-k}$, so p_k is just realized by the k-th Chebyshev polynomial T_k. Consider the fibered product of p and p_l, that is, the affine variety

$$G \times_{G//G} G//G = \{(g, [t]) \in G \times G//G \mid p(g) = p_l([t])\}.$$

Set

$$\tilde{G} = G \times_{G//G} G//G, \qquad \tilde{G}^0 = G^0 \times_{G//G} G//G.$$

Then $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ is isomorphic to $\tilde{G}^0$. In fact, by Theorem 7.7 and the defining relation (7.1) we know that $\mathcal{Z}_0(U_\epsilon)$ is isomorphic to $\mathcal{O}(G^0)$, and that $\mathcal{Z}(U_\epsilon) = \mathcal{Z}_0(U_\epsilon) \otimes_{\mathcal{Z}_0(U_\epsilon) \cap \mathbb{C}[\Omega]} \mathbb{C}[\Omega]$. By the quantum Harish-Chandra homomorphism, see [27] for details, $\mathbb{C}[\Omega] \cong \mathbb{C}[K + K^{-1}]$ and

$\mathcal{Z}_0(U_\epsilon) \cap \mathbb{C}[\Omega] \cong \mathbb{C}[K^l + K^{-l}]$. Let us identify $\mathbb{C}[K, K^{-1}]$ with $\mathcal{O}(T)$. Then $\mathbb{C}[\Omega] \cong \mathcal{O}(T)^W$ and $\mathbb{C}[K^l + K^{-l}] \cong \mathcal{O}(T/\mu_l)^W$, where μ_l is the subgroup of T corresponding to the lth-roots of unity under the isomorphism $T \cong \mathbb{C}^*$. Hence

$$\mathcal{Z}(U_\epsilon) \cong \mathcal{O}(G^0) \otimes_{\mathcal{O}(T/\mu_l)^W} \mathcal{O}(T)^W.$$

The isomorphism of $\mathrm{Spec}(\mathcal{Z}(U_\epsilon))$ with $G^0 \times_{G//G} G//G$ follows by duality.

7.2.2　Extension to $\mathcal{L}_{0,n}^\epsilon$

The results of the previous section extend naturally to $\mathcal{L}_{0,n}^\epsilon$. First we consider the generalization of Theorem 7.7.

The bracket $\{\,,\,\}_{\mathrm{FR}}$ on G has been extended to G^n by Fock–Rosly [43]. From its very definition, it is readily checked to be defined by

$$\{\overset{1}{L}{}^{(i)}, \overset{2}{L}{}^{(i)}\}_{\mathrm{FR}} = \mathfrak{r}\,\overset{1}{L}{}^{(i)}\overset{2}{L}{}^{(i)} - \overset{1}{L}{}^{(i)}\overset{2}{L}{}^{(i)}\mathfrak{r}' + \overset{2}{L}{}^{(i)}\mathfrak{r}'\overset{1}{L}{}^{(i)} - \overset{1}{L}{}^{(i)}\mathfrak{r}\,\overset{2}{L}{}^{(i)}, \tag{7.37}$$

$$\{\overset{1}{L}{}^{(i)}, \overset{2}{L}{}^{(j)}\}_{\mathrm{FR}} = \mathfrak{r}\,\overset{1}{L}{}^{(i)}\overset{2}{L}{}^{(j)} + \overset{1}{L}{}^{(i)}\overset{2}{L}{}^{(j)}\mathfrak{r} - \overset{2}{L}{}^{(j)}\mathfrak{r}\,\overset{1}{L}{}^{(i)} - \overset{1}{L}{}^{(i)}\mathfrak{r}\,\overset{2}{L}{}^{(j)}, \tag{7.38}$$

where $L^{(i)} = \underline{\overset{V_2}{M}}{}^{(i)}$ (see (7.12)), $i, j \in \{1, \dots, n\}$, and $i < j$.

The formula (7.23) extends naturally to define derivations $D_a \colon {}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon \to {}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon$ preserving $\mathcal{Z}({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon)$, for every $a \in \mathcal{Z}({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon)$. Hence we can define a Poisson bracket $\{\,,\,\}_{\mathrm{QCA}}$ on $\mathcal{Z}({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon)$ (keeping voluntarily the same notation as on $\mathcal{Z}(U_\epsilon)$) by

$$\{a, b\}_{\mathrm{QCA}} = D_a(b)$$

for every $a, b \in \mathcal{Z}({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon)$. As Φ_n is an isomorphism of algebras, both derivations and Poisson bracket can be defined and computed by means of their pushforwards on $U_\epsilon^{\otimes n}$. Indeed, for any $a \in \mathcal{Z}({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon)$, $\Phi_n(a) \in \mathcal{Z}(U_\epsilon^{\otimes n})$, so the derivation $D_{\Phi_n(a)} \colon U_\epsilon^{\otimes n} \to U_\epsilon^{\otimes n}$ is defined and we can put $D_a = \Phi_n^{-1} D_{\Phi_n(a)} \Phi_n$. For every $a \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$, by the inclusion $\mathcal{L}_{0,n}^\epsilon \subset {}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon$ it yields a derivation $D_a \colon \mathcal{L}_{0,n}^\epsilon \to \mathcal{L}_{0,n}^\epsilon$ preserving $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$.

Denote by $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ (resp. $\mathcal{Z}_0({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon)$) the subalgebras of $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$ (resp. $\mathcal{Z}({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon)$) generated by $T_l(\omega^{(i)})$, $b^{(i)l}$, $c^{(i)l}$, $d^{(i)l}$ (resp. $T_l(\omega^{(i)})$, $b^{(i)l}$, $c^{(i)l}$, $d^{(i)l}$ and $\delta^{(i)\pm l}$) for $i = 1, \dots, n$. Recall the Frobenius morphism $\mathrm{Fr} \colon (\mathcal{L}_{0,1}^1)^{\otimes n} \to \mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ (see (7.12)), and the identification $\mathcal{L}_{0,1}^1 = \mathcal{O}(G)$.

Theorem 7.17. *The map*

$$\Phi := (\bar{\sigma}^{-1*} \circ \bar{\psi}^*)^{\otimes n} \circ \Phi_1^{\otimes n} \colon \ (\mathcal{Z}_0({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon), \{\,,\,\}_{\mathrm{QCA}}) \to (\mathcal{O}(G^0)^{\otimes n}, \{\,,\,\}_{\mathrm{FR}})$$

is an isomorphism of Poisson algebras, and Φ^{-1} restricted to $\mathcal{O}(G)^{\otimes n} \subset \mathcal{O}(G^0)^{\otimes n}$ coincides with the Frobenius homomorphism. Hence $\mathrm{Fr} \colon (\mathcal{O}(G)^{\otimes n}, \{\,,\,\}_{\mathrm{FR}}) \to (\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon), \{\,,\,\}_{\mathrm{QCA}})$ is an isomorphism of Poisson algebras.

Proof. The map $\bar{\sigma}^{-1*} \circ \bar{\psi}^* \colon (\mathcal{Z}_0(U_\epsilon), \{\,,\,\}_{\mathrm{QCA}}) \to (\mathcal{O}(G^0), \{\,,\,\}_{\mathrm{FR}})$ is an isomorphism of Poisson algebras by the results recalled in the previous section. Since $\Phi_1 \colon {}_{\mathrm{loc}}\mathcal{L}_{0,1}^A \to U_A'$ is an algebra isomorphism by Lemma 5.7, when specializing at $q = \epsilon$ it yields a Poisson isomorphism between $(\mathcal{Z}_0({}_{\mathrm{loc}}\mathcal{L}_{0,1}^\epsilon), \{\,,\,\}_{\mathrm{QCA}})$ and $(\mathcal{Z}_0(U_\epsilon), \{\,,\,\}_{\mathrm{QCA}})$. This proves the case $n = 1$.

When $n \geq 2$, the map is well-defined because $\mathcal{Z}_0({}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon) = {}_{\mathrm{loc}}\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ (the localization by the powers $\delta^{(i)lk}$, $k \in \mathbb{Z}$), $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon) = \mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon)^{\otimes n}$ by Proposition 7.2, and $\Phi_1^{\otimes n}$ extends from $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ to ${}_{\mathrm{loc}}\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$ for obvious reasons (the algebra being commutative). On another hand, since $\Phi_n \colon {}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon \to U_\epsilon^{\otimes n}$ is an equivariant algebra isomorphism (see (6.26)) and $\Phi_n(\omega^{(i)}) = \Omega^{(i)}$,

it yields a Poisson isomorphism between $\big(\mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon),\{\,,\,\}_{\mathrm{QCA}}\big)$ and $(\mathcal{Z}_0(U_\epsilon),\{\,,\,\}_{\mathrm{QCA}})^{\otimes n}$, i.e., the algebra $\mathcal{Z}_0(U_\epsilon)^{\otimes n}$ with the product Poisson structure. Hence it is enough to prove that

$$\Phi \circ \Phi_n^{-1}\colon\ \big(\mathcal{Z}_0(U_\epsilon),\{\,,\,\}_{\mathrm{QCA}}\big)^{\otimes n} \to \big(\mathcal{O}(G^0)^{\otimes n},\{\,,\,\}_{\mathrm{FR}}\big)$$

is an isomorphism of Poisson algebras. This can be checked on generators, which is most easily done by using the inverse map. Put $\Pi = \mathrm{id}_{V_2} \otimes (\Phi_n \circ (\Phi_1^{\otimes n})^{-1})$. By Corollary 7.4 we have

$$\Pi\big(\mathcal{M}^{(i)}\big) = \mathcal{R}^{(i)}\mathcal{M}^{(i)}\mathcal{R}^{(i)-1}. \tag{7.39}$$

