

Contractions of Degenerate Quadratic Algebras, Abstract and Geometric

Mauricio A. ESCOBAR RUIZ [†], Willard MILLER Jr. [‡] and Eyal SUBAG [§]

[†] *Centre de Recherches Mathématiques, Université de Montréal,
C.P. 6128, succ. Centre-Ville, Montréal, QC H3C 3J7, Canada*
E-mail: mauricio.escobar@nuclares.unam.mx

[‡] *School of Mathematics, University of Minnesota, Minneapolis, Minnesota, 55455, USA*
E-mail: miller@ima.umn.edu
URL: <https://www.ima.umn.edu/~miller/>

[§] *Department of Mathematics, Pennsylvania State University, State College,
Pennsylvania, 16802 USA*
E-mail: eus25@psu.edu

Received August 09, 2017, in final form December 26, 2017; Published online December 31, 2017
<https://doi.org/10.3842/SIGMA.2017.099>

Abstract. Quadratic algebras are generalizations of Lie algebras which include the symmetry algebras of 2nd order superintegrable systems in 2 dimensions as special cases. The superintegrable systems are exactly solvable physical systems in classical and quantum mechanics. Distinct superintegrable systems and their quadratic algebras can be related by geometric contractions, induced by Bôcher contractions of the conformal Lie algebra $\mathfrak{so}(4, \mathbb{C})$ to itself. In 2 dimensions there are two kinds of quadratic algebras, nondegenerate and degenerate. In the geometric case these correspond to 3 parameter and 1 parameter potentials, respectively. In a previous paper we classified all abstract parameter-free nondegenerate quadratic algebras in terms of canonical forms and determined which of these can be realized as quadratic algebras of 2D nondegenerate superintegrable systems on constant curvature spaces and Darboux spaces, and studied the relationship between Bôcher contractions of these systems and abstract contractions of the free quadratic algebras. Here we carry out an analogous study of abstract parameter-free degenerate quadratic algebras and their possible geometric realizations. We show that the only free degenerate quadratic algebras that can be constructed in phase space are those that arise from superintegrability. We classify all Bôcher contractions relating degenerate superintegrable systems and, separately, all abstract contractions relating free degenerate quadratic algebras. We point out the few exceptions where abstract contractions cannot be realized by the geometric Bôcher contractions.

Key words: Bôcher contractions; quadratic algebras; superintegrable systems; conformal superintegrability; Poisson structures

2010 Mathematics Subject Classification: 22E70; 16G99; 37J35; 37K10; 33C45; 17B60; 81R05; 33C45

1 Introduction

An abstract *degenerate (quantum) quadratic algebra* Q is a noncommutative multiparameter associative algebra generated by linearly independent operators X, H, L_1, L_2 , with parameters a_j , such that H is in the center and the following commutation relations hold [17]:

$$[X, L_j] = \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 1} P_{e_1, e_2, e_3, e_4}^{(j)} L_1^{e_1} L_2^{e_2} H^{e_3} X^{2e_4}, \quad j = 1, 2, \quad (1.1)$$

$$[L_1, L_2] = \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 1} T_{e_1, e_2, e_3, e_4} \{L_1^{e_1}, L_2^{e_2}, X\} H^{e_3} X^{2e_4}. \quad (1.2)$$

Finally, there is the relation:

$$G \equiv \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 2} S_{e_1, e_2, e_3, e_4} \{L_1^{e_1}, L_2^{e_2}, X^{2e_4}\} H^{e_3} + c_1 X L_1 X + c_2 X L_2 X = 0, \quad (1.3)$$

$$X^0 = H^0 = I,$$

where $\{L_1^{e_1}, L_2^{e_2}, X^{2e_4}\}$ is the 6-term symmetrizer of three operators. The constants $P_{e_1, e_2, e_3, e_4}^{(j)}$, T_{e_1, e_2, e_3, e_4} and S_{e_1, e_2, e_3, e_4} are polynomials in the parameters a_i of degrees $1 - e_1 - e_2 - e_3 - e_4$, $1 - e_1 - e_2 - e_3 - e_4$ and $2 - e_1 - e_2 - e_3 - e_4$, respectively, while c_1, c_2 are of degree 0. If all parameters $a_j = 0$ the algebra is *free*. For these quantum quadratic algebras there is a natural grading such that the operators H, L_j are 2nd order and X is 1st order. The field of scalars can be either \mathbb{R} or \mathbb{C} .

