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Elliptic Hypergeometric Sum/Integral Transformations and Supersymmetric Lens Index

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Abstract. We prove a pair of transformation formulas for multivariate elliptic hypergeometric sum/integrals associated to the A_n and BC_n root systems, generalising the formulas previously obtained by Rains. The sum/integrals are expressed in terms of the lens elliptic gamma function, a generalisation of the elliptic gamma function that depends on an additional integer variable, as well as a complex variable and two elliptic nomes. As an application of our results, we prove an equality between $S^1 \times S^3/\mathbb{Z}_r$ supersymmetric indices, for a pair of four-dimensional $\mathcal{N}=1$ supersymmetric gauge theories related by Seiberg duality, with gauge groups $\mathrm{SU}(n+1)$ and $\mathrm{Sp}(2n)$. This provides one of the most elaborate checks of the Seiberg duality known to date. As another application of the A_n integral, we prove a star-star relation for a two-dimensional integrable lattice model of statistical mechanics, previously given by the second author.

Key words: elliptic hypergeometric; elliptic gamma; supersymmetric; Seiberg duality; integrable; exactly solvable; Yang–Baxter; star-star

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1 Introduction

Elliptic hypergeometric series and integrals comprise the top-level class of hypergeometric functions, and provide generalisations of many of the well-known classical and basic hypergeometric functions and their corresponding identities. In contrast to the latter classical and basic counterparts whose study was initiated centuries ago, the area of elliptic hypergeometric functions has only been developed in relatively recent times, following their initial discovery from the Boltzmann weights of the integrable fused RSOS models of statistical mechanics [13, 17]. Central to this paper are the elliptic hypergeometric integrals that are expressed in terms of the elliptic gamma function, and satisfy many remarkable identities [32], some of which have been found to have important applications in different areas of mathematical physics.

One of these areas is exactly solved models, where the elliptic beta integral [29], a central identity for the theory of elliptic hypergeometric functions, is known to be equivalent to a Yang–Baxter equation [11] (more specifically a star-triangle relation), which is a fundamental identity for integrability of two-dimensional lattice models of statistical mechanics [3]. Specifically, the Yang–Baxter equation implies that the row-to-row transfer matrices of the lattice model commute, and following the method of Baxter [2], this can be used to solve for the partition function in the thermodynamic limit. Such lattice models related to elliptic hypergeometric integrals are

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quite general [7, 8, 9, 11, 18, 19, 21, 22, 33], and reduce to many important integrable lattice models of "Ising type" (e.g., [1, 5, 15, 16, 20, 39]) as special limiting cases.

The above lattice models satisfying the star-triangle relation, involve interactions between single-component spins. There exist also multi-component spin models that satisfy a so-called star-star relation, another fundamental identity for integrability of lattice models [4]. In this case a solution of the star-star relation introduced by Bazhanov and Sergeev [10], was recently shown [6] to be equivalent to a special case of Rains' multivariate transformation formula associated to the A_n root system, due to Rains [25]. This example establishes another link between integrable lattice models and elliptic hypergeometric integrals.

Another important connection arises between identities of elliptic hypergeometric integrals, and Seiberg duality [28] of the indices of four-dimensional supersymmetric gauge theories. This was first observed by Dolan and Osborn [14], who showed that supersymmetric indices for a pair of Seiberg-dual theories on $S^1 \times S^3$ [23], are equivalent as a consequence of Rains transformation formulas [25]. This connection provides a rigorous verification of the proposed dualities for supersymmetric gauge theories, by matching the equivalence of supersymmetric indices with the mathematically proven identities of elliptic hypergeometric integrals. This particular connection is quite powerful, as in principle it provides a way to generate a large number of complicated, and generally new, identities of elliptic hypergeometric integrals (and in some cases the Yang–Baxter equation), from systematic analysis of dualities among supersymmetric gauge theories [35, 36].

One interesting direction is to generalise the above indices on $S^1 \times S^3$, by replacing the S^3 with a lens space $S^3/\mathbb{Z}_r$ [12]. The resulting expressions for the indices, depend on sets of both complex and integer variables, and involve a summation over some discrete variable, as well as integration, while for $r = 1$, the expressions reduce to known elliptic hypergeometric integrals [14]. Consequently, such an expression that involves both the summation and integration will be referred to here as an "elliptic hypergeometric sum/integral" (there has also been proposed the name "rarefied elliptic hypergeometric integral" [34]). The sum/integrals are expressed in terms of a generalisation of the elliptic gamma function [27], known as the lens elliptic gamma function [12, 22]. Compared to the elliptic gamma function, the lens elliptic gamma function depends on an extra integer parameter $r = 1, 2, \ldots$, and an extra integer variable m mod r, and reduces to the regular elliptic gamma function for the case $r = 1$.

Equivalence of Seiberg dual indices on $S^1 \times S^3/\mathbb{Z}_r$, implies corresponding identities between different elliptic hypergeometric sum/integrals. The simplest example of such an identity is the elliptic beta sum/integral proven by the first author [22], which corresponds to "electric-magnetic" duality between two particular $\mathcal{N} = 1$ theories [18]. This elliptic beta sum/integral depends on six integer variables, as well as six complex variables and two complex elliptic nomes, and reduces to Spiridonov's elliptic beta integral [29] for $r = 1$. Spiridonov has also recently proven sum/integral evaluation formulas associated to the BC_n root system [34], and considered further mathematical properties of these sum/integral expressions, for example, deriving the sum/integral analogue of the elliptic Gauss hypergeometric equation [32].

As expected, the above sum/integral identities also have relevance to integrable lattice models. In this context, the second author found a solution of the star-star relation [41], that generalises the multi-spin model of Bazhanov and Sergeev [10], to the case of continuous, as well as discrete, spin variables. The simplest case of this model corresponds to the above elliptic beta sum/integral [22]. The star-star relation [41] provides one of the most general solutions of the Yang–Baxter equation known in the literature, and one of the motivations of this paper was to obtain the corresponding elliptic hypergeometric sum/integral transformation formula for the general case.

The main result of this paper, is proving a pair of new sum/integral transformation formulas that are associated respectively to the A_n and BC_n root systems. It is shown how these transformation formulas imply lens index duality for supersymmetric gauge theories on $S^1 \times S^3/\mathbb{Z}_r$ [12], and imply the above-mentioned star-star relation [41] (in the A_n case). The sum/integral transformation formulas generalise the corresponding transformation formulas of Rains [25], where the latter are equivalent to the choice $r = 1$. As expected, the BC_n sum/integral transformation formula derived in this paper, reduces to Spiridonov's BC_n sum/integral identity [34] as a special case.

The method of proof for the sum/integral formulas basically follows from Rains' proofs [25] of the $r = 1$ cases. That is, the transformations are first proven for some special choice of the variables, for which the transformations can be written in the form of a determinant of univariate sum/integrals of (lens) theta functions. Taking limits of these special cases provides a dense set of cases for which the transformations hold, and thus the transformations hold in general. There are some essential modifications in the details of the proofs, due to the appearance of the additional integer variables, and also due to the use of the lens elliptic gamma function, which has a non-trivial normalisation factor given in terms of multiple Bernoulli polynomials [24]. Otherwise the steps of the proofs are quite analogous to the case $r = 1$.

The paper is arranged as follows. Section 2 defines the lens elliptic gamma function, and gives a number of identities that it satisfies, which are used throughout the paper. Sections 3 and 4 present the respective A_n and BC_n sum/integral transformation formulas, along with their proofs. These sections also contain an overview of the related elliptic hypergeometric integrals that are obtained as special cases of the sum/integral transformations. Section 5 discusses a supersymmetric gauge theory interpretation of the A_n and BC_n transformation formulas as Seiberg dualities. Finally, Section 6 introduces the aforementioned lattice model with multi-component real and integer valued spin variables [41], and shows that the star-star relation of the model is equivalent to a special case of the A_n sum/integral transformation formula.

2 Definitions

The multiple Bernoulli polynomials $B_{n,k}(z;\omega_1,\ldots,\omega_n)$ are defined through the generating function

$$\frac{x^n e^{zx}}{\prod_{j=1}^{n}(e^{\omega_j x}-1)} = \sum_{k=0}^{\infty} B_{n,k}(z;\omega_1,\ldots,\omega_n)\frac{x^k}{k!},$$

where $z \in \mathbb{C}$, and $\omega_1,\ldots,\omega_n \in \mathbb{C}-\{0\}$.

These functions previously appeared in relation to the modular properties of multiple gamma functions [24]. For this paper only a particular multiple Bernoulli polynomial $B_{3,3}(z;\omega_1,\omega_2,\omega_3)$, is needed, which is given explicitly by

$$\begin{aligned} B_{3,3}(z;\omega_1,\omega_2,\omega_3) = {} & \frac{z^3}{\omega_1\omega_2\omega_3} - \frac{3z^2\sum_{i=1}^{3}\omega_i}{2\omega_1\omega_2\omega_3} + \frac{z\left(\sum_{i=1}^{3}\omega_i^2 + 3\sum_{1\le i<j\le 3}\omega_i\omega_j\right)}{2\omega_1\omega_2\omega_3} \\ & - \frac{\left(\sum_{i=1}^{3}\omega_i\right)\left(\sum_{1\le i<j\le 3}\omega_i\omega_j\right)}{4\omega_1\omega_2\omega_3}. \end{aligned} \tag{2.1}$$

Let us also introduce the two complex parameters $\sigma,\tau \in \mathbb{C}$ that satisfy

$$\mathrm{Im}(\sigma),\ \mathrm{Im}(\tau) > 0,$$

and define $R(z;\sigma,\tau)$, and $R_2(z,m;\sigma,\tau)$, as the following combinations of (2.1):

$$R(z;\sigma,\tau) := \frac{B_{3,3}(z;\sigma,\tau,-1) + B_{3,3}(z-1;\sigma,\tau,-1)}{12},$$

$$
\begin{aligned}
R_2(z,m;\sigma,\tau) &:= R(z+m\sigma;r\sigma,\sigma+\tau)+R(z+(r-m)\tau;r\tau,\sigma+\tau)\\
&= \frac{(\sigma+\tau-2z)(2z^2-2z(\sigma+\tau)+\sigma\tau(r^2+6(m-r)m)+1)}{24r\sigma\tau}\\
&\quad -\frac{(\sigma-\tau)(2m-r)(m-r)m}{12r},
\end{aligned}
\tag{2.2}
$$

where $z\in\mathbb{C}$, $m\in\mathbb{Z}$, and

$$
r=1,2,\ldots,
$$

is an integer parameter.

The lens elliptic gamma function is defined here as [12, 18, 22, 26]

$$
\Gamma(z,m;\sigma,\tau):=\mathrm{e}^{\phi_e(z,m;\sigma,\tau)}\gamma(z,m;\sigma,\tau),\qquad z\in\mathbb{C},\qquad m\in\{0,1,\ldots,r-1\},
\tag{2.3}
$$

where

$$
\begin{aligned}
\phi_e(z,m;\sigma,\tau) &= 2\pi i\left(R_2\left(z,0;\sigma-\frac{1}{2},\tau+\frac{1}{2}\right)-R_2\left(z,m;\sigma-\frac{1}{2},\tau+\frac{1}{2}\right)\right)\\
&= 2\pi i\left(R_2(z,0;\sigma,\tau)+R_2\left(0,m;\frac{1}{2},-\frac{1}{2}\right)-R_2(z,m;\sigma,\tau)\right),
\end{aligned}
\tag{2.4}
$$

and

$$
\gamma(z,m;\sigma,\tau)=\prod_{j,k=0}^{\infty}\frac{1-\mathrm{e}^{-2\pi i z}\mathsf{p}^{-m}(\mathsf{pq})^{j+1}\mathsf{p}^{r(k+1)}}{1-\mathrm{e}^{2\pi i z}\mathsf{p}^{m}(\mathsf{pq})^{j}\mathsf{p}^{rk}}\frac{1-\mathrm{e}^{-2\pi i z}\mathsf{q}^{-r+m}(\mathsf{pq})^{j+1}\mathsf{q}^{r(k+1)}}{1-\mathrm{e}^{2\pi i z}\mathsf{q}^{r-m}(\mathsf{pq})^{j}\mathsf{q}^{rk}}.
\tag{2.5}
$$

The elliptic nomes in (2.5) are defined in terms of σ and τ as

$$
\mathsf{p}=\mathrm{e}^{2i\pi\sigma},\qquad \mathsf{q}=\mathrm{e}^{2i\pi\tau}.
$$

The lens elliptic gamma function (2.3) is periodic in the complex and integer arguments respectively, satisfying

$$
\Gamma(z+2kr,m;\sigma,\tau)=\Gamma(z,m;\sigma,\tau),\qquad \Gamma(z,m+kr;\sigma,\tau)=\Gamma(z,m;\sigma,\tau),\qquad k\in\mathbb{Z}.
\tag{2.6}
$$

Accordingly, throughout the paper the notation a mod r, for an integer a, is always taken to be the corresponding element of $\{0,1,\ldots,r-1\}$, as in (2.3). Note also that $\gamma(z+k,m;\sigma,\tau)=\gamma(z,m;\sigma,\tau)$ for integer k, and the $2r$-periodicity of $\Gamma(z,m;\sigma,\tau)$ comes from the factor ϕ_e.

For $r=1$ (in which case we may take $m=0$), (2.3) is just the usual elliptic gamma function [27], defined as

$$
\Gamma(z,m;\sigma,\tau)\,|_{(r=1)}=\Gamma_1(z;\sigma,\tau)=\prod_{j,k=0}^{\infty}\frac{1-\mathrm{e}^{-2\pi i z}\mathsf{p}^{j+1}\mathsf{q}^{k+1}}{1-\mathrm{e}^{2\pi i z}\mathsf{p}^{j}\mathsf{q}^{k}}.
\tag{2.7}
$$

The lens elliptic gamma function (2.3) may be written as a product of two regular elliptic gamma functions (2.7), as

$$
\Gamma(z,m;\sigma,\tau)=\mathrm{e}^{\phi_e(z,m;\sigma,\tau)}\Gamma_1(z+\sigma m;r\sigma,\sigma+\tau)\Gamma_1(z+\tau(r-m);r\tau,\sigma+\tau).
$$

In this paper, the lens elliptic gamma function (2.3) will usually be written as $\Gamma(z,m)$, where

$$
\Gamma(z,m):=\Gamma(z,m;\sigma,\tau),
$$

with implicit dependence on the parameters σ and τ.