Moreover, (7.14) and (7.29) show that $\bar{\sigma}^{-1*}\circ\bar{\psi}^*$ identifies the matrix coefficients of $\Phi_1(\mathcal{Z}_0(\mathcal{L}_{0,1}^\epsilon))$ in the fundamental representation V_2 of U_ϵ, and the matrix coefficients of the fundamental representation of G on $\mathbb{C}^2$. Therefore it is enough to check that

$$\big\{\Pi^*\big(\overset{1}{L}{}^{(i)}\big),\Pi^*\big(\overset{2}{L}{}^{(j)}\big)\big\}_{\mathrm{QCA}}^{\otimes n} = \big\{\overset{1}{L}{}^{(i)},\overset{2}{L}{}^{(j)}\big\}_{\mathrm{FR}} \tag{7.40}$$

for every $i,j \in \{1,\ldots,n\}$, where $\Pi^*\big(L^{(i)}\big)$ is the pull-back of $L^{(i)}$ via Π, i.e., its expression as a function of the matrices $\mathcal{M}^{(i)}$. Now we have

$$\mathcal{M}^{(i)} = \mathcal{M}_+^{(i)}\mathcal{M}_-^{(i)-1}, \tag{7.41}$$

where

$$\mathcal{M}_+^{(i)} = \begin{pmatrix} z'^{(i)} & z'^{(i)}y^{(i)} \\ 0 & z'^{(i)-1} \end{pmatrix}, \qquad \mathcal{M}_-^{(i)} = \begin{pmatrix} z'^{(i)-1} & 0 \\ z'^{(i)}x^{(i)} & z'^{(i)} \end{pmatrix}.$$

Note also that (7.15) and (7.16) imply

$$\mathcal{R}^{(i)} = \mathcal{M}_+^{(n)}\cdots\mathcal{M}_+^{(i+1)}. \tag{7.42}$$

Using the map $\sigma^{-1*}\circ\psi^*$ to identify $\mathcal{M}_+^{(i)}$, $\mathcal{M}_-^{(i)}$ with matrices L_+, L_- of coordinate functions in G^*, we see that they satisfy the bracket identities (7.22). Since G^* is a Poisson–Lie group, products of matrices L_+ have the same bracket as L_+. In particular this applies to $\mathcal{R}^{(i)}$, and we can write

$$\Pi^*(L^{(i)}) = \mathcal{R}^{(i)}\mathcal{M}_+^{(i)}\mathcal{M}_-^{(i)-1}\mathcal{R}^{(i)-1},$$

where all matrices in the product have known brackets. Then (7.40) can be straightforwardly compared with (7.37) and (7.38), using the Leibniz rule for simplifications, and the fact that $r + r'$ is ad-invariant. We leave the verifications to the reader.

The equality of Φ^{-1} with Fr on $\mathcal{O}(G)^{\otimes n}$ follows immediatly from the fact that $\bar{\sigma}^{-1*}\circ\bar{\psi}^*$ identifies matrix coefficients, as discussed above. The image of Fr is $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)$, by Proposition 7.2. This achieves the proof. ∎

Next we turn to our generalization of Theorem 7.14. First we define a partial action of the group $\mathcal{G}$ on $\mathrm{Spec}(\mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon))$ and $\mathrm{Spec}(\mathcal{Z}(_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon))$ by generalizing the method we used in the $n = 1$ case.

Recall the derivations $D_a\colon {}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon \to {}_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon$, defined for every $a \in \mathcal{Z}(_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon)$. Denote by $\hat{x}^{(i)}, \hat{y}^{(i)}, \hat{z}^{(i)} \in \mathcal{Z}(_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon)$ the inverse images by Φ_n of the elements $x^{(i)}$, $y^{(i)}$, $z^{(i)}$, and let $\mathcal{E}^{(i)}$, $\mathcal{F}^{(i)}$, $\mathcal{H}^{(i)}$ be the derivations of $_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon$ defined by

$$\mathcal{E}^{(i)} = \hat{z}^{(i)}D_{\hat{x}^{(i)}}, \qquad \mathcal{F}^{(i)} = -\hat{z}^{(i)}D_{\hat{y}^{(i)}}, \qquad \mathcal{H}^{(i)} = -2\hat{z}^{(i)-1}D_{\hat{z}^{(i)}}.$$

Note that because of the relations (7.1) and the definition of $\delta^{(i)}$ we can obtain simple formulas for $\hat{z}^{(i)}$ and $\hat{x}^{(i)}$, namely:

$$\hat{z}^{(i)\pm 1} = \delta^{(i)\mp l}, \qquad \hat{x}^{(i)} = -c^{(i)l} \prod_{k=i+1}^{n} \delta^{(k)-l}.$$

Set

$$\mathcal{E}^{\Delta} = \sum_{i=1}^{n} \mathcal{E}^{(i)}, \qquad \mathcal{F}^{\Delta} = \sum_{i=1}^{n} \mathcal{F}^{(i)}, \qquad \mathcal{H}^{\Delta} = \sum_{i=1}^{n} \mathcal{H}^{(i)}.$$

Proposition 7.18.

(1) *The derivations $\mathcal{E}^{\Delta}$, $\mathcal{F}^{\Delta}$ and $\mathcal{H}^{\Delta}$ satisfy the following commutation relations, generating the Lie algebra $\mathfrak{sl}(2)$:*

$$\left[\mathcal{H}^{\Delta}, \mathcal{E}^{\Delta}\right] = 2\mathcal{E}^{\Delta}, \qquad \left[\mathcal{H}^{\Delta}, \mathcal{F}^{\Delta}\right] = -2\mathcal{F}^{\Delta}, \qquad \left[\mathcal{E}^{\Delta}, \mathcal{F}^{\Delta}\right] = \mathcal{H}^{\Delta}.$$

(2) *The power series $\exp\left(t\mathcal{E}^{\Delta}\right)$, $\exp\left(t\mathcal{F}^{\Delta}\right)$ define morphisms of algebras from $_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon}$ to $_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon}[[t]]$, sending $\mathcal{Z}(_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon})$ to $\mathcal{Z}(_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon})[[t]]$ and fixing the elements $\omega^{(i)}$. Moreover*

$$\exp\left(t\Phi_n(\mathcal{E}^{\Delta})\right)\left(\mathcal{M}^{(i)}\right) = \exp(-tX)\mathcal{M}^{(i)}\exp(tX),$$
$$\exp\left(t\Phi_n(\mathcal{F}^{\Delta})\right)\left(\mathcal{M}^{(i)}\right) = \exp(-tY)\mathcal{M}^{(i)}\exp(tY).$$

As a result $\exp\left(t\mathcal{E}^{\Delta}\right)$ and $\exp\left(t\mathcal{F}^{\Delta}\right)$ are sending $\mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon})$ to $\mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon})[[t]]$.

Proof. (1) Straighforward from (7.32) and from the local Poisson-commutativity, i.e., the property that $\{a, b\} = 0$ when $a \in \{x^{(i)}, y^{(i)}, z^{(i)}\}$, $b \in \{x^{(j)}, y^{(j)}, z^{(j)}\}$, $i \neq j$.

(2) The first claim is straightforward, the second is a direct application of (7.34) and the local Poisson-commutativity. ∎

As in (7.35) let $s \in \mathbb{C}$, and denote by $D\left(\phi_s^{(i)}\right)$ the set of points $\chi \in \mathrm{Spec}(\mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon}))$ such that for all $u \in \mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon})$ the series $\left(\exp(s\mathcal{E}^{(i)})(u)\right)(\chi)$ is normally convergent in a small neighborhood of s. Thus $D\left(\phi_s^{(i)}\right) = \left\{\chi \in \mathrm{Spec}\left(\mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon})\right), \left|s\hat{x}_{\chi}^{(i)}\hat{z}_{\chi}^{(i)}\right| < 1\right\}$. Define an action of the element $\phi_s^{(i)}$ on $D\left(\phi_s^{(i)}\right)$ by

$$u\left(\phi_s^{(i)}.\chi\right) = \left(\exp\left(-s\mathcal{E}^{(i)}\right)(u)\right)(\chi). \tag{7.43}$$

Similarly, put $D\left(\psi_s^{(i)}\right) = \left\{\chi \in \mathrm{Spec}\left(\mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon})\right), \left|s\hat{y}_{\chi}^{(i)}\hat{z}_{\chi}^{(i)}\right| < 1\right\}$, and define an action of $\psi_s^{(i)}$ on $D\left(\psi_s^{(i)}\right)$ by

$$u\left(\psi_s^{(i)}.\chi\right) = \left(\exp\left(-s\mathcal{F}^{(i)}\right)(u)\right)(\chi). \tag{7.44}$$

Denote by $\mathcal{G}^{(i)}$ the group generated by $\phi_s^{(i)}, \psi_s^{(i)}, s \in \mathbb{C}$. It is isomorphic to $\mathcal{G}$. Denote by $\mathcal{G}^{\mathrm{tot}}$ the direct product of the groups $\mathcal{G}^{(i)}$, and by $\mathcal{G}^{\Delta}$ the subgroup of $\mathcal{G}^{\mathrm{tot}}$ generated by the diagonal elements $\phi_s^{\Delta} = \left(\phi_s^{(1)}, \ldots, \phi_s^{(n)}\right)$ (resp. $\psi_s^{\Delta} = \left(\psi_s^{(1)}, \ldots, \psi_s^{(n)}\right)$). These elements act on $D\left(\phi_s^{\Delta}\right) = \cap_{i=1}^n D\left(\phi_s^{(i)}\right)$, (resp. $D\left(\psi_s^{\Delta}\right) = \cap_{i=1}^n D\left(\psi_s^{(i)}\right)$), by acting dually with the series $\exp\left(s\mathcal{E}^{\Delta}\right)$ (resp. $\exp\left(s\mathcal{F}^{\Delta}\right)$).

As in the case $n = 1$, the results of Exel [37, 36] imply that (7.43) and (7.44) define partial action of the group $\mathcal{G}^{\mathrm{tot}} \cong \mathcal{G}^n$ and $\mathcal{G}^{\Delta} \cong \mathcal{G}$ on $\mathrm{Spec}(\mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^{\epsilon}))$. We call the first one the *total*

partial quantum coadjoint action and the second one the *diagonal partial quantum coadjoint action.*

Next we need the following result of Fock–Rosly. Recall that we denote by $X_G(\Sigma)$ the variety of G-characters of the sphere Σ with $n + 1$ punctures, and by $\{\,,\,\}_{\mathrm{Gold}}$ the Goldman Poisson bracket on $X_G(\Sigma)$. Denote by $G^n /\!/ G$ the algebraic quotient of G^n by the adjoint action of G. It is the affine variety with coordinate ring $\mathcal{O}(G^n)^G$, the ring of regular functions on G^n invariant under the coadjoint action of G. The points of $X_G(\Sigma)$ are in one-to-one correspondence with the trace equivalence classes of representations $\pi_1(\Sigma) \to G$. Therefore, choosing a basepoint and generators of the fundamental group of Σ affords an isomorphism of algebraic sets

$$\mathfrak{c}\colon\ X_G(\Sigma) \to G^n /\!/ G.$$

Theorem 7.19 ([43])**.** *The adjoint action of the Poisson–Lie group $(G, \{\,,\,\})$ on the Poisson manifold $(G^n, \{\,,\,\}_{\mathrm{FR}})$ is a Poisson map. Hence $\{\,,\,\}_{\mathrm{FR}}$ defines a Poisson bracket on $G^n /\!/ G$. Moreover the map $\mathfrak{c}$ is a Poisson isomorphism: $\mathfrak{c}_*\{\,,\,\}_{\mathrm{Gold}} = \{\,,\,\}_{\mathrm{FR}}$.*

The Theorems 7.17 and 7.19 relate $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})$ with $\mathcal{O}(G^n)$ and $\mathcal{O}(G^n)^G$ with $\mathcal{O}(X_G(\Sigma))$. We need to "lift" these results to the whole center $\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})$ and corresponding rings of regular functions.

