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A Note on the Formal Groups of Weighted Delsarte Threefolds

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Abstract. One-dimensional formal groups over an algebraically closed field of positive characteristic are classified by their height. In the case of $K3$ surfaces, the height of their formal groups takes integer values between 1 and 10, or ∞. For Calabi–Yau threefolds, the height is bounded by $h^{1,2}+1$ if it is finite, where $h^{1,2}$ is a Hodge number. At present, there are only a limited number of concrete examples for explicit values or the distribution of the height. In this paper, we consider Calabi–Yau threefolds arising from weighted Delsarte threefolds in positive characteristic. We describe an algorithm for computing the height of their formal groups and carry out calculations with various Calabi–Yau threefolds of Delsarte type.

Key words: Artin–Mazur formal groups; Calabi–Yau threefolds; weighted Delsarte varieties

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Dedicated to Professor Noriko Yui, from whom I learned, as a graduate student and a collaborator now, enthusiasm, dynamics and humanity in mathematical research.

1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$. Let X be a Calabi–Yau threefold over k, by which we mean a smooth projective variety over k of dimension 3 with a trivial canonical sheaf and $\dim H^1(X, \mathcal{O}_X) = \dim H^2(X, \mathcal{O}_X) = 0$. In [2], Artin and Mazur defined a functor Φ_X for X on the category of finite local k-algebras A with residue field k by

$$\Phi_X(A) = \ker\big(H^3_{\rm et}(X_A, \mathbb{G}_m) \longrightarrow H^3_{\rm et}(X, \mathbb{G}_m)\big),$$

where $X_A = X \times \operatorname{Spec} A$ and $\mathbb{G}_m$ is the sheaf of multiplicative groups. It is proved in [2] that this functor (more generally, those for Calabi–Yau varieties of any dimension) is representable by a smooth formal group of dimension equal to the geometric genus of X, which is one in our case. By abuse of notation, we also use Φ_X for this formal group and called it the (*Artin–Mazur*) *formal group* of X.

A formal group in positive characteristic p is endowed with the multiplication-by-p map. The p-rank of its kernel is called the *height* of Φ_X and denoted by $h := \operatorname{ht}\Phi_X$, namely

$$p^h = \#\ker([p]\colon \Phi_X \longrightarrow \Phi_X).$$

It is known that one-dimensional formal groups in positive characteristic are determined up to isomorphism by the height. In the case of $K3$ surfaces, the height takes integer values between 1

This paper is a contribution to the Special Issue on Modular Forms and String Theory in honor of Noriko Yui. The full collection is available at http://www.emis.de/journals/SIGMA/modular-forms.html and 10, or ∞ (cf. [1, 8, 19]). In the case of Calabi–Yau threefolds, it is proved in [18] that h is bounded above by $h^{1,2}+1$ if $h \neq \infty$, where $h^{1,2}$ is a Hodge number of X. Note that it is still a conjecture that $h^{1,2}$ is bounded for Calabi–Yau threefolds.

In this paper, we consider Calabi–Yau threefolds arising from weighted projective hypersurfaces of Delsarte type and give an algorithm to compute the height of their Artin–Mazur formal groups. The algorithm involves only combinatorial argument so that one can actually do calculations to find numerical data for the height of Calabi–Yau threefolds. Among others, we find a Calabi–Yau threefold with height 42 or 82 in some characteristic.

2 Quotient maps and the height of formal groups

In this section, we show that the formal groups are invariant under the quotient map by a group of *symplectic* automorphisms (by which we mean the automorphisms that preserve the dualizing sheaf of a variety). The Calabi–Yau threefolds we consider in Sections 4 and 5 often have large groups of symplectic automorphisms. In this section, we focus our attention to a very special case, namely, orbifold Calabi–Yau threefolds (X, Y) which form a mirror pair (i.e., $h^{1,1}(X) = h^{2,1}(Y)$ and $h^{2,1}(X) = h^{1,1}(Y)$), and discuss their formal groups briefly. More details about relationships between quotient maps and formal groups will be discussed elsewhere.

Lemma 2.1. *Let X be a Calabi–Yau threefold over k and G be a finite group acting on X symplectically (i.e., preserving the dualizing sheaf of X). Write $Y := X/G$. Assume that p is coprime to the order of G and that there exists a crepant resolution $\widetilde{Y}$ of Y. Then $\widetilde{Y}$ is a Calabi–Yau threefold with $\Phi_X \cong \Phi_{\widetilde{Y}}$. In particular, the formal groups of X and $\widetilde{Y}$ have the same height:* $\operatorname{ht}\Phi_X = \operatorname{ht}\Phi_{\widetilde{Y}}$.

Proof. Since p is coprime to the order of G, Y has at most rational singularities and by Theorem 3.1 (and the paragraph 3.10) of [16], we have $\Phi_Y \cong \Phi_{\widetilde{Y}}$ for a crepant resolution $\widetilde{Y}$ of Y. Write $f\colon X \longrightarrow Y$ for the quotient map. As G acts on X symplectically, $g^*\omega_X \cong \omega_X$ for every $g \in G$ and thus $\omega_X^G \cong \omega_X$ for the dualizing sheaf ω_X of X. Since X is Calabi–Yau, we find $\mathcal{O}_Y \cong f_*\mathcal{O}_X^G \cong f_*\omega_X^G \cong f_*\omega_X \cong f_*\mathcal{O}_X$. Again by Theorem 3.1 of [16], we see $\Phi_X \cong \Phi_Y$. Hence $\Phi_X \cong \Phi_{\widetilde{Y}}$. ∎

Proposition 2.2. *Let X be a threefold over k with $\mathcal{O}_X \cong \omega_X$ and G be a finite group acting on X symplectically. Write $\widetilde{X}$ and $\widetilde{Y}$ for crepant resolutions of X and Y, respectively, constructed as in the following diagram:*

$$\begin{array}{ccc} X & \longleftarrow & \widetilde{X} \\ \downarrow & & \\ X/G \;= Y & \longleftarrow & \widetilde{Y}. \end{array} \tag{2.1}$$

