A Faithful Braid Group Action on the Stable Category of Tricomplexes

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Abstract. Bicomplexes of vector spaces frequently appear throughout algebra and geometry. In Section 2 we explain how to think about the arrows in the spectral sequence of a bicomplex via its indecomposable summands. Polycomplexes seem to be much more rare. In Section 3 of this paper we rethink a well-known faithful categorical braid group action via an action on the stable category of tricomplexes.

Key words: braid group; categorical action; bicomplexes; spectral sequence; tricomplexes; stable category

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To Dmitry Fuchs on his 80th birthday

1 Introduction

> "The impact of spectral sequences on algebraic topology was tremendous: Many major problems of topology, both solved and unsolved, became exercises for students ..."
> A. Fomenko and D. Fuchs [6, Preface]

Representation theory, which has been established for over a century, deals with linear actions of groups and algebras. Much more recent is the discovery of interesting categorical actions of groups, primarily discrete groups. In these examples discrete groups act by symmetries of categories, which in many cases are triangulated, and the action preserves the triangular structure. One of the first nontrivial examples appeared in [16], see also [10, 22]. There the n-strand braid group Br_n acts on the homotopy category of complexes of modules over a particular finite-dimensional algebra A_{n-1}. The action is by exact functors and on the Grothendieck group the action descends either to the Burau representation of the braid group (if one keep tracks of an additional grading on modules, in addition to the homological grading) or to the reduced permutation action of the symmetric group. Neither of these linear actions is faithful, but its categorical lifting was shown to be faithful in [16].

The algebra A_{n-1} (the *zigzag* algebra) is the quotient of the path algebra of the quiver with $n-1$ vertices and edges connecting adjacent vertices in both directions (assuming $n>3$, with

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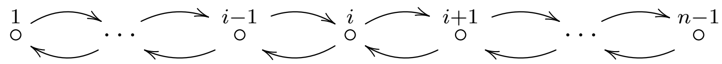
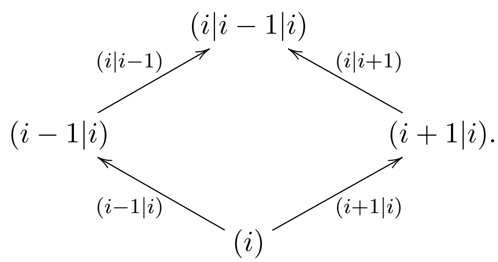
minor changes necessary for $n = 2, 3$)



Generators of A_{n-1} corresponding to arrows in the quiver are denoted $(i|i \pm 1)$. The defining relations

$$(i|i+1|i+2) = 0, \qquad (i|i-1|i-2) = 0, \qquad (i|i-1|i) = (i|i+1|i)$$

(for i's for which both sides of a relation make sense) are quadratic, A_{n-1} is finite dimensional, with a basis consisting of idempotents (i), edges $(i, i \pm 1)$ and length two paths $(i|i \pm 1|i)$. For $1 < i < n$ indecomposable projective A_{n-1} module $P_i = A_{n-1}(i)$ is four-dimensional, with the basis $\{(i), (i-1|i), (i+1|i), (i|i-1|i)\}$ and can be visualized as a diamond.



The defining relations in A_{n-1} can be interpreted as the defining relations in the category of bicomplexes. Namely, let

$$\partial_1 = \sum_{i=1}^{n-2} (i|i+1), \qquad \partial_2 = \sum_{i=2}^{n-1} (i|i-1).$$

Then the defining relations in A_{n-1} can be rewritten as

$$\partial_1^2 = 0, \qquad \partial_2^2 = 0, \qquad \partial_1 \partial_2 = \partial_2 \partial_1.$$

These are exactly the relations on the two differentials in a bicomplex. A bicomplex is built out of vector spaces placed in the vertices of an integral lattice $\mathbb{Z}^2$, with the differentials going along the two coordinates, with the unit step each.

One can introduce a grading on A_{n-1} by making, for instance, left-pointing arrows (edges) in the quiver to have degree one and right-pointing edges degree zero. The unit element of A_{n-1} decomposes as the sum of $n-1$ idempotents, one for each vertex of the graph, $1 = (1) + (2) + \cdots + (n-1)$, inducing the decomposition of an A_{n-1}-module into a sum of vector spaces

$$M = \bigoplus_{i=1}^{n-1} (i)M,$$

and the additional grading on M leads to the bigrading, with the left and right directed edges changing the bigrading by $(1, 0)$ and $(0, 1)$, respectively.

In this way, graded A_{n-1}-modules may be identified with bicomplexes with nonzero terms restricted to a suitable area of the lattice $\mathbb{Z}^2$. Changing the indexing of quiver vertices from $\{1, 2, \ldots, n-1\}$ to $\mathbb{Z}$ by passing to the quiver that is infinite in both directions (see figure in equation (2.2)) results in a non-unital algebra A_∞ with a system of idempotents $\{(i)\}_{i \in \mathbb{Z}}$ such that graded A_∞-modules naturally correspond to bicomplexes.

The braid group Br_n acts on the homotopy category of (either graded or ungraded) A_{n-1}-modules by tensoring with a suitable complex of A_{n-1}-bimodules. This works as well in the limit of A_∞-modules, with the braid group Br_∞ with strands (and generators σ_i) enumerated by integers.

Passing from modules over an algebra B to complexes of modules means working with suitably graded modules over the algebra $B[d]/(d^2)$. In our case, graded A_{n-1} or A_∞ modules can be identified with bicomplexes (more precisely, there is an equivalence of corresponding abelian categories). Consequently, complexes of A_{n-1} and A_∞-modules may be identified with tricomplexes, with the homological grading in $A_{n-1}[d]/(d^2)$ corresponding to the additional, third, grading in tricomplexes.

Passing from complexes to the homotopy category of complexes (of modules over an algebra B) means modding out by null-homotopic morphisms. If one restricts to complexes of projective B-modules, which is a common and important subcategory of the category of complexes, this means killing morphisms which factor through a direct sum of objects of the form

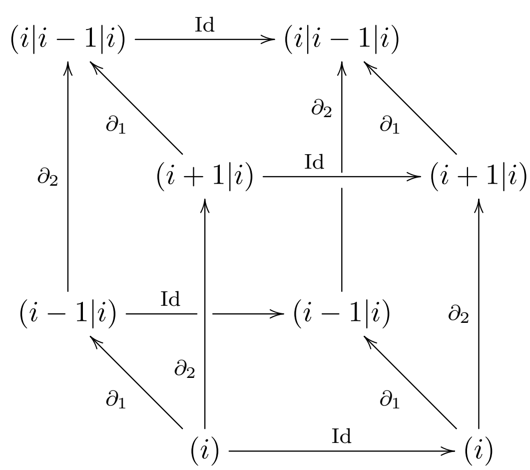
$$0 \longrightarrow B \xrightarrow{\mathrm{Id}} B \longrightarrow 0$$

in various homological degrees. Specializing B to A_∞, the above complex decomposes as a direct sum of terms of the form

$$0 \longrightarrow A_\infty(i) \xrightarrow{\mathrm{Id}} A_\infty(i) \longrightarrow 0 \tag{1.1}$$

for various $i \in \mathbb{Z}$ (cf. the next diagram below). By keeping track of the additional grading, one can further shift these copies of $A_\infty(i)$ and parametrize them by a pair of integers (i, j). Together with the homological grading k, one gets a 3-parameter family of possible indecomposable summands that each represent the zero complex in the homotopy category.

If this setup is converted into the language of bicomplexes and tricomplexes, the module $A_\infty(i)$ corresponds to a free rank one bicomplex in the bigrading associated to the idempotent (i) and another independent grading j, see Definition 2.8. The complex (1.1) corresponds to a free rank one tricomplex, with its generator placed in tridegree $(i, 0, 0)$. We refer the reader to (3.15) and (3.16) for the precise matching of trigradings and shifts:



Here in the diagram, the (basis elements of the) first copy of $A_\infty(i)$ in (1.1) is exhibited as the left-most square in the cube, while the second copy of A_∞ is displayed as the right-most square. These two squares are connected by the homological differential labelled with Id maps.

In the homotopy category of projective graded A_∞-modules, a morphism is zero if it factors through the object which is a direct sum of complexes (1.1) over various i, j, and k, where i labels the idempotent, j is the additional grading parameter in A_∞, and k is the homological grading. Converting this to tricomplexes, one unites the three integer grading parameters i, j, k of different origins into a single trigrading on tricomplexes. The complex (1.1) becomes a free tricomplex of rank one that can sit in any position relative to the trigrading. Killing morphisms that factor through sums of such free rank one tricomplexes is equivalent to the condition that one is working in the stable category of tricomplexes, that is in the category of tricomplexes modulo the ideal of morphisms that factor through a free tricomplex.

Tricomplexes can be described as trigraded modules over the algebra Λ_3 with generators ∂_1, ∂_2, ∂_3 and relations

$$\partial_i^2 = 0, \qquad i = 1, 2, 3, \qquad \partial_i\partial_j = \partial_j\partial_i, \qquad i \neq j.$$

This 8-dimensional algebra is Frobenius, and it is even a Hopf algebra in the category of super-vector spaces. Consequently, its stable category of trigraded modules is triangulated (and monoidal, due to the Hopf algebra structure).

The braid group action on the homotopy category of A_{n-1} and A_∞-modules transfers to the stable category of tricomplexes. Note that the homotopy category of A_∞-modules and the stable category of tricomplexes are not equivalent, but rather admit equivalent subcategories with matching actions of the braid group. On the A_∞ side, it is the homotopy category of complexes of projective modules, and on the tricomplex side, the stable subcategory generated by tricomplexes that restrict to free bicomplexes relative to the subalgebra generated by differentials ∂_1, ∂_2. The braid group action respects these subcategories and the equivalence between them.

The braid group acts by exact functors on this triangulated category of tricomplexes. The actions does not respect the monoidal structure, though, and choosing the action requires singling out one differential out of three. Choosing different differentials gives three commuting braid group actions.

For now, we view this example as a curiosity. One natural question is whether our example fits into the more general framework of Hopfological algebra [14, 21], where stable categories of modules over Hopf algebras, such as Λ_3, are used as base categories for new constructions of categorifications (see, e.g., [15]) or, perhaps, some other algebro-geometric structures. Another open problem is whether homotopy categories of complexes over other algebras of importance in categorification, such as arc algebras [12], can be rethought through some generalization of the stable category of tricomplexes.

Tricomplexes seem to appear exceedingly rarely in mathematics. Currently, they have made appearances in the BRST theory [23], in the deformation theory of Hopf algebras [28], and in the algebraic K-theory [2]. A modified notion of a tricomplex, called quasi-tricomplex, occurs in the theory of variation [20].

Beyond tricomplexes, polycomplexes can be related to $(\mathbb{C}^*)^n$-equivariant coherent sheaves on $\mathbb{CP}^{n-1}$ via a version of Beilinson–Gelfand–Gelfand–Koszul duality.

The braid group action on the stable category of tricomplexes is constructed in Section 3 of this paper. In Section 2 we explain a way to think about arrows in the spectral sequence of a bicomplex of vector spaces via indecomposable modules over the rings A_{n-1} and A_∞. This relation was independently discovered by Stelzig [25].

2 Spectral sequences via indecomposable bicomplexes

> "The subject of spectral sequences is elementary, but the notion of the spectral sequence of a double complex involves so many objects and indices that it seems at first repulsive." D. Eisenbud [5, Appendix 3.13]

The standard textbook approach to spectral sequences makes them seem sophisticated and mysterious gadgets [1, 7, 18, 26, 27] and [5, Appendix 3.13]. Timothy Chow, in the introduction to his article on spectral sequences [3], quotes the opinions of experts who, essentially, say that the definition is so complicated that you just have to get used to it.

The goal of this section is to explain spectral sequences, restricted to bicomplexes of vector spaces, in a simple and straightforward way. Most of this section has appeared in lectures to graduate students by the first author, see for instance the informal lecture notes [13]. Similar results also appeared in Stelzig [25]. We warn the reader that this elementary approach works only for a bicomplex of vector spaces. Bicomplexes and filtered complexes that appear in spectral sequences in algebraic topology carry an enormous amount of extra structure, such as an action of the Steenrod algebra when working over $\mathbb{Z}/p$, and cannot be easily understood in this elementary way. The complexity and beauty of these structures are captured in the Fomenko and Fuchs classic [6] and other books, see McCleary [18].

2.1 Cohomology

Let $\Bbbk$ be a field and M

$$\dots \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \dots$$

a complex of $\Bbbk$-vector spaces. We allow unbounded complexes and infinite-dimensional vector spaces. It is easy to see that M decomposes into the direct sum of length zero complexes

$$0 \longrightarrow \mathrm{H}^i \longrightarrow 0,$$

with a vector space H^i in degree i, and length one complexes

$$0 \longrightarrow W^i \xrightarrow{\mathrm{id}} W^i \longrightarrow 0, \tag{2.1}$$

with two copies of a vector space W^i in degrees i and $i+1$. Thus,

$$M^i \cong \mathrm{H}^i \oplus W^i \oplus W^{i-1},$$

although this direct sum decomposition of the vector space M^i is not canonical. The inclusion of the direct sum $\mathrm{H}^i \oplus W^{i-1} \subset M^i$ is canonical, being the inclusion $\ker(d^i) \subset M^i$. The i-th cohomology group $\mathrm{H}^i(M)$ of M is canonically isomorphic to H^i. Direct summands (2.1) are contractible (recall that a complex is called *contractible* if the identity endomorphism is null-homotopic).

Example 2.1. Let X be a smooth compact manifold and $(\Omega(X), d)$ the de Rham complex of smooth forms on X. In this case $\mathrm{H}^i(X, \mathbb{R})$ are finite-dimensional vector spaces, while the vector spaces $\Omega^i(X)$ and hence W^i are infinite-dimensional. The bulk of the complex $\Omega(X)$ is occupied by contractible "junk", while the "valuable part" (cohomology) has small size. If we equip X with a Riemannian metric g, the operator $d^* = \pm * d *$ adjoint to d gives rise to the Laplace operator

$$\Delta\colon\ \Omega^i(X) \longrightarrow \Omega^i(X), \qquad \Delta = dd^* + d^*d.$$

The Laplace operator provides a *canonical* embedding of each complex $0 \longrightarrow \mathrm{H}^i(X, \mathbb{R}) \longrightarrow 0$ into the complex $(\Omega(X), d)$, via the isomorphism $\mathrm{H}(X, \mathbb{R}) \cong \ker(\Delta)$.