At first, recall the isomorphism $\mathcal{O}(\tilde{G}^0) \cong \mathcal{Z}(U_\epsilon)$ of Remark 7.16. It maps $\mathcal{O}(T)^W$ to $\mathbb{C}[\Omega]$, the $\mathcal{G}$-invariant subalgebra of $\mathcal{Z}(U_\epsilon)$. Composing it with Φ_1^{-1}, we get an isomorphism of $\mathcal{O}(\tilde{G}^0)$ with $\mathcal{Z}(_{\mathrm{loc}}\mathcal{L}^\epsilon_{0,1})$, mapping the subalgebras $\mathcal{O}(\tilde{G})$ and $\mathcal{O}(G)$ to $\mathcal{Z}(\mathcal{L}^\epsilon_{0,1})$ and $\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,1})$ respectively. It follows straightforwardly from the arguments of Theorem 7.17 that Fr extends to an isomorphism

$$\widetilde{\mathrm{Fr}}\colon\ \big(\mathcal{O}(\tilde{G})^{\otimes n}, \{\,,\,\}_{\mathrm{FR}}\big) \to \big(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n}), \{\,,\,\}_{\mathrm{QCA}}\big)$$

mapping the n copies of $\mathcal{O}(T)^W$ associated to the factors of $\mathcal{O}(\tilde{G}^n)$ to $\mathbb{C}[\omega^{(1)}], \ldots, \mathbb{C}[\omega^{(n)}]$. Note in particular that $\{\,,\,\}_{\mathrm{FR}}$ extends trivially from $\mathcal{O}(G)^{\otimes n}$ to $\mathcal{O}(\tilde{G})^{\otimes n}$. Also, the restriction map (keeping voluntarily the notation of (7.19))

$$\tau\colon\ \mathrm{Spec}(\mathcal{Z}(\mathcal{L}^\epsilon_{0,n})) \longrightarrow \mathrm{Spec}(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})),$$

which by Proposition 7.2 is a regular map of degree l^n, corresponds under $\widetilde{\mathrm{Fr}}$ to the projection map $\tilde{G}^n \to G^n$.

Now, let us identify as above G^n with the space of representation $R_G(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), G)$ (fixing a basepoint and generators of $\pi_1(\Sigma)$). The conjugation action extends trivially from G^n to $\tilde{G}^n$. Then the projection map $\tilde{G}^n \to G^n$ provides an identification of $\tilde{G}^n$ with a branched covering space $\tilde{R}'_G(\Sigma)$ of $R_G(\Sigma)$, endowed with the conjugation action of G. The points of $\tilde{R}'_G(\Sigma)$ are given by representations $\rho\colon \pi_1(\Sigma) \to G$ endowed with a choice of solution $x \in \mathbb{C}$ of the equation $T_l(x) = \mathrm{Tr}(\rho(\gamma_i))$, $1 \le i \le n$, where $\gamma_1, \ldots, \gamma_n$ are the n chosen generators of $\pi_1(\Sigma)$. Taking algebraic quotients yields a branched covering map $\tilde{X}'_G(\Sigma) \to X_G(\Sigma)$ of the same degree l^n, and one can lift Theorem 7.19 to an isomorphism (with, again, $\{\,,\,\}_{\mathrm{Gold}}$ trivially lifted from $X_G(\Sigma)$):

$$\tilde{\mathfrak{c}}\colon\ \big(\tilde{X}'_G(\Sigma), \{\,,\,\}_{\mathrm{Gold}}\big) \to \big(\tilde{G}^n /\!/ G, \{\,,\,\}_{\mathrm{FR}}\big).$$

We can now state and prove our generalization of Theorem 7.14. When considering the (partial) action of $\mathcal{G}$ on $\mathrm{Spec}(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n}))$ it will always be meant to be the diagonal action, by means of $\mathcal{G}^\Delta$.

Corollary 7.20.

(1) *The dual diffeomorphism $\mathrm{Fr}^*\colon \big(\mathrm{Spec}(\mathcal{Z}_0(\mathcal{L}^\epsilon_{0,n})), \{\,,\,\}_{\mathrm{QCA}}\big) \to (G^n, \{\,,\,\}_{\mathrm{FR}})$ maps the action of $\mathcal{G}$ on the tangent spaces of $(0, 0, \pm 1)^n$ onto the coadjoint action of G on $\mathfrak{sl}(2, \mathbb{C})^{*n}$.*

(2) *For any conjugacy class Γ in G^n, $(\mathrm{Fr}^*)^{-1}(\Gamma)$ is a (non empty) $\mathcal{G}$-orbit in $\mathrm{Spec}(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$, and τ^{-1} of this orbit has l^n connected components, all of whose are $\mathcal{G}$-orbits in $\mathrm{Spec}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$.*

(3) *The map $\widetilde{\mathrm{Fr}} \circ \tilde{\mathfrak{c}}^{-1*}$ takes values in $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$, and therefore affords an isomorphism of Poisson algebras*

$$\widetilde{\mathrm{Fr}} \circ \tilde{\mathfrak{c}}^{-1*}\colon \ \left(\mathcal{O}(\tilde{X}_G'(\Sigma)), \{\,,\,\}_{\mathrm{Gold}}\right) \to \left(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}, \{\,,\,\}_{\mathrm{QCA}}\right).$$

(4) *The orbits of the group $\mathcal{G}^{\mathrm{tot}}$ in $\mathrm{Spec}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon))$ are the symplectic leaves of $\{\,,\,\}_{\mathrm{QCA}}$. These project onto the symplectic leaves of $\{\,,\,\}_{\mathrm{QCA}}$ in $\mathrm{Spec}(\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}})$.*

Proof. (1) Using as usual the isomorphism $\bar{\sigma}^{-1*} \circ \bar{\psi}^*$ to identify $\mathcal{Z}_0(U_\epsilon)$ with $\mathcal{O}(G^0)$, it follows easily from (7.39), (7.41) and (7.42) that the automorphism $\Phi_n \circ \left(\Phi_1^{-1}\right)^{\otimes n}$ of $\mathcal{O}(G^0)^{\otimes n}$ is equivariant with respect to the coadjoint action of G. Since $\Phi_n^{-1*}((0,0,\pm 1)^n) = (\pm I, \ldots, \pm I)$, the formulas in Proposition 7.18(2) show that $d\Phi_n^{-1*}$ maps the action of $\mathcal{E}^\Delta$, $\mathcal{F}^\Delta$ on the tangent spaces of $(0,0,\pm 1)^n$ to the coadjoint action of X, Y on $\mathfrak{sl}(2,\mathbb{C})^{*n}$. Post-composing Φ_n^{-1*} with $\left(\Phi_1^{-1*}\right)^{\otimes n} \circ \Phi_n^*$, Proposition 7.18(1) proves that Φ^{-1*} maps the action of $\mathcal{G}$ on the tangent spaces of $(0,0,\pm 1)^n \in \mathrm{Spec}(\mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon))$ onto the coadjoint action of G on $\mathfrak{sl}(2,\mathbb{C})^{*n} \cong T^*_{(\pm I,\ldots,\pm I)}\left(G^0\right)^n$. The result follows, Fr^* being an extension of Φ^{-1*}.

(2) This is an integrated version of (1) above. It follows from the arguments used to prove Theorem 7.14(c), by replacing (7.34) with the formulas in Proposition 7.18(2), and Lemma 7.10 with Proposition 7.18(1).

(3) It is enough to prove that $\mathrm{Fr} \circ \mathfrak{c}^{-1*}$ establishes a Poisson isomorphism between $\mathcal{O}(X_G(\Sigma))$ and $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$. By Theorem 7.17, $\mathrm{Fr} = \Phi_{|\mathcal{O}(G^n)}^{-1}\colon \left(\mathcal{O}(G)^{\otimes n}, \{\,,\,\}_{\mathrm{FR}}\right) \to \left(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon), \{\,,\,\}_{\mathrm{QCA}}\right)$ is an isomorphism of Poisson algebras. By (2) above Fr maps invariants functions to invariant functions. That the bracket $\{\,,\,\}_{\mathrm{QCA}}$ is well-defined on $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$ is an immediate consequence of its definition and the structure of module algebra of $\mathcal{L}_{0,n}^\epsilon$. Then the conclusion follows from the last claim of Theorem 7.19.

(4) The first claim follows from Theorem 7.14(d), and the facts that Φ_n is a Poisson isomorphism from $\left(\mathcal{Z}_0(_{\mathrm{loc}}\mathcal{L}_{0,n}^\epsilon), \{\,,\,\}_{\mathrm{QCA}}\right)$ to $\left(\mathcal{Z}_0(U_\epsilon), \{\,,\,\}_{\mathrm{QCA}}\right)^{\otimes n}$ (see the proof of Theorem 7.17), and that Φ_n^{-1*} maps the partial action of $\mathcal{G}^{\mathrm{tot}}$ on $\mathrm{Spec}(\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon))$ to the partial action of $\mathcal{G}^n$ (the n-fold direct product) on $\mathrm{Spec}(\mathcal{Z}_0(U_\epsilon))^n$ (by the definition of $\mathcal{G}^{\mathrm{tot}}$). The second claim follows from the first one in Theorem 7.19. ∎

8   Topological formulation

8.1   The Wilson loop functor

Recall that we denote by $\mathcal{C}_A$ the category of U_A^{res}-modules of type 1, and that $\mathcal{C}_A \otimes \mathbb{C}[q^{1/D}, q^{-1/D}]$ is a ribbon category.