Write $h := \operatorname{ht}\Phi_{\widetilde{X}}$. Assume that $(\widetilde{X}, \widetilde{Y})$ is a mirror pair of Calabi–Yau threefolds and h is finite. Then

$$h \le \min\left\{h^{1,1}(\widetilde{X}) + 1,\, h^{1,2}(\widetilde{X}) + 1\right\}.$$

Proof. Note that $h = \operatorname{ht}\Phi_{\widetilde{X}} = \operatorname{ht}\Phi_{\widetilde{Y}}$ by Lemma 2.1. Since h is finite, it follows from Corollary 2.3 of [18] that

$$h \le h^{1,2}(\widetilde{X}) + 1 \qquad \text{and} \qquad h \le h^{1,2}(\widetilde{Y}) + 1.$$

As $(\widetilde{X}, \widetilde{Y})$ is a mirror pair of Calabi–Yau threefolds, we find $h^{1,2}(\widetilde{Y}) + 1 = h^{1,1}(\widetilde{X}) + 1$. Combining these relations, we obtain the asserted inequality. ∎

Remark 2.3. It is still an open problem whether or not $h^{1,1}$ or $h^{1,2}$ is bounded for Calabi–Yau threefolds.

In later sections of this paper, we see examples of Calabi–Yau threefolds with many symplectic automorphisms. One may then expect to construct their mirror partners by using some group actions G as in Proposition 2.2. In some cases, however, we do not find such G as described in the following statement; a concrete example for this can be found in Example 6.3.

Corollary 2.4. *Let $\widetilde{X}$ be a Calabi–Yau threefold defined as a crepant resolution of X and write $h = \operatorname{ht}\Phi_{\widetilde{X}}$. Assume that $h > \min\{h^{1,1}(\widetilde{X})+1,\, h^{1,2}(\widetilde{X})+1\}$. Then either $h=\infty$ or there exists no group G of symplectic automorphisms on X such that $(\widetilde{X},\widetilde{Y})$ of Proposition 2.2 forms a mirror pair.*

Proof. Take the contrapositive of Proposition 2.2. ■

3 Weighted Delsarte varieties

In order to compute the cohomology groups of weighted Delsarte varieties, we explain some geometric properties of them (see also [7]).

Let $Q = (q_0, \dots, q_n)$ be an $(n+1)$-tuple of positive integers such that $p \nmid q_i$, $0 \le i \le n$, and $\gcd(q_0, \dots, \hat{q}_i, \dots, q_n) = 1$ for every $0 \le i \le n$, where $\hat{q}_i$ means that q_i is omitted. The weighted projective n-space over k of type Q, denoted by $\mathbb{P}^n(Q)$, is the projective variety $\mathbb{P}^n(Q) := \operatorname{Proj} k[x_0, \dots, x_n]$, where the polynomial algebra is graded by the condition $\deg(x_i) = q_i$ for $0 \le i \le n$ (cf. [5]).

Let m be a positive integer such that $p \nmid m$. Let $A = (a_{ij})$ be an $(n+1)\times(n+1)$ matrix of integer entries having the properties

(i) $a_{ij} \ge 0$ and $p \nmid a_{ij}$ for every (i,j) with $a_{ij} \ne 0$,

(ii) $p \nmid \det A$,

(iii) $\sum_{j=0}^{n} q_j a_{ij} = m$ for $0 \le i \le n$,

(iv) given j, $a_{ij} = 0$ for some i.

We define an $(n-1)$-dimensional *weighted Delsarte variety in $\mathbb{P}^n(Q)$ of degree m with matrix A* (cf. [3, 7, 14]) to be the weighted projective hypersurface, X_A, defined by

$$\sum_{i=0}^{n} x_0^{a_{i0}} x_1^{a_{i1}} \cdots x_n^{a_{in}} = 0 \subset \mathbb{P}^n(Q).$$

When A is a diagonal matrix, the equation has the form

$$x_0^{d_0} + x_1^{d_1} + \cdots + x_n^{d_n} = 0$$

and we call it a *weighted Fermat variety of degree d*.

Weighted Delsarte varieties are birational to finite quotients of Fermat varieties and many properties of their cohomology groups can be extracted from those of Fermat varieties (cf. [7, 15, 20]). For instance, write $d = |\det A|$ and let F_d be the $(n-1)$-dimensional Fermat variety of degree d in the usual projective space $\mathbb{P}^n$:

$$F_d\colon\ y_0^d + y_1^d + \cdots + y_n^d = 0.$$

Write μ_d for the group of d-th roots of unity in $k^\times$, which consists of d elements as $p \nmid d$. Set $\Gamma = (\mu_d \times \cdots \times \mu_d)/(\text{diagonal elements})$ for the product of $n+1$ copies of μ_d modulo diagonal elements and define

$$\Gamma_A = \left\{ \boldsymbol{\gamma} = \left(\prod_{j=0}^{n} \gamma_j^{a_{0j}}, \prod_{j=0}^{n} \gamma_j^{a_{1j}}, \ldots, \prod_{j=0}^{n} \gamma_j^{a_{nj}} \right) \in \Gamma \;\middle|\; (\gamma_0, \ldots, \gamma_n) \in \mu_d^{n+1} \right\}. \tag{3.1}$$

Then Γ_A is a subgroup of Γ and it acts on F_d by

$$\boldsymbol{\gamma} \cdot (y_0 : y_1 : \cdots : y_n) = \left(\left(\prod_{j=0}^{n} \gamma_j^{a_{0j}} \right) y_0 : \left(\prod_{j=0}^{n} \gamma_j^{a_{1j}} \right) y_1 : \cdots : \left(\prod_{j=0}^{n} \gamma_j^{a_{nj}} \right) y_n \right)$$

for $\boldsymbol{\gamma} \in \Gamma_A$ and $(y_0 : y_1 : \cdots : y_n) \in F_d$.