A complex of $\Bbbk$-vector spaces is the same as a graded module over the exterior $\Bbbk$-algebra Λ_1 on one generator d of degree 1:

$$\Lambda_1 := \Bbbk[d]/\big(d^2\big).$$

The i-th homogeneous piece of a graded Λ_1-module M is a vector space M^i, and the action of d is exactly the differential $d\colon M^i \longrightarrow M^{i+1}$.

The category $\mathcal{M}_1$ of graded modules over Λ_1 is Krull–Schmidt, and any module (even infinite-dimensional) decomposes into a direct sum of indecomposable modules S^i and P^i. Here S^i is the one-dimensional $\Bbbk$-vector space placed in degree i, and corresponds to the complex $0\longrightarrow\Bbbk\longrightarrow 0$. The differential acts by 0 and the module S^i is simple. The module $P^i = \Lambda_1\{i\}$ is free and corresponds to the complex

$$0\longrightarrow\Bbbk \xrightarrow{\ 1\ } \Bbbk\longrightarrow 0.$$

Thus,

$$M \cong \bigoplus_{i\in\mathbb{Z}} \big(\mathrm{H}^i \otimes S^i\big) \oplus \big(W^i \otimes P^i\big),$$

and the cohomology of M only catches the first terms in the sum. Recall that an object M of an additive category is called *indecomposable* if M is not isomorphic to a direct sum $N_1 \oplus N_2$ with both N_1, N_2 nontrivial.

2.2 Bicomplexes

Let us now move on to bicomplexes. A bicomplex M over a field $\Bbbk$ is a family $\{M^{i,j}\}$ of vector spaces, for $i, j \in \mathbb{Z}$, and maps

$$\partial_1\colon\ M^{i,j}\longrightarrow M^{i+1,j}, \qquad \partial_2\colon\ M^{i,j}\longrightarrow M^{i,j+1}$$

subject to the equations

$$\partial_1^2 = 0, \qquad \partial_2^2 = 0, \qquad \partial_1\partial_2 + \partial_2\partial_1 = 0,$$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle \partial_2} & & \uparrow{\scriptstyle \partial_2} & & \\
 & \xrightarrow{\partial_1} & M^{i,j+1} & \xrightarrow{\partial_1} & M^{i+1,j+1} & \xrightarrow{\partial_1} & \\
 & & \uparrow{\scriptstyle \partial_2} & & \uparrow{\scriptstyle \partial_2} & & \\
 & \xrightarrow{\partial_1} & M^{i,j} & \xrightarrow{\partial_1} & M^{i,j+1} & \xrightarrow{\partial_1} & \\
 & & \uparrow{\scriptstyle \partial_2} & & \uparrow{\scriptstyle \partial_2} & &
\end{array}$$

Let Λ_2 be the exterior $\Bbbk$-algebra on two generators ∂_1, ∂_2, so that the above equations are the defining relations for the generators. Λ_2 has a natural bigrading by

$$\deg(\partial_1) = (1, 0), \qquad \deg(\partial_2) = (0, 1).$$

A bicomplex M is the same as a bigraded left Λ_2-module. We denote the category of bicomplexes by $\mathcal{M}_2$.

We say that a bicomplex M is *bounded* if only finitely many $M^{i,j}$ are not 0.

Example 2.2. Let us describe some bounded indecomposable bicomplexes.

(1) The bicomplex $S^{i,j}$ is one-dimensional with a copy of $\mathbb{k}$ sitting in the (i,j)-th bidegree:

$$S^{i,j}\colon\quad \begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{k} & \longrightarrow & 0. \\ & & \uparrow & & \\ & & 0 & & \end{array}$$

In other words, $S^{i,j}$ is the simple Λ_2-module sitting in bidegree (i,j).

(2) The indecomposable bicomplex $P^{i,j} \cong \Lambda_2\{i,j\}$ is a free rank one module (looking like a square on a planar lattice), a copy of Λ_2 with bigrading shifted, so that the nonzero term in the southwest corner sits in (i,j)-th degree:

$$P^{i,j}\colon\quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{k} & \longrightarrow & \mathbb{k} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{k} & \longrightarrow & \mathbb{k} & \longrightarrow & 0. \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

(3) The bicomplex $Z^{i,j}_{\rightarrow,l}$ has the top leftmost term in bidegree (i,j) and goes zigzag to the right and down. The number $l \in \mathbb{N}$ denotes the number of nonzero arrows, $l+1$ is the dimension of the vector space underlying this bicomplex

$$Z^{i,j}_{\rightarrow,l}\colon\quad \begin{array}{ccccccc} \mathbb{k} & \longrightarrow & \mathbb{k} & & & & \\ & & \uparrow & & & & \\ & & \mathbb{k} & \longrightarrow & \mathbb{k} & & \\ & & & & \uparrow & & \\ & & & & \ddots & \longrightarrow & \mathbb{k} \\ & & & & & & \uparrow \\ & & & & & & \mathbb{k}. \end{array}$$

(4) Likewise, the bicomplex $Z^{i,j}_{\uparrow,l}$ starts from the bidegree (i,j) and goes zigzag down and to the right.

$$Z^{i,j}_{\uparrow,l}\colon\quad \begin{array}{ccccccc} \mathbb{k} & & & & & & \\ \uparrow & & & & & & \\ \mathbb{k} & \longrightarrow & \mathbb{k} & & & & \\ & & \uparrow & & & & \\ & & \ddots & \longrightarrow & \mathbb{k} & & \\ & & & & \uparrow & & \\ & & & & \mathbb{k} & \longrightarrow & \mathbb{k}. \end{array}$$

Theorem 2.3. *Any bounded bicomplex $M \in \mathcal{M}_2$ (possibly with infinite-dimensional vector spaces $M^{i,j}$) breaks up into a direct sum of indecomposable bicomplexes $S^{i,j}$, $P^{i,j}$, $Z^{i,j}_{\leftarrow,l}$, $Z^{i,j}_{\uparrow,l}$ described above.*

We will postpone the proof of the theorem until Section 2.4.

Let $\mathrm{Tot}(M)$ be the total complex of the bicomplex M, with the differential $d = \partial_1 + \partial_2$ and the terms given by direct sums over $M^{i,j}$ for $i + j$ fixed,

$$\cdots \xrightarrow{d} \mathrm{Tot}^k(M) \xrightarrow{d} \mathrm{Tot}^{k+1}(M) \xrightarrow{d} \cdots,$$

where $\mathrm{Tot}^k(M) = \bigoplus_{i+j=k} M^{i,j}$.

A common situation is that we want to compute the homology of $\mathrm{Tot}(M)$ with respect to the differential d and already know the homology of M with respect to, say, differential ∂_2 (the upward differential in our conventions). These homology groups $\mathrm{H}(M, \partial_2)$ are bigraded,

$$\mathrm{H}(M, \partial_2) = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{H}^{i,j}(M, \partial_2),$$

and we would like to understand the relation between them and $\mathrm{H}(\mathrm{Tot}(M), d)$. If we write M as a (possibly infinite) direct sum of indecomposable bicomplexes M_α, for α in some index set A, then both $\mathrm{H}(M, \partial_2)$ and $\mathrm{H}(\mathrm{Tot}(M), d)$ decompose as direct sums of cohomology groups of M_α:

$$\mathrm{H}(M, \partial_2) \cong \bigoplus_{\alpha\in A} \mathrm{H}(M_\alpha, \partial_2),$$
$$\mathrm{H}(\mathrm{Tot}(M), d) \cong \bigoplus_{\alpha\in A} \mathrm{H}(\mathrm{Tot}(M_\alpha), d).$$

Hence, we will compare $\mathrm{H}(M, \partial_2)$ and $\mathrm{H}(\mathrm{Tot}(M), d)$ for all types of indecomposable summands of M, case by case.

Case 1. $S^{i,j}$ contributes a copy of $\Bbbk$ to $\mathrm{H}^{i,j}(M, \partial_2)$ and a copy of $\Bbbk$ to $\mathrm{H}^{i+j}(\mathrm{Tot}(M), d)$.

Case 2. $\mathrm{H}(P^{i,j}, \partial_2) = 0$ and $\mathrm{H}(\mathrm{Tot}(P^{i,j}), d) = 0$. Thus, the "square" indecomposable bicomplex $P^{i,j}$ contributes nothing to both $\mathrm{H}(M, \partial_2)$ and $\mathrm{H}(\mathrm{Tot}(M), d)$.

For the module $Z^{i,j}_{\uparrow,l}$, there are two sub-cases.

Case 3.a. Firstly, let l, the number of nonzero arrows in the zigzag, be odd in $Z^{i,j}_{\uparrow,l}$,

$$Z^{i,j}_{\uparrow,l}: \quad \begin{array}{ccccc} \Bbbk & & & & \\ \uparrow & & & & \\ \Bbbk & \longrightarrow & \Bbbk & & \\ & & \uparrow & & \\ & & \ddots & \longrightarrow & \Bbbk \\ & & & & \uparrow \\ & & & & \Bbbk. \end{array}$$

Cohomology of $Z^{i,j}_{\uparrow,l}$ with respect to the vertical differential ∂_2 is zero. The total complex of this zigzag has the form

$$0 \longrightarrow \Bbbk^r \xrightarrow{d} \Bbbk^r \longrightarrow 0,$$

where d is an isomorphism and $2r = l + 1$. Hence, cohomology of the total complex is zero as well.

Case 3.b. Suppose now that l in $Z^{i,j}_{\uparrow,l}$ is even, $l = 2r$,

$$Z^{i,j}_{\uparrow,l}: \quad \begin{array}{ccccccc} \Bbbk & & & & & & \\ \uparrow & & & & & & \\ \Bbbk & \longrightarrow & \Bbbk & & & & \\ & & \uparrow & & & & \\ & & \ddots & \longrightarrow & \Bbbk & & \\ & & & & \uparrow & & \\ & & & & \Bbbk & \longrightarrow & \Bbbk. \end{array}$$

Cohomology with respect to ∂_2 produce a single $\Bbbk$ in bidegree $(i+r, j-r)$. The total complex has the form

$$0 \longrightarrow \Bbbk^r \xrightarrow{d} \Bbbk^{r+1} \longrightarrow 0$$

with d injective. Cohomology of the total complex is $\Bbbk$ in degree $i+j$ and zero elsewhere.

Case 4.a. For the module $Z^{i,j}_{\rightarrow,l}$, there are two sub-cases as well. We start with even $l = 2r$,

$$Z^{i,j}_{\rightarrow,l} \quad (l = 2r): \quad \begin{array}{ccccccc} \Bbbk & \longrightarrow & \Bbbk & & & & \\ & & \uparrow & & & & \\ & & \Bbbk & \longrightarrow & \Bbbk & & \\ & & & & \uparrow & & \\ & & & & \ddots & \longrightarrow & \Bbbk \\ & & & & & & \uparrow \\ & & & & & & \Bbbk. \end{array}$$

Cohomology with respect to ∂_2 give a single $\Bbbk$ in bidegree (i, j). The total complex is

$$0 \longrightarrow \Bbbk^{r+1} \xrightarrow{d} \Bbbk^{r} \longrightarrow 0$$

with a surjective d, and it has cohomology $\Bbbk$ in degree $i+j$ and zero elsewhere.

Before we treat the last case, observe that in each of the above cases cohomology of the total complex is given by simply collapsing the bigrading of $\mathrm{H}(M, \partial_2)$ into a single grading by adding i and j. Thus, if M does not contain any direct summands isomorphic to $Z^{i,j}_{\rightarrow,l}$ with odd l,

$$\mathrm{H}^k(\mathrm{Tot}(M), d) = \bigoplus_{i+j=k} \mathrm{H}^{i,j}(M, \partial_2).$$

Case 4.b. Lastly, consider $Z^{i,j}_{\rightarrow,l}$ with odd $l = 2r+1$,

$$Z^{i,j}_{\rightarrow,l} \quad (l = 2r+1): \quad \begin{array}{ccccccccc} \Bbbk & \longrightarrow & \Bbbk & & & & & & \\ & & \uparrow & & & & & & \\ & & \Bbbk & \longrightarrow & \Bbbk & & & & \\ & & & & \uparrow & & & & \\ & & & & \ddots & \longrightarrow & \Bbbk & & \\ & & & & & & \uparrow & & \\ & & & & & & \Bbbk & \longrightarrow & \Bbbk. \end{array}$$

Taking cohomology with respect to ∂_2 produces two copies of $\Bbbk$, in bigradings that differ by $(r, 1-r)$:

$$\mathrm{H}(Z^{i,j}_{\to,l}, \partial_2) = \begin{array}{ccccccc} \Bbbk & \longrightarrow & 0 & & & & \\ & & \uparrow & & & & \\ & & 0 & \longrightarrow & 0 & & \\ & & & & \uparrow & & \\ & & & & \ddots & \longrightarrow & 0 \\ & & & & & & \uparrow \\ & & & & & & 0 \longrightarrow \Bbbk. \end{array}$$

Collapsing the bigrading in cohomology gives us two copies of $\Bbbk$ in adjacent degrees $i+j$ and $i+j+1$.

The total complex has the form

$$0 \longrightarrow \Bbbk^{r+1} \xrightarrow{d} \Bbbk^{r+1} \longrightarrow 0$$

with d an isomorphism, and the cohomology of the total complex is zero. Thus, for a general bounded bicomplex M, the cohomology $\mathrm{H}(M, \partial_2)$, after the bigrading collapsed into a single grading, is isomorphic to the cohomology of the total complex of M, plus pairs of copies of the ground field in adjacent degrees $(i+j, i+j+1)$, for each direct summand of M isomorphic to $Z^{i,j}_{\to,l}$ with odd l.