An abstract *degenerate (classical) quadratic algebra* \mathcal{Q} is a Poisson algebra with linearly independent generators $\mathcal{X}, \mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$, and parameters a_i , satisfying relations (1.1), (1.2), (1.3) with the commutator replaced by the Poisson bracket, H, L_j, X by $\mathcal{H}, \mathcal{L}_j, \mathcal{X}$, and the symmetrizer $\{L_1^{e_1}, L_2^{e_2}, X^{e_3}\}$ by the product $\mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{X}^{e_3} / 3!$.

These structures arise naturally in the study of classical and quantum superintegrable systems in two dimensions, e.g., [23, 22], and, in the case of zero potential systems, they are examples of Poisson structures, on which there is a considerable literature [4, 6, 20]. A quantum 2D superintegrable system is an integrable Hamiltonian system on a 2-dimensional real or complex Riemannian manifold with potential: $H = \Delta_2 + V$, that admits 3 algebraically independent partial differential operators commuting with H , the maximum possible:

$$[H, L_j] = 0, \quad L_3 = H, \quad j = 1, 2, 3.$$

Here Δ_2 is the Laplace operator on the manifold. (We call this a *Helmholtz superintegrable system* with eigenvalue equation $H\Psi = E\Psi$ to distinguish it from a Laplace conformally superintegrable system, $H\Psi = (\Delta_2 + V)\Psi = 0$ [18].) A system is of order K if the maximum order of the symmetry operators L_j (other than H) is K ; all such systems are known for $K = 1, 2$ [3, 11, 14, 15]. Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\Psi = E\Psi$ to be solved exactly, analytically and algebraically. A classical 2D superintegrable system is an integrable Hamiltonian system on a real or complex 2-dimensional Riemannian manifold with potential: $\mathcal{H} = \sum_{j,k=1}^2 g^{jk}(\mathbf{x}) p_j p_k + V(\mathbf{x})$ in

local coordinates x_1, x_2, p_1, p_2 that admit 3 functionally independent phase space functions $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ in involution with \mathcal{H} , the maximum possible.

$$\{\mathcal{H}, \mathcal{L}_j\} = 0, \quad \mathcal{L}_3 = \mathcal{H}, \quad j = 1, 2, 3.$$

A system is of order K if the maximum order of the constants of the motion $L_j, j \neq 3$, as polynomials in p_1, p_2 is K . Again all such systems are known for $K = 1, 2$, and, for them, there is a 1-1 relationship between classical and quantum 2nd order 2D superintegrable systems [13], i.e., the quantum system can be computed from the classical system, and vice versa.

The possible superintegrable systems divide into six classes:

1. First order systems. These are the (zero-potential) Laplace–Beltrami eigenvalue equations on constant curvature spaces. The symmetry algebras close under commutation to form the Lie algebras $\mathfrak{e}(2, \mathbb{R}), \mathfrak{e}(1, 1), \mathfrak{o}(3, \mathbb{R})$ or $\mathfrak{o}(2, 1)$. Such systems have been studied in detail, using group theory methods.

2. Free triplets. These are superintegrable systems with zero potential and all generators of 2nd order. The possible spaces for which these systems can occur were classified by Koenigs (1896). They are: constant curvature spaces, the four Darboux spaces, and eleven 4-parameter Koenigs spaces [19]. In most cases the symmetry operators will not generate a quadratic algebra, i.e., the algebra will not close. If the system generates a nondegenerate quadratic algebra we call it a *free quadratic triplet*.
3. Nondegenerate systems. These are superintegrable systems with a non-zero potential and the generating symmetries are all of 2nd order. The space of potentials is 4-dimensional:

$$V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2 V_{(2)}(\mathbf{x}) + a_3 V_{(3)}(\mathbf{x}) + a_4.$$

The symmetry operators generate a nondegenerate quadratic algebra with parameters a_j .

4. Degenerate systems. There are 4 generators: one 1st order X and 3 second order H , L_1 , L_2 . Here, X^2 is not contained in the span of H , L_1 , L_2 . The space of potentials is 2-dimensional: $V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2$. The symmetry operators generate a degenerate quadratic algebra with parameters a_j . Relation (1.3) is an expression of the fact that 4 symmetry operators cannot be algebraically independent. The possible degenerate systems, classified up to conjugacy with respect to the symmetry groups of their underlying spaces, are listed in Appendix A.
5. Exceptional system. $E15$: $V = f(x - iy)$, f an arbitrary function.