Lemma 3.1. *Let X_A be a weighted Delsarte variety in $\mathbb{P}^n(Q)$ with matrix A. Then X_A is birational to the quotient F_d/Γ_A.*

Proof. Write $B = (b_{ij})$ for the cofactor matrix of A. Then there is a dominant rational map $f\colon F_d \cdots\to X_A$ defined by

$$f((y_0 : y_1 : \cdots : y_n)) = \left(\prod_{j=0}^{n} y_j^{b_{0j}} : \prod_{j=0}^{n} y_j^{b_{1j}} : \cdots : \prod_{j=0}^{n} y_j^{b_{nj}} \right).$$

Hence X_A is birational to F_d/Γ_A. ∎

We describe the ℓ-adic étale cohomology of the varieties involved, where ℓ is a prime different from $p = \operatorname{char} k$. It is known that the cohomology of Fermat variety F_d is decomposed into one-dimensional pieces parameterized by the characters of Γ. In fact, note first that F_d is of dimension $n-1$ and

$$H^{n-1}_{\text{et}}(F_d, \mathbb{Q}_\ell) \cong \begin{cases} H^{n-1}_{\text{prim}}(F_d, \mathbb{Q}_\ell) & \text{if } n \text{ is even}, \\ V(0) \oplus H^{n-1}_{\text{prim}}(F_d, \mathbb{Q}_\ell) & \text{if } n \text{ is odd}, \end{cases}$$

where $V(0)$ denotes the subspace corresponding to the hyperplane section and $H^{n-1}_{\text{prim}}(F_d, \mathbb{Q}_\ell)$ is the primitive part of $H^{n-1}(F_d, \mathbb{Q}_\ell)$ (cf. [11, 13]). Define

$$\mathfrak{A}(F_d) = \left\{ \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n) \;\middle|\; \alpha_i \in \mathbb{Z}/d\mathbb{Z},\ \alpha_i \neq 0,\ 0 \le i \le n,\ \sum_{i=0}^{n} \alpha_i = 0 \right\}$$

and

$$V(\boldsymbol{\alpha}) = \left\{ v \in H^{n-1}_{\text{prim}}(F_d, \mathbb{Q}_\ell) \mid \boldsymbol{\gamma}^*(v) = \gamma_0^{\alpha_0} \gamma_1^{\alpha_1} \cdots \gamma_n^{\alpha_n} \cdot v,\ \forall \boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_n) \in \Gamma \right\},$$

where $\boldsymbol{\gamma}^*$ is the endomorphism of $H^{n-1}_{\text{et}}(F_d, \mathbb{Q}_\ell)$ induced from $\boldsymbol{\gamma}$. Then the primitive part $H^{n-1}_{\text{prim}}(F_d, \mathbb{Q}_\ell)$ of the cohomology for F_d is given as

$$H^{n-1}_{\text{prim}}(F_d, \mathbb{Q}_\ell) = \bigoplus_{\boldsymbol{\alpha} \in \mathfrak{A}(F_d)} V(\boldsymbol{\alpha}).$$

A similar property holds for X_A since it is birational to the quotient variety F_d/Γ_A. Here we describe the cohomology of F_d/Γ_A.

Lemma 3.2. *Let X_A be a weighted Delsarte variety in $\mathbb{P}^n(Q)$ with matrix A. Let Γ_A be the group defined in (3.1) and put*

$$\mathfrak{A}(X_A) = \left\{ \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathfrak{A}(F_d) \;\middle|\; \sum_{i=0}^{n} a_{ij}\alpha_i = 0,\ 0 \le j \le n \right\}.$$

Then

$$H^{n-1}_{\mathrm{et}}(F_d/\Gamma_A, \mathbb{Q}_\ell) \cong \begin{cases} \bigoplus_{\boldsymbol{\alpha}\in\mathfrak{A}(X_A)} V(\boldsymbol{\alpha}) & \text{if } n \text{ is even,} \\ V(0) \oplus \bigoplus_{\boldsymbol{\alpha}\in\mathfrak{A}(X_A)} V(\boldsymbol{\alpha}) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The proof is similar to the case of weighted Delsarte (or diagonal) surfaces; see, for instance, [6] and [7] for details. The assertion follows from the isomorphism

$$H^{n-1}_{\mathrm{et}}(F_d/\Gamma_A, \mathbb{Q}_\ell) \cong H^{n-1}_{\mathrm{et}}(F_d, \mathbb{Q}_\ell)^{\Gamma_A}$$

as $\mathbb{Q}_\ell$-vector spaces and by considering the Γ_A-action on $H^{n-1}_{\mathrm{et}}(F_d, \mathbb{Q}_\ell)$. ■

It is a bit lengthy to describe the set $\mathfrak{A}(X_A)$ etc. here. The reader may refer, for instance, to [6] or [19] for some concrete examples of $\mathfrak{A}(X_A)$.

4 Calabi–Yau threefolds of Delsarte type

In this section, we discuss Calabi–Yau threefolds arising from weighted Delsarte threefolds. Using their cohomology groups, we describe an algorithm for computing the height of their formal groups.

Let X be a weighted projective variety in $\mathbb{P}^n(Q)$ with $Q = (q_0, \ldots, q_n)$. X is said to be *quasi-smooth* (cf. [5]) if its affine quasi-cone is smooth outside the origin. For instance, weighted Fermat varieties are quasi-smooth. As a special case of weighted quasi-smooth varieties, we observe the following property for quasi-smooth weighted Delsarte varieties.

Lemma 4.1. *Let X_A be a quasi-smooth weighted Delsarte variety in $\mathbb{P}^n(Q)$ of degree m with matrix A. Write $d = |\det A|$. Let Γ_A be the group defined in (3.1) acting on the Fermat variety F_d. Then the following assertions hold:*

(1) *The quotient variety F_d/Γ_A has at most rational abelian quotient singularities.*

(2) *X_A has at most rational cyclic quotient singularities.*

Proof. (1) Since F_d is a smooth variety and Γ_A is an abelian group, F_d/Γ_A has at most abelian quotient singularities. In characteristic 0, quotient singularities are known to be rational. Hence we only need to show that F_d/Γ_A is liftable to characteristic 0 and this follows from conditions (i) and (ii) on matrix A.