Since we want to know the cohomology of the total complex, the extraneous terms need to be eliminated. Ideally, we would locate all direct summands $Z^{i,j}_{\to,2r+1}$ and kill off pairs of $\Bbbk$, one for each summand, in the relative bigrading position $(r, 1-r)$. For a general r, we need to eliminate pairs in the relative positions (i,j) and $(i+r, j-r+1)$ by a map $d_r^{i,j}$:

$$\begin{array}{ccccc} \Bbbk & 0 & & & \\ & \searrow & & & \\ & 0 & 0 & & \\ & & \searrow \; {\scriptstyle d_r^{i,j}=1} & & \\ & & \ddots & 0 & \\ & & & \searrow & \\ & & & 0 & \Bbbk \end{array}$$

on the square lattice. This is exactly what the spectral sequence does. The E_1-term of the spectral sequence of the bicomplex $(M, \partial_1, \partial_2)$ is the cohomology of M with respect to ∂_2:

$$E_1^{i,j} = \mathrm{H}^{i,j}(M, \partial_2).$$

To pass to the E_2-term, we remove contributions to $\mathrm{H}(M, \partial_2)$ from the direct summands $Z^{i,j}_{\to,1}$, which are $\Bbbk \xrightarrow{1} \Bbbk$. Notice that the E_2-term is simply the cohomology of $\mathrm{H}(M, \partial_2)$ with respect to the differential ∂_1 (more accurately, differential ∂_1 on M descends to a differential on $\mathrm{H}(M, \partial_2)$, which we also call ∂_1):

$$E_2 = \mathrm{H}(\mathrm{H}(M, \partial_2), \partial_1).$$

Going from E_2 to the E_3-term, we remove pairs of one-dimensional vector spaces $\Bbbk$ which come from summands $Z^{i,j}_{\to,3}$ and differ by $(2,-1)$-bigrading. In general, in the E_r-term there are no contributions from summands $Z^{i,j}_{\to,l}$ for all odd $l \leq 2r-1$.

The reader can find an accurate definition of spaces $E_r^{i,j}$ and differentials $d_r^{i,j}$ in almost any textbook on homological algebra, often done in a slightly different framework of a filtered complex rather than a bicomplex. However, we find the above approach via indecomposable bicomplexes more clarifying and intuitive than the standard textbook definition of the pages E_r and differentials $d_r^{i,j}$ of a spectral sequence.

2.3 Bicomplexes and Hodge theory

The Hodge bicomplex [4, 8, 27]. Let X be a closed almost complex manifold. This means X is a smooth closed manifold equipped with an endomorphism J of its real tangent bundle $T_{\mathbb{R}}(X)$ such that $J^2 = -1$. The complexified tangent bundle $T(X) = T_{\mathbb{R}}(X) \otimes_{\mathbb{R}} \mathbb{C}$ of X decomposes into the direct sum of i and $-i$ eigenspaces of J,

$$T(X) = T^{1,0}(X) \oplus T^{0,1}(X).$$

This induces a direct sum decomposition of all exterior powers $\wedge^k T^*$ of the complexified cotangent bundle $T^*(X)_c$:

$$\wedge^k T^* = \bigoplus_{i+j=k} \wedge^{i,j} T^*.$$

Let $\Omega^k_{\mathbb{C}}(X)$ be the space of smooth sections of $\wedge^k T^*$ and $(\Omega_{\mathbb{C}}(X), d)$ the complex with d the complexified de Rham differential:

$$\cdots \xrightarrow{d} \Omega^k_{\mathbb{C}}(X) \xrightarrow{d} \Omega^{k+1}_{\mathbb{C}}(X) \xrightarrow{d} \cdots .$$

Let $\Omega^{i,j}(X)$ be the vector space of smooth sections of $\wedge^{i,j} T^*$. In general, d shows no respect for the direct sum decomposition

$$\Omega^k_{\mathbb{C}}(X) = \mathop{\oplus}_{i+j=k} \Omega^{i,j}(X).$$

However, Newlander and Nirenberg proved [19] that d takes $\Omega^{i,j}(X)$ to $\Omega^{i+1,j}(X) \oplus \Omega^{i,j+1}(X)$ for all i,j if and only if the almost complex structure J of X comes from a complex structure on X. In this case $d = \partial + \bar\partial$, where

$$\partial\colon\ \Omega^{i,j}(X) \longrightarrow \Omega^{i+1,j}(X)$$

is the composition of d with the projection onto the $(i+1,j)$-component, and

$$\bar\partial\colon\ \Omega^{i,j}(X) \longrightarrow \Omega^{i,j+1}(X)$$

is the composition of d with the projection onto the $(i,j+1)$-component. The relation $d^2 = 0$ splits into the relations

$$\partial^2 = 0, \qquad \bar\partial^2 = 0, \qquad \partial\bar\partial + \bar\partial\partial = 0.$$

Thus, to a complex manifold X there is assigned the Hodge bicomplex $\big(\Omega_{\mathbb{C}}(X), \partial, \bar\partial\big)$. Its cohomology groups with respect to $\bar\partial$ is known as the *Dolbeault cohomology*, while the cohomology with respect to $d = \partial + \bar\partial$ is the *de Rham cohomology of* X with coefficients in $\mathbb{C}$. The spectral sequence of this bicomplex, called the *Hodge to de Rham spectral sequence*, has the Dolbeault cohomology as the E_1-term and converges to the de Rham cohomology of X.

Assume now that X is a Kähler manifold. Then the $\partial\bar\partial$-lemma holds.

Lemma 2.4. *If $\omega \in \Omega_{\mathbb{C}}(X)$ is a d-closed form and either ∂-exact or $\bar{\partial}$-exact, then*

$$\omega = \partial\bar{\partial}\alpha$$

for some $\alpha \in \Omega_{\mathbb{C}}(X)$.

Since the lemma is true for $\Omega_{\mathbb{C}}(X)$, it also holds for each indecomposable summand of X. A simple examination shows that the lemma fails for any zigzag $Z^{i,j}_{\rightarrow,l}$ and $Z^{i,j}_{\uparrow,l}$ for $l > 0$ (when $l = 0$, the zigzag degenerates to the simple bicomplex $S^{i,j}$). We obtain immediately the following.

Proposition 2.5. *For a compact Kähler manifold X, every indecomposable summand of the bicomplex $\Omega_{\mathbb{C}}(X)$ is isomorphic to either $S^{i,j}$ or $P^{i,j}$ for some i, j.*

Equivalently, $\Omega_{\mathbb{C}}(X)$ has no zigzags (including no zigzags of length 1, that is $\Bbbk \xrightarrow{1} \Bbbk$ and its vertical conterpart).

Thus, the bicomplex $\Omega_{\mathbb{C}}(X)$ decomposes into the direct sum

$$\Omega_{\mathbb{C}}(X) \cong \Omega_s(X) \oplus \Omega_p(X),$$

where $\Omega_s(X)$ is a finite-dimensional semisimple bicomplex (a direct sum of one-dimensional simple bicomplexes $S^{i,j}$), while $\Omega_p(X)$ is an infinite-dimensional free bicomplex (a direct sum of free bicomplexes $P^{i,j}$). The first summand is finite-dimensional since $\Omega_{\mathbb{C}}(X)$ has finite-dimensional cohomology groups, and

$$\Omega_s(X) \cong \mathrm{H}(\Omega_{\mathbb{C}}(X), \partial) \cong \mathrm{H}(\Omega_{\mathbb{C}}(X), \bar{\partial}) \cong \mathrm{H}(\Omega_{\mathbb{C}}(X), d) \cong \mathrm{H}(X, \mathbb{C}).$$

The first three terms are bigraded vector spaces, and the second isomorphism says that, after collapsing the bigrading to a single grading, the groups become the usual de Rham cohomology groups of X.

We see that the cohomology groups of a compact Kähler manifold X with respect to ∂, $\bar{\partial}$, and d are isomorphic; they are also isomorphic to the largest semisimple summand of $\Omega_{\mathbb{C}}(X)$. The Hodge to de Rham spectral sequence for X degenerates at E_1 ($E_1 = E_\infty$). Likewise, the ∂ counterpart of the Hodge to de Rham spectral sequence degenerates at $E_1 = \mathrm{H}(\Omega_{\mathbb{C}}(X), \partial)$.

2.4 Proof of Theorem 2.3

Let M be a graded module over Λ_2. Suppose $m \in M$ is a homogeneous vector of bidegree (i, j) such that $\partial_1\partial_2(m) \neq 0$. Then, it is clear that the submodule generated by m and spanned by vectors in the diagram below

$$\begin{array}{ccc} \partial_2(m) & \xrightarrow{\partial_1} & \partial_2\partial_1(m) = -\partial_1\partial_2(m) \\ {\scriptstyle \partial_2}\uparrow & & \uparrow{\scriptstyle \partial_2} \\ m & \xrightarrow{\partial_1} & \partial_1(m) \end{array}$$

is isomorphic to Λ_2, up to a grading shift, and thus is a projective submodule inside M.

Recall that over a Frobenius algebra, any projective module is injective and vice versa [17, Theorem 15.9]. The same proof shows that over a graded Frobenius algebra, in the category of graded modules, projective objects coincide with injective objects. Since Λ_2 is graded Frobenius, the submodule above is also graded injective, and therefore must be a direct summand of M. We can decompose $M \cong P \oplus N$, where P is a graded direct sum of projective-injectives of the form $P^{i,j}$ (case (2) of Example 2.2), and N is annihilated by the element $\partial_1\partial_2 = -\partial_2\partial_1 \in \Lambda_2$. Further, we may regard N as a module over the bigraded quotient algebra

$$\widehat{\Lambda}_2 := \frac{\Lambda_2}{(\partial_1\partial_2)} \cong \frac{\Bbbk[\partial_1, \partial_2]}{\left(\partial_1^2, \partial_2^2, \partial_1\partial_2\right)}.$$

Now assume N is a bounded bigraded Λ_2-module which does not contain any projective-injective summands. By the above discussion, N is a bigraded module over $\widehat{\Lambda}_2'$. Write for each term

$$N^{i,j} \cong D^{i,j} \oplus C^{i,j},$$

where

$$D^{i,j} = \mathrm{Ker}(\partial_1) \cap \mathrm{Ker}(\partial_2) \cap N^{i,j}$$

is the subspace annihilated by both ∂_1 and ∂_2, and $C^{i,j}$ is an arbitrary complementary vector subspace to $D^{i,j}$ inside $N^{i,j}$. Necessarily,

$$\partial_1\left(C^{i,j}\right) \subset D^{i+1,j}, \qquad \partial_2\left(C^{i,j}\right) \subset D^{i,j+1}$$

since $\partial_1\partial_2|_N \equiv 0$. Thus, there are two direct summands of N containing the subspaces $C^{i,j}$ and $D^{i,j}$:

$$\left(\begin{array}{ccccccc}
\ddots & & & & & & \\
\uparrow{\scriptstyle \partial_2} & & & & & & \\
C^{i-1,j+1} & \xrightarrow{\partial_1} & D^{i,j+1} & & & & \\
& & \uparrow{\scriptstyle \partial_2} & & & & \\
& & C^{i,j} & \xrightarrow{\partial_1} & D^{i+1,j} & & \\
& & & & \uparrow{\scriptstyle \partial_2} & & \\
& & & & C^{i+1,j-1} & \xrightarrow{\partial_1} & \ddots
\end{array}\right),$$

$$\left(\begin{array}{ccccccc}
\ddots & \xrightarrow{\partial_1} & D^{i-1,j+1} & & & & \\
& & \uparrow{\scriptstyle \partial_2} & & & & \\
& & C^{i,j-1} & \xrightarrow{\partial_1} & D^{i,j} & & \\
& & & & \uparrow{\scriptstyle \partial_2} & & \\
& & & & C^{i,j-1} & \xrightarrow{\partial_1} & D^{i+1,j-1} \\
& & & & & & \uparrow{\scriptstyle \partial_2} \\
& & & & & & \ddots
\end{array}\right).$$

In particular, if we further assume that N as above is indecomposable, then there must be $(i,j) \in \mathbb{Z}^2$ such that N is isomorphic to one of the above "zig-zag" modules, and either $C^{i,j} = N^{i,j}$ or $D^{i,j} = N^{i,j}$. Flattening out the zig-zag, say, the first one, we may identify N with an indecomposable finite-dimensional representation of the A quiver with the alternating orientation

$$\cdots \xleftarrow{\partial_2} C^{i-1,j+1} \xrightarrow{\partial_1} D^{i,j+1} \xleftarrow{\partial_2} C^{i,j} \xrightarrow{\partial_1} D^{i+1,j} \xleftarrow{\partial_2} C^{i+1,j-1} \xrightarrow{\partial_1} \cdots.$$

By the classical result of Gabriel (see, for instance, [24]), such an indecomposable module must be of the form

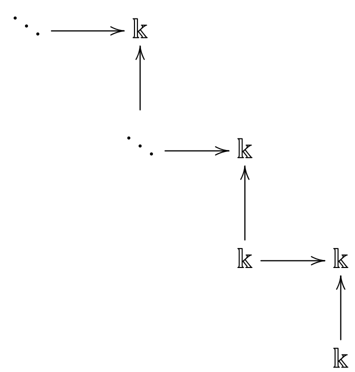
$$\cdots \xleftarrow{=} \Bbbk \xrightarrow{=} \Bbbk \xleftarrow{=} \Bbbk \xrightarrow{=} \Bbbk \xleftarrow{=} \Bbbk \xrightarrow{=} \cdots.$$

Such an indecomposable module translates back into either the simple module or a zig-zag module listed in Example 2.2 (cases (1), (3) and (4)). The theorem follows.

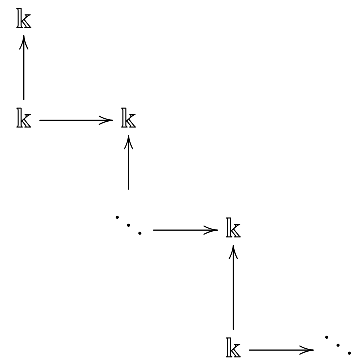
Remark 2.6 (unbounded complexes)**.** As the proof reveals above, one may extend Theorem 2.3 to the case of unbounded bicomplexes as well.

Case 5. Initially vertical and bounded from "below" or "above"; the bounded corner sitting in bidegree (i,j):

$Z^{i,j}_{\uparrow,+}$:



$Z^{i,j}_{\uparrow,-}$:



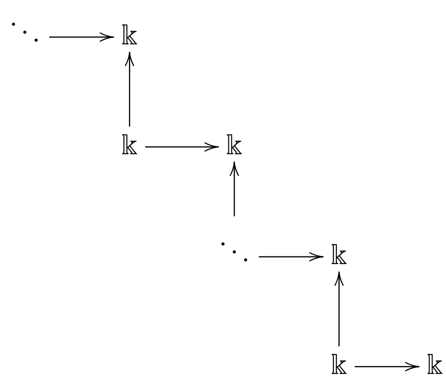
Cohomology spaces of $Z^{i,j}_{\uparrow,\pm}$ with respect to the vertical differential ∂_2 are both zero. But the collapsed total complexes, which both have the form

$$0 \longrightarrow \Bbbk^{\infty} \xrightarrow{d} \Bbbk^{\infty} \longrightarrow 0,$$

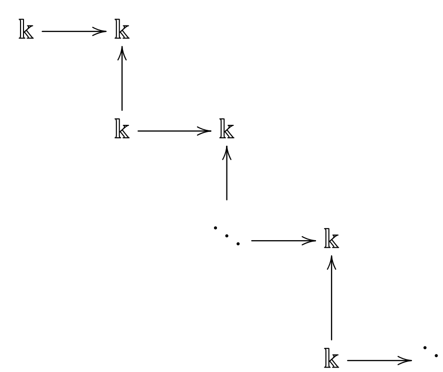
have different total cohomologies. It is readily seen that, for $Z^{i,j}_{\uparrow,+}$, the total differential is both injective and surjective. However, for $Z^{i,j}_{\uparrow,-}$, the total differential is injective, but not surjective. The cokernel of d is given by $\Bbbk$ sitting in the bidegree (i,j).