The exceptional case is characterized by the fact that the symmetry generators are functionally linearly dependent [10, 12, 13, 15]. This is the only 2nd order functionally linearly dependent 2D system but there are many such systems in 3D, including the Calogero 3-body system on the line. In 3D such systems have not yet been classified.

Every degenerate superintegrable system occurs as a restriction of the 3-parameter potentials to 1-parameter ones, such that one of the symmetries becomes a perfect square: $L = X^2$. Here X is a first order symmetry and a new 2nd order symmetry appears so that this restriction admits more symmetries than the original system, see Remark A.1. Basic results that relate these superintegrable systems are the closure theorems:

Theorem 1.1. *A free triplet, classical or quantum, extends to a superintegrable system with potential if and only if it generates a free quadratic algebra \tilde{Q} , degenerate or nondegenerate.*

Theorem 1.2. *A superintegrable system, degenerate or nondegenerate, classical or quantum, with quadratic algebra Q , is uniquely determined by its free quadratic algebra \tilde{Q} .*

These theorems were proved for systems in [16]. The proofs are constructive: Given a free quadratic algebra \tilde{Q} one can compute the potential V and the symmetries of the quadratic algebra Q . Thus as far as superintegrable systems are concerned, all information about the systems is contained in the free classical quadratic algebras.

Remark 1.3. This paper is a companion to [5] where we studied nondegenerate quadratic algebras, and we assume that the reader has some familiarity with this prior work. In particular, Bôcher contractions, their properties and associated notation, are treated there and we use them in this paper without detailed comment.

The layout of this paper is as follows: In Section 2 we show how degenerate Helmholtz superintegrable systems can be split into Stäckel equivalence classes of Laplace conformally superintegrable systems and we determine how each Helmholtz system can be characterized in its equivalence class. In Section 3 we determine all Bôcher, i.e., geometrical, contractions of the Laplace systems and obtain complete lists of the possible Helmholtz contractions. In Section 4

we classify all abstract free quadratic algebras and determine which of these can be realized as the quadratic algebra of a Helmholtz degenerate superintegrable system. In Section 5 we classify all abstract contractions of abstract free quadratic algebras and determine which of these can be realized as Bôcher and Heisenberg contractions of the quadratic algebras of Helmholtz degenerate superintegrable systems. In Fig. 2 we describe how restrictions of nondegenerate superintegrable systems to degenerate ones, and contractions of degenerate superintegrable systems account for the lower half of the Askey scheme. The upper half of the Askey scheme is described by contractions of nondegenerate systems [5]. In Section 6 we assess our results. A list of all Helmholtz degenerate superintegrable systems can be found in Appendix A.

2 Stäckel transforms and Laplace equations

Distinct degenerate classical or quantum superintegrable systems can be mapped to one another by Stäckel transforms, invertible transforms that preserve the structure of the quadratic algebra. This divides the 15 systems into 6 Stäckel equivalence classes [22]. The most convenient way to understand the equivalence classes is in terms of Laplace-like equations [18]. Since every 2D space is conformally flat there always exist “Cartesian-like” coordinates x, y such that the Hamilton–Jacobi equation can be expressed in the form $H = E$ where $H = \frac{p_x^2 + p_y^2}{\lambda(x,y)} + \alpha V$ and α is a parameter. This is equivalent to the Laplace-like equation $p_x^2 + p_y^2 + a_1 V_1 + a_2 V_2 = 0$ where $V_1 = \lambda V$, $V_2 = \lambda$, $a_1 = \alpha$, $a_2 = -E$, now with 2 parameters. Symmetries (constants of the motion) for the Helmholtz equation correspond to conformal symmetries of the Laplace equation. The Hamilton–Jacobi equation is defined on one of a variety of conformally flat spaces but the Laplace equation is always defined on flat space with conformal symmetry algebra $\mathfrak{so}(4, \mathbb{C})$ [18]. An important observation is that the Laplace equations are Stäckel equivalence classes: two Helmholtz systems are Stäckel equivalent if and only if they correspond to the same Laplace equation.