Recall also the following notions (see [78]). Denote by $\mathrm{Rib}_{\mathcal{C}_A}$ the category whose morphisms are the isotopy classes rel(∂) of oriented ribbon graphs in $[0,1]^3$ *colored* over $\mathcal{C}_A$ (ie. with each component labelled by an object of $\mathcal{C}_A$), with boundary segments (if any) in $]0,1[\times\{1/2\}\times\{i\}$, where $i \in \{0,1\}$. The objects of $\mathrm{Rib}_{\mathcal{C}_A}$ are the tuples $((V_1, \varepsilon_1), \ldots, (V_k, \varepsilon_k))$, where $\varepsilon_1, \ldots, \varepsilon_k = \pm$ and $V_1, \ldots, V_k$ are objects of $\mathcal{C}_A$. The source and target objects of a morphism of $\mathrm{Rib}_{\mathcal{C}_A}$ thus correspond to tuples of segments in $]0,1[\times\{1/2\}\times\{i\}$, $i \in \{0,1\}$, endowed with normal co-orientations specifying the associated signs $\pm$. We denote by $\mathbb{RT}$ the Reshetikhin–Turaev functor

$$\mathbb{RT}\colon \ \mathrm{Rib}_{\mathcal{C}_A} \to \mathcal{C}_A \otimes \mathbb{C}[q^{1/D}, q^{-1/D}].$$

Now, fix points $p_1 < \cdots < p_n$ in $]0, 1/2[$, and define $\mathrm{Rib}_{n,\mathcal{C}_A}$ as the category with the same objects as $\mathrm{Rib}_{\mathcal{C}_A}$ but morphisms the isotopy classes rel(∂) of oriented ribbon graphs in $[0,1]^3 \setminus (\{p_1, \ldots, p_n\} \times \{1/2\} \times [0,1])$, colored over $\mathcal{C}_A$ and with boundary segments (if any) in $]1/2, 1[\times \{1/2\} \times \{i\}$, $i \in \{0, 1\}$. Figure 8.1 shows an example.
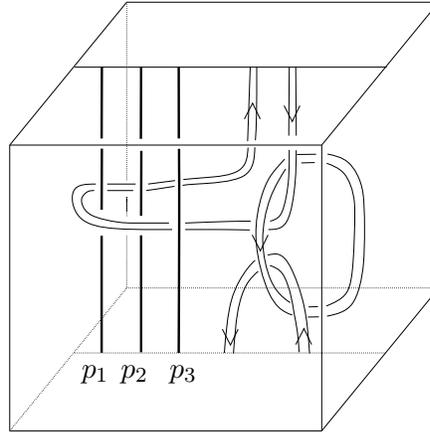
Figure 8.1. A morphism of $\mathrm{Rib}_{n,\mathcal{C}_A}$.

The composition of morphisms of $\mathrm{Rib}_{n,\mathcal{C}_A}$ is defined as for $\mathrm{Rib}_{\mathcal{C}_A}$. That is, given morphisms T_1, T_2 with a same pattern of co-oriented boundary segments on the bottom of T_1 and top of T_2, $T_1 \circ T_2$ is obtained by placing T_1 atop T_2, gluing the corresponding boundary segments, and deforming the result by isotopy into $[0,1]^3 \setminus (\{p_1, \ldots, p_n\} \times \{1/2\} \times [0,1])$. Identifying $[0,1]^3$ with the "right half" cube $[1/2, 1] \times [0,1]^2$ in $[0,1]^3 \setminus (\{p_1, \ldots, p_n\} \times \{1/2\} \times [0,1])$ yields an obvious faithful functor

$$\iota \colon \ \mathrm{Rib}_{\mathcal{C}_A} \to \mathrm{Rib}_{n,\mathcal{C}_A}.$$

The morphisms of $\mathrm{Rib}_{n,\mathcal{C}_A}$ are obtained by composing morphisms of $\mathrm{Rib}_{\mathcal{C}_A}$ and elementary morphisms as shown in the following picture, for $a = 1, \ldots, n$.
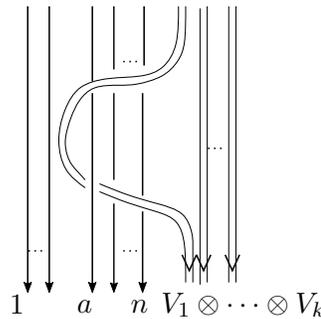
Figure 8.2. A generating morphism of $\mathrm{Rib}_{n,\mathcal{C}_A}$ not in $\mathrm{Rib}_{\mathcal{C}_A}$.

Recall that we denote by Σ the sphere with $n+1$ open disks removed. Denote by $\mathcal{L}_{\mathcal{C}_A}(\Sigma)$ the A-module freely generated by the empty set and the isotopy classes of closed oriented ribbon graphs in $\Sigma \times [0,1]$ colored over $\mathcal{C}_A$. The stacking product $L.L'$ of elements $L, L' \in \mathcal{L}_{\mathcal{C}_A}(\Sigma)$ is defined as the isotopy class of the disjoint union of representatives of L and L', pushed in $\Sigma \times [0, 1/2]$ and $\Sigma \times [1/2, 1]$ respectively. The stacking product makes $\mathcal{L}_{\mathcal{C}_A}(\Sigma)$ an associative A-algebra. Since Σ is diffeomorphic to the closure of $[0,1]^2 \setminus (\{p_1, \ldots, p_n\} \times \{1/2\})$, $\mathcal{L}_{\mathcal{C}_A}(\Sigma)$ can

be identified with the A-algebra generated by the morphisms $\varnothing \to \varnothing$ of $\mathrm{Rib}_{n,\mathcal{C}_A}$, the stacking product being the A-linear extension of the composition of morphisms.

Define a new category $\mathcal{C}_A\big(\mathcal{L}_{0,n}^A\big)$ with objects the couples (n, V), where V is an object of $\mathcal{C}_A \otimes \mathbb{C}[q^{1/D}, q^{-1/D}]$, and with spaces of morphisms

$$\mathrm{Hom}((n, V), (n, W)) = \big(\mathrm{Hom}_{U_A^{\mathrm{res}}}(V, W) \otimes \mathbb{C}\big[q^{1/D}, q^{-1/D}\big]\big) \otimes \mathcal{L}_{0,n}^A,$$

where the composition is the tensor product of the composition of U_A^{res}-module homomorphisms and the product in $\mathcal{L}_{0,n}^A$.

The following result follows from the arguments in Sections 6.1 and 6.2 of Faitg's PhD Thesis [39], to which we refer for full details (see also [41]). These arguments generalize and simplify those of [20]. In these works only the Wilson loop map W was considered, but the extension to $\mathbb{W}$ is straightforward.

Theorem 8.1. *There is unique functor* $\mathbb{W} \colon \mathrm{Rib}_{n,\mathcal{C}_A} \to \mathcal{C}_A\big(\mathcal{L}_{0,n}^A\big)$ *such that on objects we have* $\mathbb{W}((V_1, \varepsilon_1), \ldots, (V_k, \varepsilon_k)) = \big(n, V_1^{\varepsilon_1} \otimes \cdots \otimes V_k^{\varepsilon_k}\big)$, *where* $V_j^+ = V_j$ *and* $V_j^- = V_j^*$ *(the dual of* V_j*), and on morphisms:*

- $\mathbb{W}(\iota(T)) = \mathbb{R}\mathbb{T}(T) \otimes 1$ *for every morphism* T *of* $\mathrm{Rib}_{\mathcal{C}_A}$,

- $\mathbb{W}$ *gives the value* $\overset{V}{M}{}^{(a)}$ *to the morphism shown in Figure* 8.2*, in the case where there is a single ribbon colored by* $V := V_1$.

Moreover the morphism of algebras $W \colon \mathcal{L}_{\mathcal{C}_A}(\Sigma) \to \mathcal{L}_{0,n}^A \otimes \mathbb{C}\big[q^{1/D}, q^{-1/D}\big]$ *obtained by restricting* $\mathbb{W}$ *to* $\mathcal{L}_{\mathcal{C}_A}(\Sigma)$ *takes values in the invariant subalgebra* $\big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes \mathbb{C}\big[q^{1/D}, q^{-1/D}\big]$.

We call $\mathbb{W}$ the Wilson loop functor, and W the Wilson loop map. By construction, for every element $\hat{T} \in \mathcal{L}_{\mathcal{C}_A}(\Sigma)$, and any morphism T of $\mathrm{Rib}_{n,\mathcal{C}_A}$ obtained from $\hat{T}$ by cutting open ribbons with colors $V_1, \ldots, V_k$, setting $V = V_1 \otimes \cdots \otimes V_k$ we have

$$W(\hat{T}) = \mathrm{qTr}_V\left(\mathbb{W}(T)\right). \tag{8.1}$$

Here is an alternative way of defining $\mathbb{W}$, purely in terms of the Reshetikhin–Turaev functor $\mathbb{R}\mathbb{T}$. Define a category $\mathcal{C}_A\big(\tilde{U}_A^{\otimes n}\big)$ with same objects (n, V) as $\mathcal{C}_A\big(\mathcal{L}_{0,n}^A\big)$ but with spaces of morphisms

$$\mathrm{Hom}((n, V), (n, W)) = \big(\mathrm{Hom}_{U_A^{\mathrm{res}}}(V, W) \otimes \mathbb{C}\big[q^{1/D}, q^{-1/D}\big]\big) \otimes \tilde{U}_A^{\otimes n},$$

where the composition is the tensor product of the composition of U_A^{res}-module homomorphisms and the product in $\tilde{U}_A^{\otimes n}$.