For integers $m \in \{0, 1, \ldots, r-1\}$, the poles and zeroes of lens elliptic gamma function (2.3) are respectively located at the points

$$z = -(\sigma+\tau)j - \sigma(rk+m) + n,\ -(\sigma+\tau)j - \tau(r(k+1)-m) + n,$$
$$z = (\sigma+\tau)(j+1) + \sigma(r(k+1)-m) + n,\ (\sigma+\tau)(j+1) + \tau(rk+m) + n,$$

where $j, k = 0, 1, \ldots$, and $n \in \mathbb{Z}$.

The lens elliptic gamma function satisfies some useful relations:

$$\Gamma((\sigma+\tau) - z, -m)\Gamma(z, m) = 1,$$
$$\Gamma(z + n\sigma, m - n) = \Gamma(z, m) \prod_{j=0}^{n-1} \theta_1(z + j\sigma, m - j),$$
$$\Gamma(z + n\tau, m + n) = \Gamma(z, m) \prod_{j=0}^{n-1} \theta_2(z + j\tau, m + j),$$

for integers $n = 1, 2, \ldots$. Here the theta functions are defined as

$$\theta_1(z, m) = \mathrm{e}^{\phi_1(z,m)} \theta\big(\mathrm{e}^{-2\pi i z}\mathrm{e}^{2\pi i \tau m} \,|\, \mathrm{e}^{2\pi i \tau r}\big),$$
$$\theta_2(z, m) = \mathrm{e}^{\phi_2(z,m)} \theta\big(\mathrm{e}^{2\pi i z}\mathrm{e}^{2\pi i \sigma m} \,|\, \mathrm{e}^{2\pi i \sigma r}\big), \tag{2.8}$$

where the normalisation factors are

$$\phi_1(z, m) = \phi_e(z + \sigma, m - 1; \sigma, \tau) - \phi_e(z, m; \sigma, \tau)$$
$$= \frac{\pi i}{12r}\big(3(r + 1 - 2m)(2z + 1) - (r^2 - 1)(\sigma - \tau - 1) - 6m(r - m)(\tau + 1)\big),$$
$$\phi_2(z, m) = \phi_e(z + \tau, m + 1; \sigma, \tau) - \phi_e(z, m; \sigma, \tau)$$
$$= \frac{-\pi i}{12r}\big(3(r - 1 - 2m)(2z - 1) - (r^2 - 1)(\sigma - \tau - 1) + 6m(r - m)(\sigma - 1)\big), \tag{2.9}$$

with ϕ_e as defined in (2.4), and where $\theta(z \,|\, \mathsf{q})$ is the usual theta function

$$\theta(z \,|\, \mathsf{q}) = (z; \mathsf{q})_\infty \left(\frac{\mathsf{q}}{z}; \mathsf{q}\right)_\infty, \qquad (z; \mathsf{q})_\infty = \prod_{j=0}^{\infty} \big(1 - z\mathsf{q}^j\big).$$

The theta functions θ_1, and θ_2, in (2.8), have non-trivial dependence on both of the parameters σ, and τ, through the normalisation functions (2.9).

The theta functions (2.8), each satisfy the same periodicities (2.6) as the lens elliptic gamma function, i.e., for any integer k

$$\theta_1(z + 2kr, m) = \theta_1(z, m), \qquad \theta_1(z, m + kr) = \theta_1(z, m),$$
$$\theta_2(z + 2kr, m) = \theta_2(z, m), \qquad \theta_2(z, m + kr) = \theta_2(z, m).$$

The theta functions (2.8) also satisfy

$$\theta_1(-z, -m) = -\theta_1(z, m)\mathrm{e}^{-2\pi i(z-m)/r}, \qquad \theta_2(-z, -m) = -\theta_2(z, m)\mathrm{e}^{-2\pi i(z-m)/r},$$

and

$$\theta_1(z + n\tau, m + n) = \theta_1(z, m)\mathrm{e}^{-n\pi i(2z+(n-1)\tau+r-2m-n+1)/r},$$
$$\theta_2(z + n\sigma, m - n) = \theta_2(z, m)\mathrm{e}^{-n\pi i(2z+(n-1)\sigma+r-2m+n-1)/r},$$

$$\theta_1(z+rn\tau,m)=\theta_1(z,m)\mathrm{e}^{-n\pi i(2z+\tau(rn-1)+1)},$$
$$\theta_2(z+rn\sigma,m)=\theta_2(z,m)\mathrm{e}^{-n\pi i(2z+\sigma(rn-1)+1)},$$

for integers n.

In this paper, a set of complex variables $t_1,\dots,t_n\in\mathbb{C}$ will frequently be represented as a vector

$$\boldsymbol{t}=(t_1,\dots,t_n),$$

and addition with a complex number $\gamma\in\mathbb{C}$ is given by

$$\gamma+\boldsymbol{t}:=(\gamma+t_1,\dots,\gamma+t_n).$$

An analogous notation also applies to sets of integer variables $a_1,\dots,a_n\in\mathbb{Z}$, and addition with integers.

3 The $A_n\leftrightarrow A_m$ transformation

3.1 Main theorem

Let us introduce the complex variables σ, τ, t_i, s_i, and integer variables a_i, b_i, for $i=0,1,\dots,m+n+1$, satisfying

$$\mathrm{Im}(\sigma),\mathrm{Im}(\tau)>0,$$
$$\sum_{i=0}^{m+n+1}(t_i+s_i)\equiv(m+1)(\sigma+\tau)\pmod{2r},\qquad\sum_{i=0}^{m+n+1}(a_i+b_i)\equiv0\pmod{r}.\tag{3.1}$$

In terms of these variables, we define $I^m_{A_n}(Z,Y\,|\,\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b})$ as the following elliptic hypergeometric sum/integral

$$I^m_{A_n}(Z,Y\,|\,\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b})=\frac{\lambda^n}{(n+1)!}\sum_{\substack{y_0,\dots,y_{n-1}=0\\ \sum_{i=0}^{n}y_i=Y}}^{r-1}\int_{\sum_{i=0}^{n}z_i=Z}\Delta^m_{A_n}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b})\prod_{i=0}^{n-1}\mathrm{d}z_i,\tag{3.2}$$

where $m,n=0,1,\dots$,

$$\lambda=\left(\mathsf{p}^r;\mathsf{p}^r\right)_\infty\left(\mathsf{q}^r;\mathsf{q}^r\right)_\infty,$$
$$\Delta^m_{A_n}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b})=\frac{\prod_{i=0}^{n}\prod_{j=0}^{m+n+1}\Gamma(t_j+z_i,a_j+y_i)\Gamma(s_j-z_i,b_j-y_i)}{\prod_{0\le i<j\le n}\Gamma(z_i-z_j,y_i-y_j)\Gamma(z_j-z_i,y_j-y_i)},$$

and

$$\begin{aligned}&\boldsymbol{z}=(z_0,z_1,\dots,z_n),&&\boldsymbol{t}=(t_0,t_1,\dots,t_{m+n+1}),&&\boldsymbol{s}=(s_0,s_1,\dots,s_{m+n+1}),\\&\boldsymbol{y}=(y_0,y_1,\dots,y_n),&&\boldsymbol{a}=(a_0,a_1,\dots,a_{m+n+1}),&&\boldsymbol{b}=(b_0,b_1,\dots,b_{m+n+1}).\end{aligned}\tag{3.3}$$

Due to the periodicities of the lens elliptic gamma function (2.6), the condition on the summation variables in (3.2) is to be understood as $\sum_{i=0}^{n}y_i=Y\pmod{r}$, and the condition on the integration variables is to be understood as $\sum_{i=0}^{n} z_i = Z \pmod{2r}$. However in the following we will avoid writing the latter mod conditions on the summation and integration variables for conciseness.

For the values satisfying $\mathrm{Im}(s_i) > \frac{\mathrm{Im}(Z)}{n+1} > -\mathrm{Im}(t_i)$, the contour in (3.2) may be chosen to be C^n, where C is a straight line that connects the two points $i\frac{\mathrm{Im}(Z)}{n+1}$, and $1 + i\frac{\mathrm{Im}(Z)}{n+1}$. Otherwise the sum/integral (3.2) is defined by meromorphic continuation from the latter case, with appropriately chosen contours connecting the points $z_j = k_j i$, respectively to the points $z_j = 1 + k_j i$, where k_j are real numbers, for $j = 0, 1, \ldots, n-1$.

The particular case $n = 0$ of (3.2) is given by

$$I^m_{A_0}(Z, Y \,|\, \boldsymbol{t}, \boldsymbol{a}; \boldsymbol{s}, \boldsymbol{b}) = \prod_{j=0}^{m+1} \Gamma(t_j + Z, a_j + Y)\Gamma(s_j - Z, b_j - Y).$$

The integrand $\Delta^m_{A_n}(\boldsymbol{z}, \boldsymbol{y}; \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{s}, \boldsymbol{b})$ is obviously r-periodic in each the discrete variables y_i, a_i, b_i (because of r-periodicity (2.6) of the lens elliptic gamma function (2.3)). The integrand is also non-trivially periodic under the shift of any integration variable z_i by $z_i + k_i$ for integers k_i, where $\sum_{i=0}^{n} k_i = 0$ (due to the condition $\sum_{i=0}^{n} z_i = Z$). The latter periodicity in the integration variables z_i follows from the balancing condition $\sum_{i=0}^{m+n+1} (a_i + b_i) \equiv 0 \pmod{r}$. Finally, the integrand satisfies the usual $2r$-periodicity in the complex variables t_i and s_i.

In the following let us define

$$T = \sum_{i=0}^{m+n+1} t_i, \qquad S = \sum_{i=0}^{m+n+1} s_i, \qquad A = \sum_{i=0}^{m+n+1} a_i, \qquad B = \sum_{i=0}^{m+n+1} b_i. \tag{3.4}$$

The main result of this section is the following elliptic hypergeometric sum/integral transformation formula.

Theorem 3.1. *The sum/integral* (3.2)*, under the balancing condition* (3.1)*, satisfies*

$$I^m_{A_n}(Z, Y \,|\, \boldsymbol{t}, \boldsymbol{a}; \boldsymbol{s}, \boldsymbol{b}) = I^n_{A_m}(Z + T, Y + A \,|\, \tilde{\boldsymbol{t}}, \tilde{\boldsymbol{a}}; \tilde{\boldsymbol{s}}, \tilde{\boldsymbol{b}}) \prod_{i,j=0}^{m+n+1} \Gamma(t_i + s_j, a_i + b_j), \tag{3.5}$$

where

$$\tilde{\boldsymbol{t}} = -\boldsymbol{t}, \qquad \tilde{\boldsymbol{s}} = \sigma + \tau - \boldsymbol{s}, \qquad \tilde{\boldsymbol{a}} = -\boldsymbol{a}, \qquad \tilde{\boldsymbol{b}} = -\boldsymbol{b}. \tag{3.6}$$

3.2 Corollaries

Theorem 3.1 contains as special cases several existing results in the literature, which are summarised in the diagram below:

$$\begin{array}{ccccc}
\text{Theorem 3.1} & \xrightarrow{m=0} & \text{Corollary 3.3} & \xrightarrow{n=0} & \text{Corollary 3.5 [22]} \\
\Big\downarrow r=1 & & \Big\downarrow r=1 & & \Big\downarrow r=1 \\
\text{Corollary 3.2 [25]} & \xrightarrow{m=0} & \text{[25, 30]} & \xrightarrow{n=0} & \text{Elliptic beta integral [29].}
\end{array}$$

First, the case $r = 1$ of Theorem 3.1 is equivalent to the following $A_n \leftrightarrow A_m$ elliptic hypergeometric integral transformation formula proven by Rains [25]:

Corollary 3.2 ([25])**.**

$$\frac{\lambda^n}{(n+1)!}\int_{\sum_{i=0}^{n} z_i=Z} \Delta^m_{A_n}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{t},\boldsymbol{s})\prod_{i=0}^{n-1}\mathrm{d}z_i = \prod_{i,j=0}^{m+n+1}\Gamma_1(t_i+t_j)$$
$$\times \frac{\lambda^m}{(m+1)!}\int_{\sum_{i=0}^{m} z_i=Z} \Delta^n_{A_m}(\boldsymbol{z},\boldsymbol{y};\tilde{\boldsymbol{t}},\tilde{\boldsymbol{s}})\prod_{i=0}^{m-1}\mathrm{d}z_i,$$

where

$$\Delta^m_{A_n}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{t},\boldsymbol{s}) = \frac{\prod_{i=0}^{n}\prod_{j=0}^{m+n+1}\Gamma_1(t_j+z_i)\Gamma_1(s_j-z_i)}{\prod_{0\le i<j\le n}\Gamma_1(z_i-z_j)\Gamma_1(z_j-z_i)},$$

and $\Gamma_1(z) := \Gamma_1(z;\sigma,\tau)$ *is the usual elliptic gamma function defined in* (2.7)*.*

Next, the $m=0$ case of (3.5), for $\mathrm{Im}(t_i), \mathrm{Im}(s_i) > 0$, and $Z=0$, $Y=0$, gives the following result:

Corollary 3.3.

$$I^0_{A_n}(0,0\,|\,\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b}) = \prod_{j=0}^{n+1}\Gamma(T-t_j,A-a_j)\Gamma(S-s_j,B-b_j)\prod_{i=0}^{n+1}\Gamma(t_i+s_j,a_i+b_j). \tag{3.7}$$

Remark 3.4. For $r=1$, equation (3.7) is equivalent to an elliptic beta integral identity associated to the A_n root system [25, 30, 31].