(2) A quasi-smooth variety is locally isomorphic to the quotient of a smooth variety by some cyclic group action and this cyclic group is a subgroup of μ_{q_i} for some weight q_i (cf. [5]). Since $p \nmid q_i$ for every i, the group action by a subgroup of μ_{q_i} can be lifted to characteristic 0. Hence X_A has at most cyclic quotient singularities and they are rational. ■

Now we consider weighted Delsarte threefolds.

Lemma 4.2. *Let X_A be a weighted Delsarte threefold in $\mathbb{P}^4(Q)$ with matrix A. Assume that X_A is quasi-smooth and $m = q_0 + q_1 + q_2 + q_3 + q_4$. Then the dualizing sheaf of X_A is trivial and there exists a crepant resolution for X_A.*

Proof. Since X_A is a quasi-smooth hypersurface and of dimension 3, it is known (cf. [4, Proposition 6]) to be in general position relative to $\mathbb{P}^4(Q)_{\rm sing}$ (i.e., $\operatorname{codim}_{X_A}\left(X_A \cap \mathbb{P}^4(Q)_{\rm sing}\right) \geq 2$, where $\mathbb{P}^4(Q)_{\rm sing}$ is the singular locus of $\mathbb{P}^4(Q)$). Hence the dualizing sheaf of X_A is computed as $\omega_{X_A} \cong \mathcal{O}_{X_A}(m - q_0 - q_1 - q_2 - q_3 - q_4) \cong \mathcal{O}_{X_A}$ (see [5]). The existence of a crepant resolution for X_A is proved in [10]. ■

Definition 4.3. If X_A is a quasi-smooth weighted Delsarte threefold with matrix A of degree m with $m = q_0 + \cdots + q_4$, then a crepant resolution $\widetilde{X}_A$ of X_A is called a *Calabi–Yau threefold of* (*weighted*) *Delsarte type in* $\mathbb{P}^4(Q)$ *with matrix* A. When A is a diagonal matrix, X_A is also called a *Calabi–Yau threefold of* (*weighted*) *Fermat type*.

Since the quotient F_d/Γ_A is birational to X_A, it is also birational to $\widetilde{X}_A$. Using the cohomological information of F_d/Γ_A, we write several birational properties of $\widetilde{X}_A$. Recall that $A = (a_{ij})$, $d = |\det A|$ and that F_d is the Fermat threefold of degree d in $\mathbb{P}^4$. We have

$$H^3_{\rm et}(F_d/\Gamma_A, \mathbb{Q}_\ell) \cong \bigoplus_{\boldsymbol{\alpha} \in \mathfrak{A}(X_A)} V(\boldsymbol{\alpha}),$$

where

$$\mathfrak{A}(X_A) = \left\{ (\alpha_0, \dots, \alpha_4) \in (\mathbb{Z}/d\mathbb{Z})^5 \;\middle|\; \begin{array}{l} \alpha_i \neq 0,\ 0 \leq i \leq 4,\ \sum_{i=0}^4 \alpha_i = 0 \\ \sum_{i=0}^4 a_{ij}\alpha_i = 0 \text{ for } 0 \leq j \leq 4 \end{array} \right\}.$$

For each $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_4) \in \mathfrak{A}(X_A)$, define an integer

$$\|\boldsymbol{\alpha}\| = \sum_{i=0}^4 \left\langle \frac{\alpha_i}{d} \right\rangle - 1,$$

where $< \alpha_i/d >$ denotes the fractional part of α_i/d. It takes values $\|\boldsymbol{\alpha}\| = 0, 1, 2$ or 3. If $\widetilde{X}_A$ is a Calabi–Yau threefold of Delsarte type in $\mathbb{P}^4(Q)$ with matrix A, then there exists a unique element

$$\boldsymbol{\alpha}_0 = (\alpha_0, \alpha_1, \dots, \alpha_4) \in \mathfrak{A}(X_A)$$

with $\|\boldsymbol{\alpha}_0\| = 0$ (cf. [17, 19]). Note that $\|-\boldsymbol{\alpha}_0\| = (4-1) - \|\boldsymbol{\alpha}_0\| = 3 - 0 = 3$.

Recall that p $(= \operatorname{char} k)$ is relatively prime to d. Let f be the order of p modulo d. Put

$$H = \left\{ p^i \pmod d \,|\, 0 \leq i < f \right\},$$

which is a subgroup of $(\mathbb{Z}/d\mathbb{Z})^\times$. For $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_4) \in \mathfrak{A}(X_A)$, we define a non-negative integer

$$\mathcal{A}_H(\boldsymbol{\alpha}) = \sum_{t \in H} \|t\boldsymbol{\alpha}\|.$$

It is known that the part (line segments) of the Newton polygon of $\Phi_{\widetilde{X}_A}$ with slope less than 1 corresponds to the height of the formal group of $\widetilde{X}_A$; if there is no such part, then $\operatorname{ht}\Phi_{\widetilde{X}_A} = \infty$ (cf. [2, 18]). In our situation, X_A, $\widetilde{X}_A$ and F_d/Γ_A are birational to each other (by resolving rational singularities) and the formal groups are invariant under resolution of rational singularities; see Lemma 2.1. Hence the height of $\Phi_{\widetilde{X}_A}$ can be computed from the Newton polygon of F_d/Γ_A. The slopes of this polygon is given by $\mathcal{A}_H(\boldsymbol{\alpha})/f$ and the length of the part with slope less than 1 is equal to the number of $\boldsymbol{\alpha} \in \mathfrak{A}(X_A)$ satisfying $\mathcal{A}_H(\boldsymbol{\alpha})/f < 1$. In summary, the height of $\Phi_{\widetilde{X}_A}$ can be calculated by evaluating $\mathcal{A}_H(\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in \mathfrak{A}(X_A)$.