Remark 2.1. Indeed, If the Laplace conformally superintegrable equation can be split in the form $p_x^2 + p_y^2 + V_0 - \tilde{E}W = 0$, where \tilde{E} is an arbitrary parameter, W is a nonconstant function, and V_0, W are independent of E , then W , by division, defines a conformal Stäckel transform to the superintegrable Helmholtz system $\tilde{H} = \frac{1}{W}(p_x^2 + p_y^2 + V_0) = \tilde{E}$. If the Laplace system admits another splitting $p_x^2 + p_y^2 + V'_0 - \tilde{E}'W' = 0$, it determines another superintegrable Helmholtz system $\tilde{H}' = \frac{1}{W'}(p_x^2 + p_y^2 + V'_0) = \tilde{E}'$ and \tilde{H}' can be obtained from \tilde{H} by an invertible Stäckel transform $\frac{W'}{W}$. Thus all Helmholtz systems that can be obtained from the Laplace equation by splitting the potential are Stäckel equivalent to one another.

The Laplace equations for nondegenerate systems were derived in [18], see Table 1. The Laplace equations for degenerate systems are listed in Table 2. The notation a_i in Table 2 describes how these systems can be obtained as restrictions of systems in Table 1, but with added symmetry.

The Helmholtz systems corresponding to each Laplace system are:

Stäckel equivalence classes: Here the notation refers to the Helmholtz superintegrable systems listed in the Appendix.

1. Class $A (a_3, a_4)$: System $S3$ corresponds to $(1, 0)$ and $(0, 1)$. System $S6$ corresponds to $(1, 1)$. System $D4(b)D$ corresponds to (a_3, a_4) with $a_3 a_4 (a_3 - a_4) \neq 0$.
2. Class $B (a_1, a_4)$: System $S5$ corresponds to $(1, 0)$. System $E6$ corresponds to $(0, 1)$. System $D2D$ corresponds to (a_1, a_4) with $a_1 a_4 \neq 0$.
3. Class $C (a_3, a_4)$: System $E3$ corresponds to $(1, 0)$. System $E18$ corresponds to $(0, 1)$. System $D3E$ corresponds to (a_3, a_4) with $a_3 a_4 \neq 0$.

System	Non-degenerate potentials $V(x, y)$
[1, 1, 1, 1]	$\frac{a_1}{x^2} + \frac{a_2}{y^2} + \frac{4a_3}{(x^2+y^2-1)^2} - \frac{4a_4}{(x^2+y^2+1)^2}$
[2, 1, 1]	$\frac{a_1}{x^2} + \frac{a_2}{y^2} - a_3(x^2 + y^2) + a_4$
[2, 2]	$\frac{a_1}{(x+iy)^2} + \frac{a_2(x-iy)}{(x+iy)^3} + a_3 - a_4(x^2 + y^2)$
[3, 1]	$a_1 - a_2x + a_3(4x^2 + y^2) + \frac{a_4}{y^2}$
[4]	$a_1 - a_2(x + iy) + a_3(3(x + iy)^2 + 2(x - iy)) - a_4(4(x^2 + y^2) + 2(x + iy)^3)$
[0]	$a_1 - (a_2x + a_3y) + a_4(x^2 + y^2)$
(1)	$\frac{a_1}{(x+iy)^2} + a_2 - \frac{a_3}{(x+iy)^3} + \frac{a_4}{(x+iy)^4}$
(2)	$a_1 + a_2(x + iy) + a_3(x + iy)^2 + a_4(x + iy)^3$

Table 1. Four parameter Laplace systems.

System	Degenerate potentials $V(x, y)$
<i>A</i>	$\frac{4a_3}{(x^2+y^2-1)^2} - \frac{4a_4}{(x^2+y^2+1)^2}$
<i>B</i>	$\frac{a_1}{x^2} + a_4$
<i>C</i>	$a_3 - a_4(x^2 + y^2)$
<i>D</i>	$a_1 - a_2x$
<i>E</i>	$\frac{a_1}{(x+iy)^2} + a_3$
<i>F</i>	$a_1 - a_2(x + iy)$

Table 2. Two-parameter Laplace systems.

4. Class *D* (a_1, a_2): System *E5* corresponds to $(1, 0)$. System *D1D* corresponds to (a_1, a_2) with $a_1a_2 \neq 0$.
5. Class *E* (a_1, a_3): System *E14* corresponds to $(0, 1)$. System *E12* corresponds to (a_1, a_3) with $a_1a_3 \neq 0$.
6. Class *F* (a_1, a_2): System *E13* corresponds to (a_1, a_2) with $a_2 \neq 0$. System *E4* corresponds to $(1, 0)$.

Here, for example, system *D3E* belongs to class *C* and is obtained from the Laplace equation by dividing it by $a_3 - a_4(x^2 + y^2)$ where $a_3a_4 \neq 0$, whereas *E18* is obtained by the same division with $a_3 = 0, a_4 = 1$.