By specializing further to the case $m=0$, $n=1$ in Theorem 3.1, we obtain the following sum/integral analogue of elliptic beta integral of Spiridonov:

Corollary 3.5 ([22])**.** *For complex variables* σ*,* τ*,* t_i*, and integer variables* a_i*,* $i=1,2,\dots,6$*, satisfying*

$$\mathrm{Im}(\sigma),\ \mathrm{Im}(\tau),\ \mathrm{Im}(t_i) > 0, \qquad \sum_{i=1}^{6} t_i \equiv \sigma+\tau \ (\mathrm{mod}\ 2r), \qquad \sum_{i=1}^{6} a_i \equiv 0 \ (\mathrm{mod}\ r),$$

the following identity holds

$$\frac{\lambda}{2}\sum_{y=0}^{r-1}\int_0^1 \mathrm{d}z \frac{\prod_{i=1}^{6}\Gamma(t_i+z,a_i+y)\Gamma(t_i-z,a_i-y)}{\Gamma(2z,2y)\Gamma(-2z,-2y)} = \prod_{1\le i<j\le 6}\Gamma(t_i+t_j,a_i+a_j). \tag{3.8}$$

Remark 3.6. For $r=1$, equation (3.8) is equivalent to the elliptic beta integral [29].

Note that there is a symmetry of the integrand in (3.8) under $z\to -z$, $y\to r-y$, which may be used to truncate the sum to values $0\le y\le \lfloor r/2\rfloor$ [18].

The hyperbolic limit of (3.8) was also recently studied [18] in connection with two-dimensional integrable lattice models of statistical mechanics, where it generalises the Faddeev–Volkov model [15, 39], and in connection with supersymmetric gauge theory, where it describes duality of three-dimensional $\mathcal{N}=2$ theories on squashed lens spaces. As expected, the hyperbolic limit of (3.8) provides a sum/integral generalisation of the hyperbolic beta integral [37].

Finally we note that in a previous paper [18], it was shown that the formula (3.8) is also satisfied when the normalisation function (2.2), is defined as

$$R_2(z,m;\sigma,\tau) := R(z+m\sigma;\zeta\sigma,\sigma+\tau) + R(z+(\zeta-m)\tau;\zeta\tau,\sigma+\tau), \tag{3.9}$$

where ζ is a non-zero integer. The case $\zeta = r$ corresponds to the normalisation (2.2), while the case $\zeta = 1$ corresponds to the normalisation of the lens elliptic gamma function used in [34]. We note that the main result in Theorem 3.1 (and also Theorem 4.1) is also satisfied when the normalisation function is chosen as (3.9), which can be checked by explicitly expanding both sides of (3.5), and seeing that there is no dependence on ζ. However the properties of the lens elliptic gamma function given in Section 2, are only true for the case of $\zeta = r$. Particularly, in this paper we always consider $\zeta = r$ unless explicitly stated otherwise.

3.3 Proof of Theorem 3.1

The proof of Theorem 3.1 in the $r = 1$ case [25] generalises to the $r > 1$ case considered here. This involves first proving a special case of Theorem 3.1, using a Frobenius type determinant formula for the lens theta functions (2.8), and then taking limits of this special case, which are then used to show that a dense set of cases hold for the general transformation (3.5).

First, we have the following determinant formula which follows directly from Lemma 4.3 of [25]:

Lemma 3.7.

$$\det_{0\le i,j<n}\left(\frac{\theta_k(t+x_i+w_j,c_i+d_j)}{\theta_k(t,0)\,\theta_k(x_i+w_j,c_i+d_j)}\right) = \frac{\theta_k(t+X+W,C+D)}{\theta_k(t,0)\prod\limits_{i,j=0}^{n-1}\theta_k(x_i+w_j,c_i+d_j)}$$
$$\times\prod_{0\le i<j<n} e^{2\pi i(x_j+w_j-c_j-d_j)/r}\theta_k(x_i-x_j,c_i-c_j)\theta_k(w_i-w_j,d_i-d_j), \tag{3.10}$$

where $\theta_{k=1,2}(z,m)$ represents either of $\theta_1(z,m)$ or $\theta_2(z,m)$ in (2.8), and

$$X=\sum_{i=0}^{n-1}x_i, \qquad W=\sum_{i=0}^{n-1}w_i, \qquad C=\sum_{i=0}^{n-1}c_i, \qquad D=\sum_{i=0}^{n-1}d_i.$$

Note that both sides of the equation (3.10) are periodic in both the complex and integer variables respectively, i.e., they each are invariant under the shifts $c_i \to c_i + k_c r$, $d_i \to d_i + k_d r$, $x_i \to x_i + k_x$, $w_j \to w_j + k_w$, for integers k_c, k_d, k_x, k_w.

The determinant (3.10) will be used to prove the following special case of (3.5).

Lemma 3.8. *Theorem 3.1 holds for the case $m \to n-1, n \to n-1$, with the following choice of variables:*

$$\begin{aligned} &t_i = x_i, && t_{n+i} = \tau - w_i, && s_i = \sigma - x_i, && s_{n+i} = w_i,\\ &a_i = c_i, && a_{n+i} = 1 - d_i, && b_i = -1 - c_i, && b_{n+i} = d_i, \end{aligned} \tag{3.11}$$

where $i = 0, 1, \ldots, n-1$.

Proof. Consider the following univariate sum/integral related to (3.10)

$$\sum_{y=0}^{r-1}\int \mathrm{d}z\frac{\theta_1(s+w-z,d-y)}{\theta_1(s,0)\theta_1(w-z,d-y)}\frac{\theta_2(t+x+z,c+y)}{\theta_2(t,0)\theta_2(x+z,c+y)}. \tag{3.12}$$

This sum/integral is invariant upon exchanging $x \leftrightarrow w$, and $c \leftrightarrow d$, which follows from the change of integration and summation variables $z = z - x + w$, and $y = y - c + d$, respectively.

It then follows that a determinant of instances of (3.12)

$$\det_{0\le i,j<n}\left(\sum_{y=0}^{r-1}\int \mathrm{d}z \frac{\theta_1(s+w_j-z,d_j-y)}{\theta_1(s,0)\theta_1(w_j-z,d_j-y)}\frac{\theta_2(t+x_i+z,c_i+y)}{\theta_2(t,0)\theta_2(x_i+z,c_i+y)}\right), \tag{3.13}$$

is invariant under the exchange

$$x_j \leftrightarrow w_j, \qquad c_j \leftrightarrow d_j. \tag{3.14}$$

Since the row and column indices in (3.13) are not coupled, it may be written in terms of a multivariate sum/integral of a product of determinants of the form (3.10), as

$$\begin{aligned}
&n!\det_{0\le i,j<n}\left(\sum_{y=0}^{r-1}\int \mathrm{d}z \frac{\theta_1(s+w_j-z,d_j-y)}{\theta_1(s,0)\theta_1(w_j-z,d_j-y)}\frac{\theta_2(t+x_i+z,c_i+y)}{\theta_2(t,0)\theta_2(x_i+z,c_i+y)}\right)\\
&\quad=\sum_{y_0,\dots,y_{n-1}=0}^{r-1}\int \det_{0\le i,j<n}\left(\frac{\theta_1(s+w_i-z_j,d_i-y_j)}{\theta_1(s,0)\theta_1(w_i-z_j,d_i-y_j)}\right)\\
&\quad\quad\times\det_{0\le i,j<n}\left(\frac{\theta_2(t+x_i+z_j,c_i+y_j)}{\theta_2(t,0)\theta_2(x_i+z_j,c_i+y_j)}\right)\prod_{i=0}^{n-1}\mathrm{d}z_i\\
&\quad=\prod_{0\le i<j<n}\mathrm{e}^{2\pi i(x_j+w_j-c_j-d_j)/r}\sum_{y_0,\dots,y_{n-1}=0}^{r-1}\int \tilde{\Delta}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{x},\boldsymbol{c};\boldsymbol{w},\boldsymbol{d})\prod_{i=0}^{n-1}\mathrm{d}z_i.
\end{aligned}$$

The integrand in the last line is

$$\tilde{\Delta}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{x},\boldsymbol{c};\boldsymbol{w},\boldsymbol{d}) = \frac{\theta_1(s+W-Z,D-Y)\theta_2(t+X+Z,C+Y)}{\theta_1(s,0)\theta_2(t,0)}\Delta(\boldsymbol{z},\boldsymbol{y};\boldsymbol{x},\boldsymbol{c};\boldsymbol{w},\boldsymbol{d}),$$

where

$$\begin{aligned}
&\Delta(\boldsymbol{z},\boldsymbol{y};\boldsymbol{x},\boldsymbol{c};\boldsymbol{w},\boldsymbol{d})\\
&=\frac{\prod\limits_{0\le i<j<n}\theta_1(w_i-w_j,d_i-d_j)\theta_2(x_i-x_j,c_i-c_j)\theta_1(z_i-z_j,y_i-y_j)\theta_2(z_j-z_i,y_j-y_i)}{\prod\limits_{i,j=0}^{n-1}\theta_1(w_j-z_i,d_j-y_i)\theta_2(x_j+z_i,c_j+y_i)},
\end{aligned}$$

and

$$Z=\sum_{i=0}^{n-1}z_i, \qquad Y=\sum_{i=0}^{n-1}y_i.$$

The symmetry under (3.14) implies the following equality

$$\sum_{y_1,\dots,y_{n-1}=0}^{r-1}\int \tilde{\Delta}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{x},\boldsymbol{c};\boldsymbol{w},\boldsymbol{d})\prod_{i=0}^{n-1}\mathrm{d}z_i = \sum_{y_1,\dots,y_{n-1}=0}^{r-1}\int \tilde{\Delta}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{w},\boldsymbol{d};\boldsymbol{x},\boldsymbol{c})\prod_{i=0}^{n-1}\mathrm{d}z_i.$$

Substituting $t \to t + rk\sigma$ in this relation gives

$$\sum_{y_1,\dots,y_{n-1}=0}^{r-1}\int \frac{\tilde{\Delta}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{x},\boldsymbol{c};\boldsymbol{w},\boldsymbol{d})}{\mathrm{e}^{2\pi i(X+Z)k}}\prod_{i=0}^{n-1}\mathrm{d}z_i = \sum_{y_1,\dots,y_{n-1}=0}^{r-1}\int \frac{\tilde{\Delta}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{w},\boldsymbol{d};\boldsymbol{x},\boldsymbol{c})}{\mathrm{e}^{2\pi i(W+Z)k}}\prod_{i=0}^{n-1}\mathrm{d}z_i,$$

for integers k. This in turn means that

$$\sum_{y_1,\dots,y_{n-1}=0}^{r-1} \int f(\mathrm{e}^{2\pi i(X+Z)})\tilde{\Delta}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{x},\boldsymbol{c};\boldsymbol{w},\boldsymbol{d}) \prod_{i=0}^{n-1} \mathrm{d}z_i$$
$$= \sum_{y_1,\dots,y_{n-1}=0}^{r-1} \int f(\mathrm{e}^{2\pi i(W+Z)})\tilde{\Delta}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{w},\boldsymbol{d};\boldsymbol{x},\boldsymbol{c}) \prod_{i=0}^{n-1} \mathrm{d}z_i,$$

for any function $f(z)$ holomorphic in a neighbourhood of the contour. Writing this in terms of integration over Z, and summation over Y, it is seen that the following equality must hold

$$\sum_{\substack{y_0,\dots,y_{n-2}=0 \\ \sum_{i=0}^{n-1} y_i=Y}}^{r-1} \int_{\sum_{i=0}^{n-1} z_i=Z} \Delta(\boldsymbol{z},\boldsymbol{y};\boldsymbol{x},\boldsymbol{c};\boldsymbol{w},\boldsymbol{d}) \prod_{i=0}^{n-2} \mathrm{d}z_i$$
$$= \sum_{\substack{y_0,\dots,y_{n-2}=0 \\ \sum_{i=0}^{n-1} y_i=Y+C-D}}^{r-1} \int_{\sum_{i=0}^{n-1} z_i=Z+X-W} \Delta(\boldsymbol{z},\boldsymbol{y};\boldsymbol{w},\boldsymbol{d};\boldsymbol{x},\boldsymbol{c}) \prod_{i=0}^{n-2} \mathrm{d}z_i.$$

This is equivalent to the transformation (3.5) with the variables (3.11). ■

The above special case (3.11) will be used, along with the following general limit of the A_n sum/integral (3.2).

Lemma 3.9. *For $a_0=-b_0$, the limit $t_0\to -s_0$ of (3.2) is given by*

$$\lim_{t_0\to -s_0} \frac{I^m_{A_n}(Z,Y\,|\,\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b})\big|_{a_0=-b_0}}{\Gamma(t_0+s_0,0)\prod_{i=1}^{m+n+1}\Gamma(t_0+s_i,-b_0+b_i)\Gamma(t_i+s_0,a_i+b_0)}$$
$$= I^m_{A_{n-1}}(Z+s_0,Y+b_0\,|\,\bar{\boldsymbol{t}},\bar{\boldsymbol{a}};\bar{\boldsymbol{s}},\bar{\boldsymbol{b}}), \tag{3.15}$$

where $\boldsymbol{t}$, $\boldsymbol{a}$, $\boldsymbol{s}$, $\boldsymbol{b}$ are as defined in (3.3), and

$$\bar{\boldsymbol{t}}=(t_1,t_2,\dots,t_{m+n+1}), \qquad \bar{\boldsymbol{a}}=(a_1,a_2,\dots,a_{m+n+1}),$$
$$\bar{\boldsymbol{s}}=(s_1,s_2,\dots,s_{m+n+1}), \qquad \bar{\boldsymbol{b}}=(b_1,b_2,\dots,b_{m+n+1}).$$

Note here that the value some of the other variables in $\boldsymbol{t}$, $\boldsymbol{a}$, $\boldsymbol{s}$, $\boldsymbol{b}$, should also be changed, if the balancing condition (3.1) is to be satisfied both before and after taking the limit (3.15).

Proof. To take the limit (3.15), the contour will need to be deformed across the poles at

$$z_i=-t_0 \pmod{2r}, \qquad y_i=-a_0 \pmod{r}=b_0 \pmod{r}, \qquad i=0,1,\dots,n,$$

for $i=0,1,\dots,n$. Then since in this limit the numerator remains finite and the denominator has a divergent factor, $\lim\limits_{t_0\to -s_0}\Gamma(t_0+s_0,0)$, the only non-zero contribution in the limit (3.15) comes from the residues calculated at the above poles. By symmetry, the residue of each z_i at the poles must contribute the same value to (3.15), resulting in a factor $n+1$. In the limit, the terms in the denominator of (3.15), cancel the required terms in $I^m_{A_n}$ to give the integrand of $I^m_{A_{n-1}}$. The remaining factors come from using $\lim\limits_{z_0\to -t_0}(t_0+z_0)\Gamma(t_0+z_0,0)=i/(2\pi\lambda)$, in calculating the residue. ■

We can now prove Theorem 3.1.