Case 6. The module $Z^{i,j}_{\rightarrow,\pm}$, which starts horizontally and is bounded from below or above, whose bounded corner lies in bidegree (i,j):

$Z^{i,j}_{\rightarrow,+}$:



$Z^{i,j}_{\rightarrow,-}$:

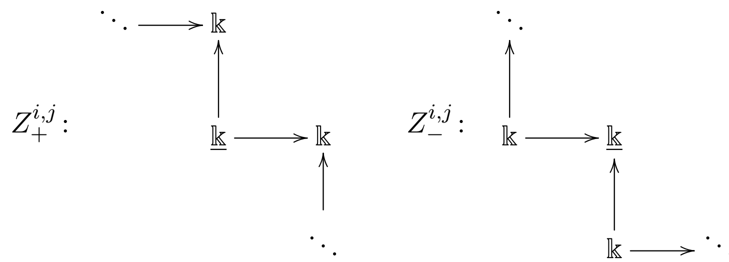


Cohomology spaces with respect to ∂_2 give a single $\Bbbk$ in bidegree (i,j). However, the total cohomology of the collapsed complexes

$$0 \longrightarrow \Bbbk^{\infty} \xrightarrow{d} \Bbbk^{\infty} \longrightarrow 0$$

behaves differently. For $Z^{i,j}_{\rightarrow,+}$, the total differential is clearly injective, but not surjective. The cohomology classes represented by the vectors 1 sitting in bidegrees $(i-r, j+r)$, $r \in \mathbb{N}$, are all cohomologous, and their images in the total complex represent the same cohomology class in degree $i+j$. On the other hand, the total differential of $Z^{i,j}_{\rightarrow,-}$ is an isomorphism, and thus there is no total cohomology.

Case 7. The module $Z^{i,j}_{\pm}$, which is unbounded in both directions. The underlined copy of $\Bbbk$ sits in bidegree (i,j). The modules are taken to be the same up to shifting (i,j) to $(i+r, j-r)$, where $r \in \mathbb{Z}$, and identifying $Z_+^{i,j}$ with $Z_-^{i+1,j}$:



Again, the vertical cohomology with respect to ∂_2 of $Z_\pm^{i,j}$ are both zero. The total cohomology for the collapsed complexes both have one-dimensional cohomology sitting in the cokernel of d. In this case, the spectral sequences will not converge.

Let us call a bicomplex $M = \oplus_{i,j\in\mathbb{Z}} M^{i,j}$ *bounded from Southeast* when $M^{i,j} = 0$ if $i \gg 0$ and $j \ll 0$. A bicomplex M is called *bounded from Northwest* when $M^{i,j} = 0$ if $i \ll 0$ and $j \gg 0$. Combining with the observations in Section 2.2, we see that if a bicomplex M is bounded from Southeast, then, together with finite-dimensional summands, M may contain additional summands of the form $Z^{i,j}_{\uparrow,+}$ and $Z^{i,j}_{\rightarrow,+}$. However, taking ∂_2-cohomology first does not create additional classes that need to be killed off in the total cohomology. Similarly, bicomplexes that are bounded from Northwest may contain infinite-dimensional summands of type $Z^{i,j}_{\uparrow,-}$ and $Z^{i,j}_{\rightarrow,-}$. Taking ∂_1 cohomology contributes nothing towards total cohomology.

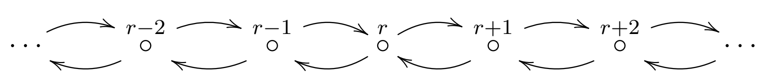
Corollary 2.7. *If M is a bicomplex bounded from Southeast, then there is a spectral sequence whose E_1 page equals* $(\mathrm{H}(M,\partial_2),\partial_1)$*, converging to the total cohomology of M. Likewise, if M is a bicomplex bounded from Northwest, then there is a spectral sequence starting at* $(\mathrm{H}(M,\partial_1),\partial_2)$ *converging to the total cohomology of M.*

Let us call a complex semibounded if it bounded from either Northwest or Southeast. A semibounded complex cannot contain summands $Z_+^{i,j}$ or $Z_-^{i,j}$ that prevent either spectral sequence from converging.

2.5 Connection to zig-zag algebras

Let us point out the connection between the category $\mathcal{M}_2$ of bicomplexes with the module category over (an infinite version of) the zig-zag algebra considered in [16].

Let Q_∞ be the following quiver whose vertices are labelled by $r \in \mathbb{Z}$:



(2.2)

Set $\Bbbk Q_\infty$ to be the path algebra associated to Q_∞ over the ground field. We use, for instance, notation $(i|j|k)$, where i, j, k are vertices of the quiver Q_∞, to denote the path which starts at vertex i, then goes through j (necessarily $j = i \pm 1$) and ends at k. The composition of paths is given by

$$(i_i|i_2|\cdots|i_r)\cdot(j_1|j_2|\cdots|j_s) = \begin{cases} (i_i|i_2|\cdots|i_r|j_2|\cdots|j_s), & \text{if } i_r = j_1, \\ 0, & \text{otherwise,} \end{cases}$$

where $i_1, \ldots, i_r$ and $j_1, \ldots, j_s$ are sequences of neighboring vertices in Q_∞.

Definition 2.8. The *zig-zag algebra* $A = A_\infty$ is the quotient of the path algebra $\Bbbk Q_\infty$ by the relations, for any $r \in \mathbb{Z}$,

$$(r|r+1|r+2) = 0, \qquad (r|r-1|r-2) = 0, \qquad (r|r-1|r) = (r|r+1|r).$$

We make the zig-zag algebra graded by setting[1]

$$\deg(r) = \deg(r|r+1) = 0, \qquad \deg(r|r-1) = 1,$$

for all $r \in \mathbb{Z}$. It is a non-unital algebra with a system of mutually orthogonal idempotents $\{(r)|r \in \mathbb{Z}\}$. There is an obvious automorphism T on A, defined by

$$T(r) := (r+1), \qquad T(r|r+1) := (r+1|r+2), \qquad T(r|r-1) := (r+1|r).$$

For a fixed pair of integers $(r,i) \in \mathbb{Z}^2$, there is a graded projective module $P_r\langle i\rangle$ which is generated by the idempotent (r), whose degree is shifted up by i. More explicitly, $P_r\langle i\rangle$ is the four-dimensional vector space with the basis

$$\big\{(r)\sigma_i,\, (r+1|r)\sigma_i,\, (r-1|r)\sigma_i,\, (r|r+1|r)\sigma_i\big\},$$

where σ_i stands for the module generator sitting in degree i.

We will consider the category of graded modules over A, which we denote by $\mathcal{M}(A)$, in what follows. The automorphism T of A induces an autoequivalence $\mathcal{T}$ of $\mathcal{M}(A)$, defined by $\mathcal{T} := \big(T^{-1}\big)^*$. Clearly $\mathcal{T}(P_r\langle i\rangle) = P_{r+1}\langle i\rangle$ holds for all $r, i \in \mathbb{Z}$.

Given a module $M = \oplus_{i,j\in\mathbb{Z}} M^{i,j}$ in $\mathcal{M}_2$, we place the homogeneous bigraded component of $M^{i,j}$ at (i,j) in the corresponding node of the two-dimensional lattice $\mathbb{Z}^2$. For each $r \in \mathbb{Z}$, we collect together $M^{i,j}$s on the line of slope one (depicted as the dashed line in the picture below):

$$M_r := \bigoplus_{i-j=r} M^{i,j}.$$

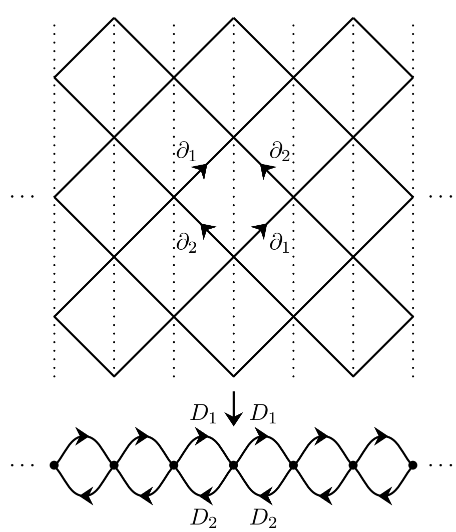
Note that M_r is singly graded, with its homogeneous degree j part M_r^j set to be $M^{r+j,j}$.

Since ∂_1 and ∂_2 have bidegrees $(1,0)$ and $(0,1)$, respectively, they induce maps

$$D_1 := \partial_1\colon\ M_r^j \longrightarrow M_{r+1}^j, \qquad D_2 := (-1)^r\partial_2\colon\ M_r^j \longrightarrow M_{r-1}^{j+1},$$

These maps satisfy $D_1^2 = 0$, $D_2^2 = 0$ and $D_1D_2 = D_2D_1$. We put the vector space M_r at the rth vertex of A and declare the rightward (resp. leftward) going arrows to be the induced map D_1 (resp. D_2). We have thus obtained a graded A-module by summing over the r-degrees $M_\infty := \oplus_{r\in\mathbb{Z}} M_r$.

Schematically, we depict the correspondence as follows:



[1]The grading is chosen to match with the convention of [16].

Furthermore, a morphism $f\colon M\longrightarrow N$ in $\mathcal{M}_2$ componentwise given by

$$f=\bigoplus_{i,j\in\mathbb{Z}} f^{i,j}\colon \quad \bigoplus_{i,j\in\mathbb{Z}} M^{i,j}\longrightarrow \bigoplus_{i,j\in\mathbb{Z}} N^{i,j},$$

satisfies

$$\partial_1 f^{i,j}=f^{i+1,j}\partial_1, \qquad \partial_2 f^{i,j}=f^{i,j+1}\partial_2.$$

One has the associated morphism of bigraded A-modules, which is defined as

$$f_r:=\bigoplus_{i-j=r} f^{i,j}\colon\ M_r\longrightarrow N_r, \qquad f_\infty:=\bigoplus_{r\in\mathbb{Z}} f_r.$$

Clearly $D_1 f_r=f_{r+1}D_1$ and $D_2 f_r=f_{r-1}D_2$ holds for all $r\in\mathbb{Z}$, so that f_∞ is a morphism of bigraded A-modules. This defines a functor $\mathcal{F}_\infty\colon \mathcal{M}_2\longrightarrow\mathcal{M}(A)$.

As the above functorial assignment is clearly reversible, the functor $\mathcal{F}_\infty$ is invertible.

Proposition 2.9. *The functor $\mathcal{F}_\infty\colon \mathcal{M}_2\longrightarrow\mathcal{M}(A)$ is an equivalence of abelian categories. Furthermore, the functor satisfies*

$$\mathcal{F}_\infty(M\{1,0\})=\mathcal{T}(\mathcal{F}_\infty(M)),$$
$$\mathcal{F}_\infty(M\{0,1\})=\mathcal{T}^{-1}(\mathcal{F}_\infty(M))\langle 1\rangle.$$

3 Tricomplexes and braid group actions

3.1 The monoidal category of tricomplexes

We denote by $\mathbf{i}=(i_1,i_2,i_3)$ an ordered triple of integers, and write $\mathbf{i}=i_1e_1+i_2e_2+i_3e_3$ where

$$e_1=(1,0,0), \qquad e_2=(0,1,0), \qquad e_3=(0,0,1).$$

In particular, we write $\mathbf{0}:=(0,0,0)$ as the additive unit element.

Let Λ_3 be the exterior algebra over $\Bbbk$ with three generators ∂_1, ∂_2, ∂_3:

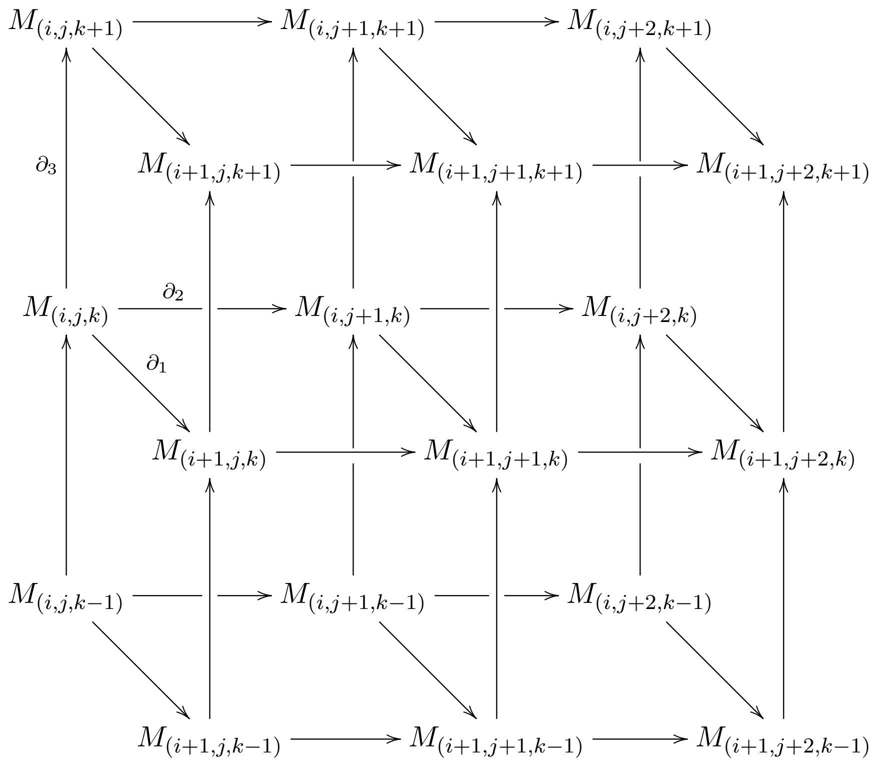
$$\partial_j^2=0, \qquad j=1,2,3, \qquad \partial_j\partial_k+\partial_k\partial_j=0, \qquad j\neq k.$$

We make Λ_3 a triply-graded $\Bbbk$-algebra, by assigning degree e_j to ∂_j. Let $\mathcal{M}_3$ be the category of triply-graded left Λ_3-modules with respect to tri-degree preserving maps. A module M consists of a collection of $\Bbbk$-vector spaces $M_{\mathbf{i}}$,

$$M=\bigoplus_{\mathbf{i}\in\mathbb{Z}^3} M_{\mathbf{i}},$$

together with linear maps $\partial_j\colon M_{\mathbf{i}}\longrightarrow M_{\mathbf{i}+e_j}$ subject to the exterior algebra relations. It is useful to visualize M as a 3-dimensional object: the vector space $M_{\mathbf{i}}$ sits in the $\mathbf{i}$ node of a 3-dimensional lattice and the maps ∂_j go along oriented edges of the lattice. Below is a portion of M

depicted:



The grading shift by $\mathbf{i}$, denoted $\{\mathbf{i}\}$, is an automorphism of $\mathcal{M}_3$. Any simple object of $\mathcal{M}_3$ is isomorphic to $S_{\mathbf{i}} := \underline{\Bbbk}\{\mathbf{i}\}$ for a unique $\mathbf{i}$. Here $\underline{\Bbbk}$ is a one-dimensional $\Bbbk$-vector space, in tridegree $\mathbf{0}$, viewed as a Λ_3-module with the trivial action of ∂_1, ∂_2, ∂_3.