Lemma 4.4. *Let $\widetilde{X}_A$ be a Calabi–Yau threefold of Delsarte type in $\mathbb{P}^4(Q)$ with matrix A. If there exists $\boldsymbol{\alpha}'$ with $\mathcal{A}_H(\boldsymbol{\alpha}') < f$, then $\mathcal{A}_H(\boldsymbol{\alpha}') = \mathcal{A}_H(\boldsymbol{\alpha}_0)$; furthermore, the height of $\Phi_{\widetilde{X}_A}$ is finite and equal to the length (cardinality) of the H-orbit of $\boldsymbol{\alpha}_0$.*

Proof. Suppose that $\|t\boldsymbol{\alpha}'\| \geq 1$ for all $t \in H$. Then as $\#H = f$, we have $\mathcal{A}_H(\boldsymbol{\alpha}') \geq f$; but, this is against the assumption. Hence $\|t\boldsymbol{\alpha}'\| = 0$ for some $t \in H$ and by the uniqueness of $\boldsymbol{\alpha}_0$, we find $t\boldsymbol{\alpha}' = \boldsymbol{\alpha}_0$. Since H is a subgroup of $(\mathbb{Z}/d\mathbb{Z})^\times$, we find $\mathcal{A}_H(\boldsymbol{\alpha}') = \mathcal{A}_H(\boldsymbol{\alpha}_0)$ and this holds for every element in the H-orbit of $\boldsymbol{\alpha}_0$. Therefore the number of $\boldsymbol{\alpha}$'s with $\mathcal{A}_H(\boldsymbol{\alpha}) < f$ is equal to the length of the H-orbit of $\boldsymbol{\alpha}_0$ and so is the height of $\Phi_{\widetilde{X}_A}$. ■

We are going to calculate the length of the H-orbit of $\boldsymbol{\alpha}_0$, namely the number of distinct elements in $\{t\boldsymbol{\alpha}_0 \,|\, t \in H\}$. Given $\boldsymbol{\alpha}_0 = (\alpha_0, \alpha_1, \dots, \alpha_4)$ with $\|\boldsymbol{\alpha}_0\| = 0$, we may choose every α_i as $0 < \alpha_i < d$ so that $\alpha_0 + \alpha_1 + \cdots + \alpha_4 = d$; then we set

$$e = \gcd(\alpha_0, \alpha_1, \dots, \alpha_4, d) \qquad \text{and} \qquad d_A = \frac{d}{e}. \tag{4.1}$$

We immediately see $e < d$ and $d_A \geq 2$. Further, suppose $d_A = 2$. Then it follows from $0 < \alpha_i < d$ that $\alpha_i = e$ for all i, which implies $5e = d = 2e$. Since this is absurd, we find $d_A \geq 3$.

Lemma 4.5. *Let $\widetilde{X}_A$ be a Calabi–Yau threefold of Delsarte type in $\mathbb{P}^4(Q)$ with matrix A. Let d_A be the integer defined in (4.1). Write f (resp. f_A) for the order of p modulo d (resp. modulo d_A). Then the following assertions hold.*

(1) $\mathcal{A}_H(\boldsymbol{\alpha}_0) = \frac{f}{f_A} \sum_{i=0}^{f_A - 1} \|p^i \boldsymbol{\alpha}_0\|$.

(2) $\mathcal{A}_H(\boldsymbol{\alpha}_0) < f$ *if and only if* $\|p^i \boldsymbol{\alpha}_0\| \leq 1$ *for* $0 \leq i < f_A$.

Proof. (1) Since $e = \gcd(\gcd(\alpha_0, \alpha_1, \dots, \alpha_4), d)$, write $\gcd(\alpha_0, \alpha_1, \dots, \alpha_4) = eg$ for some g with $\gcd(g, d_A) = 1$. Then

$$\begin{aligned} p^i\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_0 &\Leftrightarrow p^i \alpha_j \equiv \alpha_j \pmod{d} \\ &\Leftrightarrow d \,|\, \big(p^i - 1\big) \gcd(\alpha_0, \alpha_1, \dots, \alpha_4) \\ &\Leftrightarrow d_A \,|\, \big(p^i - 1\big) g \\ &\Leftrightarrow p^i \equiv 1 \pmod{d_A}. \end{aligned}$$

Hence

$$\mathcal{A}_H(\boldsymbol{\alpha}_0) = \frac{f}{f_A} \big(\|\boldsymbol{\alpha}_0\| + \|p\boldsymbol{\alpha}_0\| + \cdots + \big\|p^{f_A - 1}\boldsymbol{\alpha}_0\big\|\big) = \frac{f}{f_A} \sum_{i=0}^{f_A - 1} \big\|p^i \boldsymbol{\alpha}_0\big\|.$$

(2) Let $H' = \{p^i \pmod{d} \,|\, 0 \leq i < f_A\}$ and write a (resp. b) for the multiplicity of 1 (resp. 2) among the $\|t\boldsymbol{\alpha}_0\|$'s for $t \in H'$. Depending on whether $H' \ni -1$ or not, we divide the proof into two cases:

(i) If $H' \ni -1$, then $a + b + 2 = f_A$ and we find

$$\sum_{i=0}^{f_A - 1} \big\|p^i \boldsymbol{\alpha}_0\big\| = a + 2b + 3 = f_A + b + 1 > f_A.$$

Hence by (1), $\mathcal{A}_H(\boldsymbol{\alpha}_0) > f$.

(ii) If $H \not\ni -1$, then $a+b+1 = f_A$ and one sees

$$\frac{f}{f_A}\sum_{i=0}^{f_A-1}\left\|p^i\boldsymbol{\alpha}_0\right\| = \frac{f}{f_A}(0+1+\cdots+1+2+\cdots+2) = \frac{f(f_A+b-1)}{f_A}.$$

Hence $\mathcal{A}_H(\boldsymbol{\alpha}_0) < f$ is equivalent to $b-1<0$ (i.e., $b=0$).