Proof of Theorem 3.1. Note that the special case of the transformation in (3.11) has the sets of $2n$ variables t_i, s_i, a_i, b_i (subject to balancing condition (3.1)) parameterised in terms of the sets of n independent variables x_i, w_i, c_i, d_i. Similarly, in the following, the general sets of variables t_i, s_i, a_i, b_i in (3.2) will be parameterised by the $m+n+2$ pairs of sums of variables, as

$$\{\alpha_i, \beta_i\} = \{t_i + s_i, a_i + b_i\}, \qquad i = 0, 1, \dots, m+n+1. \tag{3.16}$$

Let also $\mathcal{C}_{mn}$ denote the set of points $(\{\alpha_0, \beta_0\}, \{\alpha_1, \beta_1\}, \dots, \{\alpha_{m+n+1}, \beta_{m+n+1}\})$, for which the transformation (3.5) holds. For example, in this notation, the special case (3.11) derived above corresponds to $(\{\sigma, -1\}, \{\sigma, -1\}, \dots, \{\sigma, -1\}, \{\tau, 1\}, \{\tau, 1\}, \dots, \{\tau, 1\}) \in \mathcal{C}_{nn}$.

Consider first a limit $t_0 \to -s_1$, for $a_0 = -b_1$, on both sides of the general transformation (3.5), for some point $(\{\alpha_0, \beta_0\}, \{\alpha_1, \beta_1\}, \dots, \{\alpha_{m+n+1}, \beta_{m+n+1}\}) \in \mathcal{C}_{mn}$. The limit of the right hand side requires no contour deformation, and has a simple cancellation of all factors of lens elliptic gamma functions appearing in the integrand, which contain any of the t_0, a_0, or s_1, b_1, variables. This produces a sum/integral of the type $I_{A_m}^{n-1}$. The limit of the left hand side follows from the above limit (3.15), resulting in a sum/integral of the type $I_{A_{n-1}}^{m}$. Overall, in terms of (3.16) this limit produces the transformation corresponding to the point $(\{\alpha_0 + \alpha_1, \beta_0 + \beta_1\}, \{\alpha_2, \beta_2\}, \dots, \{\alpha_{m+n+1}, \beta_{m+n+1}\}) \in \mathcal{C}_{m(n-1)}$.

Next consider the limit $t_0 \to -s_1 + \sigma + \tau$, for $a_0 = -b_1$, again on both sides of the transformation (3.5), for $(\{\alpha_0, \beta_0\}, \{\alpha_1, \beta_1\}, \dots, \{\alpha_{m+n+1}, \beta_{m+n+1}\}) \in \mathcal{C}_{mn}$. This case is rather similar to the case in the previous paragraph, but here the left hand side of (3.5) now involves a simple cancellation, resulting in $I_{A_n}^{m-1}$, and the right hand side gives the limit via (3.15), resulting in $I_{A_{m-1}}^{n}$. Overall, this limit results in the transformation that corresponds to the point $(\{\alpha_0 + \alpha_1 - \sigma - \tau, \beta_0 + \beta_1\}, \{\alpha_2, \beta_2\}, \dots, \{\alpha_{m+n+1}, \beta_{m+n+1}\}) \in \mathcal{C}_{(m-1)n}$.

Now starting at the point $(\{\sigma, -1\}, \{\sigma, -1\}, \dots, \{\sigma, -1\}, \{\tau, 1\}, \{\tau, 1\}, \dots, \{\tau, 1\}) \in \mathcal{C}_{nn}$, corresponding to the special case (3.11), the above two limits may be used to show that

$$\begin{aligned}
&(\{2\sigma, -2\}, \{\sigma, -1\}, \dots, \{\sigma, -1\}, \{\tau, 1\}, \{\tau, 1\}, \dots, \{\tau, 1\}) \in \mathcal{C}_{n(n-1)},\\
&(\{\sigma - \tau, -2\}, \{\sigma, -1\}, \dots, \{\sigma, -1\}, \{\tau, 1\}, \{\tau, 1\}, \dots, \{\tau, 1\}) \in \mathcal{C}_{(n-1)n},\\
&(\{\sigma, -1\}, \{\sigma, -1\}, \dots, \{\sigma, -1\}, \{2\tau, 2\}, \{\tau, 1\}, \dots, \{\tau, 1\}) \in \mathcal{C}_{n(n-1)},\\
&(\{\sigma, -1\}, \{\sigma, -1\}, \dots, \{\sigma, -1\}, \{\tau - \sigma, 2\}, \{\tau, 1\}, \dots, \{\tau, 1\}) \in \mathcal{C}_{(n-1)n}.
\end{aligned} \tag{3.17}$$

Iterating the above limits a further $r-1$ times, gives

$$\begin{aligned}
&(\{(r+1)\sigma, -1\}, \{\sigma, -1\}, \dots, \{\sigma, -1\}, \{\tau, 1\}, \{\tau, 1\}, \dots, \{\tau, 1\}) \in \mathcal{C}_{n(n-r)},\\
&(\{\sigma - r\tau, -1\}, \{\sigma, -1\}, \dots, \{\sigma, -1\}, \{\tau, 1\}, \{\tau, 1\}, \dots, \{\tau, 1\}) \in \mathcal{C}_{(n-r)n},\\
&(\{\sigma, -1\}, \{\sigma, -1\}, \dots, \{\sigma, -1\}, \{(r+1)\tau, 1\}, \{\tau, 1\}, \dots, \{\tau, 1\}) \in \mathcal{C}_{n(n-r)},\\
&(\{\sigma, -1\}, \{\sigma, -1\}, \dots, \{\sigma, -1\}, \{\tau - r\sigma, 1\}, \{\tau, 1\}, \dots, \{\tau, 1\}) \in \mathcal{C}_{(n-r)n}.
\end{aligned} \tag{3.18}$$

In iterating the relations (3.17) r times, the integer component β_i of the $\{\tau, 1\}$ or $\{\sigma, -1\}$, cycle through r different values $+1 + k$, and $-1 - k$, respectively, for $k = 0, 1, \dots, r-1$.

Now consider starting from an arbitrary value of the n integer components β_i, which may be arrived at by using (3.17) up to r times for each pair $\{\alpha_i, \beta_i\}$. The relations (3.18) can then be repeatedly iterated to form arbitrary combinations (that are subject to the balancing condition (3.1)) of the form $\alpha_i = j_1 r\tau - j_2 r\sigma + k_1\tau - k_2\sigma$, or $\alpha_i = j_1 r\sigma - j_2 r\tau + k_1\sigma - k_2\tau$, for some integers $j_1, j_2 = 0, 1, \dots$, and $k_1, k_2 = 0, 1, \dots, r-1$. Taking into account the $2r$-periodicity of the α_i, as $n \to \infty$ this gives a dense set of points for the α_i in C_{mn} (for any choice of the β_i), and thus Theorem 3.1 holds in general. ■

4 The $BC_n \leftrightarrow BC_m$ transformation

4.1 Main theorem

Let us introduce complex variables σ, τ, t_i, and integer variables a_i, for $i = 0, 1, \ldots, 2m+2n+3$, satisfying

$$\begin{aligned}&\mathrm{Im}(\sigma), \mathrm{Im}(\tau) > 0,\\ &\sum_{i=0}^{2m+2n+3} t_i \equiv (m+1)(\sigma+\tau) \pmod{2r}, \qquad \sum_{i=0}^{2m+2n+3} a_i \equiv 0 \pmod{r}.\end{aligned} \tag{4.1}$$

In terms of these variables, $I^m_{BC_n}(\boldsymbol{t}, \boldsymbol{a})$ is defined to be the following elliptic hypergeometric sum/integral

$$I^m_{BC_n}(\boldsymbol{t}, \boldsymbol{a}) := \frac{\lambda^n}{2^n\, n!} \sum_{y_1,\ldots,y_n=0}^{r-1} \int_{C^n} \Delta^m_{BC_n}(\boldsymbol{z}, \boldsymbol{y}; \boldsymbol{t}, \boldsymbol{a}) \prod_{i=1}^{n} \mathrm{d}z_i, \tag{4.2}$$

where

$$\begin{aligned}&\boldsymbol{z} = (z_1, z_2, \ldots, z_n), \qquad \boldsymbol{t} = (t_0, t_1, \ldots, t_{2m+2n+3}),\\ &\boldsymbol{y} = (y_1, y_2, \ldots, y_n), \qquad \boldsymbol{a} = (a_0, a_1, \ldots, a_{2m+2n+3}),\end{aligned} \tag{4.3}$$

and

$$\Delta^m_{BC_n}(\boldsymbol{z}, \boldsymbol{y}; \boldsymbol{t}, \boldsymbol{a}) := \frac{\prod_{i=1}^{n} \prod_{j=0}^{2m+2n+3} \Gamma(t_j + z_i, a_j + y_i)\Gamma(t_j - z_i, a_j - y_i)}{\prod_{i=1}^{n} \Gamma(\pm 2z_i, \pm 2y_i) \prod_{1\le i<j\le n} \Gamma(\pm z_i \pm z_j, \pm y_i \pm y_j)}. \tag{4.4}$$

Here the compact notation for the lens elliptic gamma function is now used, where for example

$$\begin{aligned}&\Gamma(\pm z_1, \pm y_1) = \Gamma(z_1, y_1)\Gamma(-z_1, -y_1),\\ &\Gamma(\pm z_1 \pm z_2, \pm y_1 \pm y_2) = \Gamma(z_1 + z_2, y_1 + y_2)\Gamma(z_1 - z_2, y_1 - y_2)\\ &\qquad\qquad \times \Gamma(-z_1 + z_2, -y_1 + y_2)\Gamma(-z_1 - z_2, -y_1 - y_2),\end{aligned} \tag{4.5}$$

for complex numbers z_1, z_2, and integers y_1, y_2, respectively.

For the values $\mathrm{Im}(t_i) > 0$, the contour in (4.2) may be chosen to be C^n, where $C = [0, 1]$. Otherwise, C is a contour that connects the two points ik, and $1 + ik$, where k is a real number, and C also separates all points

$$\begin{aligned}&\{t_j + (\sigma+\tau)l_1 + \sigma(rl_2 + (a_j - y_i) \bmod r) + n,\\ &\quad t_j + (\sigma+\tau)l_1 + \tau(r(l_2+1) - (a_j - y_i) \bmod r) + n\},\end{aligned}$$

in the strip $0 \le \mathrm{Re}(z) \le 1$, from the points

$$\begin{aligned}&\{-t_j - (\sigma+\tau)l_1 - \sigma(rl_2 + (a_j + y_i) \bmod r) + n,\\ &\quad -t_j - (\sigma+\tau)l_1 - \tau(r(l_2+1) - (a_i + y_i) \bmod r) + n\},\end{aligned}$$

in the same strip, where $j = 0, \ldots, 2m+2n+3$, $l_1, l_2 = 0, 1, \ldots$, and $n \in \mathbb{Z}$.

The integrand (4.4) is periodic in the summation and integration variables y_i, and z_i, respectively, i.e., $\Delta^m_{BC_n}$ is invariant under either $z_i \to z_i + k_i$, or $y_i \to y_i + rk_i$, $i = 1, 2, \ldots, n$, for any

$k_i \in \mathbb{Z}$. The periodicity of (4.4) in the z_i follows from the balancing condition $\sum_{i=0}^{2m+2n+3} a_i \equiv 0$ (mod r). The integrand is also $2r$-periodic in the complex variables t_i.

Note also the particular case $n = 0$ of (4.2) gives

$$I^m_{BC_0}(\boldsymbol{t}, \boldsymbol{a}) = 1,$$

while the case $n = 1$ of (4.2), is equivalent to the A_n integral (3.2)

$$I^m_{BC_1}(\boldsymbol{t}, \boldsymbol{a}) = I^m_{A_1}(\boldsymbol{t}_1, \boldsymbol{a}_1; \boldsymbol{t}_2, \boldsymbol{a}_2), \tag{4.6}$$

where a choice of the variables in (4.6) is

$$\boldsymbol{t}_1 = (t_0, t_1, \dots, t_{m+n+1}), \qquad \boldsymbol{t}_2 = (t_{m+n+2}, t_{m+n+3}, \dots, t_{2m+2n+3}),$$
$$\boldsymbol{a}_1 = (a_0, a_1, \dots, a_{m+n+1}), \qquad \boldsymbol{a}_2 = (a_{m+n+2}, a_{m+n+3}, \dots, a_{2m+2n+3}).$$

The BC_n sum/integral (4.2) satisfies the following transformation formula which is the main result of this section.

Theorem 4.1. *The sum/integral (4.2), under the balancing condition (4.1), satisfies*

$$I^m_{BC_n}(\boldsymbol{t}, \boldsymbol{a}) = I^n_{BC_m}(\tilde{\boldsymbol{t}}, \tilde{\boldsymbol{a}}) \prod_{0 \le i < j \le 2m+2n+3} \Gamma(t_i + t_j, a_i + a_j), \tag{4.7}$$

where

$$\tilde{\boldsymbol{t}} = \frac{\sigma + \tau}{2} - \boldsymbol{t}, \qquad \tilde{\boldsymbol{a}} = -\boldsymbol{a}.$$

4.2 Colloraries

The $r = 1$ case of Theorem 4.1 is equivalent to the $BC_n \leftrightarrow BC_m$ elliptic hypergeometric integral transformations given by Rains [25].

Corollary 4.2 ([25])**.** *For $r = 1$, (4.7) is*

$$\frac{\lambda^n}{2^n n!} \int_{C^n} \Delta^m_{BC_n}(\boldsymbol{z}, \boldsymbol{t}) \prod_{i=1}^n \mathrm{d}z_i = \frac{\lambda^m}{2^m m!} \int_{C^m} \Delta^n_{BC_m}(\boldsymbol{z}, \tilde{\boldsymbol{t}}) \prod_{i=1}^m \mathrm{d}z_i \prod_{0 \le i < j \le 2m+2n+3} \Gamma_1(t_i + t_j),$$

where

$$\Delta^m_{BC_n}(\boldsymbol{z}, \boldsymbol{t}) = \frac{\prod_{i=1}^n \prod_{j=0}^{2m+2n+3} \Gamma_1(t_j + z_i)\Gamma_1(t_j - z_i)}{\prod_{i=1}^n \Gamma_1(\pm 2z_i) \prod_{1 \le i < j \le n} \Gamma_1(\pm z_i \pm z_j)},$$

and $\Gamma_1(z) = \Gamma_1(z; \sigma, \tau)$ is the usual elliptic gamma function defined in (2.7).