Any indecomposable projective in $\mathcal{M}_3$ is isomorphic to $P_{\mathbf{i}} := \Lambda_3\{\mathbf{i}\}$, for a unique $\mathbf{i}$. Any projective in $\mathcal{M}_3$ is isomorphic to the direct sum of $P_{\mathbf{i}}$'s, possibly with infinite multiplicities. Since Λ_3 is a trigraded Frobenius algebra, the $P_{\mathbf{i}}$ are also injective objects of $\mathcal{M}_3$. A module M contains $P_{\mathbf{i}}$ as a direct summand (and not just as a submodule) if and only if $\partial_1\partial_2\partial_3 m \neq 0$ for some $m \in M_{\mathbf{i}}$.

Let $Q = \Lambda_3\omega/\Lambda_3\partial_3\omega$ be the cyclic module with one generator ω in tri-degree $\mathbf{0}$ and relation $\partial_3\omega = 0$. We depict Q as a square

$$\begin{array}{ccc} \Bbbk\partial_2\omega & \xrightarrow{\partial_1} & \Bbbk\partial_2\partial_1\omega \\ {\scriptstyle\partial_2}\uparrow & & \uparrow{\scriptstyle\partial_2} \\ \Bbbk\omega & \xrightarrow{\partial_1} & \Bbbk\partial_1\omega. \end{array}$$

There is a graded isomorphism of modules $Q \cong \Lambda_3/\partial_3\Lambda_3$.

The algebra Λ_3 is a Hopf algebra in the category of trigraded (super) vector spaces, where the (super) $\mathbb{Z}/2\mathbb{Z}$-grading is given by reducing $i_1+i_2+i_3$ modulo 2, and $\Delta(\partial_r) = \partial_r \otimes 1 + 1 \otimes \partial_r$. Consequently, the tensor product $M \otimes N$ of trigraded Λ_3-modules is a trigraded Λ_3-module, with ∂_r acting by

$$\partial_r(m \otimes n) = \partial_r(m) \otimes n + (-1)^{i_r} m \otimes \partial_r(n), \qquad r = 1, 2, 3,$$

where m is in degree (i_1, i_2, i_3).

Similarly, there is a trigraded *inner-hom* on $\mathcal{M}_3$, defined by

$$\mathrm{HOM}_{\Bbbk}(M, N) := \bigoplus_{\mathbf{i} \in \mathbb{Z}^3} \mathrm{Hom}_{\Bbbk}(M, N\{\mathbf{i}\}),$$

where the right hand side is the direct sum of homogeneous linear maps from M to $N\{\mathbf{i}\}$. The inner hom space carries a natural Λ_3 action defined by, for any $f \in \mathrm{Hom}_{\Bbbk}(M, N\{i_1, i_2, i_3\})$

$$\partial_r(f)(m) = \partial_r(f(m)) - (-1)^{i_r} f(\partial_r(m)), \qquad r = 1, 2, 3. \tag{3.1}$$

The spaces of Λ_3-invariants under this action consist of morphisms in $\mathcal{M}_3$ of all degrees:

$$\mathrm{HOM}_{\Bbbk}(M, N)^{\Lambda_3} = \bigoplus_{\mathbf{i} \in \mathbb{Z}^3} \mathrm{Hom}_{\mathcal{M}_3}(M, N\{\mathbf{i}\}).$$

It is useful to regard Λ_2 and Λ_1 as certain graded Hopf subalgebras in Λ_3. To do this, we break the apparent symmetry and define Λ_2 to be the subalgebra generated by ∂_1 and ∂_2, while setting Λ_1' to be the subalgebra generated by ∂_3. The natural algebra inclusions

$$\iota\colon\ \Lambda_2 \hookrightarrow \Lambda_3, \qquad \jmath\colon\ \Lambda_1' \subset \Lambda_3$$

admit retractions

$$\mu\colon\ \Lambda_3 \longrightarrow \Lambda_2, \qquad \nu\colon\ \Lambda_3 \longrightarrow \Lambda_1', \tag{3.2}$$

which are respectively given by setting ∂_3 or ∂_1, ∂_2 to be zero.

Using these subquotient algebras, we define a functor by taking "partial graded-hom" with respect to Λ_1', as follows. Fix i and j degrees. Given any $M \in \mathcal{M}_3$, set

$$M_{i,j} := \nu^* \left(\bigoplus_{k \in \mathbb{Z}} M_{i,j,k} \right),$$

where in the last term, we only keep the Λ_1'-module structure on $\oplus_k M_{i,j,k}$. The functor extends naturally to morphisms in $\mathcal{M}_3$, and has the effect, on objects, of taking the direct sum of $M_{i,j,k}$ over $k \in \mathbb{Z}$. It remembers the ∂_3-complex structure inherited from that of M, while making ∂_1, ∂_2 act by 0.

3.2 A braid group action

In this section, we exhibit a braid group action on the stable category of trigraded Λ_3-modules.

The tensor product $Q \otimes M_{i,j}$ is an object of $\mathcal{M}_3$, with ∂_1, ∂_2 acting only along Q (since their actions on $M_{i,j}$ are trivial) and ∂_3 acting along $M_{i,j}$.

Consider the functor

$$\mathcal{U}_r(M) := \bigoplus_{i-j=r} Q \otimes M_{i,j}.$$

Geometrically, we take the plane $P_r = \{(i,j,k) \,|\, i - j = r\}$ in $\mathbb{Z}^3$, with vector spaces $M_{\mathbf{i}}$ sitting in the nodes, and form four copies of the plane (the tensor product with Q) related by the differentials ∂_1 and ∂_2. The differential ∂_3 acts along edges $(\mathbf{i}, \mathbf{i}+e_3)$ contained in the plane P_r. We depict the summand $Q \otimes M_{i,j}$ in the next diagram. For a fixed e_3-degree k, $Q \otimes M_{i,j,k}$ has four copies of $M_{i,j,k}$ sitting in degrees (i,j,k), $(i+1,j,k)$, $(i,j+1,k)$ and $(i+1,j+1,k)$ respectively. They correspond to

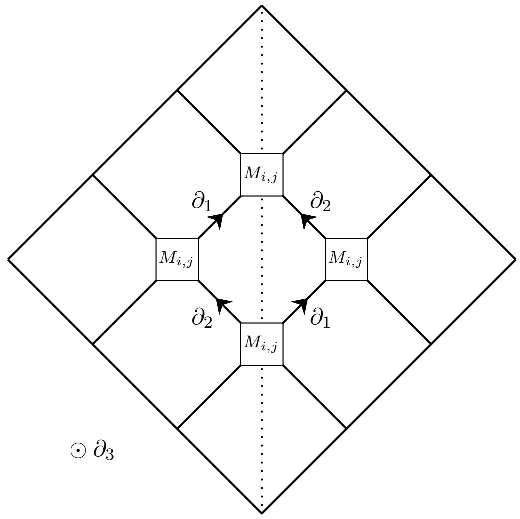
$$\Bbbk\omega \otimes M_{i,j,k}, \qquad \Bbbk\partial_1\omega \otimes M_{i,j,k}, \qquad \Bbbk\partial_2\omega \otimes M_{i,j,k}, \qquad \Bbbk\partial_2\partial_1\omega \otimes M_{i,j,k}.$$

All maps except for

$$\partial_1\colon\ M_{i,j,k} \cong \Bbbk\partial_2\omega \otimes M_{i,j,k} \longrightarrow \Bbbk\partial_2\partial_1 \otimes M_{i,j,k} \cong M_{i,j,k}$$

act as identity maps, which is the negative identity map.

Now, summing over k and keeping track of the differential ∂_3, we obtain the diagram



(3.3)

Here the differential ∂_3 points perpendicularly out of the plane.

Proposition 3.1. *The following isomorphisms between endofunctors of $\mathcal{M}_3$ hold:*

$$\begin{aligned}&\mathcal{U}_r^2 \cong \mathcal{U}_r\{1,1,0\} \oplus \mathcal{U}_r,\\ &\mathcal{U}_r\mathcal{U}_{r\pm1}\mathcal{U}_r \cong \mathcal{U}_r\{1,1,0\},\\ &\mathcal{U}_r\mathcal{U}_s = 0 \qquad \text{if} \quad |r-s|>1.\end{aligned}$$

Proof. We start with the first equation. We compute the left hand side as

$$\begin{aligned}\mathcal{U}_r^2(M) &= \bigoplus_{i-j=r} \mathcal{U}_r(Q\otimes M_{i,j}) = \bigoplus_{i-j=r}\left(\bigoplus_{k-l=r} Q\otimes (Q\otimes M_{i,j})_{k,l}\right)\\ &= \bigoplus_{i-j=r} Q\otimes(\Bbbk\omega\otimes M_{i,j}\oplus \Bbbk\partial_2\partial_1\omega\otimes M_{i,j})\\ &\cong \left(\bigoplus_{i-j=r} Q\otimes \Bbbk\omega\otimes M_{i,j}\right)\oplus\left(\bigoplus_{i-j=r} Q\otimes\Bbbk\partial_2\partial_1\omega\otimes M_{i,j}\right)\\ &\cong \mathcal{U}_r(M)\oplus\mathcal{U}_r(M)\{1,1,0\}.\end{aligned}$$

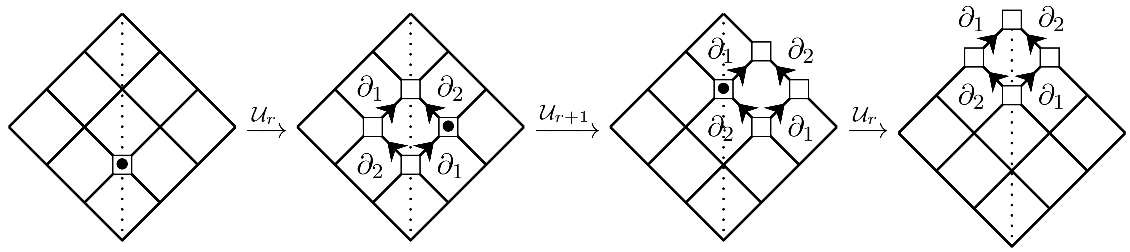
Here, in the third equality, we have used that $Q\otimes M_{i,j}$ has only two terms concentrated on the line $k-l=r$ (see the above picture (3.3)).

For the second isomorphism, we have (taking the $r+1$ case)

$$\begin{aligned}\mathcal{U}_r\mathcal{U}_{r+1}\mathcal{U}_r(M) &= \bigoplus_{i-j=r}\mathcal{U}_r\mathcal{U}_{r+1}(Q\otimes M_{i,j}) = \bigoplus_{i-j=r}\mathcal{U}_r\left(Q\otimes(\Bbbk\partial_1\omega\otimes M_{i,j})\right)\\ &= \bigoplus_{i-j=r} Q\otimes\Bbbk\partial_2\omega\otimes\Bbbk\partial_1\omega\otimes M_{i,j} = \bigoplus_{i-j=r} Q\otimes M_{i,j}\{1,1,0\}\\ &= \mathcal{U}_r(M)\{1,1,0\}.\end{aligned} \tag{3.4}$$

The last isomorphism is easy, and we leave it as an exercise to the reader. ∎

Remark 3.2. Perhaps the cartoon below, in the scheme of equation (3.3), helps visualizing the equalities in the above proof. We show this for equation (3.4) as an example. Depict a copy of $M_{i,j}$ by a box in the lattices below. A black dot in a box indicates the term contributing to the functor on the outward arrow:



There exists a unique morphism in $\mathcal{M}_3$

$$Q \otimes M_{i,j} \longrightarrow M, \tag{3.5}$$

which takes $\omega \otimes m$ to m. This morphism takes $\partial_1\omega \otimes m$ to $\partial_1 m$, etc.

Summing over i, j such that $i-j=r$, morphisms (3.5) combine into a module homomorphism

$$\mathrm{in}_r\colon\ \mathcal{U}_r(M) \longrightarrow M$$

natural in M. Thus, $\mathrm{in}_r\colon \mathcal{U}_r \Longrightarrow \mathrm{Id}$ is a natural transformation of functors on $\mathcal{M}_3$.

Next, we construct a module homomorphism

$$M \xrightarrow{\mathrm{out}_r} \mathcal{U}_r(M)\{-1,-1,0\}.$$

Denote by M^ν the underlying trigraded vector space of M, while only remembering the Λ_1'-module structure. Consider the map

$$\begin{aligned} \mathrm{out}\colon M &\longrightarrow Q \otimes M^\nu\{-1,-1,0\}, \\ m &\mapsto (-1)^{i+j}(\omega \otimes \partial_1\partial_2(m) + \partial_1\omega \otimes \partial_2(m) - \partial_2\omega \otimes \partial_1(m) + \partial_1\partial_2\omega \otimes m), \end{aligned}$$

where $m \in M_{i,j,k}$ is a homogeneous element.

Lemma 3.3. *The map* out: $M \longrightarrow Q \otimes M^\nu\{-1,-1,0\}$ *is a morphism of trigraded* Λ_3*-modules.*

Proof. The map clearly commutes with ∂_3-actions on both sides, as ∂_3 kills ω and anti-commutes with ∂_1 and ∂_2. To verify that out also commutes with ∂_1 and ∂_2 requires a small computation. We check, for instance, that it commutes with ∂_1, and leave the ∂_2-computation to the reader.