Therefore the inequality $\mathcal{A}_H(\boldsymbol{\alpha}_0) < f$ holds if and only if the case (ii) occurs with $b=0$; this is the case where $\|\boldsymbol{\alpha}_0\| = 0$ and $\|p^i\boldsymbol{\alpha}_0\| = 1$ for all i with $1 \le i < f_A$. ∎

Theorem 4.6. *Let $\widetilde{X}_A$ be a Calabi–Yau threefold of Delsarte type in $\mathbb{P}^4(Q)$ with matrix A. Let d_A be the integer defined in (4.1) and f_A be the order of p modulo d_A. Write $h := \operatorname{ht}\Phi_{\widetilde{X}_A}$. Then the following assertions hold.*

(1) *h is finite if and only if $\|p^i\boldsymbol{\alpha}_0\| \le 1$ for $0 \le i < f_A$.*

(2) *If h is finite, then $h = f_A$.*

Proof. (1) Recall first that the height of $\Phi_{\widetilde{X}_A}$ can be computed from the Newton polygon of F_d/Γ_A and its slope-less-than-1 part has length equal to the number of $\boldsymbol{\alpha} \in \mathfrak{A}(X_A)$ satisfying $\mathcal{A}_H(\boldsymbol{\alpha}) < f$. Write K for the quotient field of the ring $W(k)$ of Witt vectors over k. Then by [2],

$$\begin{aligned} h < \infty &\Leftrightarrow \dim_K\left(H^3_{\text{cris}}(\widetilde{X}_A)\otimes K_{[0,1[}\right) \ge 1 \\ &\Leftrightarrow \#\{\boldsymbol{\alpha}\in\mathfrak{A}(X_A)\,|\,\mathcal{A}_H(\boldsymbol{\alpha}) < f\} \ge 1 \\ &\Leftrightarrow \mathcal{A}_H(\boldsymbol{\alpha}_0) < f \\ &\Leftrightarrow \left\|p^i\boldsymbol{\alpha}_0\right\| \le 1 \quad \text{for all } i \text{ with } 0 \le i < f_A. \end{aligned}$$

The last equivalence follows from Lemma 4.5.

(2) Let H be the orbit of p in $(\mathbb{Z}/d\mathbb{Z})^\times$. From the proof of Lemma 4.5, one sees that the length of the H-orbit of $\boldsymbol{\alpha}_0$ is equal to f_A. Hence together with Lemma 4.4, we have the following:

$$\begin{aligned} h &= \dim_K\left(H^3_{\text{cris}}(\widetilde{X}_A)\otimes K_{[0,1[}\right) = \#\{\boldsymbol{\alpha}\in\mathfrak{A}(X_A)\,|\,\mathcal{A}_H(\boldsymbol{\alpha}) < f\} \\ &= \text{the length of the } H\text{-orbit of } \boldsymbol{\alpha}_0 = f_A. \end{aligned}$$ ∎

For actual calculations, it is simpler to work modulo d_A rather than modulo d. We thus reformulate Lemma 4.5 and Theorem 4.6 in terms of modulo d_A.

Corollary 4.7. *With the assumptions as in Theorem 4.6, define*

$$\boldsymbol{\alpha}_A = \frac{1}{e}\boldsymbol{\alpha}_0 = \left(\frac{\alpha_0}{e},\ldots,\frac{\alpha_4}{e}\right).$$

Let $H_A = \{p^i \pmod{d_A}\,|\,0 \le i < f_A\}$, and for $\boldsymbol{\beta} = (\beta_0,\ldots,\beta_4) \in (\mathbb{Z}/d_A\mathbb{Z})^5$, write

$$\|\boldsymbol{\beta}\|_{d_A} = \sum_{i=0}^{4}\left\langle\frac{\beta_i}{d_A}\right\rangle - 1, \qquad \mathcal{A}_{H_A}(\boldsymbol{\beta}) = \sum_{t\in H_A}\|t\boldsymbol{\beta}\|_{d_A}.$$

Then we may regard $\boldsymbol{\alpha}_A$ as an element of $(\mathbb{Z}/d_A\mathbb{Z})^5$ and have the following.

(1) $\|p^i\boldsymbol{\alpha}_0\| = \|p^i\boldsymbol{\alpha}_A\|_{d_A}$.

(2) $\mathcal{A}_H(\boldsymbol{\alpha}_0) = \frac{f}{f_A}\mathcal{A}_{H_A}(\boldsymbol{\alpha}_A)$.

(3) *h is finite if and only if $\|p^i\boldsymbol{\alpha}_A\|_{d_A} \le 1$ for $0 \le i < f_A$.*

Proof. (1) We write $\boldsymbol{\alpha}_A = (\beta_0, \ldots, \beta_4)$. Then

$$\|p^i\boldsymbol{\alpha}_0\| = \sum_{j=0}^{4}\left\langle\frac{p^i\alpha_j}{d}\right\rangle - 1 = \sum_{j=0}^{4}\left\langle\frac{p^i e\beta_j}{ed_A}\right\rangle - 1 = \sum_{j=0}^{4}\left\langle\frac{p^i\beta_j}{d_A}\right\rangle - 1 = \|p^i\boldsymbol{\alpha}_A\|_{d_A}.$$

(2) This follows from Lemma 4.5.
(3) This follows from (1) and Theorem 4.6. ■

Corollary 4.8. *With the assumptions and notation as in Theorem 4.6 and Corollary 4.7, we have the following.*

(1) *If $p \equiv 1 \pmod{d_A}$, then $h = 1$.*
(2) *If there exists μ such that $p^\mu \equiv -1 \pmod{d_A}$, then h is infinite.*

Proof. (1) If $p \equiv 1 \pmod{d_A}$, then $f_A = 1$ and $\|\boldsymbol{\alpha}_A\|_{d_A} = 0 \leq 1$. By Theorem 4.6, $h = 1$.
(2) p^μ is in H_A and we find $\|p^\mu\boldsymbol{\alpha}_A\|_{d_A} = \|-\boldsymbol{\alpha}_A\|_{d_A} = 3 - \|\boldsymbol{\alpha}_A\|_{d_A} = 3$. Hence by Theorem 4.6, h is infinite. ■

Compared with the formal groups of $K3$ surfaces of Delsarte type in [8], it seems rather restrictive to have $\|p^i\boldsymbol{\alpha}_A\|_{d_A} = 1$ for all $1 \leq i < f_A$. This may be a reason why the infinite height occurs more often than finite height for Calabi–Yau threefolds $\widetilde{X}_A$; see examples in Section 5.