The $m = 0$ case of Theorem 4.1 is equivalent to a BC_n elliptic hypergeometric sum/integral identity proven by Spiridonov [34] (named there "rarefied elliptic hypergeometric integral").

Corollary 4.3 ([34])**.** *For $m = 0$, (4.7) is*

$$\frac{\lambda^n}{2^n n!} \sum_{y_1, \dots, y_n = 0}^{r-1} \int_{C^n} \Delta^m_{BC_n}(\boldsymbol{z}, \boldsymbol{y}; \boldsymbol{t}, \boldsymbol{a}) \prod_{i=1}^n \mathrm{d}z_i = \prod_{0 \le i < j \le 2n+3} \Gamma(t_i + t_j, a_i + a_j). \tag{4.8}$$

Remark 4.4. The $n = 1$ case of (4.8) is equivalent to the elliptic beta/sum integral (3.8).

As was mentioned in the previous section, Theorem 3.1, and Theorem 4.1, remain satisfied when the normalisation (2.2) is replaced with (3.9). The normalisation of (4.8) in [34] corresponds to $\zeta = 1$ in (3.9).

4.3 Proof of Theorem 4.1

The proof of Theorem 4.1 basically follows the same idea as in the proof of Theorem 3.1, with minor differences. In fact a special case of Theorem 3.1 is first used to prove the following Lemma.

Lemma 4.5. *Theorem 4.1 holds for $m = 1$, $n = 1$, with the following choice of the variables*

$$\begin{aligned}
&t_0 = x_0, && t_1 = \tau - x_0, && t_2 = x_1, && t_3 = \tau - x_1,\\
&t_4 = w_0, && t_5 = \sigma - w_0, && t_6 = w_1, && t_7 = \sigma - w_1,\\
&a_0 = c_0, && a_1 = 1 - c_0, && a_2 = c_1, && a_3 = 1 - c_1,\\
&a_4 = d_0, && a_5 = -1 - d_0, && a_6 = d_1, && a_7 = -1 - d_1.
\end{aligned} \tag{4.9}$$

Proof. The case (4.9) of the transformation (4.7), is explicitly given by

$$\begin{aligned}
&\sum_{y=0}^{r-1} \int_C \frac{\theta_1(2z, 2y)\theta_2(-2z, -2y)}{\prod\limits_{j\in\{0,1\}} \theta_1(x_j \pm z, c_j \pm y)\theta_2(w_j \pm z, d_j \pm y)} \mathrm{d}z\\
&\quad = \sum_{y=0}^{r-1} \int_C \frac{\theta_1(2z, 2y)\theta_2(-2z, -2y)}{\prod\limits_{j\in\{0,1\}} \theta_2(\frac{\sigma+\tau}{2} - x_j \pm z, -c_j \pm y)\theta_1(\frac{\sigma+\tau}{2} - w_j \pm z, -d_j \pm y)} \mathrm{d}z\\
&\quad \times \frac{\theta_2(x_0 - x_1, c_0 - c_1)\theta_2(x_0 + x_1 - \tau, c_0 + c_1 - 1)}{\theta_1(x_0 - x_1, c_0 - c_1)\theta_1(x_0 + x_1 - \tau, c_0 + c_1 - 1)}\\
&\quad \times \frac{\theta_1(w_0 - w_1, d_0 - d_1)\theta_1(w_0 + w_1 - \sigma, d_0 + d_1 + 1)}{\theta_2(w_0 - w_1, d_0 - d_1)\theta_2(w_0 + w_1 - \sigma, d_0 + d_1 + 1)}.
\end{aligned} \tag{4.10}$$

This is equivalent to an $m = n = 1$ case of Theorem 3.1, with the following choice of variables satisfying (3.1)

$$\begin{aligned}
&t_0 = x_0, && t_1 = \tau - x_0, && t_2 = w_0, && t_3 = \sigma - w_0,\\
&s_0 = x_1, && s_1 = \tau - x_1, && s_2 = w_1, && s_3 = \sigma - w_1,\\
&a_0 = c_0, && a_1 = +1 - c_0, && a_2 = d_0, && a_3 = -1 - d_0,\\
&b_0 = c_1, && b_1 = +1 - c_1, && b_2 = d_1, && b_3 = -1 - d_1.
\end{aligned}$$

■

Note that Lemma 4.5 appears to be a special case of an $m = n = 1$ identity given in [34], corresponding to the normalisation of the lens elliptic gamma function where $\zeta = 1$ in (3.9).

To prove a more general case of Lemma 4.5, the special case of (4.10) will be used, along with the following determinant identity, which follows directly from equation (3.18) in [25].

Lemma 4.6.

$$\begin{aligned}
\det_{1\le i,j\le n} \left(\frac{1}{\theta_k(x_i \pm w_j, c_i \pm d_j)} \right) &= \prod_{1\le i<j\le n} \frac{\theta_k(x_i \pm x_j, c_i \pm c_j)\theta_k(w_i \pm w_j, d_i \pm d_j)}{e^{-2\pi i(x_j - w_i - c_j + d_i + r/2)/r}}\\
&\quad \times \prod_{i,j=1}^{n} \frac{1}{\theta_k(x_i \pm w_j, c_i \pm d_j)},
\end{aligned} \tag{4.11}$$

where $\theta_k(z, m)$ represents either of $\theta_1(z, m)$ or $\theta_2(z, m)$ in (2.8).

In (4.11), the compact notation for the lens theta functions follows analogously to (4.5).

Note that both sides of the equation (4.11) are periodic in both the complex and integer variables respectively, i.e., they each are invariant under the shifts $c_i \to c_i + k_1 r$, $d_i \to d_i + k_2 r$, $x_i \to x_i + k_3$, $w_j \to w_j + k_4$, for integers k_1, k_2, k_3, k_4.

The results (4.10), (4.11), are used to prove the following special case of Theorem 4.1.

Lemma 4.7. *The transformation* (4.7) *holds for* $m = n$*, with the following choice of variables*

$$\begin{aligned} &t_{2i} = x_i, && t_{2i+1} = \tau - x_i, && t_{2n+2i+2} = w_i, && t_{2n+2i+3} = \sigma - w_i, \\ &a_{2i} = c_i, && a_{2i+1} = 1 - c_i, && a_{2n+2i+2} = d_i, && a_{2n+2i+3} = -1 - d_i. \end{aligned} \tag{4.12}$$

Proof. Consider taking a determinant of particular instances of (4.10). Using (4.11), the relevant determinant coming from the left hand side of (4.10) may be written as

$$\begin{aligned} & n! \det_{1\le i,j\le n} \left(\sum_{y=0}^{r-1} \int_C \frac{\theta_1(2z,2y)\theta_2(-2z,-2y)\mathrm{d}z}{\prod_{k\in\{0,i\}} \theta_1(x_k \pm z, c_k \pm y) \prod_{k\in\{0,j\}} \theta_2(w_k \pm z, d_k \pm y)} \right) \\ &\quad = \sum_{y_1,\dots,y_n=0}^{r-1} \int_{C^n} \det_{1\le i,j\le n} \left(\frac{\theta_1(2z_j,2y_j)}{\prod_{k\in\{0,i\}} \theta_1(x_k \pm z_j, c_k \pm y_j)} \right) \\ &\quad\quad \times \det_{1\le i,j\le n} \left(\frac{\theta_2(-2z_j,-2y_j)}{\prod_{k\in\{0,i\}} \theta_2(w_k \pm z_j, d_k \pm y_j)} \right) \prod_{i=1}^n \mathrm{d}z_i \\ &\quad = \sum_{y_1,\dots,y_n=0}^{r-1} \int_{C^n} \Delta(\boldsymbol{z},\boldsymbol{y};\boldsymbol{x},\boldsymbol{c};\boldsymbol{w},\boldsymbol{d}) \prod_{i=1}^n \mathrm{d}z_i, \end{aligned} \tag{4.13}$$

where the integrand in the last line is

$$\begin{aligned} \Delta(\boldsymbol{z},\boldsymbol{y};\boldsymbol{x},\boldsymbol{c};\boldsymbol{w},\boldsymbol{d}) &= \frac{\prod_{i=1}^n \theta_1(2z_i,2y_i)\theta_2(-2z_i,-2y_i)}{\prod_{i=1}^n \prod_{j=0}^n \theta_1(x_j \pm z_i, c_j \pm y_i)\theta_2(w_j \pm z_i, d_j \pm y_i)} \\ \times \prod_{1\le i<j\le n} & \frac{\theta_1(x_i \pm x_j, c_i \pm c_j)\theta_2(w_i \pm w_j, d_i \pm d_j)}{\mathrm{e}^{-2\pi i(x_j+w_j-c_j-d_j)/r}} \theta_1(z_i \pm z_j, y_i \pm y_j)\theta_2(-z_i \pm z_j, -y_i \pm y_j). \end{aligned} \tag{4.14}$$

Similarly, the corresponding determinant coming from the right hand side of (4.10) is

$$\begin{aligned} & n! \det_{1\le i,j\le n} \left(\sum_{y=0}^{r-1} \int_C \frac{\theta_1(2z,2y)\theta_2(-2z,-2y)\mathrm{d}z}{\prod_{k\in\{0,j\}} \theta_1(\frac{\sigma+\tau}{2} - w_k \pm z, -d_k \pm y) \prod_{k\in\{0,i\}} \theta_2(\frac{\sigma+\tau}{2} - x_k \pm z, -c_k \pm y)} \right. \\ &\quad\quad \times \frac{\theta_2(x_0 - x_i, c_0 - c_i)\theta_2(x_0 + x_i - \tau, c_0 + c_i - 1)}{\theta_1(x_0 - x_i, c_0 - c_i)\theta_1(x_0 + x_i - \tau, c_0 + c_i - 1)} \\ &\quad\quad \left. \times \frac{\theta_1(w_0 - w_j, d_0 - d_j)\theta_1(w_0 + w_j - \sigma, d_0 + d_j + 1)}{\theta_2(w_0 - w_j, d_0 - d_j)\theta_2(w_0 + w_j - \sigma, d_0 + d_j + 1)} \right) \\ &\quad = \prod_{i=1}^n \frac{\theta_2(x_0 - x_i, c_0 - c_i)\theta_2(x_0 + x_i - \tau, c_0 + c_i - 1)}{\theta_1(x_0 - x_i, c_0 - c_i)\theta_1(x_0 + x_i - \tau, c_0 + c_i - 1)} \\ &\quad\quad \times \frac{\theta_1(w_0 - w_i, d_0 - d_i)\theta_1(w_0 + w_i - \sigma, d_0 + d_i + 1)}{\theta_2(w_0 - w_i, d_0 - d_i)\theta_2(w_0 + w_i - \sigma, d_0 + d_i + 1)} \\ &\quad\quad \times \sum_{y_1,\dots,y_n=0}^{r-1} \int_{C^n} \Delta\left(\boldsymbol{z},\boldsymbol{y};\frac{\sigma+\tau}{2} - \boldsymbol{w}, -\boldsymbol{d}; \frac{\sigma+\tau}{2} - \boldsymbol{x}, -\boldsymbol{c}\right) \prod_{i=1}^n \mathrm{d}z_i, \end{aligned} \tag{4.15}$$

where Δ is defined in (4.14).

Since they were constructed from instances of (4.10), the expressions (4.13) and (4.15) must be equal, and this is exactly the transformation (4.7) with variables (4.12). ■

Consider next the following limit of the BC_n sum/integral (4.2).

Lemma 4.8. *For $a_0 = -a_1$, the limit $t_0 \to -t_1$ of (4.2) is given by*

$$\lim_{t_0\to -t_1} \frac{I^m_{BC_n}(\boldsymbol{t},\boldsymbol{a})\big|_{a_0=-a_1}}{\Gamma(t_0+t_1,0)\prod\limits_{2\le i\le 2m+2n+3}\Gamma(t_0+t_i,-a_1+a_i)\Gamma(t_1+t_i,a_1+a_i)} = I^m_{BC_{n-1}}(\bar{\boldsymbol{t}},\bar{\boldsymbol{a}}), \quad (4.16)$$

where $\boldsymbol{t},\boldsymbol{a}$, are as given in (4.3), and

$$\bar{\boldsymbol{t}} = (t_2,t_3,\dots,t_{2m+2n+3}), \qquad \bar{\boldsymbol{a}} = (a_2,a_3,\dots,a_{2m+2n+3}).$$

Note here that the value some of the other variables in $\boldsymbol{t}$, $\boldsymbol{a}$, should also be changed, if the balancing condition (4.1) is to be satisfied both before and after taking the limit (4.16).

Proof. Similarly to (3.15), the contour needs be deformed to cross the poles at

$$\begin{aligned} z_i &= -t_0 \ (\mathrm{mod}\ 2r), \qquad \text{for} \quad y_i = -a_0 \ (\mathrm{mod}\ r) = +a_1 \ (\mathrm{mod}\ r),\\ z_i &= +t_0 \ (\mathrm{mod}\ 2r), \qquad \text{for} \quad y_i = +a_0 \ (\mathrm{mod}\ r) = -a_1 \ (\mathrm{mod}\ r), \end{aligned}$$

for $i = 0,1,\dots,n$. The numerator remains finite in the limit, and the only non-zero contribution to (4.16) comes from the residues at the above poles, which by symmetry each have the same value, resulting in a factor $2n$. At the poles, the terms in the denominator of (4.16), cancel the required terms in $I^m_{BC_n}$ to give the integrand of $I^m_{BC_{n-1}}$. The remaining factors come from using $\lim\limits_{z_0\to -t_0}(t_0+z_0)\Gamma(t_0+z_0,0) = i/(2\pi\lambda)$, and similarly for $z_0 \to t_0$, in calculating the residues. ■

The proof of Theorem 4.1, now follows analogously to Theorem 3.1, however with $\mathcal{C}_{mn}$ now consisting of the points of the type $(\{\alpha_0,\beta_0\},\{\alpha_1,\beta_1\},\dots,\{\alpha_{m+n+1},\beta_{m+n+1}\})$, where

$$\alpha_i = t_{2m+2n+2-2i} + t_{2m+2n+3-2i}, \qquad \beta_i = a_{2m+2n+2-2i} + a_{2m+2n+3-2i},$$

where $i = 0,1,\dots,m+n+1$.