On the one hand, if $m \in M_{i,j,k}$, and using that ∂_1 acts trivially on M^ν, we have

$$\partial_1(\mathrm{out}(m)) = (-1)^{i+j}(\partial_1\omega \otimes \partial_1\partial_2(m) - \partial_1\partial_2\omega \otimes \partial_1(m)).$$

On the other hand,

$$\begin{aligned} \mathrm{out}(\partial_1(m)) &= (-1)^{i+j+1}\left(\partial_1\omega \otimes \partial_2(\partial_1(m)) + \partial_1\partial_2\omega \otimes \partial_1(m)\right) \\ &= (-1)^{i+j+1}\left(-\partial_1\omega \otimes \partial_1\partial_2(m) + \partial_1\partial_2\omega \otimes \partial_1(m)\right). \end{aligned}$$

Comparing these expressions, the commutativity with the ∂_1-actions follows. ■

Since $Q \otimes M^\nu$ naturally decomposes into a direct sum of Λ_3-modules

$$Q \otimes M^\nu \cong \bigoplus_{i,j\in\mathbb{Z}} Q \otimes M_{i,j},$$

for each $r \in \mathbb{Z}$, we have a natural projection map of Λ_3-modules

$$\pi_r\colon\ Q\otimes M \longrightarrow \bigoplus_{i-j=r} Q\otimes M_{i,j}.$$

We can thus define the composition map

$$\mathrm{out}_r := \pi_r \circ \mathrm{out}\colon\ M \longrightarrow \mathcal{U}_r(M)\{-1,-1,0\}.$$

Componentwise, out_r has the effect, for a homogeneous $m \in M_{i,j,k}$,

$$\mathrm{out}_r(m) := \begin{cases} (-1)^{i+j}(\omega\otimes\partial_1\partial_2 m + \partial_1\partial_2\omega\otimes m), & \text{if } i-j=r,\\ (-1)^{i+j}\partial_1\omega\otimes\partial_2(m), & \text{if } i-j=r+1,\\ (-1)^{i+j+1}\partial_2\omega\otimes\partial_1(m), & \text{if } i-j=r-1,\\ 0, & \text{otherwise.}\end{cases} \tag{3.6}$$

We have thus obtained out_r as a tri-grading preserving homomorphism of Λ_3-modules, functorial in M. In other words, similarly as for in_r, the map $\mathrm{out}_r\colon \mathrm{Id} \Rightarrow \mathcal{U}_r\{-1,-1,0\}$ is a natural transformation of functors.

Let $\mathcal{SM}_3$ be the stable category of trigraded left Λ_3-modules. It has the same objects as $\mathcal{M}_3$ and the morphisms are those in $\mathcal{M}_3$ modulo morphisms that factor through a projective object of $\mathcal{M}_3$. In particular, a projective trigraded Λ_3-module is isomorphic to the zero object in $\mathcal{SM}_3$. The stable category is triangulated, with the shift functor $[1]_{\mathcal{SM}}$ taking M to the cokernel of an inclusion $M \subset P$, where P is a projective module. For concreteness, we can choose P to be $\Lambda_3 \otimes M\{-1,-1,-1\}$, with the inclusion taking m to $\partial_1\partial_2\partial_3 \otimes m$. The shift by $\{-1,-1,-1\}$ makes the inclusion grading-preserving. Then $M[1]_{\mathcal{SM}} = \widehat{\Lambda}\otimes M$ where

$$\widehat{\Lambda} = \Lambda_3/(\partial_1\partial_2\partial_3)\{-1,-1,-1\}.$$

The cone of a morphism $f\colon M \longrightarrow N$ is defined as the cokernel of the inclusion

$$M \subset N \oplus (\Lambda_3 \otimes M\{-1,-1,-1\}),$$

which takes m to $(f(m), \partial_1\partial_2\partial_3(m))$. For more details, we refer the reader to Happel [9]. We will need the following result computing morphism spaces in $\mathcal{SM}_3$, bearing in mind the Λ_3 action defined in equation (3.1).

Lemma 3.4. *Given two objects $M, N \in \mathcal{SM}_3$, there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{SM}_3}(M,N) = \frac{\mathrm{Hom}_{\mathcal{M}_3}(M,N)}{\partial_1\partial_2\partial_3\,\mathrm{Hom}_{\Bbbk}(M, N\{-1,-1,-1\})}.$$

Proof. See [21, Corollary 5.5]. ∎

We introduce another cone construction defined for morphisms in the abelian category $\mathcal{M}_3$. Given a morphism $f\colon M \longrightarrow N$ in $\mathcal{M}_3$, the ∂_3-cone $\mathrm{C}_3(f)$, as a trigraded vector space, is the object $M\{0,0,-1\} \oplus N$, on which the Λ_3-generators act by

$$\partial_3(m,n) = (-\partial_3 m, f(m) + \partial_3(n)), \tag{3.7}$$

and $\partial_j(m,n) = (\partial_j m, \partial_j n)$ for $j = 1, 2$.

Alternatively, regard $\Lambda_1' = \Bbbk[\partial_3]/(\partial_3^2)$ as a trigraded Λ_3-module via the homomorphism ν (see equation (3.2)), the ∂_3-cone is defined as the push-out of $f\colon M \longrightarrow N$ and $\partial_3 \otimes \mathrm{Id}_M\colon M \longrightarrow \Lambda_1' \otimes M$.

This is the top square of the following diagram, whose columns are short exact in the abelian category because of the push-out property:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle \partial_3 \otimes \mathrm{Id}_M}\downarrow & & \downarrow \\ \Lambda_1' \otimes M\{0,0,-1\} & \longrightarrow & \mathrm{C}_3(f) \\ \downarrow & & \downarrow \\ M\{0,0,-1\} & = & M\{0,0,-1\}. \end{array} \tag{3.8}$$

Define $\mathcal{R}_r := \mathrm{C}_3(\mathrm{in}_r)$, i.e., it is the functor in $\mathcal{M}_3$ that takes a module M to the ∂_3-cone of the homomorphism $\mathrm{in}_r \colon \mathcal{U}_r(M) \longrightarrow M$. Let $\mathcal{R}_r' := \mathrm{C}_3(\mathrm{out}_r)\{0,0,1\}$, which takes M to the ∂_3-cone of $\mathrm{out}_r \colon M \longrightarrow \mathcal{U}_r(M)\{-1,-1,0\}$, with the grading shifted by $\{0,0,1\}$, so that the vector spaces in the nodes of M stay in their original tridegrees, and ∂_3 changes sign in its action on $\mathcal{U}_r(M)$, not M.

Lemma 3.5. *The functors $\mathcal{R}_r$, $\mathcal{R}_r'$ descend to well-defined functors on the stable category $\mathcal{SM}_3$.*

Proof. It suffices to show that, if M is a projective Λ_3-module, then $\mathcal{R}_r(M)$ and $\mathcal{R}_r'(M)$ are both projective. Let us do this for $\mathcal{R}_r$, and the $\mathcal{R}_r'$ case is similar.

By (3.8), $\mathcal{R}_r(M)$ fits into a short exact sequence of Λ_3-modules

$$0 \longrightarrow M \longrightarrow \mathcal{R}_r(M) \longrightarrow \mathcal{U}_r(M)\{0,0,-1\} \longrightarrow 0.$$

Since Λ_3 is Frobenius, M is also injective and the above sequence splits. We are thus reduced to showing that $\mathcal{U}_r(M)\{0,0,-1\}$ is graded projective. Without loss of generality, we may assume that $M \cong \Lambda_3\{i,j,k\}$ is indecomposable. As Λ_1'-modules, there is a direct sum decompostion

$$\Lambda_3 \cong \Lambda_1' \oplus \Lambda_1'\{1,0,0\} \oplus \Lambda_1'\{0,1,0\} \oplus \Lambda_1'\{1,1,0\}.$$

Using this decomposition and the fact that $Q \otimes \nu^*(\Lambda_1') \cong \Lambda_3$, we have

$$\mathcal{U}_r(\Lambda_3\{i,j,k\}) = \begin{cases} \Lambda_3\{i,j,k\} \oplus \Lambda_3\{i+1,j+1,k\}, & i-j=r, \\ \Lambda_3\{i+1,j,k\}, & i-j=r+1, \\ \Lambda_3\{i,j+1,k\}, & i-j=r-1, \\ 0, & |i-j-r|>1. \end{cases}$$

The result follows. ■

Theorem 3.6.

(i) *The functors $\mathcal{R}_r, \mathcal{R}_r'$ are invertible mutually-inverse endofunctors on the stable category $\mathcal{SM}_3$.*

(ii) *The following functor isomorphisms hold:*

$$\mathcal{R}_r\mathcal{R}_{r+1}\mathcal{R}_r \cong \mathcal{R}_{r+1}\mathcal{R}_r\mathcal{R}_{r+1}, \tag{3.9a}$$
$$\mathcal{R}_r\mathcal{R}_s \cong \mathcal{R}_s\mathcal{R}_r \qquad \text{if} \quad |r-s|>1. \tag{3.9b}$$

Consequently, the collection of functors $\{\mathcal{R}_r \mid r \in \mathbb{Z}\}$ gives rise to an action of the infinite braid group of infinitely many strands Br_∞ on the triangulated category $\mathcal{SM}_3$.

The proof of the theorem will occupy the next subsection.

Remark 3.7. In this section, we have interpreted the three differentials of Λ_3 in two different ways: the $\Lambda_2 \subset \Lambda_3$ plays the role of the algebra A (cf. Section 2.5), while ∂_3 behaves more like a "homological differential". This apparent symmetry breaking allows one to construct three equivalent braid group actions on $\mathcal{SM}_3$ as in Theorem 3.6, by the automorphism of Λ_3 permuting the indices $\{1, 2, 3\}$.

3.3 Proof of Theorem 3.6

Invertibility of $\mathcal{R}_r$. First we show $\mathcal{R}'_r\mathcal{R}_r \cong \mathrm{Id}$. We check the effect of the left hand side on a trigraded Λ_3-module M.

$$\mathcal{R}'_r(\mathcal{R}_r(M)) \cong \mathcal{R}'_r\left(\mathcal{U}_r(M)\{0,0,-1\} \xrightarrow{\mathrm{in}_r} M\right)$$
$$\cong \left(\begin{array}{ccc} \mathcal{U}_r(M)\{0,0,-1\} & \xrightarrow{-\mathrm{in}_r} & M \\ \mathrm{out}_r\big\downarrow & & \big\downarrow\mathrm{out}_r \\ \mathcal{U}_r^2(M)\{-1,-1,0\} & \xrightarrow{\mathrm{in}_r} & \mathcal{U}_r(M)\{-1,-1,1\} \end{array}\right). \tag{3.10}$$

Here, in the diagram, the horizontal arrows are interpreted as the ∂_3-differential arising from the ∂_3-cone of in_r, while the vertical arrows indicate that of out_r. The differential action by ∂_1, ∂_2 preserves the position of the node, while the ∂_3 acts both internally at the nodes and transfer elements long the arrows (see equation (3.7)).

By Proposition 3.1, we may decompose

$$\mathcal{U}_r^2(M)\{-1,-1,0\} \cong \mathcal{U}_r(M) \oplus \mathcal{U}_r(M)\{-1,-1,0\}. \tag{3.11a}$$

As in the proof of the proposition, we further identify

$$\mathcal{U}_r(M) \cong \left(\bigoplus_{i-j=r} Q \otimes \partial_1\partial_2\omega \otimes M_{i,j}\right)\{-1,-1,0\}, \tag{3.11b}$$

$$\mathcal{U}_r(M)\{-1,-1,0\} \cong \left(\bigoplus_{i-j=r} Q \otimes \omega \otimes M_{i,j}\right)\{-1,-1,0\}. \tag{3.11c}$$

By the definition of the ∂_3-cone, the sum of terms on the lower horizontal line of (3.10) consitutes a Λ_3-submodule of $\mathcal{R}'_r(\mathcal{R}_r(M))$. The morphism in_r on the lower horizontal line of (3.10) maps the summand (3.11c) isomorphically onto $\mathcal{U}_r\{-1,-1,1\}$. Hence we have in $\mathcal{R}'_r(\mathcal{R}_r(M))$ a Λ_3-submodule

$$\left(\mathcal{U}_r(M)\{-1,-1,0\} \xrightarrow{\mathrm{in}_r} \mathcal{U}_r(M)\{-1,-1,1\}\right)$$
$$\cong \mathcal{U}_r(M) \otimes \nu^*(\Lambda'_1) \cong \bigoplus_{i-j=r} (Q \otimes \nu^*(\Lambda'_1)) \otimes M_{i,j}. \tag{3.12}$$

As $Q \otimes \Lambda'_1 \cong \Lambda_3$ is a tri-graded free Λ_3-module, it is a not only a submodule in $\mathcal{R}'_r(\mathcal{R}_r(M))$ but also a direct summand, which is annihilated when passing to the stable category $\mathcal{SM}_3$. We thus may safely identify $\mathcal{R}'_r(\mathcal{R}_r(M))$ with the quotient of it by this submodule, which we denote by M_1.

Now M is clearly a Λ_3-submodule in M_1. We claim that M_1/M is also a free Λ_3-module, and hence is a direct summand in M_1 whose complement is isomorphic to M. It then follows that the natural inclusion map $M \hookrightarrow M_1$ is an isomorphism in $\mathcal{SM}_3$.

To prove the claim, note that

$$M_1/M \cong \big(\mathcal{U}_r(M)\{0,0,-1\} \xrightarrow{\mathrm{out}_r} \mathcal{U}_r(M)\big),$$

where the right hand side denotes a ∂_3-cone. If $m \in M_{i,j}$ is a homogeneous element, the map out_r has, by equation (3.6), the effect

$$\begin{aligned}
\mathrm{out}_r(\omega\otimes m) &= (-1)^{i+j}\left(\omega\otimes\partial_1\partial_2\omega\otimes m + \partial_1\partial_2\omega\otimes\omega\otimes m\right),\\
\mathrm{out}_r(\partial_1\omega\otimes m) &= (-1)^{i+j+1}\partial_1\omega\otimes\partial_2\partial_1\omega\otimes m,\\
\mathrm{out}_r(\partial_2\omega\otimes m) &= (-1)^{i+j+2}\partial_2\omega\otimes\partial_1\partial_2\omega\otimes m,\\
\mathrm{out}_r(\partial_1\partial_2\omega\otimes m) &= (-1)^{i+j+2}\partial_1\partial_2\omega\otimes\partial_1\partial_2\omega\otimes m.
\end{aligned}$$

The right hand side of the first equation contains elements in $\mathcal{U}_r(M)\{-1,-1,0\}$ (see equation (3.11c)), which has already been mod out in M_1. The rest of the terms on the right hand side of the equations have their middle term $\partial_1\partial_2\omega$. It follows that out_r maps $\mathcal{U}_r(M)\{0,0,-1\}$ isomorphically onto $\mathcal{U}_r(M)$. The claim follows.