Remark 4.9. For $K3$ surfaces of Delsarte type in $\mathbb{P}^3(Q)$, an analogous statement to Corollary 4.8 holds, where the converse to (2) is also true (cf. [8]). But, for a threefold $\widetilde{X}_A$, the converse to (2) does not hold in general. In fact, Example 5.4 below shows that the orbit of $p \equiv 5 \pmod 8$ does not contain -1, while h is infinite.

5 Calabi–Yau threefolds of weighted Fermat type

In this section, we apply the results of the previous section to weighted Fermat threefolds and compute the height of the formal group of a crepant resolution $\widetilde{X}_A$. Let X_A be a weighted Fermat threefold defined by the equation

$$X_A\colon\ x_0^{d_0} + x_1^{d_1} + x_2^{d_2} + x_3^{d_3} + x_4^{d_4} = 0$$

with $d_iq_i = m$ for $0 \leq i \leq 4$. When $q_0 + q_1 + q_2 + q_3 + q_4 = m$, a crepant resolution $\widetilde{X}_A$ of X_A is Calabi–Yau. Here Yui [20] has observed that there are 147 possibilities for $Q = (q_0, \ldots, q_4)$. First we restate Theorem 4.6 for weighted Fermat threefolds.

Proposition 5.1. *Let $\widetilde{X}_A$ be a Calabi–Yau threefold of Fermat type in $\mathbb{P}^4(Q)$ of degree m with matrix A. Then $\boldsymbol{\alpha}_A = (q_0, q_1, q_2, q_3, q_4)$ and $d_A = m$. Furthermore, if $h = \operatorname{ht}\Phi_{\widetilde{X}_A}$ is finite, then h is equal to the order of p modulo m.*

Proof. For weighted Fermat threefolds, we find that $d = \det A = d_0d_1d_2d_3d_4$ and $\boldsymbol{\alpha}_0 = (d_1d_2d_3d_4, \ldots, d_0d_1d_2d_3)$ from the definition of $\mathfrak{A}(X_A)$. Since $d_iq_i = m$ and $\gcd(q_0, \ldots, q_4) = 1$, we see $\boldsymbol{\alpha}_A = (q_0, q_1, q_2, q_3, q_4)$ and $d_A = m$. The rest of the claim follows from Theorem 4.6. ■

Following are some results obtained from our calculations.

Proposition 5.2. *Let $\widetilde{X}_A$ be a Calabi–Yau threefold of weighted Fermat type. Then the following is a complete list of possible finite values for the height of the formal group of $\widetilde{X}_A$, where "possible" means that the values appear for some $\widetilde{X}_A$ in some characteristic p:*

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 18, 20, 21, 22, 42.$$

Example 5.3. Let $m = 5$ and $Q = (1,1,1,1,1)$. Let X_A be the weighted Fermat threefold defined by $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$. Assume $p \neq 5$. Then we have $h^{1,1} = 1$, $h^{1,2} = 101$, $d_A = 5$ and

$$h = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 5, \\ \infty & \text{otherwise.} \end{cases}$$

Example 5.4. Let $m = 8$ and $Q = (1,1,1,1,4)$. Let X_A be the weighted Fermat threefold defined by $x_0^8 + x_1^8 + x_2^8 + x_3^8 + x_4^2 = 0$. Assume $p \neq 2$. Then we have $h^{1,1} = 1$, $h^{1,2} = 149$, $d_A = 8$ and

$$h = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8, \\ 2 & \text{if } p \equiv 3 \pmod 8, \\ \infty & \text{if } p \equiv 5, 7 \pmod 8. \end{cases}$$

Example 5.5. Let $m = 966$ and $Q = (2, 21, 138, 322, 483)$. Let X_A be the weighted Fermat threefold defined by $x_0^{483} + x_1^{46} + x_2^7 + x_3^3 + x_4^2 = 0$. Assume $p \nmid 966$. Then we have $h^{1,1} = h^{1,2} = 143$, and $d_A = 966$. For instance, if $p \equiv 43 \pmod{966}$, then $h = 22$.

Example 5.6. Let $m = 1806$ and $Q = (1, 42, 258, 602, 903)$. (This is the largest degree for the Fermat type.) Let X_A be the weighted Fermat threefold defined by $x_0^{1806} + x_1^{43} + x_2^7 + x_3^3 + x_4^2 = 0$. Assume $p \nmid 1806$. Then we have $h^{1,1} = h^{1,2} = 251$, $d_A = 1806$ and

$$h = \begin{cases} 1 & \text{if } p \equiv 1 \pmod m \quad (1 \text{ class}), \\ 2 & \text{if } p \equiv 85, \dots \pmod m \quad (3 \text{ classes}), \\ 3 & \text{if } p \equiv 79, \dots \pmod m \quad (6 \text{ classes}), \\ 6 & \text{if } p \equiv 295, \dots \pmod m \quad (6 \text{ classes}), \\ 7 & \text{if } p \equiv 127, \dots \pmod m \quad (6 \text{ classes}), \\ 14 & \text{if } p \equiv 211, \dots \pmod m \quad (6 \text{ classes}), \\ 21 & \text{if } p \equiv 169, \dots \pmod m \quad (12 \text{ classes}), \\ 42 & \text{if } p \equiv 421, \dots \pmod m \quad (12 \text{ classes}), \\ \infty & \text{otherwise} \quad (452 \text{ classes}). \end{cases}$$