The limit (4.16) implies that if $(\{\alpha_0,\beta_0\},\{\alpha_1,\beta_1\},\dots,\{\alpha_{m+n+1},\beta_{m+n+1}\}) \in \mathcal{C}_{mn}$, then

$$\begin{aligned} &(\{\alpha_0+\alpha_1,\beta_0+\beta_1\},\{\alpha_2,\beta_2\},\dots,\{\alpha_{m+n+1},\beta_{m+n+1}\}) \in \mathcal{C}_{m(n-1)},\\ &(\{\alpha_0+\alpha_1-(\sigma+\tau),\beta_0+\beta_1\},\{\alpha_2,\beta_2\},\dots,\{\alpha_{m+n+1},\beta_{m+n+1}\}) \in \mathcal{C}_{(m-1)n}. \end{aligned} \quad (4.17)$$

Then starting from the special case (4.12), corresponding to the point

$$(\{\sigma,-1\},\{\sigma,-1\},\dots,\{\sigma,-1\},\{\tau,1\},\{\tau,1\},\dots,\{\tau,1\}) \in \mathcal{C}_{nn},$$

the above limits (4.17) imply that analogous relations to (3.17), (3.18) hold for Theorem 4.1. Then following the same argument of the A_n case, this implies that for $n\to\infty$ we have a dense set of points for the α_i in $\mathcal{C}_{mn}$ (for any β_i), and thus Theorem 4.1 holds in general.

5 Application to supersymmetric gauge theories

We now come to some applications of the A_n and BC_n elliptic hypergeometric sum/integral transformation formulas, given in Theorems 3.1 and 4.1 respectively.

The first application, discussed in this section, is in the area of the supersymmetric gauge theories: quantitative checks of the Seiberg duality [28] for supersymmetric QCD (SQCD) with gauge groups $\mathrm{SU}(N_c)$ and $\mathrm{Sp}(2N_c)$, at the level of the lens index [12]. This generalises the similar considerations for the $r = 1$ case [14], to more general cases with $r > 1$.

5.1 SU(N_c)

Let us first consider the gauge group SU(N_c). The Seiberg duality claims an equivalence between the following two theories, electric and magnetic.

Electric theory. The electric theory is the SU(N_c) SQCD with N_f flavors. In addition to an $\mathcal{N}=1$ vector multiplet V associated with the SU(N_c) gauge group, we also have $\mathcal{N}=1$ chiral multiplets q and $\bar{q}$, which are in the fundamental and anti-fundamental representation of SU(N_c) gauge symmetry, respectively. This theory has no superpotential, $W_{\text{electric}}=0$.

This theory has $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$ non-anomalous flavor symmetries, where $\mathrm{U}(1)_R$ is the R-symmetry. The charge assignment of the fields under the gauge/flavor symmetries is listed in Table 1.

Table 1. Electric theory for SU(N_c) Seiberg duality.

	SU(N_c)	SU(N_f)$_L$	SU(N_f)$_R$	U(1)$_B$	U(1)$_R$
q	$\square$	$\square$	$\mathbf{1}$	1	$r_q = 1 - N_c/N_f$
$\bar{q}$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	−1	$r_q = 1 - N_c/N_f$
V	adj.	$\mathbf{1}$	$\mathbf{1}$	0	1

To write down the lens index for this theory, let us prepare continuous/discrete fugacities for each flavor symmetry. We have $(\boldsymbol{z},\boldsymbol{y})$ for SU(N_c) gauge symmetry, where both $\boldsymbol{z}$ and $\boldsymbol{y}$ are N_c-component vectors and the components of $\boldsymbol{z}$ ($\boldsymbol{y}$) are complex parameters (integers taking values in $\mathbb{Z}_r$); these represent the discrete holonomies of the gauge fields along the S^1 and the torsion cycle of $S^3/\mathbb{Z}_r$ of the geometry $S^1 \times S^3/\mathbb{Z}_r$. Similarly, we have $(\bar{\boldsymbol{t}},\bar{\boldsymbol{a}})$ and $(\bar{\boldsymbol{s}},\bar{\boldsymbol{b}})$ for SU(N_f)$_L$ and SU(N_f)$_R$ flavor symmetries. Since these correspond to SU symmetries (and not U), we have the constraints

$$\sum_{i=1}^{N_c-1} z_i = \sum_{i=1}^{N_f-1} \bar{t}_i = \sum_{i=1}^{N_f-1} \bar{s}_i = 0, \qquad \sum_{i=1}^{N_c-1} y_i \equiv \sum_{i=1}^{N_f-1} \bar{a}_i \equiv \sum_{i=1}^{N_f-1} \bar{b}_i \equiv 0 \pmod{r}.$$

We also need a pair (B, n_B) for the U(1)$_B$ symmetry, where again B is a continuous parameter and n_B is an integer. Finally, R-symmetry plays a distinguished role and is associated with the pair $(R,0)$, where R is determined by σ and τ as

$$R = \frac{\tau+\sigma}{2}.$$

Note that integer-valued fugacity for R-symmetry is absent.

The lens index of this electric theory is then written as [12, 26]

$$\begin{aligned}(\mathcal{I}_e)^{N_f}_{\mathrm{SU}(N_c)}&(\boldsymbol{z},\boldsymbol{y};\bar{\boldsymbol{t}},\bar{\boldsymbol{a}};\bar{\boldsymbol{s}},\bar{\boldsymbol{b}};B,n_B)\\ &= \sum_{\substack{y_0,\ldots,y_{n-1}=0\\ \sum_{i=0}^{n} y_i=0}}^{r-1} \int_{\sum_{i=0}^{n} z_i=0} \mathcal{I}_V(\boldsymbol{z},\boldsymbol{y})\mathcal{I}_q(\boldsymbol{z},\boldsymbol{y};\bar{\boldsymbol{t}},\bar{\boldsymbol{a}};\bar{\boldsymbol{s}},\bar{\boldsymbol{b}};B,n_B). \end{aligned} \tag{5.1}$$

The variables $(\boldsymbol{z},\boldsymbol{y})$ are integrated/summed over since they are associated with the gauge symmetry. Inside the integrand, $\mathcal{I}_V$ and $\mathcal{I}_C$ are the one-loop contributions from the fields V and q, $\bar{q}$, respectively, and are given by

$$\mathcal{I}_V(\boldsymbol{z},\boldsymbol{y}) = \frac{\lambda^{N_c-1}}{N_c!} \frac{1}{\prod\limits_{0\le i<j\le N_c-1} \Gamma(z_i-z_j,y_i-y_j)\,\Gamma(z_j-z_i,y_j-y_i)},$$

$$\mathcal{I}_q(\boldsymbol{z},\boldsymbol{y};\bar{\boldsymbol{t}},\bar{\boldsymbol{a}};\bar{\boldsymbol{s}},\bar{\boldsymbol{b}};B,n_B) = \prod_{i=0}^{N_c-1}\prod_{j=0}^{N_f-1} \overbrace{\Gamma(\bar{t}_j+z_i+B+r_qR,\bar{a}_j+y_i+n_B)}^{q}\,\overbrace{\Gamma(\bar{s}_j-z_i-B+r_qR,\bar{b}_j-y_i-n_B)}^{\bar{q}}.$$

As indicated above, the gamma function factors inside the expression for $\mathcal{I}_q$ are contributions from the quarks q, $\bar{q}$, and their symmetry charges are reflected in the arguments of gamma functions. For example, the $\mathrm{U}(1)_B$ and $\mathrm{U}(1)_R$ charges of the quark q are $+1$ and r_q, and hence the combination $B+r_qR$ appears inside the gamma functions for the quark q.

Let us define the unbarred vectors $\boldsymbol{t}$, $\boldsymbol{s}$, $\boldsymbol{a}$, $\boldsymbol{b}$ by adding trace parts to the barred vectors $\bar{\boldsymbol{t}}$, $\bar{\boldsymbol{s}}$, $\bar{\boldsymbol{a}}$, $\bar{\boldsymbol{b}}$:

$$\boldsymbol{t}=\bar{\boldsymbol{t}}+r_qR+B,\qquad \boldsymbol{s}=\bar{\boldsymbol{s}}+r_qR-B,\qquad \boldsymbol{a}=\bar{\boldsymbol{a}}+n_B,\qquad \boldsymbol{b}=\bar{\boldsymbol{b}}-n_B. \tag{5.2}$$

After this rewriting B and n_B are now included in the trace parts of $\boldsymbol{t}$, $\boldsymbol{s}$, $\boldsymbol{a}$, $\boldsymbol{b}$. The one-loop determinant for the quark fields q, $\bar{q}$ is written as

$$\mathcal{I}_q(\boldsymbol{z},\boldsymbol{y};\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b}) = \prod_{i=0}^{N_c-1}\prod_{j=0}^{N_f-1}\Gamma(t_j+z_i,a_j+y_i)\,\Gamma(s_j-z_i,b_j-y_i),$$

and the constraints (5.9) are written as

$$\begin{aligned}&\sum_{i=0}^{N_c-1} z_i=0,\qquad \sum_{i=0}^{N_f-1}(t_i+s_i)=2N_fr_qR=(N_f-N_c)(\tau+\sigma),\\ &\sum_{i=0}^{N_c-1} y_i\equiv\sum_{i=0}^{N_f-1}(a_i+b_i)\equiv 0 \pmod r.\end{aligned} \tag{5.3}$$

We can now easily verify that the lens index (5.1) coincides with the sum/integral (3.2) with $Z=Y=0$, with the identification $n=N_c-1$ and $m=N_f-N_c-1$:

$$(\mathcal{I}_e)^{N_f}_{\mathrm{SU}(N_c)}(\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b}) = I^{N_f-N_c-1}_{A_{N_c-1}}(0,0\,|\,\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b}). \tag{5.4}$$

Notice that the constraint (5.3) also matches with (3.1) under this parameter identification.

Magnetic theory. Let us next come to the magnetic theory. This is $\mathrm{SU}(\tilde{N}_c)$ SQCD with N_f flavors, where the N_c of the electric theory is replaced by

$$\tilde{N}_c=N_f-N_c.$$

We have an $\mathrm{SU}(N_f-N_f)$ $\mathcal{N}=1$ vector multiplet $\tilde{V}$ and N_f chiral multiplets Q and $\bar{Q}$. In addition we also have a meson field M, and the non-trivial superpotential involving it:

$$W_{\text{magnetic}}=QM\bar{Q}.$$

The flavor symmetry of the magnetic theory is the same as that for the electric theory. The charge assignments of the fields under gauge/flavor symmetries is summarized in Table 2. Note in particular that the $\mathrm{U}(1)_R$-charge r_Q of the dual quarks Q, $\bar{Q}$ is the same as that for the quarks of the electric theory, under the substitution $N_c\to\tilde{N}_c$:

$$r_Q=r_q|_{N_c\to\tilde{N}_c}. \tag{5.5}$$

Let us now come to the lens index of the theory. This is similar to that of the electric theory, but there are some important differences. First we need to change $N_c\to N_f-N_c$. This is also

Table 2. Magnetic theory for $\mathrm{SU}(N_c)$ Seiberg duality.

	$\mathrm{SU}(\tilde{N}_c)$	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_R$
Q	$\square$	$\overline{\square}$	$\mathbf{1}$	$r_B = N_c/\tilde{N}_c$	$r_Q = N_c/N_f$
$\bar{Q}$	$\overline{\square}$	$\mathbf{1}$	$\square$	$-r_B = -N_c/\tilde{N}_c$	$r_Q = N_c/N_f$
M	$\mathbf{1}$	$\square$	$\overline{\square}$	0	$2r_q = 2\,(1 - N_c/N_f)$
$\tilde{V}$	adj.	$\mathbf{1}$	$\mathbf{1}$	0	1

reflected in the change of the R-charge $r_q \to r_Q$, as stated in (5.5). Second, compared with the electric case we need to invert the signs of the $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ fugacities $(\bar{\boldsymbol{t}}, \bar{\boldsymbol{s}}, \bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})$; for example the electric quark q was in the fundamental representation under the $\mathrm{SU}(N_f)_L$ symmetry, whereas the magnetic quark Q is in the anti-fundamental representation under the same symmetry. We also need to take into account the difference in the $\mathrm{U}(1)_B$-symmetry charges, which has the effect of changing $B \to r_B B$. Finally, we also need to take into account contributions from mesons, which do not exist in the electric theory:

$$\mathcal{I}_M(\bar{\boldsymbol{t}}, \bar{\boldsymbol{a}}; \bar{\boldsymbol{s}}, \bar{\boldsymbol{b}}) = \prod_{i,j=1}^{N_f} \Gamma(\bar{t}_i + \bar{s}_j + 2r_q R, \bar{a}_i + \bar{b}_j).$$

By combining all the ingredients, we have

$$(\mathcal{I}_m)^{N_f}_{\mathrm{SU}(N_c)}(\bar{\boldsymbol{t}}, \bar{\boldsymbol{a}}; \bar{\boldsymbol{s}}, \bar{\boldsymbol{b}}) = \mathcal{I}_M(\bar{\boldsymbol{t}}, \bar{\boldsymbol{a}}; \bar{\boldsymbol{s}}, \bar{\boldsymbol{b}}) (\mathcal{I}_e)^{N_f}_{\mathrm{SU}(\tilde{N}_c)}(-\bar{\boldsymbol{t}}, -\bar{\boldsymbol{a}}; -\bar{\boldsymbol{s}}, -\bar{\boldsymbol{b}}; r_B B, r_B n_B). \tag{5.6}$$

There is one subtlety here. The formula (5.6) does not make sense as it is, since r_B and hence $r_B n_B$ is in general not an integer. This is related to the fact that charges under the $\mathrm{U}(1)$ symmetry (such as the $\mathrm{U}(1)_B$ symmetry as discussed here) is not quantized – this means that the discrete holonomy for the torsion cycle of $S^1 \times S^3/\mathbb{Z}_r$ can be turned on only if the charges of all the fields under the $\mathrm{U}(1)$ symmetry is an integer. For our purposes, we can simply choose to take the integer parameter n_B to be

$$n_B \in \tilde{N}_c \mathbb{Z}. \tag{5.7}$$

Equivalently, we choose the normalization of $\mathrm{U}(1)_B$ such that the charges of all the fields under this symmetry are integers.