It is not hard to find a summand in $\mathcal{R}'_r(\mathcal{R}_r(M))$ isomorphic to M. Denote by

$$\mathrm{out}_r^{13}\colon\ M \longrightarrow Q\otimes Q^\nu\otimes M^\nu, \qquad m\mapsto \sum_i h_i\otimes\omega\otimes m_i,$$

where h_i, m_i are the components of $\mathrm{out}_r(m)=\sum_i h_i\otimes m_i \in Q\otimes M^\nu$ as in equation (3.6). The submodule

$$\left\{\left(-\mathrm{out}_r^{13}(m), m\right) \mid m\in M\right\} \subset \mathcal{U}_r^2(M)\{-1,-1,0\}\oplus M$$

constitutes, by the above discussion, a Λ_3-summand isomorphic to M. The inclusion of this summand realizes the functor isomorphism $\mathrm{Id}\cong\mathcal{R}'_r\mathcal{R}_r$.

The isomorphism $\mathcal{R}_r\mathcal{R}'_r\cong\mathrm{Id}$ follows by a similar argument. Essentially, one just needs to flip the last term of (3.10) along its northwest-southeast diagonal. We leave the details to the reader as an exercise.

Braid relations. We next check the functor relations (3.9a) and (3.9b).

The commutation relation (3.9b) is easy to check, as one can readily see that both sides are functorially isomorphic, when applied to a trigraded Λ_3-module M, to the ∂_3-cone of the morphism $\mathrm{in}_r\oplus\mathrm{in}_s$:

$$\begin{array}{ccc}
\mathcal{U}_r(M)\{0,0,-1\} & & \\
 & \searrow^{\mathrm{in}_r} & \\
 & & M. \\
 & \nearrow_{\mathrm{in}_s} & \\
\mathcal{U}_s(M)\{0,0,-1\} & &
\end{array}$$

Here we have applied Proposition 3.1 so that $\mathcal{U}_r\mathcal{U}_s(M)\cong 0\cong\mathcal{U}_s\mathcal{U}_r(M)$.

To check the functor relation (3.9a), we first compute $\mathcal{R}_r\mathcal{R}_{r+1}\mathcal{R}_r$ applied to a Λ_3-module M, which is equal to the total ∂_3-complex

$$\begin{array}{ccccccc}
 & & \mathcal{U}_r\mathcal{U}_{r+1}(M)\{0,0,-2\} & \xrightarrow{-\mathrm{in}_{r+1}} & \mathcal{U}_r(M)\{0,0,-1\} & & \\
 & \nearrow^{\mathrm{in}_r} & & {\scriptstyle \mathrm{in}_r}\searrow\!\!\!\!\!\!\nearrow{\scriptstyle -\mathrm{in}_r} & & \searrow^{\mathrm{in}_r} & \\
\mathcal{U}_r\mathcal{U}_{r+1}\mathcal{U}_r(M)\{0,0,-3\} & \xrightarrow{-\mathrm{in}_{r+1}} & \mathcal{U}_r^2(M)\{0,0,-2\} & & \mathcal{U}_{r+1}(M)\{0,0,-1\} & \xrightarrow{\mathrm{in}_{r+1}} & M. \\
 & \searrow_{\mathrm{in}_r} & & {\scriptstyle \mathrm{in}_r}\searrow\!\!\!\!\!\!\nearrow{\scriptstyle -\mathrm{in}_r} & & \nearrow_{\mathrm{in}_r} & \\
 & & \mathcal{U}_{r+1}\mathcal{U}_r(M)\{0,0,-2\} & \xrightarrow[\mathrm{in}_{r+1}]{} & \mathcal{U}_r(M)\{0,0,-1\} & &
\end{array}$$

We will gradually strip off the projective-injective summands of this module, which, for brevity, we will call M_0 in what follows.

By Proposition 3.1, we identify

$$\mathcal{U}_r\mathcal{U}_{r+1}\mathcal{U}_r(M)\{0,0,-3\} \cong \mathcal{U}_r(M)\{1,1,-3\} \cong \bigoplus_{i-j=r} Q\otimes \partial_2\omega\otimes\partial_1\omega\otimes M_{i.j}\{0,0,-3\}.$$

By the definition (3.5) of in_{r+1}, the external ∂_3-differential $-\mathrm{in}_{r+1}$ maps this term isomorphically onto the summand $\mathcal{U}_r(M)\{1,1,-2\}$ of

$$\mathcal{U}_r^2(M)\{0,0,-2\} \cong \mathcal{U}_r(M)\{1,1,-2\} \oplus \mathcal{U}_r\{0,0,-2\}. \tag{3.13}$$

Indeed, componentwise, the morphism has the effect

$$\begin{aligned} &Q\otimes\partial_2\omega\otimes\partial_1\omega\otimes M_{i.j}\{0,0,-3\} \longrightarrow Q\otimes\partial_1\partial_2\omega\otimes M_{i,j}\{0,0,-3\},\\ &h\otimes\partial_2\omega\otimes\partial_1\omega\otimes m \mapsto -h\otimes\partial_2\partial_1\omega\otimes m. \end{aligned}$$

Via these identifications, denote the direct sum of every term in M_0 other than $\mathcal{U}_r(M)\{1,1,-3\}$ and $\mathcal{U}_r(M)\{1,1,-2\}$ by M_1. Clearly M_1 is a Λ_3-submodule, whose quotient is equal to the ∂_3-cone

$$\Big(\mathcal{U}_r(M)\{1,1,-3\} \xrightarrow{-\mathrm{Id}_{\mathcal{U}_r(M)\{1,1,-2\}}} \mathcal{U}_r(M)\{1,1,-2\}\Big).$$

This cone is a projective-injective object in $\mathcal{M}_3$ (cf. equation (3.12)). Hence M_1 is isomorphic to M_0 in $\mathcal{SM}_3$.

Next, inside M_1, the second summand of (3.13) maps onto the anti-diagonal in the direct sum of two copies of $\mathcal{U}_r(M)\{0,0,-1\}$. Therefore

$$\Big(\mathcal{U}_r(M)\{0,0,-2\} \xrightarrow{-\mathrm{in}_r\oplus\mathrm{in}_r} \mathrm{Im}(-\mathrm{in}_r\oplus\mathrm{in}_r)\Big) \cong \mathrm{C}_3(\mathrm{Id}_{\mathcal{U}_r(M)\{0,0,-1\}})$$

is a projective Λ_3-module, and thus is isomorphic to zero in $\mathcal{SM}_3$. Modulo these terms, and equating the quotient as under the sum map

$$\mathcal{U}_r(M)\{0,0,-1\}\oplus\mathcal{U}_r(M)\{0,0,-1\}/\mathrm{Im}(-\mathrm{in}_r\oplus\mathrm{in}_r) \cong \mathcal{U}_r(M)\{0,0,-1\},$$

we have that M_1 is isomorphic to the following total ∂_3-cone M_2:

$$M_2 = \left(\begin{array}{ccccc} \mathcal{U}_r\mathcal{U}_{r+1}(M)\{0,0,-2\} & \xrightarrow{\mathrm{in}_r} & \mathcal{U}_{r+1}(M)\{0,0,-1\} & & \\ & {\scriptstyle -\mathrm{in}_{r+1}}\searrow\;\nearrow{\scriptstyle -\mathrm{in}_r} & & \searrow^{\mathrm{in}_{r+1}} & \\ & & & & M \\ & & & \nearrow_{\mathrm{in}_r} & \\ \mathcal{U}_{r+1}\mathcal{U}_r(M)\{0,0,-2\} & \xrightarrow{\mathrm{in}_{r+1}} & \mathcal{U}_r(M)\{0,0,-1\} & & \end{array}\right). \tag{3.14}$$

It follows by the above discussion that $\mathcal{R}_r\mathcal{R}_{r+1}\mathcal{R}_r(M)$ is isomorphic to M_2 in $\mathcal{SM}_3$, and this isomorphism is clearly functorial in M.

A similar computation for $\mathcal{R}_{r+1}\mathcal{R}_r\mathcal{R}_{r+1}(M)$ shows that it is functorially isomorphic to M_2 of equation (3.14). The braid relation follows.

3.4 Connection to homological algebra of zig-zag algebras

In view of Section 2.5, it is not surprising that trigraded Λ_3-modules are closely related to the homological algebra of the zig-zag algebra A. The main goal of this subsection is to utilize this relationship to establish the faithfulness of the braid group Br_∞ action on $\mathcal{SM}_3$ (Theorem 3.6), building on the results of [16].

Let M be a complex of graded A-modules

$$M = \Big(\cdots \longrightarrow M_{k-1} \xrightarrow{d_{k-1}} M_k \xrightarrow{d_k} M_{k+1} \longrightarrow \cdots \Big),$$

where each $M_k = \oplus_{j\in\mathbb{Z}} M_k^j$ is a graded A-module. Recall that a morphism $f\colon M \longrightarrow N$ is called *null-homotopic* if there is a collection of homogeneous A-module maps $h_k\colon M_k \longrightarrow N_{k-1}$, $k\in\mathbb{Z}$, such that $d_{k-1}h_k + h_{k+1}d_k = f_k$ holds for all k. The homotopy category $\mathcal{C}(A)$ is the quotient of the category of chain complexes of graded A-modules by the ideal of null-homotopic morphisms. A complex M is called *contractible* if Id_M is null-homotopic.

The homotopy category $\mathcal{C}(A)$ carries two commuting grading shifts denoted by $\langle 1\rangle$ and $[1]$ respectively. They are defined by

$$(M\langle 1\rangle)_k^j := M_k^{j-1}, \qquad (M[1])_k^j := M_{k+1}^j.$$

In addition, the automorphism $\mathcal{T}$ of $\mathcal{M}(A)$ extends to an automorphism of $\mathcal{C}(A)$, denoted by the same letter, defined by termwise applying $\mathcal{T}$ on complexes:

$$\mathcal{T}(M) := \Big(\cdots \longrightarrow \mathcal{T}(M_{k-1}) \xrightarrow{\mathcal{T}(d_{k-1})} \mathcal{T}(M_k) \xrightarrow{\mathcal{T}(d_k)} \mathcal{T}(M_{k+1}) \longrightarrow \cdots \Big).$$

In what follows, we will also use the notation $\mathcal{C}(A\text{-pmod})$ to stand for the full subcategory of $\mathcal{C}(A)$ consisting of complexes of graded projective A-modules up to homotopy.

We also re-interpret chain complexes of graded A-modules as differential graded modules over the graded dg algebra (A,d), where A sits in homological degree zero, and the natural grading of A is orthogonal to the homological grading. A chain complex of graded A-modules is equivalent to the data of a differential graded (A,d)-module

$$M = \bigoplus_{j,k\in\mathbb{Z}} M_k^j, \qquad d\big(M_k^j\big) \subset M_{k+1}^j.$$

Extending the (inverse) equivalence of Proposition 2.9, there is an auto-equivalence of abelian categories

$$\mathcal{G}_\infty\colon\ (A,d)\text{-mod} \longrightarrow \mathcal{M}_3,$$

where, on the object $\mathcal{G}_\infty(M)\in\mathcal{M}_3$ for a given $M\in(A,d)$-mod, the generator ∂_3 acts by the differential $(-1)^k d\colon M_k \longrightarrow M_{k+1}$. It follows from Proposition 2.9 that $\mathcal{G}_\infty$ commutes with the translation by the various shift functors as follows:

$$\begin{aligned}
&\mathcal{G}_\infty(M\langle 1\rangle) = \mathcal{G}_\infty(M)\{1,1,0\},\\
&\mathcal{G}_\infty(M[1]) = \mathcal{G}_\infty(M)\{0,0,-1\},\\
&\mathcal{G}_\infty(\mathcal{T}(M)) = \mathcal{G}_\infty(M)\{1,0,0\}.
\end{aligned} \tag{3.15}$$

As a result, one can deduce that

$$\mathcal{G}_\infty(P_r\langle j\rangle[k]) \cong Q\{r+j, j, -k\}. \tag{3.16}$$

Lemma 3.8. *A morphism of chain complexes* $f\colon M\longrightarrow N$ *of graded* A*-modules is null-homotopic if and only if it factors through the canonical embedding of graded dg modules over* A

$$\lambda_M\colon\ M\longrightarrow M\otimes \Bbbk[d]/\big(d^2\big)[1], \qquad m\mapsto m\otimes d.$$

Consequently, M *is contractible if and only if* M *is a dg summand of* $M\otimes\Bbbk[d]/\big(d^2\big)[1]$.

Proof. Suppose $f=dh+hd\colon M\longrightarrow N$ is null-homotopic, with $h\colon M_k\longrightarrow N_{k-1}$ the null-homotopy map. We define

$$\widehat{h}\colon\ M\otimes\Bbbk[d]/\big(d^2\big)[1]\longrightarrow N$$

by, for any homogeneous $m\in M_k$,

$$\widehat{h}(m\otimes 1):=(-1)^k h(m), \qquad \widehat{h}(m\otimes d):=dh(m)+hd(m).$$

It is an easy exercise to check that $\widehat{h}$ is a map of dg A-modules. Then, we clearly have a factorization $f=\widehat{h}\circ\lambda_M$.

Conversely, if there is a factorization of dg A-modules

$$\begin{array}{ccc} M & \xrightarrow{\quad f\quad} & N \\ & {\scriptstyle\lambda_M}\searrow \quad \nearrow{\scriptstyle\widehat{h}} & \\ & M\otimes\Bbbk[d]/\big(d^2\big)[1], & \end{array}$$

define $h\colon M\longrightarrow N$ by $h(m):=(-1)^k\widehat{h}(m\otimes 1)$ for any $m\in M_k$. Another easy computation shows that $f=dh+hd$ is indeed null-homotopic. ∎

For the next result, for any graded dg module M over A, denote by M^μ the corresponding dg module with the same underlying bigraded A-module as M, but the differential acting by zero instead. Under the equivalence $\mathcal{G}_\infty$, this corresponds to the μ-pull-back (see equation (3.2)) of a trigraded Λ_3-module.