6 More examples

Here we consider another type of polynomials. Let X_A be a weighted Delsarte threefold defined by the equation:

$$x_0^{m_0} x_1 + x_1^{m_1} + x_2^{m_2} + x_3^{m_3} + x_4^{m_4} = 0 \subset \mathbb{P}^4(Q).$$

It has degree $m = q_0 m_0 + q_1 = m_1 q_1 = \cdots = m_4 q_4$ and it may be called a *weighted quasi-diagonal threefold of degree* m (cf. [20]). There are 137 weights to realize weighted quasi-diagonal threefolds. Several quantities associated with them are computed as follows.

Proposition 6.1. *Let X_A be a weighted quasi-diagonal threefold in $\mathbb{P}^4(Q)$ defined above. Let $\widetilde{X}_A$ be a Calabi–Yau threefold given as a crepant resolution of X_A. With the notation as in Corollary 4.7, set*

$$M = \operatorname{lcm}(m_0, m_2, m_3, m_4),$$

$M_i = M/m_i$, $i = 0, 2, 3, 4$, and $M_1 = M - M_0 - M_2 - M_3 - M_4$. Then

(1) $d_A = M$ *and* $\boldsymbol{\alpha}_A = (M_0, M_1, M_2, M_3, M_4)$.

(2) *Let* f_A *be the order of* p *modulo* M. *Then* $h := \operatorname{ht}\Phi_{\widetilde{X}_A}$ *is finite if and only if* $\|p^i\boldsymbol{\alpha}_A\|_{d_A} \le 1$ *for all* i, $0 \le i < f_A$.

(3) *If* h *is finite, then* $h = f_A$.

Proof. See Theorem 9.2 of [9] for claim (1). The rest is a consequence of Theorem 4.6. ■

Proposition 6.2. *Let* $\widetilde{X}_A$ *be a Calabi–Yau threefold arising from weighted quasi-diagonal threefold. Then the following is a complete list of possible finite values for the height of the formal group of* $\widetilde{X}_A$*, where "possible" means that the values appear in some characteristic* p*:*

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 22, 23, 24, 27, 28, 30, 41, 42, 46, 82.$$

Example 6.3. Let $m = 84$ and $Q = (1, 1, 12, 28, 42)$. Let X_A be the weighted quasi-diagonal threefold defined by

$$x_0^{83}x_1 + x_1^{84} + x_2^7 + x_3^3 + x_4^2 = 0$$

in $\mathbb{P}^4(1, 1, 12, 28, 42)$. Assume $p \nmid 84$. Then $d_A = 3486$, $h^{1,1} = 11$, $h^{2,1} = 491$ and

$$h = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{3486}, \\ 2 & \text{if } p \equiv 1163, 3319 \pmod{3486}, \\ & \dots \\ 41 & \text{if } p \equiv 127, 169, 253, \dots \pmod{3486}, \\ 82 & \text{if } p \equiv 43, 85, 211, \dots \pmod{3486}, \\ \infty & \text{otherwise.} \end{cases}$$

Remark 6.4. In Example 6.3, we find that $h > \min\{h^{1,1}(\widetilde{X}_A)+1, h^{1,2}(\widetilde{X}_A)+1\} = \min\{11+1, 491+1\} = 12$ for some characteristic p. According to Corollary 2.4, there is no group G of symplectic actions on X_A such that a crepant resolution of $Y = X_A/G$ becomes a mirror partner of $\widetilde{X}_A$. (We note, however, that the construction in Corollary 2.4 is very restrictive. A more general construction of mirror pairs in this direction is the Berglund–Hübsch–Krawitz mirror symmetry, where we may find a mirror partner of $\widetilde{X}_A$.)

Remark 6.5. One may also compute the height $\operatorname{ht}\Phi_{\widetilde{X}_A}$ for the following quasi-diagonal threefolds, but none of them gives a new value for the height beyond the lists of Propositions 5.2 and 6.2

$$x_0^{m_0}x_2 + x_1^{m_1} + x_2^{m_2} + x_3^{m_3} + x_4^{m_4} = 0,$$
$$x_0^{m_0} + x_1^{m_1}x_2 + x_2^{m_2} + x_3^{m_3} + x_4^{m_4} = 0,$$
$$x_0^{m_0} + x_1^{m_1} + x_2^{m_2} + x_3^{m_3} + x_3x_4^{m_4} = 0.$$

From a view point of the Kreuzer–Skarke classification [12] of invertible polynomials, these polynomials and the one in Proposition 6.2 are of the same type (i.e., a chain of length 2 and a Fermat of length 3). Here we computed their height individually because the arithmetic properties of these polynomials are different. Since there are a few more polynomials of the same type, we should try to calculate the height for others as well.

Remark 6.6. There are also slightly more general types of weighted Delsarte threefolds. But, the height of $\Phi_{\widetilde{X}_A}$ for the following three cases is also within the lists of Propositions 5.2 and 6.2:

$$x_0^{m_0}x_1 + x_0x_1^{m_1} + x_2^{m_2} + x_3^{m_3} + x_4^{m_4} = 0,$$
$$x_0^{m_0}x_2 + x_1^{m_1} + x_0x_2^{m_2} + x_3^{m_3} + x_4^{m_4} = 0,$$
$$x_0^{m_0}x_1 + x_1^{m_1}x_2 + x_0x_2^{m_2} + x_3^{m_3} + x_4^{m_4} = 0.$$

We note that the first two of these polynomials are of the same type in the Kreuzer–Skarke classification. Since there are still other types of polynomials, it would be interesting to discuss the height of formal groups of weighted Delsarte threefolds from a view point of the Kreuzer–Skarke classification.

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