We can now include the trace part into the definitions of the vectors, as in the case of the electric theory:

$$\begin{aligned} &\tilde{\boldsymbol{t}} = -\bar{\boldsymbol{t}} + r_Q R + r_B B, \qquad \tilde{\boldsymbol{s}} = -\bar{\boldsymbol{s}} + r_Q R - r_B B, \\ &\tilde{\boldsymbol{a}} = -\bar{\boldsymbol{a}} + r_B n_B, \qquad \tilde{\boldsymbol{b}} = -\bar{\boldsymbol{b}} - r_B n_B, \end{aligned} \tag{5.8}$$

and these variables obey the constraints

$$\begin{aligned} &\sum_i z_i = 0, \qquad \sum_i (\tilde{t}_i + \tilde{s}_i) = N_f r_Q(\tau + \sigma) = N_c(\tau + \sigma), \\ &\sum_i y_i \equiv \sum_i (a_i + b_i) \equiv 0 \pmod{r}. \end{aligned} \tag{5.9}$$

The magnetic lens index (5.6) now reads

$$(\mathcal{I}_m)^{N_f}_{\mathrm{SU}(N_c)} = \left(\prod_{i,j=1}^{N_f} \Gamma(t_i + s_j, a_i + b_j) \right) (\mathcal{I}_e)^{N_f}_{\mathrm{SU}(\tilde{N}_c)}(\tilde{\boldsymbol{t}}, \tilde{\boldsymbol{a}}; \tilde{\boldsymbol{s}}, \tilde{\boldsymbol{b}})$$

$$= \left(\prod_{i,j=1}^{N_f} \Gamma(t_i + s_j, a_i + b_j)\right) I_{A_{N_f-N_c-1}}^{N_c-1}(0,0\,|\,\tilde{\boldsymbol{t}}, \tilde{\boldsymbol{a}}; \tilde{\boldsymbol{s}}, \tilde{\boldsymbol{b}}). \tag{5.10}$$

By comparing (5.2) and (5.8), we can directly express the variables $(\tilde{\boldsymbol{t}}, \tilde{\boldsymbol{s}}, \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}})$ in terms of variables $(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{a}, \boldsymbol{b})$ without tildes:

$$\begin{aligned} \tilde{\boldsymbol{t}} &= \frac{T}{N_f - N_c} - \boldsymbol{t}, \qquad & \tilde{\boldsymbol{s}} &= \frac{S}{N_f - N_c} - \boldsymbol{s}, \\ \tilde{\boldsymbol{a}} &= \frac{A}{N_f - N_c} - \boldsymbol{a}, \qquad & \tilde{\boldsymbol{b}} &= \frac{B}{N_f - N_c} - \boldsymbol{b}, \end{aligned} \tag{5.11}$$

with

$$\begin{aligned} T &= \sum_{i=1}^{N_f} t_i = N_f B + (N_f - N_c)R, \qquad & S &= \sum_{i=1}^{N_f} t_i = -N_f B + (N_f - N_c)R, \\ A &= \sum_{i=1}^{N_f} a_i = N_f n_B, \qquad & B &= \sum_{i=1}^{N_f} b_i = -N_f n_B. \end{aligned}$$

Notice that $A/(N_f - N_c)$ and $B/(N_f - N_c)$ are integers thanks to the quantization condition (5.7).

Duality. We are now ready to state the equality of the lens indices of the electric and magnetic theory, which simply states

$$(\mathcal{I}_e)_{\mathrm{SU}(N_c)}^{N_f} = (\mathcal{I}_m)_{\mathrm{SU}(N_c)}^{N_f}. \tag{5.12}$$

In view of (5.4) and (5.10) we have by now shown that this equation coincides with the transformation formula (3.5), with $Z = Y = 0$. Indeed, the relations between parameters as stated in (5.11) can be seen to coincide with the transformation rules (3.6), after a straightforward change of variables, as follows from the quantization condition (5.7). This concludes our proof of the relation (5.12).

The identity (5.12) has previously been checked only up to certain orders in a series-expansion with respect to fugacities. The results of this section settles the problem of proving the identity mathematically in general, and provides the one of the most elaborate quantitative check of the Seiberg duality known to date.

5.2 $\mathrm{Sp}(2N_c)$

Let us next discuss Seiberg duality for $\mathrm{Sp}(2N_c)$ gauge groups.

Electric theory. The electric theory is $\mathrm{Sp}(2N_c)$ theory[1] with N_f flavors, which in practice means that we have a matter field q in the fundamental $2N_c$-dimensional representation of $\mathrm{Sp}(2N_c)$. The superpotential is absent, as in the $\mathrm{SU}(N_c)$ case. This theory has $\mathrm{SU}(2N_f) \times \mathrm{U}(1)_R$ global symmetry, and the charge assignment of the fields are listed in Table 3.

As in the case of $\mathrm{SU}(N_c)$ theory, we need fugacities for the lens index, which are $(\boldsymbol{z}, \boldsymbol{y})$ for $\mathrm{SU}(N_c)$ gauge symmetry and $(\bar{\boldsymbol{t}}, \boldsymbol{a})$ for $\mathrm{SU}(2N_f)$ flavor symmetry, with the constraints

$$\sum_{i=1}^{N_c-1} z_i = \sum_{i=1}^{N_f-1} \bar{t}_i = 0, \qquad \sum_{i=1}^{N_c-1} y_i \equiv \sum_{i=1}^{N_f-1} a_i \equiv 0 \pmod{r}. \tag{5.13}$$

[1]The convention for the $\mathrm{Sp}(2N_c)$ gauge group here is that its argument is always even, for example $\mathrm{Sp}(2) = \mathrm{SU}(2)$.

Table 3. Electric theory for $\mathrm{Sp}(2N_c)$ Seiberg duality.

	$\mathrm{Sp}(2N_c)$	$\mathrm{SU}(2N_f)$	$\mathrm{U}(1)_R$
q	$\square$	$\square$	$r'_q = 1-(N+1)/N_f$
V	adj.	1	1

The lens index of this electric theory is then written as

$$(\mathcal{I}_e)^{N_f}_{\mathrm{Sp}(2N_c)}(\bar{\boldsymbol{t}},\boldsymbol{a}) = \sum_{\substack{y_0,\dots,y_{n-1}=0\\ \sum_{i=0}^{n} y_i=0}}^{r-1} \int_{\sum_{i=0}^{n} z_i=0} \mathcal{I}_V(\boldsymbol{z},\boldsymbol{y})\mathcal{I}_q(\boldsymbol{z},\boldsymbol{y};\bar{\boldsymbol{t}},\boldsymbol{a}), \tag{5.14}$$

where $\mathcal{I}_V$ and $\mathcal{I}_C$ are the one-loop contributions from the fields V and q, respectively, and are given by

$$\mathcal{I}_V(\boldsymbol{z},\boldsymbol{y}) = \frac{\lambda^{N_c}}{2^{N_c}N_c!}\frac{1}{\prod_{i=0}^{N_c-1}\Gamma(\pm 2z_i,\pm 2y_i)}\frac{1}{\prod_{0\le i<j\le N_c-1}\Gamma(\pm z_i\pm z_j,\pm y_i\pm y_j)},$$

$$\mathcal{I}_q(\boldsymbol{z},\boldsymbol{y};\bar{\boldsymbol{t}},\boldsymbol{a}) = \prod_{i=0}^{N_c-1}\prod_{j=0}^{2N_f-1}\Gamma(\bar{t}_j\pm z_i + r'_q R, a_j\pm y_i),$$

where the factor $2^{N_c}N_c!$ comes from the Weyl group for $C_{N_c} = \mathrm{Sp}(2N_c)$.

We can define the unbarred vector $\bar{\boldsymbol{t}}$ by

$$\boldsymbol{t} = \bar{\boldsymbol{t}} + r_q R, \qquad \boldsymbol{a} = \bar{\boldsymbol{a}},$$

and the constraints (5.13) are written as

$$\sum_{i=0}^{N_c-1} z_i = 0, \qquad \sum_{i=0}^{2N_f-1} t_i = 2N_f r'_q R = (N_f-N_c-1)(\tau+\sigma),$$
$$\sum_{i=0}^{N_c-1} y_i \equiv \sum_{i=0}^{N_f-1} a_i \equiv 0 \pmod{r}.$$

We can now easily verify that the lens index (5.14) coincides with the sum/integral (4.2), with the identification $n = N_c$ and $m = N_f - N_c - 2$:

$$(\mathcal{I}_e)^{N_f}_{\mathrm{Sp}(2N_c)}(\boldsymbol{t},\boldsymbol{a}) = I^{N_f-N_c-2}_{BC_{N_c}}(\boldsymbol{t},\boldsymbol{a}). \tag{5.15}$$

Notice that the constraint (5.3) also matches with (4.1) under this parameter identification.

Magnetic theory. Let us next discuss the magnetic theory. We will be brief here since the analysis is similar to previous cases.

The fields, symmetries and charge assignments are summarized in Table 4. Similar to the case of the $\mathrm{SU}(N_c)$ theory we have a meson field M with the superpotential $W = MQQ$, except now M is in the anti-symmetric representation under the flavor $\mathrm{SU}(2N_f)$ symmetry. Another important difference from the $\mathrm{SU}(N_c)$ case is that the value of N_c in the magnetic theory is given by

$$\tilde{N}_c = N_f - N_c - 2.$$

Table 4. Magnetic theory for $\mathrm{Sp}(2N_c)$ Seiberg duality.

	$\mathrm{Sp}(2\tilde{N}_c)$	$\mathrm{SU}(2N_f)$	$\mathrm{U}(1)_R$
Q	$\square$	$\overline{\square}$	$r'_Q = (N_c+1)/N_f$
$\tilde{V}$	adj.	1	1
M	1	anti-symm.	$2(\tilde{N}_c+1)/N_f$

By repeating the similar manipulations as in the previous cases, we obtain the lens index of the magnetic theory to be

$$(\mathcal{I}_m)^{N_f}_{\mathrm{Sp}(2N_c)}(\tilde{\boldsymbol{t}},\boldsymbol{a}) = \left(\prod_{i,j=1}^{N_f}\Gamma(t_i+t_j,a_i+a_j)\right) I^{N_c}_{BC_{N_f-N_c-2}}(\boldsymbol{t},-\boldsymbol{a}), \tag{5.16}$$

where we defined

$$\tilde{\boldsymbol{t}} = -\bar{\boldsymbol{t}} + r'_Q R = -\boldsymbol{t} + (r'_q + r'_Q)R = -\boldsymbol{t} + \frac{\tau+\sigma}{2}.$$

Duality. We can now easily check from (5.15) and (5.16) that the duality relation

$$(\mathcal{I}_e)^{N_f}_{\mathrm{Sp}(2N_c)} = (\mathcal{I}_m)^{N_f}_{\mathrm{Sp}(2N_c)},$$

reduces to the BC_n elliptic hypergeometric sum/integral transformation formula as stated in (4.7). This is what we wanted to show.

It would be interesting to prove identities for more general Seiberg dualities, for more general gauge groups and more general matters (for example matters in spinor representations). The case of Seiberg dualities for $\mathrm{SO}(N)$ and $\mathrm{Spin}(N)$ gauge groups is currently under investigation.

6 Application to integrable lattice models

Let us now come to our second application for the A_n elliptic hypergeometric sum/integral transformation formula (3.5), this time to integrable lattice models.

A particular case of Theorem 3.1 (when $m=n$) is equivalent to an identity in statistical mechanics known as the star-star relation. The star-star relation is a particular condition of integrability for lattice models of statistical mechanics, which implies that the Boltzmann weights of the model satisfy the Yang–Baxter equation. This in turn implies that the row-to-row transfer matrices of the lattice model commute in pairs [4], allowing for an exact solution of the model.

The $r=1$ case of Theorem 3.1 [25] was previously shown [6] to imply a multi-spin solution of the star-star relation obtained by Bazhanov and Sergeev [10]. The general $r\geq 1$ solution of the star-star relation corresponding to Theorem 3.1 was discovered by the second author [41] using the gauge/YBE correspondence [38, 40, 41] between 2d integrable lattice models and 4d $\mathcal{N}=1$ supersymmetric gauge theories. In this section this lattice model and corresponding star-star relation are introduced, and it is explicitly shown that the latter star-star relation reduces to the $m=n$ case of Theorem 3.1.

6.1 Square lattice model

Let us first define the lattice model of statistical mechanics. Denote the square lattice by L, consisting of a set of vertices and a set of edges, the latter denoted respectively by $V(L)$ and $E(L)$. An edge $(ij)\in E(L)$ connects two vertices $i,j\in V(L)$.

The square lattice L is shown in Fig. 1, along with a directed rapidity lattice, which will be denoted by $\mathscr{L}$. The rapidity lattice $\mathscr{L}$ is made up of four different types of directed rapidity lines, which are distinguished by their orientation (horizontal or vertical), and by whether they are solid or dashed lines. The four types of rapidity lines are labelled by four real valued rapidity variables u, u', v, v'.

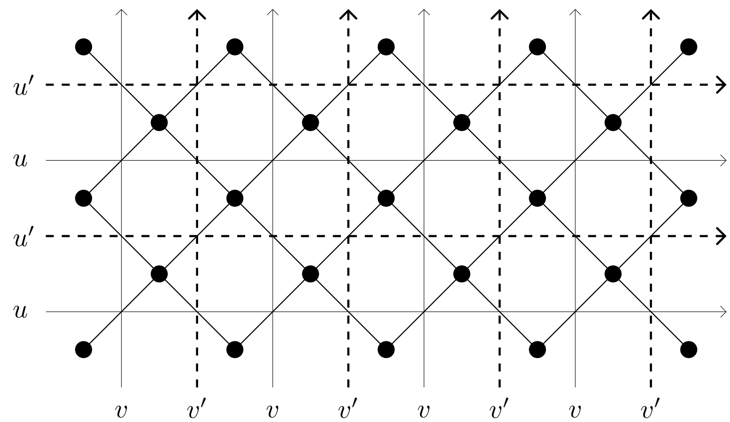

Figure 1. The square lattice L drawn diagonally, and its medial rapidity lattice $\mathscr{L}$, consisting of directed lines labelled by the four rapidity variables u, u', v, v'.