The Alekseev map being a morphism of algebra, it defines a functor

$$\Phi_n \colon \ \mathcal{C}_A\big(\mathcal{L}_{0,n}^A\big) \to \mathcal{C}_A\big(\tilde{U}_A^{\otimes n}\big)$$

by setting $\Phi_n(f \otimes a) = f \otimes \Phi_n(a)$ for every $a \in \mathcal{L}_{0,n}^A$, $f \in \mathrm{Hom}_{U_A^{\mathrm{res}}}(V, W)$. To any morphism T of $\mathrm{Rib}_{n,\mathcal{C}_A}$ we can associate a colored oriented ribbon graph in $[0, 1]^3$,

$$T^\sharp = \left(\bigcup_{i=1}^{n} [p_i - \delta, p_i + \delta] \times \{1/2\} \times [0, 1]\right)_{\tilde{U}_A} \cup T,$$

where $\delta > 0$ is small and the cores of the ribbons $[p_i - \delta, p_i + \delta] \times \{1/2\} \times [0, 1]$ are oriented from $\{p_i\} \times \{1/2\} \times \{1\}$ to $\{p_i\} \times \{1/2\} \times \{0\}$, and the subscript $\tilde{U}_A$ means that these ribbons are colored by the regular representation of $\tilde{U}_A$. Denote by $\overline{\mathrm{Rib}}_{\mathcal{C}_A}$ the category with morphisms given by the ribbon graphs $T^\sharp$; clearly the map $T \mapsto T^\sharp$ yields a functor

$$\sharp \colon \ \mathrm{Rib}_{n,\mathcal{C}_A} \to \overline{\mathrm{Rib}}_{n,\mathcal{C}_A}.$$

By associating $\sigma \circ (\mathrm{id} \otimes \pi_V)(R^{\pm})\colon \mathrm{End}(V) \otimes \tilde{U}_A \to \tilde{U}_A \otimes \mathrm{End}(V)$ to crossings of index ± 1 colored over $\mathcal{C}_A$ and $\tilde{U}_A$, in the same way as $\sigma \circ (\pi_W \otimes \pi_V)(R^{\pm 1})$ is associated by $\mathbb{RT}$ to crossings colored by objects V, W of $\mathcal{C}_A$, we extend $\mathbb{RT}$ to a functor

$$\mathbb{RT}\colon \ \overline{\mathrm{Rib}}_{n,\mathcal{C}_A} \to \mathcal{C}_A\big(\tilde{U}_A^{\otimes n}\big).$$

The next fact is a direct consequence of the definition of Φ_n, and its representation by Figure 6.1. Note that Φ_n being injective, it defines $\mathbb{W}$ uniquely from $\mathbb{RT}$.

Proposition 8.2. *We have a commutative diagram of functors:*

$$
\begin{array}{ccc}
\mathrm{Rib}_{n,\mathcal{C}_A} & \xrightarrow{\ \sharp\ } & \overline{\mathrm{Rib}}_{n,\mathcal{C}_A} \\
{\scriptstyle \mathbb{W}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{RT}} \\
\mathcal{C}_A\big(\mathcal{L}_{0,n}^A\big) & \xrightarrow{\ \Phi_n\ } & \mathcal{C}_A\big(\tilde{U}_A^{\otimes n}\big).
\end{array}
$$

8.2 The Wilson loop isomorphism

One can adjoin to $\mathcal{L}_{\mathcal{C}_A}(\Sigma)$ the A-span of closed oriented ribbon graphs in $\Sigma \times [0,1]$ colored over $\mathcal{C}_A$ *and possibly having coupons* (see [78] for this notion). Denote the resulting set by $\mathcal{L}_{\mathcal{C}_A}(\Sigma)^c$. It is an A-algebra for the stacking product, containing $\mathcal{L}_{\mathcal{C}_A}(\Sigma)$ as a subalgebra. Define a *one-coupon multicurve* as an element of $\mathcal{L}_{\mathcal{C}_A}(\Sigma)^c$ that can be represented by a (oriented, $\mathcal{C}_A$-colored) ribbon graph embedded in $\Sigma \times \{0\}$ (whence edges are unlinked and unknotted) and having a single coupon. An example of one-coupon multicurve with the coupon colored by a_X is shown in Figure 8.3, where Σ is the four-holed sphere (i.e., $n = 3$).
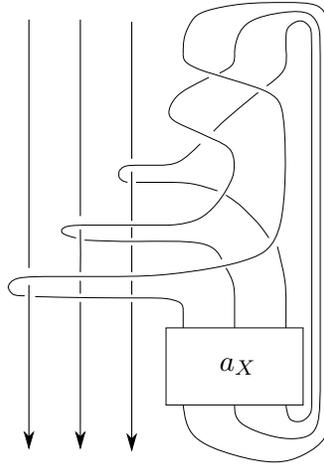
Figure 8.3. A one-coupon multicurve in $\Sigma_{0,4} \times [0,1]$.

Recall the matrices $\overset{X}{\mathbb{M}}$ and the invariant elements $v_X(a_X)$ defined for any U_A^{res}-module $X = X_1 \otimes \cdots \otimes X_n$ of type 1 and any $a_X \in \mathrm{End}_{U_A^{\mathrm{res}}}(X)$ (see the comments after Proposition 6.22). Denote by $L(a_X)$ the one-coupon multicurve in Figure 8.3; so $n = 3$. By (8.1) it is clear that

$$W(L(a_X)) = v_X(a_X). \tag{8.2}$$

This generalizes immediately to any $n \geq 1$. By the comments after Proposition 6.22 elements of the form $v_X\big(a_X^{(k)}\big)$ form a basis of $\mathcal{L}_{0,n}^{U_q}$. So Theorem 8.1 gives $\big($with coefficients in $q^{1/D}\big)$:

Lemma 8.3. *The Wilson loop map* $W \colon \mathcal{L}_{\mathcal{C}_A}(\Sigma)^c \otimes_A \mathbb{C}(q^{1/D}) \to \mathcal{L}_{0,n}^{U_q} \otimes_{\mathbb{C}(q)} \mathbb{C}(q^{1/D})$ *is surjective.*

In the sequel we assume that $\mathfrak{g} = \mathfrak{sl}(2)$. We are going to see that in this case we can strengthen Lemma 8.3.

Set $\zeta := \mathrm{i}q^{1/2}$, and denote by $\mathcal{L}_\zeta(\Sigma) \subset \mathcal{L}_{\mathcal{C}_A}(\Sigma) \otimes \mathbb{C}[\zeta, \zeta^{-1}]$ the subalgebra freely generated as a $\mathbb{Z}[\zeta, \zeta^{-1}]$-module by the empty set and the isotopy classes of oriented ribbon links in $\Sigma \times [0,1]$ colored by the fundamental representation V_2. Recall that the Kauffman bracket skein algebra $K_\zeta(\Sigma)$ is the $\mathbb{Z}[\zeta, \zeta^{-1}]$-algebra obtained from $\mathcal{L}_\zeta(\Sigma)$ by forgetting the link orientations, and taking the quotient by the ideal generated by the relations

$$L = \zeta L_+ + \zeta^{-1} L_-, \tag{8.3}$$

$$L \sqcup \bigcirc = -(\zeta^2 + \zeta^{-2})L, \tag{8.4}$$

where in the second identity $\bigcirc$ is the trivial ribbon link in a ball disjoint from L, and in the first identity $L, L_+, L_- \in \mathcal{L}_\zeta(\Sigma)$ are identical up to isotopy except in a ball in which they look like (the strands representing flat ribbons on a meridional projection disk):

$$L \qquad\qquad L_+ \qquad\qquad L_-$$
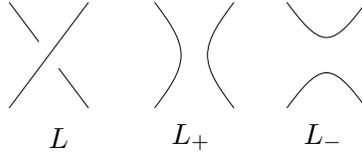
Figure 8.4. Skein related ribbon links.

The following result is [23, Theorem 1] (see also [22, Theorem 10]). For completeness we give a proof by using the notions we have introduced.

Theorem 8.4. *The linear map defined by* $\mathcal{W}(L) = \mathrm{i}^{\mathrm{lk}(L)} W(L)$, *where* $L \in \mathcal{L}_\zeta(\Sigma)$ *and* $\mathrm{lk}(L)$ *is the linking number of* L, *descends to an isomorphism of* $\mathbb{C}(\zeta)$-*algebras* $\big($*where* $\zeta := \mathrm{i}q^{1/2}\big)$:

$$\mathcal{W} \colon \ K_\zeta(\Sigma) \otimes \mathbb{C}(\zeta) \to \mathcal{L}_{0,n}^{U_q} \otimes \mathbb{C}(\zeta).$$

Proof. The facts that the values of $\mathcal{W}$ do not depend on the link orientations, and that $\mathcal{W}$ maps to 0 the ideal of $\mathcal{L}_\zeta(\Sigma)$ generated by the skein relations (8.3) and (8.4), follow from Proposition 8.2, the injectivity of Φ_n, and the properties of $\mathbb{RT}$ and the R-matrix R_{V_2, V_2} proved, e.g., in [56, Lemma 3.18, Theorem 4.3 and Corollary 4.13]. We stress that, using as usual the standard pivotal element K in the definition of $\mathbb{RT}$, the choice of variable $\zeta := \mathrm{i}q^{1/2}$ and the normalisation of the map W by $i^{\mathrm{lk}(L)}$ are necessary for the skein relations to be in the kernel of $\mathcal{W}$. Therefore $\mathcal{W}$ descends to a module map defined on $K_\zeta(\Sigma)$. Since the stacking product is induced from $\mathcal{L}_\zeta(\Sigma)$, $\mathcal{W}$ is a morphism of algebras.

To show that $\mathcal{W}$ is an isomorphism we use the following basis of $K_\zeta(\Sigma) \otimes \mathbb{C}(\zeta)$ (see, e.g., [55, 63]). Let Γ be a trivalent ribbon graph onto which Σ retracts by deformation. Recall that an admissible coloring of Γ is an assignment of a nonnegative integer to each edge, called the color of the edge, so that the colors adjacent to each vertex form admissible triples. A triple of colors (a, b, c) is admissible if $a \le b + c$, $b \le a + c$, $c \le a + b$ and $a + b + c$ is even. Admissible colorings γ of Γ parametrize multicurves carried by Γ, a color on an edge e giving the number of components running parallel to e. Given any admissible coloring γ of Γ, there is a skein Γ_γ in $K_\zeta(\Sigma) \otimes \mathbb{C}(\zeta)$ corresponding to γ, obtained by replacing an edge with color m by the m-th Jones–Wenzl idempotent and vertices with Kauffman triads (these being defined in, e.g., [63, p. 136 and Figure 14.7]). The set of multicurves on Σ forms a basis of $K_\zeta(\Sigma)$; since Γ is a spine of Σ, the set of skeins Γ_γ forms a basis of $K_\zeta(\Sigma) \otimes \mathbb{C}(\zeta)$ as well. Note that, because the Jones–Wenzl idempotents have denominators, the skeins Γ_γ must be defined over $\mathbb{C}(\zeta)$.