Lemma 3.9. *Let* M *be a graded dg module over* A*. There is an isomorphism of dg modules*

$$\phi_M\colon\ M\otimes\Bbbk[d]/\big(d^2\big)\longrightarrow M^\mu\otimes\Bbbk[d]/\big(d^2\big),$$

defined by

$$\phi_M(m\otimes 1):=m\otimes 1, \qquad \phi_M(m\otimes d):=(-1)^{k+1}d(m)\otimes 1+m\otimes d.$$

for any $m\in M_k$.

Proof. It is an easy computation to verify that ϕ_M commutes with the respective differentials on both sides. The inverse of ϕ_M is given by

$$\psi_M\colon\ M^\mu\otimes\Bbbk[d]/\big(d^2\big)\longrightarrow M\otimes\Bbbk[d]/\big(d^2\big),$$

where, for $m\in M_k$,

$$\psi_M(m\otimes 1):=m\otimes 1, \qquad \psi_M(m\otimes d):=(-1)^k d(m)\otimes 1+m\otimes d.$$

Clearly, both ϕ_M and ψ_M are homogeneous A-module maps. The lemma follows. ∎

Corollary 3.10. *The functor* $\mathcal{G}_\infty\colon (A,d)\text{-mod}\longrightarrow\mathcal{M}_3$ *descends to an exact functor*

$$\mathcal{G}\colon\ \mathcal{C}(A\text{-pmod})\longrightarrow\mathcal{S}\mathcal{M}_3.$$

Proof. By Lemma 3.8, $\mathcal{C}(A\text{-pmod})$ is the categorical quotient of the category of chain complexes of graded projective A-modules by the ideal of morphisms factoring through objects of the form $M\otimes\Bbbk[d]/(d^2)$, where M ranges over all chain-complexes of graded projective A-modules. For $\mathcal{G}$ to be well-defined, it suffices to check that, under the functor $\mathcal{G}_\infty$, such objects are sent to the class of projective-injective objects in $\mathcal{M}_3$.

By Lemma 3.9, there is an isomorphism of graded dg modules

$$M\otimes\Bbbk[d]/(d^2)\cong M^\mu\otimes\Bbbk[d]/(d^2)\cong\bigoplus_k M_k^\mu\otimes\Bbbk[d]/(d^2).$$

As each M_k^μ is a projective A-module, the result follows since, by Proposition 2.9,

$$\mathcal{G}\big(P_r\langle j\rangle\otimes\Bbbk[d]/(d^2)[-k]\big)\cong Q\{r,r+j,k\}\otimes\Lambda_1'\cong\Lambda_3\{r,r+j,k\}$$

holds for any $r,j,k\in\mathbb{Z}$.

It remains to show that $\mathcal{G}$ is exact, i.e., it commutes with homological shifts and takes distinguished triangles to distinguished triangles.

Given a complex M of graded projective modules over A, there is a short exact sequence

$$0\longrightarrow M\xrightarrow{\lambda_M}M\otimes\Bbbk[d]/(d^2)[1]\longrightarrow M[1]\longrightarrow 0.$$

Applying $\mathcal{G}_\infty$ to the short exact sequence, we obtain a short exact sequence of $\mathcal{M}_3$:

$$0\longrightarrow\mathcal{G}_\infty(M)\xrightarrow{\mathcal{G}_\infty(\lambda_M)}\mathcal{G}_\infty\big(M\otimes\Bbbk[d]/(d^2)[1]\big)\longrightarrow\mathcal{G}_\infty(M[1])\longrightarrow 0,$$

which, in turn, leads to a distinguished triangle in $\mathcal{S}\mathcal{M}_3$:

$$\mathcal{G}(M)\xrightarrow{\mathcal{G}(\lambda_M)}\mathcal{G}\big(M\otimes\Bbbk[d]/(d^2)[1]\big)\longrightarrow\mathcal{G}(M[1])\xrightarrow{[1]}\mathcal{G}(M)[1]_{\mathcal{S}\mathcal{M}}.$$

By the earlier discussion in this proof, the term $\mathcal{G}_\infty\big(M\otimes\Bbbk[d]/(d^2)[1]\big)$ vanishes in $\mathcal{S}\mathcal{M}_3$, and thus there is an isomorphism

$$\mathcal{G}(M[1])\cong\mathcal{G}(M)[1]_{\mathcal{S}\mathcal{M}},$$

which is clearly functorial in M.

Lastly, notice that distinguished triangles in $\mathcal{C}(A\text{-pmod})$, up to isomorphism, arise from short exact sequences of chain-complexes of graded projective A-modules

$$0\longrightarrow M_1\longrightarrow M\longrightarrow M_2\longrightarrow 0.$$

Applying $\mathcal{G}_\infty$ to this sequence, we obtain a short exact sequence of trigraded Λ_3-modules. This sequence results in a distinguished triangle in $\mathcal{S}\mathcal{M}_3$, being the image of the original triangle in $\mathcal{C}(A\text{-pmod})$. The exactness of $\mathcal{G}$ now follows. ■

Denote by $\mathcal{C}^b(A\text{-pmod})$, $\mathcal{C}^+(A\text{-pmod})$ and $\mathcal{C}^-(A\text{-pmod})$ the full triangulated subcategories of $\mathcal{C}(A\text{-pmod})$ consisting of, respectively, bounded, bounded-from-below and bounded-from-above complexes of graded projective modules over A. The localization functor from $\mathcal{C}(A\text{-pmod})$ into $\mathcal{D}(A)$ restricts to equivalences of categories on these full-subcategories onto their respective images in the (dg) derived category $\mathcal{D}(A)$.

Theorem 3.11. *The functor $\mathcal{G}\colon \mathcal{C}(A\text{-pmod})\longrightarrow\mathcal{SM}_3$ is fully-faithful.*

Proof. The proof is divided into three steps.

As the first step, we claim that $\mathcal{G}$, when restricted to the full-subcategories $\mathcal{C}^{\pm}(A\text{-pmod})$, is fully-faithful. To do this, we identify these categories with their images in $\mathcal{D}(A)$ under localization, and use the fact that the (dg) derived category of (A,d)-mod is compactly generated by the collection of objects $\{P_r\langle j\rangle[k] | r,j,k\in\mathbb{Z}\}$. Then, in order to prove the claim, we just need to compare the morphism spaces between the generating objects $P_r\langle j\rangle[k]$, $r,j,k\in\mathbb{Z}$, and their images $\mathcal{G}(P_r\langle j\rangle[k]) = Q\{r+j,j,-k\}$ in $\mathcal{SM}_3$ [11, Lemma 4.2].

On the one hand, we have

$$\mathrm{Hom}_{\mathcal{C}(A)}(P_{r_1}\langle j_1\rangle[k_1], P_{r_2}\langle j_2\rangle[k_2]) = \begin{cases} \Bbbk(r_1), & r_1=r_2,\ j_1=j_2,\ k_1=k_2,\\ \Bbbk(r_1\,|\,r_1+1), & r_1=r_2+1,\ j_1=j_2,\ k_1=k_2,\\ \Bbbk(r_1\,|\,r_1-1), & r_1=r_2-1,\ j_1=j_2-1,\ k_1=k_2,\\ \Bbbk(r_1\,|\,r_1+1|r_1), & r_1=r_2,\ j_1=j_2+1,\ k_1=k_2,\\ 0, & \text{otherwise.}\end{cases}$$

On the other hand, we can compute the morphism spaces of $\mathcal{G}(P_r\langle j\rangle[k])$ using Lemma 3.4

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{SM}_3}(\mathcal{G}(P_{r_1}\langle j_1\rangle[k_1]), \mathcal{G}(P_{r_2}\langle j_2\rangle[k_2]))\\ &\quad\cong \mathrm{Hom}_{\Lambda_3}(Q, Q\{r_2+j_2-r_1-j_1, j_2-j_1, k_1-k_2\})\\ &\quad\cong \mathrm{Hom}_{\Lambda_2}(Q, Q\{r_2+j_2-r_1-j_1, j_2-j_1, k_1-k_2\}).\end{aligned}$$

The second isomorphism follows since ∂_3 acts trivially on Q and its grading shifts. Note that the last space is non-zero only if $k_1=k_2$, since the Λ_2 action preserves the k-grading. When $k_1=k_2$, we can compute

$$\mathrm{Hom}_{\mathcal{SM}_3}(\mathcal{G}(P_{r_1}\langle j_1\rangle[k_1])_{\mathcal{SM}}, \mathcal{G}(P_{r_2}\langle j_2\rangle[k_2]_{\mathcal{SM}})) \cong \begin{cases} \Bbbk, & r_1=r_2,\ j_1=j_2,\\ \Bbbk, & r_1=r_2+1,\ j_1=j_2,\\ \Bbbk, & r_1=r_2-1,\ j_1=j_2-1,\\ \Bbbk, & r_1=r_2,\ j_1=j_2+1.\end{cases}$$

Comparing these computations, the claim follows.

In the second step, we show that

$$\mathrm{Hom}_{\mathcal{C}(A)}(M,N) \xrightarrow{\mathcal{G}} \mathrm{Hom}_{\mathcal{SM}_3}(\mathcal{G}(M),\mathcal{G}(N))$$

is a bijection if one of M or N lies in $\mathcal{C}^{\pm}(A\text{-pmod})$. Assume, for instance, that $M\in\mathcal{C}^{+}(A\text{-pmod})$ and $N\in\mathcal{C}(A\text{-pmod})$. Without loss of generality, we may assume that $M_k=0$ for all $k<0$. Then, N fits into a distinguished triangle

$$N_{\geq -1}\longrightarrow N\longrightarrow N_{\leq -2}\xrightarrow{[1]} N_{\geq -1}[1],$$

where $N_{\geq -1}$ is the subcomplex of N of the form

$$N_{\geq -1} = \left(\cdots\longrightarrow 0\longrightarrow N_{-1}\xrightarrow{d_{-1}} N_0\xrightarrow{d_0} N_1\xrightarrow{d_1}\cdots\right),$$

and $N_{\leq -2}$ is the quotient complex

$$N_{\leq -2} = \left(\cdots\xrightarrow{d_{-5}} N_{-4}\xrightarrow{d_{-4}} N_{-3}\xrightarrow{d_{-3}} N_{-2}\longrightarrow 0\longrightarrow\cdots\right).$$

It is readily seen that, in the homotopy category, we have

$$\mathrm{Hom}_{\mathcal{C}(A)}(M, N_{<-1}) = 0, \qquad \mathrm{Hom}_{\mathcal{C}(A)}(M, N_{<-1}[-1]) = 0.$$

Likewise, as the k-degrees of the objects $\mathcal{G}(N_{<-1})$ and $\mathcal{G}(N_{<-1}[-1])$ are bounded above by -1, we have that, by Lemma 3.4,

$$\mathrm{Hom}_{\mathcal{SM}_3}(\mathcal{G}(M), \mathcal{G}(N_{\leq -2})) = 0, \qquad \mathrm{Hom}_{\mathcal{SM}_3}(\mathcal{G}(M), \mathcal{G}(N_{\leq -2})[-1]_{\mathcal{SM}}) = 0.$$

Therefore, by the previous step, we have that

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}(A)}(M, N) = \mathrm{Hom}_{\mathcal{C}(A)}(M, N_{\geq -1}) \\ &\quad \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{SM}_3}(\mathcal{G}(M), \mathcal{G}(N)) = \mathrm{Hom}_{\mathcal{SM}_3}(\mathcal{G}(M), \mathcal{G}(N_{\geq -1})).\end{aligned}$$

The other cases are similar, and we leave them as exercises to the reader.

Finally, assume both M and N are any objects of $\mathcal{C}(A\text{-pmod})$. We can truncate N as

$$N_{\geq 0} \longrightarrow N \longrightarrow N_{\leq -1} \xrightarrow{[1]} N_{\geq 0}[1],$$

Then, we have the commutative diagram

$$\begin{array}{ccccccc} \cdots \to & \mathrm{Hom}_{\mathcal{C}(A)}(M, N_{\geq 0}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}(A)}(M, N) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}(A)}(M, N_{\leq -1}) & \to \cdots \\ & \downarrow \mathcal{G} & & \downarrow \mathcal{G} & & \downarrow \mathcal{G} & \\ \cdots \to & \mathrm{Hom}_{\mathcal{SM}_3}(\mathcal{G}(M), \mathcal{G}(N_{\geq 0})) & \to & \mathrm{Hom}_{\mathcal{SM}_3}(\mathcal{G}(M), \mathcal{G}(N)) & \to & \mathrm{Hom}_{\mathcal{SM}_3}(\mathcal{G}(M), \mathcal{G}(N_{\leq -1})) & \to \cdots \end{array}$$

The middle vertical arrow is then an isomorphism by the classical five lemma and the previous case. This finishes the proof of the theorem. ■

In [16], a braid group action on the homotopy category $\mathcal{C}(A)$ is introduced. The braid group generator $\mathcal{R}_r$ acts, on a chain complex M of graded A-modules, by

$$\mathcal{R}_r(M) := \mathrm{C}\Big(P_r \otimes (r)M \xrightarrow{f} M\Big),$$

where f is the left A-module map determined by $(r) \otimes (r)m \mapsto (r)m$.

Lemma 3.12. *The functor $\mathcal{G}$ commutes with the braid group actions on $\mathcal{C}(A\text{-pmod})$ and $\mathcal{SM}_3$.*

Proof. It suffices to show that each $\mathcal{R}_r$ commutes with $\mathcal{G}$. This follows from equation (3.16), Proposition 2.9 and the fact that $\mathcal{G}$ sends the cone construction in $\mathcal{C}(A\text{-pmod})$ to that of the ∂_3-cone in $\mathcal{SM}_3$. ■

Corollary 3.13. *The action of Br_∞ on the category $\mathcal{SM}_3$ is faithful.*

Proof. In [16, Corollary 1.2], it is shown that the braid group Br_{m+1} on $m+1$ strands acts faithfully on $\mathcal{C}^b(A_m\text{-pmod})$. The Corollary then follows from Theorem 3.11, Lemma 3.12 and taking the limit $m \to \infty$. ■

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The first author learned algebraic topology for the first time from the Russian classic by Dmitry Fuchs and Anatolii Fomenko [6] (in its first edition named *Homotopic Topology*), while the second author enjoyed teaching graduate courses out of this reprinted classic at his previous work institution. It is our great pleasure to dedicate this short note to Dmitry Fuchs on the occasion of his eightieth anniversary.

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