Spin variables σ_i are assigned to vertices $i \in V(L)$, and take values

$$\sigma_i = \{(x_{i,1}, m_{i,1}), (x_{i,2}, m_{i,2}), \dots, (x_{i,n}, m_{i,n})\},$$

where $n = 1, 2, \dots$, and

$$x_{i,j} \in [0, 2\pi), \qquad m_{i,j} \in \mathbb{Z}_r, \qquad j = 1, 2, \dots, n,$$

are respectively the real and discrete valued components of a spin σ_i. These spin components are subject to the constraints

$$\sum_{a=1}^{n} x_{i,a} = 0, \qquad \sum_{a=1}^{n} m_{i,a} = 0. \tag{6.1}$$

Taking these constraints into account, the integration measure for a spin σ_i is denoted by

$$\int d\sigma_i := \underset{\sum_{j=1}^{n} m_{i,j}=0}{\sum_{m_{i,1}=0}^{r-1} \cdots \sum_{m_{i,n-1}=0}^{r-1}} \; \underset{\sum_{j=1}^{n} x_{i,j}=0}{\int_0^1 \cdots \int_0^1} \; \prod_{k=1}^{n-1} dx_{i,k}. \tag{6.2}$$

The crossing of rapidity lines on edges $(ij) \in E(L)$ in Fig. 1 distinguish the four different types of edges shown explicitly in Fig. 2. The two types of Boltzmann weights $W_\alpha(\sigma_i, \sigma_j)$, $\overline{W}_\alpha(\sigma_i, \sigma_j)$ are assigned to the four types of edges as indicated in Fig. 2, and depend on the value of the spins at the vertices, and the value of the rapidity variables crossing the edge. The ordering of spins variables matters here, i.e., in general

$$W_\alpha(\sigma_i, \sigma_j) \neq W_\alpha(\sigma_j, \sigma_i), \qquad \overline{W}_\alpha(\sigma_i, \sigma_j) \neq \overline{W}_\alpha(\sigma_j, \sigma_i).$$

The Boltzmann weights are conveniently expressed in terms of a function $\Phi(z, m)$, which is defined in terms of the lens elliptic gamma function (2.3) as

$$\Phi(z, m) = \Gamma\left(\frac{\sigma + \tau}{2} - z, -m; \sigma, \tau\right).$$

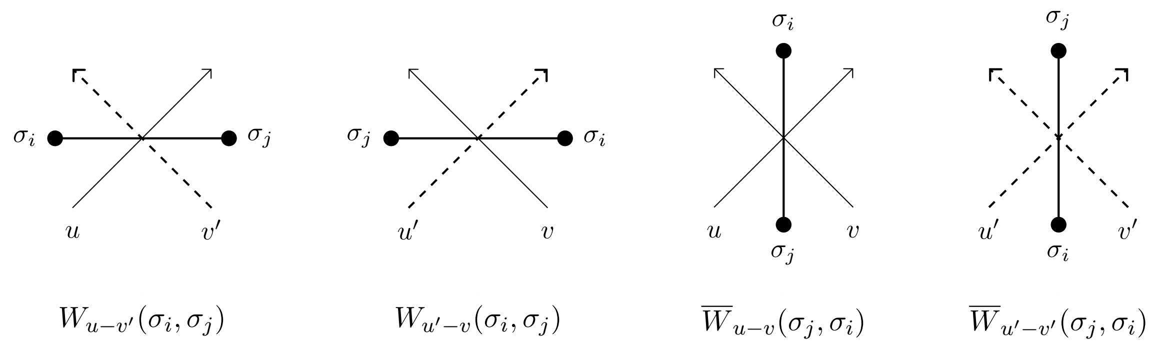

Figure 2. Four different types of edges and their associated Boltzmann weights in (6.3).

Then the Boltzmann weights are given by

$$\begin{aligned} W_\alpha(\sigma_a,\sigma_b) &= \prod_{i,j=1}^{n} \Phi(x_{a,i}-x_{b,j}+i\alpha, m_{a,i}-m_{b,j}),\\ \overline{W}_\alpha(\sigma_a,\sigma_b) &= \sqrt{S(\sigma_a)S(\sigma_b)}\, W_{\eta-\alpha}(\sigma_a,\sigma_b), \end{aligned} \tag{6.3}$$

where

$$S(\sigma_a) = \prod_{1\le i<j\le n} \Phi(-i\eta + x_{a,i}-x_{a,j}, m_{a,i}-m_{a,j})\, \Phi(-i\eta + x_{a,j}-x_{a,i}, m_{a,j}-m_{a,i}).$$

The parameter η is known as the crossing parameter, and it relates the Boltzmann weight $W_\alpha(\sigma_a,\sigma_b)$ to the Boltzmann weight $\overline{W}_\alpha(\sigma_a,\sigma_b)$. It is given here by

$$\eta = -\frac{i}{2}(\sigma+\tau).$$

Note that the Boltzmann weights (6.3) physically represent an interaction energy between the two spins connected by the edge of the lattice. It is therefore often desirable that Boltzmann weights are positive and real valued, however the conditions for a regime where this condition is satisfied are unfortunately not known.

Now let $E^{(1)}$, $E^{(2)}$, $E^{(3)}$, $E^{(4)}$, be respectively the sets of the four types of edges of L, that are depicted from left to right in Fig. 2. Then the partition function of the model is defined as

$$\begin{aligned} Z = \int \prod_{(ij)\in E^{(1)}(L)} W_{u-v'}(\sigma_i,\sigma_j) \prod_{(ij)\in E^{(2)}(L)} W_{u'-v}(\sigma_i,\sigma_j) \prod_{(ij)\in E^{(3)}(L)} \overline{W}_{u-v}(\sigma_i,\sigma_j)\\ \times \prod_{(ij)\in E^{(4)}(L)} \overline{W}_{u'-v'}(\sigma_i,\sigma_j) \prod_{i=1}^{N} \mathrm{d}\sigma_i, \end{aligned} \tag{6.4}$$

where the products are taken over the four types of edges of L given in Fig. 2, and the integration is taken over N spins σ_i interior to the lattice, with boundary spins kept fixed. Note that the integration is taken with respect to (6.2).

In addition to the edge formulation given in Fig. 2, the model may be formulated as an interaction-round-a-face (IRF) model [3], in terms of either of the four-edge stars depicted in Fig. 3. These stars are associated two different Boltzmann weights $W^{(1)}_{\boldsymbol{uv}}$, $W^{(2)}_{\boldsymbol{uv}}$, according to Fig. 2, and are given by the expressions

$$W^{(1)}_{\boldsymbol{uv}}\begin{pmatrix}\sigma_i & \sigma_j\\ \sigma_k & \sigma_l\end{pmatrix} = \int \mathrm{d}\sigma_h \overline{W}_{u-v}(\sigma_k,\sigma_h)\overline{W}_{u'-v'}(\sigma_j,\sigma_h) W_{u'-v}(\sigma_h,\sigma_i) W_{u-v'}(\sigma_h,\sigma_l),$$

$$W^{(2)}_{\boldsymbol{u}\boldsymbol{v}}\begin{pmatrix}\sigma_i & \sigma_j\\ \sigma_k & \sigma_l\end{pmatrix} = \int \mathrm{d}\sigma_h \overline{W}_{u-v}(\sigma_h,\sigma_j)\overline{W}_{u'-v'}(\sigma_h,\sigma_k)W_{u'-v}(\sigma_l,\sigma_h)W_{u-v'}(\sigma_i,\sigma_h), \tag{6.5}$$

where $\boldsymbol{u} = \{u,u'\}$, and $\boldsymbol{v} = \{v,v'\}$.

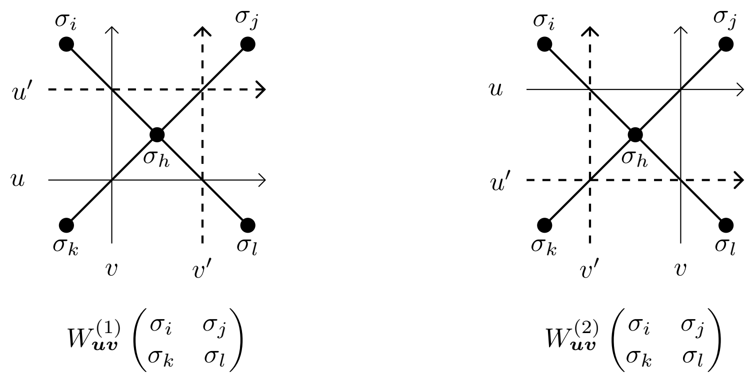

Figure 3. Two types of four-edge stars and associated Boltzmann weights in (6.5).

The lattice L may be produced by periodic translations of either one of the two four-edge stars depicted in Fig. 3. The partition function (6.4) may then be written in equivalent forms (up to boundary effects), in terms of either of the two Boltzmann weights in (6.5). For example, let $V^{(1)}$, and $V^{(2)}$, denote two disjoint subsets of V, where $V^{(1)}$ is the set of all vertices of the type associated to σ_h on the left hand side of Fig. 3, and $V^{(2)}$ is the set of vertices of the remaining type on the right hand side of Fig. 3. Then (6.4) may be written as

$$Z = \int \prod_{h\in V^{(1)}} W^{(1)}_{\boldsymbol{u}\boldsymbol{v}}\begin{pmatrix}\sigma_i & \sigma_j\\ \sigma_k & \sigma_l\end{pmatrix} \prod_{a\in V^{(2)}} \mathrm{d}\sigma_a. \tag{6.6}$$

Note that the integration over vertices $h \in V^{(1)}$ is already made through the definition of $W^{(1)}_{\boldsymbol{u}\boldsymbol{v}}$ in (6.5), and the expression (6.6) contains the integration over the remaining internal vertices $a \in V^{(2)}$.

6.2 Star-star relation and A_n sum/integral transformation

An important property of the Boltzmann weights (6.5) is that they satisfy the following *star-star relation*

$$\begin{aligned} W_{v'-v}(\sigma_l,\sigma_k)W_{u'-u}(\sigma_l,\sigma_j)W^{(1)}_{\boldsymbol{u}\boldsymbol{v}}\begin{pmatrix}\sigma_i & \sigma_j\\ \sigma_k & \sigma_l\end{pmatrix} \\ = W_{v'-v}(\sigma_j,\sigma_i)W_{u'-u}(\sigma_k,\sigma_i)W^{(2)}_{\boldsymbol{u}\boldsymbol{v}}\begin{pmatrix}\sigma_i & \sigma_j\\ \sigma_k & \sigma_l\end{pmatrix}. \end{aligned} \tag{6.7}$$

This relation is depicted graphically in Fig. 4.

The particular solution of the star-star relation (6.7) given by Boltzmann weights (6.3) and (6.5) was given by the second author [41].

The main result for this section is showing that the star-star relation (6.7) is equivalent to Theorem 3.1 in the case $m = n$. This ends up being rather straightforward.

Indeed, consider the new variables

$$\begin{aligned} t_j &= i(u-v) - x_{c,j}, & s_j &= -(u'-v-\eta) + x_{a,j},\\ t_{n+j} &= i(u'-v') - x_{b,j}, & s_{n+j} &= -(u-v'-\eta) + x_{d,j}, \end{aligned} \tag{6.8}$$

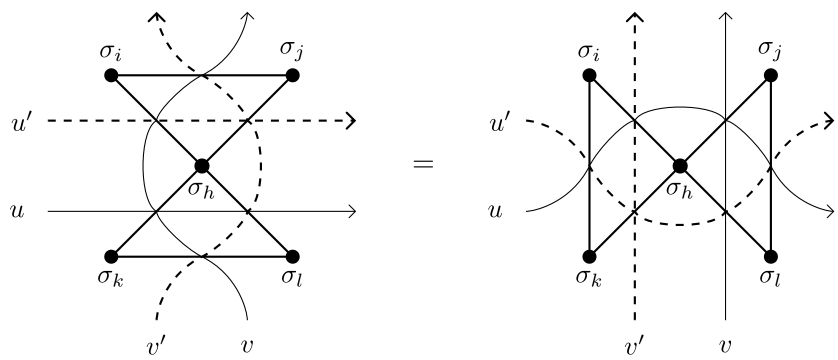

Figure 4. The star-star relation (6.7). This provides a graphical representation of the A_n transformation in Theorem 3.1, for the case $m = n$.

and

$$a_j = -m_{c,j}, \qquad a_{n+j} = -m_{b,j}, \qquad b_j = +m_{a,j}, \qquad b_{n+j} = +m_{d,j}, \tag{6.9}$$

where the x_i, and the m_i satisfy (6.1).

Then the Boltzmann weights (6.5), are seen to be equivalent to the A_n sum/integrals (3.2) in the form

$$W^{(1)}_{\boldsymbol{uv}} \begin{pmatrix} \sigma_a & \sigma_b \\ \sigma_c & \sigma_d \end{pmatrix} = I^{n-1}_{A_{n-1}}(\boldsymbol{t}, \boldsymbol{a}; \boldsymbol{s}, \boldsymbol{b}), \tag{6.10}$$

and

$$W^{(2)}_{\boldsymbol{uv}} \begin{pmatrix} \sigma_a & \sigma_b \\ \sigma_c & \sigma_d \end{pmatrix} = I^{n-1}_{A_{n-1}}(\tilde{\boldsymbol{t}}, \tilde{\boldsymbol{a}}; \tilde{\boldsymbol{s}}, \tilde{\boldsymbol{b}}), \tag{6.11}$$

where the $\tilde{\boldsymbol{t}}$, $\tilde{\boldsymbol{a}}$, $\tilde{\boldsymbol{s}}$, $\tilde{\boldsymbol{b}}$ are the variables (6.8), (6.9), transformed according to (3.6).

We also have

$$\frac{W_{v'-v}(\sigma_j, \sigma_i)\, W_{u'-u}(\sigma_k, \sigma_i)}{W_{v'-v}(\sigma_l, \sigma_k)\, W_{u'-u}(\sigma_l, \sigma_j)} = \prod_{i,j=1}^{n} \Gamma(t_j + s_i, a_j + b_i). \tag{6.12}$$

The star-star relation (6.7) then follows from Theorem 3.1 with (6.10), (6.11), and (6.12).

Acknowledgements

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