By an isotopy of $\Sigma \times [0,1] \cong [0,1]^3 \setminus (\{p_1, \ldots, p_n\} \times \{1/2\} \times [0,1])$ we can deform Γ (keeping the same notation) so that a neighborhood of the vertices lies inside a coupon embedded in $]1/2,1[\times\{1/2\} \times [0,1]$, and the n ribbons forming the portion Γ' of Γ outside the coupon are attached to its left side, encircling $\{p_i\} \times \{1/2\} \times [0,1]$, for $i = 1, \ldots, n$. Let γ be an admissible coloring of Γ. Put $\Gamma'' = \Gamma \setminus \Gamma'$, and denote by Γ'_γ, Γ''_γ the graphs Γ', Γ'' with edges colored by γ. Denote by $e_1, \ldots, e_n$ the edges of Γ' ordered by increasing height, by γ_i the color of e_i plus 1, and fix some orientation of the edges of Γ so that each e_i is oriented from top to bottom. In order to fit with the standard framework for computations with $\mathbb{R}\mathbb{T}$, let us rotate the cube $[0,1]^3$ clockwise by an angle of $\pi/2$ around the axis $\{1/2\} \times [0,1] \times \{1/2\}$. Then, using Theorem 8.1 and the fact that the m-th Jones–Wenzl idempotent projects $V_2^{\otimes m}$ onto a subspace isomorphic to V_{m+1}, we see that

$$\mathbb{W}(\Gamma'_\gamma) = \overset{V_{\gamma_1}}{M}{}^{(1)} \otimes \cdots \otimes \overset{V_{\gamma_n}}{M}{}^{(n)},$$

which is an element of $\otimes_{i=1}^n (V_{\gamma_i} \otimes V_{\gamma_i}^*) \otimes \mathcal{L}_{0,n}$ by identifying $\mathrm{End}(V)$ with $V \otimes V^*$, and

$$\mathbb{W}(\Gamma''_\gamma) \in \mathrm{Hom}_{U_q}\left(\otimes_{i=1}^n (V_{\gamma_i} \otimes V_{\gamma_i}^*), 1 \right) \otimes_{\mathbb{C}(q)} \mathbb{C}(\zeta).$$

By definitions we have $\mathcal{W}(\Gamma_\gamma) = \mathbb{W}(\Gamma''_\gamma)(\mathbb{W}(\Gamma'_\gamma))$, and the map $\gamma \mapsto \mathbb{W}(\Gamma''_\gamma)$ maps the admissible colorings of Γ to a basis of the space of invariant elements of $\otimes_{i=n}^1 \mathrm{End}(V_{\gamma_i})^* \otimes_{\mathbb{C}(q)} \mathbb{C}(\zeta)$ for the action $(\mathrm{ad}^r)^{\otimes n}$. Therefore the set $\{\mathcal{W}(\Gamma_\gamma), \gamma \text{ admissible}\}$ is a basis of the space of invariant elements of $\mathcal{L}_{0,1}^{\otimes n} \otimes_{\mathbb{C}(q)} \mathbb{C}(\zeta)$ for the action $(\mathrm{coad}^r)^{\otimes n}$. By applying to it the linear isomorphism $\Phi_n^{-1} \circ \psi \circ \Phi_1^{\otimes n} \colon \mathcal{L}_{0,1}^{\otimes n} \to \mathcal{L}_{0,n}$, with ψ the intertwiner of $(\mathrm{ad}^r)^{\otimes n}$ and ad_n^r defined in the proof of Proposition 6.7, we get a basis of $\mathcal{L}_{0,n}^{U_q} \otimes_{\mathbb{C}(q)} \mathbb{C}(\zeta)$. This concludes the proof. ∎

Remark 8.5. Above we can replace $\mathbb{W}(\Gamma'_\gamma)$ with $\overset{[\lambda]}{\mathbb{M}}$, where $[\lambda] = (\gamma_1, \ldots, \gamma_n)$, by "parsing" the ends of $e_1, \ldots, e_n$ as do the strands just above the coupon $a_X := a_{[\lambda]}$ in Figure 8.3. Correspondingly $\mathbb{W}(\Gamma''_\gamma)$ becomes an element of $\mathrm{Hom}_{U_q}\left(1, (\otimes_{i=1}^n V_{\gamma_i}) \otimes (\otimes_{i=1}^n V_{\gamma_i})^* \right) \otimes_{\mathbb{C}(q)} \mathbb{C}(\zeta)$. With these choices, the pairing of $\mathbb{W}(\Gamma'_\gamma)$ and $\mathbb{W}(\Gamma''_\gamma)$ is equivariant with respect to the actions coad_n^r and ad_n^r, and it puts in duality the product of $\mathcal{L}_{0,n}$ and the comultiplication map constructed in [22].

The following result is a version of Theorem 8.4 over the ring $\mathbb{C}[\zeta, \zeta^{-1}]$:

Theorem 8.6. *We have an isomorphism of algebras $\mathcal{W} \colon K_\zeta(\Sigma) \to (\mathcal{L}_{0,n}^A)^{U_A} \otimes_A \mathbb{C}[\zeta, \zeta^{-1}]$.*

Proof. We are going to use stated skein algebras and a result of Faitg (see [41]). Denote by $\Sigma_{0,n}^{\circ,\bullet}$ our surface Σ with one point removed on its boundary, by $t^{1/2}$ an indeterminate, and by $\mathcal{S}_t^s(\Sigma_{0,n}^{\circ,\bullet})$ the stated skein algebra of $\Sigma_{0,n}^{\circ,\bullet}$, as defined in [62]. It is an algebra over $\mathbb{C}[t^{1/2}, t^{-1/2}]$, which contains $K_t(\Sigma)$ as a $\mathbb{Z}[t, t^{-1}]$-subalgebra. By a result of [26], $\mathcal{S}_t^s(\Sigma_{0,n}^{\circ,\bullet})$ is a $\mathcal{O}_{t^2}$-comodule algebra, isomorphic to the braided tensor product of n copies of $\mathcal{S}_t^s(\Sigma_{0,1}^{\circ,\bullet})$. Here $\mathcal{O}_{t^2}$ is $\mathcal{O}_A$ with variable $q = t^2$; by the duality between comodule algebras and module algebras we can canonically regard $\mathcal{S}_t^s(\Sigma_{0,n}^{\circ,\bullet})$ as a U_A^{res}-module algebra, whence a U_A-module algebra. As coefficients contain $\mathbb{C}$ we can replace $t^{1/2}$ by $\zeta^{1/2}$ so that $\zeta = it$. On the other side, by the integral form of the last claim of Proposition 6.2, which follows immediately from Proposition 6.9 and Lemma 6.10, we know that $\mathcal{L}_{0,n}^A$ is isomorphic to the braided tensor product of n copies of the U_A-module algebra $\mathcal{L}_{0,1}^A$. In [41, Theorem 5.3] it is constructed an isomorphism of U_A-module algebras $\tilde{\mathcal{W}} \colon \mathcal{S}_\zeta^s(\Sigma_{0,n}^{\circ,\bullet}) \to \mathcal{L}_{0,n}^A \otimes_A \mathbb{C}[\zeta^{1/2}, \zeta^{-1/2}]$. The restriction of $\tilde{\mathcal{W}}$ on the subalgebra $K_\zeta(\Sigma)$ is just the Wilson loop map $\mathcal{W}$. To obtain $\tilde{\mathcal{W}}$, an explicit isomorphism is constructed between $\mathcal{S}_\zeta^s(\Sigma_{0,1}^{\circ,\bullet})$ and $\mathcal{L}_{0,1}^A \otimes_A \mathbb{C}[\zeta^{1/2}, \zeta^{-1/2}]$ (see [41, Lemmas 5.7 and 5.8]).

Let us consider the isomorphism $\mathcal{S}_\zeta^s\big(\Sigma_{0,n}^{\circ,\bullet}\big)^{U_A} \to \big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes_A \mathbb{C}\big[\zeta^{1/2},\zeta^{-1/2}\big]$ induced by $\tilde{\mathcal{W}}$. Since $\big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes_A \mathbb{C}\big[\zeta^{1/2},\zeta^{-1/2}\big] = \big(\mathcal{L}_{0,n}^{U_q} \otimes \mathbb{C}\big(\zeta^{1/2}\big)\big) \cap \big(\mathcal{L}_{0,n}^A \otimes_A \mathbb{C}\big[\zeta^{1/2},\zeta^{-1/2}\big]\big)$, we have

$$\mathcal{S}_\zeta^s\big(\Sigma_{0,n}^{\circ,\bullet}\big)^{U_A} = \tilde{\mathcal{W}}^{-1}\big(\big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes_A \mathbb{C}\big[\zeta^{1/2},\zeta^{-1/2}\big]\big)$$
$$= \tilde{\mathcal{W}}^{-1}\big(\mathcal{L}_{0,n}^{U_q} \otimes \mathbb{C}\big(\zeta^{1/2}\big)\big) \cap \mathcal{S}_\zeta^s\big(\Sigma_{0,n}^{\circ,\bullet}\big) = \big(K_\zeta(\Sigma) \otimes \mathbb{C}\big(\zeta^{1/2}\big)\big) \cap \mathcal{S}_\zeta^s\big(\Sigma_{0,n}^{\circ,\bullet}\big)$$

by Theorem 8.4, which is just $K_\zeta(\Sigma) \otimes \mathbb{C}\big[\zeta^{1/2},\zeta^{-1/2}\big]$. This proves the extension of $\mathcal{W}$ to the scalars $\mathbb{C}\big[\zeta^{1/2},\zeta^{-1/2}\big]$ is an isomorphism $K_\zeta(\Sigma) \otimes \mathbb{C}\big[\zeta^{1/2},\zeta^{-1/2}\big] \to \big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes_A \otimes \mathbb{C}\big[\zeta^{1/2},\zeta^{-1/2}\big]$. Therefore $\mathcal{W}$ is an isomorphism as well. This concludes the proof.   ■

Remark 8.7.

(1) Faitg results discussed above hold for surfaces of arbitrary genus. Also, in order to have at hands a calculus based on non oriented diagrams he uses a pivotal element different from K to define the map W (whence $\mathbb{R}\mathbb{T}$) (see [41, Remark 5.2]). This leads to the choice of t in his constructions, instead of $\zeta = iq^{1/2}$ as we obtained in the proof of Theorem 8.4.

(2) As a by-product of the proof we have $K_\zeta(\Sigma) \otimes \mathbb{C}\big[\zeta^{1/2},\zeta^{-1/2}\big] = \mathcal{S}_\zeta^s\big(\Sigma_{0,n}^{\circ,\bullet}\big)^{U_A}$.

(3) The module algebra $\mathcal{S}_\zeta^s\big(\Sigma_{0,n}^{\circ,\bullet}\big)$ is defined over $\mathbb{C}\big[\zeta^{1/2},\zeta^{-1/2}\big]$, and $K_\zeta(\Sigma)$ over $\mathbb{Z}\big[\zeta,\zeta^{-1}\big]$. One can observe that $\mathcal{L}_{0,n}^A$ is in fact defined over $\mathbb{Z}\big[q,q^{-1}\big]$, for arbitrary $\mathfrak{g}$, and Faitg's isomorphism $\tilde{\mathcal{W}}$ holds true over $\mathbb{Z}\big[\zeta^{1/2},\zeta^{-1/2}\big]$. Therefore $\mathcal{W}$ in Theorem 8.6 holds true over $\mathbb{Z}\big[\zeta,\zeta^{-1}\big]$.

8.3   The threading map and $\mathcal{G}$-invariant central elements

A *multicurve* on Σ is a union of disjoint simple non trivial (ie. not bounding a disk) closed curves considered up to isotopy. Any multicurve γ is a stacking product $\prod_{i=1}^k \gamma_i^{c_i}$, where $\gamma_1,\ldots,\gamma_k$ are disjoint (hence commuting) non-isotopic simple non trivial closed curves on Σ, and $\gamma_i^{c_i}$, $c_i \in \mathbb{N}$, consists of c_i parallel copies of γ_i.

Denote by $\mathcal{S}(\Sigma)$ the set of multicurves on Σ. Recall the normalized Chebyshev polynomials T_k, $k \in \mathbb{N}$, defined in (7.2). For $\gamma \in \mathcal{S}(\Sigma)$, $\gamma = \prod_{i=1}^k \gamma_i^{c_i}$, set

$$T(\gamma) := \prod_{i=1}^k T_{c_i}(\gamma_i).$$

It is standard that $\mathcal{S}(\Sigma)$ is a $\mathbb{Z}\big[\zeta,\zeta^{-1}\big]$-basis of $K_\zeta(\Sigma)$ (see, e.g., [72, Theorem 7]) and that $\{T_k\}_k$ is a basis of $\mathbb{Z}[X]$. Therefore $\{T(\gamma), \gamma \in \mathcal{S}(\Sigma)\}$ is a $\mathbb{Z}\big[\zeta,\zeta^{-1}\big]$-basis of $K_\zeta(\Sigma)$. It is called the *Chebyshev basis* (see [46] and the references therein). Theorem 8.6 implies that $\{\mathcal{W}(T(\gamma)), \gamma \in \mathcal{S}(\Sigma)\}$ is a $\mathbb{Z}\big[\zeta,\zeta^{-1}\big]$-basis of $\big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes_A \mathbb{Z}\big[\zeta,\zeta^{-1}\big]$.

Now let as usual $l \geq 3$ be an odd integer. Define $\mathcal{S}_l(\Sigma) \subset \mathcal{S}(\Sigma)$ as the set of multicurves of the form $\gamma_\partial \prod_{i=1}^k \gamma_i^{c_i}$, where γ_∂ is peripheral, i.e., a monomial in the skein classes of the boundary components of Σ, γ_i is a non peripheral curve, and l divides c_i for every $i \in \{1,\ldots,k\}$.

Let ϵ be a primitive root of unity of odd order l. Recall the element η in (6.25), the specialization $\big(\mathcal{L}_{0,n}^A\big)_\epsilon^{U_A}$ in (6.27), and the algebra $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$ and the derivations $D_a \colon \mathcal{L}_{0,n}^\epsilon \to \mathcal{L}_{0,n}^\epsilon$, $a \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)$, defined in Section 7.2.2.

Theorem 8.8.

(1) *The set $\{\mathcal{W}(T(\gamma)), \gamma \in \mathcal{S}_l(\Sigma)\}$ is a $\mathbb{C}$-basis of the algebra generated by $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$ and η. In particular, it is a central subalgebra of $\big(\mathcal{L}_{0,n}^A\big)_\epsilon^{U_A}$.*

(2) *The derivations D_a, $a \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$, act on $\big(\mathcal{L}_{0,n}^A\big)_\epsilon^{U_A}$.*

Proof. (1) First note that for every $\gamma \in \mathcal{S}(\Sigma)$, $\gamma = \gamma_\partial \prod_{i=1}^{k} \gamma_i^{c_i}$ with γ_∂ peripheral, we have

$$\mathcal{W}(T(\gamma)) = \mathcal{W}(T(\gamma_\partial)) \prod_{i=1}^{k} T_{c_i}(\mathcal{W}(\gamma_i)). \tag{8.5}$$

Denote by $\partial_1, \ldots, \partial_{n+1}$ the boundary components of Σ, ordered so that the diffeomorphism identifying Σ with the closure of $[0,1]^2 \setminus (\{p_1, \ldots, p_n\} \times \{1/2\})$ maps ∂_{n+1} to $\partial([0,1]^2)$, and ∂_i to a small loop encircling $(p_i, 1/2)$, $1 \le i \le n$. By (8.1) we have

$$\mathcal{W}(\partial_{n+1}) = \mathrm{qTr}\,\big(\overset{V_2}{M}{}^{(1)} \cdots \overset{V_2}{M}{}^{(n)}\big) = \eta, \ \ \mathcal{W}(\partial_i) = \mathrm{qTr}\,\big(\overset{V_2}{M}{}^{(i)}\big) = \omega^{(i)}, \qquad 1 \le i \le n. \tag{8.6}$$

Therefore the elements $\mathcal{W}(T(\gamma_\partial))$ form a basis of $\mathbb{C}\big[\omega^{(1)}, \ldots, \omega^{(n)}, \eta\big]$, which is a central subalgebra of $\big(\mathcal{L}_{0,n}^A\big)_\epsilon^{U_A}$ (see Theorem 6.27). We can give formulas of the other terms as follows. Deform a ribbon neighborhood of γ in $\Sigma \times [0,1]$ so that for every $i \in \{1, \ldots, k\}$, γ_i is represented by a one-coupon multicurve like in Figure 8.3, the coupon being filled with r simple arcs (possibly pairwise intersecting), for some $r \in \mathbb{N}$. Since γ_i is colored by V_2, we see that (8.2) takes the form

$$\mathcal{W}(\gamma_i) = \mathrm{qTr}_{V_2^{\otimes r}}\big(a(\gamma_i)\mathbb{M}(\gamma_i)\big),$$

where $a(\gamma_i) \in \mathrm{End}_{U_A^{\mathrm{res}}}\big(V_2^{\otimes r}\big)$, and $\mathbb{M}(\gamma_i)$ is defined as $\overset{[\lambda]}{\mathbb{M}}$, replacing the modules $V_{\lambda_1}, \ldots, V_{\lambda_n}$ associated to the sequence $[\lambda] = (\lambda_1, \ldots, \lambda_n)$ by V_2 or the trivial module V_1 (this latter case happens when the module labels a strand which has to be removed to get the one-coupon multicurve representing γ_i). In order to get simultaneously a simple expression of all the elements $\mathcal{W}(\gamma_1), \ldots, \mathcal{W}(\gamma_k)$, note that each γ_i, being a simple closed curve, separates Σ in two disks with punctures. We can choose the above diffeomorphism of Σ with the closure of $[0,1]^2 \setminus (\{p_1, \ldots, p_n\} \times \{1/2\})$ so that the ordering of $p_1, \ldots, p_n$ makes γ_i bounding a punctured disk $D_i \subset [0,1]^2$ with successive punctures $p_{j_i}, p_{j_i+1}, \ldots, p_{j_i+n_i-1}$. Then, for every $i \in \{1, \ldots, k\}$ we have

$$\mathcal{W}(\gamma_i) = \mathrm{qTr}_{V_2}\big(\overset{V_2}{M}{}^{(j_i)} \cdots \overset{V_2}{M}{}^{(j_i+n_i-1)}\big).$$

The algebras $\big(\mathcal{L}_{0,n}^A\big)^{U_A} \otimes_A \mathbb{Z}[\zeta, \zeta^{-1}]$, and therefore the expressions of $\mathcal{W}(\gamma_i)$, associated to different presentations of Σ as above are related by isomorphisms induced by the mapping class group of Σ.

Now assume that $\gamma \in \mathcal{S}_l(\Sigma)$, so l divides c_i, $1 \le i \le k$, in (8.5). Put $c_i' := c_i/l$. Then

$$T_{c_i}(\mathcal{W}(\gamma_i)) = T_{c_i'}(T_l(\mathcal{W}(\gamma_i))) = T_{c_i'}\big(\mathrm{Tr}\,\big(\mathrm{Fr}\,\underline{\overset{V_2}{M}}{}^{(j_i)} \cdots \mathrm{Fr}\,\underline{\overset{V_2}{M}}{}^{(j_i+n_i-1)}\big)\big), \tag{8.7}$$

where we use the standard identity $T_{c_i} = T_{c_i'} \circ T_l$ in the first equality, and Proposition 7.6 in the second.

By the first fundamental theorem of classical invariant theory for SL_2 (see, e.g., [59]) and the Cayley–Hamilton identity, the set of trace functions

$$t_{j_i, \ldots, j_i+n_i-1} \colon \big(\underline{\overset{V_2}{M}}{}^{(1)}, \ldots, \underline{\overset{V_2}{M}}{}^{(n)}\big) \mapsto \mathrm{Tr}\,\big(\underline{\overset{V_2}{M}}{}^{(j_i)} \cdots \underline{\overset{V_2}{M}}{}^{(j_i+n_i-1)}\big)$$

for all possible tuples $(j_i, \ldots, j_i+n_i-1)$ are generating functions of $\mathcal{O}(G^n)^G$. By (8.7) each of these functions is sent by Fr to $T_l(\mathcal{W}(\gamma_i))$ for some γ_i. By Corollary 7.20(3), $\widetilde{\mathrm{Fr}}$ is an isomorphism from $\mathcal{O}(\tilde{G}^n)^G$ to $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^\mathcal{G}$, mapping $\mathcal{O}(G^n)^G$ to $\mathcal{Z}_0(\mathcal{L}_{0,n}^\epsilon)^\mathcal{G}$, and the n copies of $\mathcal{O}(T)^W$ embedded in the n factors of $\mathcal{O}(\tilde{G})^{\otimes n} = \mathcal{O}(\tilde{G}^n)$ to $\mathbb{C}\big[\omega^{(1)}\big], \ldots, \mathbb{C}\big[\omega^{(n)}\big] \subset \big(\mathcal{L}_{0,n}^A\big)_\epsilon^{U_A}$. Therefore $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^\mathcal{G}$ is

spanned over $\mathbb{C}$ by the elements $\mathcal{W}(T(\gamma))$ with $\gamma \in \mathcal{S}_l(\Sigma)$ and γ_∂ a monomial in $\partial_1, \ldots, \partial_n$. In particular $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$ is a (central) subalgebra of $(\mathcal{L}_{0,n}^A)_\epsilon^{U_A}$.

(2) It is enough to show that the derivations D_a, $a \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$, restrict to endomorphisms of $(\mathcal{L}_{0,n}^A)_\epsilon^{U_A}$. This follows from the inclusion $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}} \subset (\mathcal{L}_{0,n}^A)_\epsilon^{U_A}$, since $[\tilde{a}, u] \in (\mathcal{L}_{0,n}^A)^{U_A}$ for every $\tilde{a}, u \in (\mathcal{L}_{0,n}^A)^{U_A}$ ($\mathcal{L}_{0,n}^A$ being an U_A-module algebra), which by the definition of D_a implies that $D_a((\mathcal{L}_{0,n}^A)_\epsilon^{U_A}) \subset (\mathcal{L}_{0,n}^A)_\epsilon^{U_A}$ when $a \in \mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$. ∎

Remark 8.9 (threading $T_l(V_2)$). For every $1 \leq i \leq n$, $0 \leq k \leq n - i$ we have

$$T_l\big(\mathrm{qTr}_{V_2}\big(\overset{V_2}{M}{}^{(i)}\ \overset{V_2}{M}{}^{(i+1)} \cdots \overset{V_2}{M}{}^{(i+k)}\big)\big) = \mathrm{qTr}_{T_l(V_2)}\big(\overset{T_l(V_2)}{M}{}^{(i)}\ \overset{T_l(V_2)}{M}{}^{(i+1)} \cdots \overset{T_l(V_2)}{M}{}^{(i+k)}\big),$$

where $T_l(V_2)$ is the virtual representation in the Grothendieck ring of U_A^{res}-modules, obtained by plugging V_2 in the l-th Chebyshev polynomial T_l. Indeed

$$T_l\big(\mathrm{qTr}\big(\overset{V_2}{M}\big)\big) = \mathrm{qTr}\big(\overset{T_l(V_2)}{M}\big)$$

because (using, e.g., the second picture in Figure 6.2)

$$\mathrm{qTr}\big(\overset{V_2^{\otimes k}}{M}\big) = \big(\mathrm{qTr}\big(\overset{V_2}{M}\big)\big)^k.$$

This and the computations in the first half of the proof of Proposition 7.6 prove our claim.

Remark 8.10. Recall that in Theorem 8.6 we have set $\zeta := iq^{1/2}$. Let ϵ' be such that $(\epsilon')^2 = -\epsilon$, and define $K_{\epsilon'}(\Sigma) := K_\zeta(\Sigma) \otimes_{\mathbb{Z}[\zeta, \zeta^{-1}]} \mathbb{C}_{\epsilon'}$, where $\mathbb{C}_{\epsilon'}$ is the $\mathbb{Z}[\zeta, \zeta^{-1}]$-module $\mathbb{C}$, where ζ acts by multiplication by ϵ'. By a result of [45], the set $\{T(\gamma), \gamma \in \mathcal{S}_l(\Sigma)\}$ is a $\mathbb{C}$-basis of the center of $K_{\epsilon'}(\Sigma)$. Therefore Theorem 8.6 and the proof of (1) show that $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$ and η generate $\mathcal{Z}((\mathcal{L}_{0,n}^A)_\epsilon^{U_A})$.

Remark 8.11. As in the proof of (1), denote by $\partial_1, \ldots, \partial_{n+1}$ the boundary components of Σ, and fix the generators of $\pi_1(\Sigma)$ to be $\partial_1, \ldots, \partial_n$ (for some choice of basepoints). Consider the fiber product (with the notations of Remark 7.16)

$$\tilde{G}^n \times_{G//G} G//G = \big\{((\tilde{g}_1, \ldots, \tilde{g}_n), [t]) \in \tilde{G}^n \times G//G \mid p(g_1 \cdots g_n) = p_l([t])\big\}.$$

As before Corollary 7.20, one can identify $\tilde{G}^n \times_{G//G} G//G$ with an l-fold branched covering space $\tilde{R}_G(\Sigma)$ of $\tilde{R}'_G(\Sigma)$, whose points are given by representations $\rho \colon \pi_1(\Sigma) \to G$ endowed with a choice of solution $x \in \mathbb{C}$ of the equation $T_l(x) = \mathrm{Tr}(\rho(\partial_i))$, for every $i = 1, \ldots, n$ and $i = n+1$. Taking algebraic quotients yields a branched covering map $\tilde{X}_G(\Sigma) \to X_G(\Sigma)$ of degree l^{n+1}, and one can lift $\widetilde{\mathrm{Fr}} \circ \mathfrak{c}^{*-1}$ to an isomorphism from $\mathcal{O}(\tilde{X}_G(\Sigma))$ onto the algebra generated by η and $\mathcal{Z}(\mathcal{L}_{0,n}^\epsilon)^{\mathcal{G}}$.

9 Applications to skein algebras

As usual denote by $K_\zeta(\Sigma)$ the skein algebra of the sphere with $n + 1$ punctures, $n > 1$. Recall $K_\zeta(\Sigma)$ is an algebra over $\mathbb{Z}[\zeta, \zeta^{-1}]$, where ζ is an indeterminate. In this section we use Theorem 8.6 to reformulate some of our results on $(\mathcal{L}_{0,n}^A)^{U_A}$ in the case of $\mathfrak{g} = \mathfrak{sl}(2)$ in terms of $K_\zeta(\Sigma)$.

In [16, 72] it was proved that $K_\zeta(\Sigma)$ does not have non trivial zero divisors, and its center was computed by topological means. On the contrary the proof we give below of these two facts is purely algebraic, based on properties of $(\mathcal{L}_{0,n}^A)^{U_A}$ proved in Section 6, that hold true for any complex finite dimensional simple Lie algebra $\mathfrak{g}$.

Corollary 9.1. *The skein algebra $K_\zeta(\Sigma)$ does not have non trivial zero divisors, and its center is the polynomial algebra over $\mathbb{Z}[\zeta, \zeta^{-1}]$ generated by the classes $\partial_1, \ldots, \partial_{n+1}$ of the boundary components of Σ.*

Proof. The claims are direct consequences of Theorem 8.6, the formulas (8.6), and Proposition 6.11 and the last claim of Theorem 6.27 in the case of $\mathfrak{g} = \mathfrak{sl}(2)$. ∎

Next we deduce from the results of Section 7 some properties of the center of the specializations of $K_\zeta(\Sigma)$ at roots of unity of order $4l$, $l \geq 3$ odd. Let ϵ' be such that $(\epsilon')^2 = -\epsilon$. Define

$$K_{\epsilon'}(\Sigma) := K_\zeta(\Sigma) \otimes_{\mathbb{Z}[\zeta, \zeta^{-1}]} \mathbb{C}_{\epsilon'},$$

where $\mathbb{C}_{\epsilon'}$ is the $\mathbb{Z}[\zeta, \zeta^{-1}]$-module $\mathbb{C}$, where ζ acts by multiplication by ϵ'. The sets $\mathcal{S}(\Sigma)$ and $\{T(\gamma), \gamma \in \mathcal{S}(\Sigma)\}$ are $\mathbb{Z}[\zeta, \zeta^{-1}]$-basis of $K_\zeta(\Sigma)$ as well as $\mathbb{C}$-basis of $K_{\epsilon'}(\Sigma)$. By Theorem 8.6 we get an isomorphism of algebras

$$\mathcal{W}\colon \; K_{\epsilon'}(\Sigma) \to \left(\mathcal{L}_{0,n}^A\right)_\epsilon^{U_A}.$$

Denote by $\mathcal{Z}(K_{\epsilon'}(\Sigma))$ the center of $K_{\epsilon'}(\Sigma)$. Theorem 8.8(1) implies immediately:

Corollary 9.2. *The set $\{T(\gamma), \gamma \in \mathcal{S}_l(\Sigma)\}$ is a $\mathbb{C}$-basis of a central subalgebra $\mathcal{Z}'(K_{\epsilon'}(\Sigma))$ of $K_{\epsilon'}(\Sigma)$, which is generated by η and the image of the embedding*

$$\mathrm{Ch}_\mathcal{W} := \mathcal{W}^{-1} \circ \widetilde{\mathrm{Fr}} \circ \tilde{\mathfrak{c}}^{*-1}\colon \; \mathcal{O}(\tilde{X}'_G(\Sigma)) \to \mathcal{Z}(K_{\epsilon'}(\Sigma)).$$

Therefore $\mathcal{Z}'(K_{\epsilon'}(\Sigma))$ is endowed with a natural Poisson bracket, the image of $\{\,,\,\}_{\mathrm{Gold}}$, which extends to an action by derivations of $\mathcal{Z}'(K_{\epsilon'}(\Sigma))$ on $K_{\epsilon'}(\Sigma)$.

In fact $\mathcal{Z}'(K_{\epsilon'}(\Sigma)) = \mathcal{Z}(K_{\epsilon'}(\Sigma))$ (see [45]). One can check that $\mathrm{Ch}_\mathcal{W}$ is a version of the threading map $\mathrm{Ch}\colon \mathcal{K}_{\epsilon'l^2}(\Sigma) \to \mathcal{Z}_{\epsilon'}(\Sigma)$ of Bonahon–Wong (see [17], and also [46]). The point is that it affords an explicit realization of Ch in classical invariant theory terms (via the formulas (8.5)–(8.7)), and pulls the geometric tools of Section 7 onto $K_{\epsilon'}(\Sigma)$. A more symmetric statement is obtained by extending $\mathrm{Ch}_\mathcal{W}$ to the ring $\mathcal{O}(\tilde{X}_G(\Sigma))$ of Remark 8.11, so that ∂_{n+1} belongs to its image.

Acknowledgements

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