The conformal symmetry of these Laplace equations is best exploited by using tetraspherical coordinates to linearize the action of the conformal symmetry group [1, 18]. These are projective coordinates x_1, x_2, x_3, x_4 on the null cone $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$, related to flat space coordinates x, y by

$$\begin{aligned}
 x_1 &= 2XT, & x_2 &= 2YT, & x_3 &= X^2 + Y^2 - T^2, & x_4 &= i(X^2 + Y^2 + T^2), \\
 x &= \frac{X}{T} = -\frac{x_1}{x_3 + ix_4}, & y &= \frac{Y}{T} = -\frac{x_2}{x_3 + ix_4}, & x &= \frac{s_1}{1 + s_3}, & y &= \frac{s_2}{1 + s_3}, \\
 \mathcal{H} &\equiv p_x^2 + p_y^2 + V = (x_3 + ix_4)^2 \left(\sum_{k=1}^4 p_{x_k}^2 + V_B \right) = (1 + s_3)^2 \left(\sum_{j=1}^3 p_{s_j}^2 + V_S \right), \\
 V &= (x_3 + ix_4)^2 V_B, & (1 + s_3) &= -i \frac{(x_3 + ix_4)}{x_4}, & s_1 &= \frac{ix_1}{x_4}, & s_2 &= \frac{ix_2}{x_4}, & s_3 &= \frac{-ix_3}{x_4}.
 \end{aligned}$$

Thus the Laplace equation $\mathcal{H} \equiv p_x^2 + p_y^2 + V = 0$ in Cartesian coordinates becomes $\sum_{k=1}^4 p_{x_k}^2 + V_B = 0$ in tetraspherical coordinates. Here, the s_j refer to coordinates on the unit 2-sphere: $s_1^2 + s_2^2 + s_3^2 = 1$,

The possible limits of one superintegrable system to another can be derived and classified by using tetraspherical coordinates and special Bôcher contractions of $\mathfrak{so}(4, \mathbb{C})$ to itself. The method is described in detail in [5, 18]. Here we just recall the basic definition of a Bôcher contraction, Let $\mathbf{x} = \mathbf{A}(\epsilon)\mathbf{y}$, and $\mathbf{x} = (x_1, \dots, x_4)$, $\mathbf{y} = (y_1, \dots, y_4)$ be column vectors, and $\mathbf{A} = (A_{jk}(\epsilon))$ be a 4×4 matrix with matrix elements

$$A_{kj}(\epsilon) = \sum_{\ell=-N}^N a_{kj}^{\ell} \epsilon^{\ell},$$

where N is a nonnegative integer and the a_{kj}^{ℓ} are complex constants. (Here, N can be arbitrarily large, but it must be finite in any particular case.) We say that the matrix \mathbf{A} defines a *Bôcher contraction* of the conformal algebra $\mathfrak{so}(4, \mathbb{C})$ to itself provided

- 1) $\det(\mathbf{A}) = \pm 1$, constant for all $\epsilon \neq 0$,
- 2) $\mathbf{x} \cdot \mathbf{x} \equiv \sum_{j=1}^4 x_j(\epsilon)^2 = \mathbf{y} \cdot \mathbf{y} + O(\epsilon)$.

If, in addition, $\mathbf{A} \in O(4, \mathbb{C})$ for all $\epsilon \neq 0$ the matrix \mathbf{A} defines a *special Bôcher contraction*. For a special Bôcher contraction $\mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y}$, with no error term. (These contractions correspond to limit relations introduced by Bôcher to obtain all orthogonal separable coordinates for Laplace and wave equations as limits of cyclidic coordinates. There is an infinite family of such contractions, but they can be generated by 4 basic contractions.) Related contraction methods that don't make use of tetraspherical coordinates directly can be found in references such as [8, 9].

Bôcher contractions take a Laplace system to itself. The contraction process has already been described in [5, 18] and references therein, but we discuss, briefly, the main ideas. Suppose we have a degenerate Laplace superintegrable system with potential $V(\mathbf{x}, \mathbf{a}) = a_1 V_1(\mathbf{x}) + a_2 V_2(\mathbf{x})$ and generating conformal symmetries $X = \mathcal{X} + W_0$, $L_1 = \mathcal{L}_1 + W_1$, $L_2 = \mathcal{L}_2 + W_2$, where \mathcal{X} , \mathcal{L}_1 , \mathcal{L}_2 are free 2nd order conformal symmetries and W_0 , W_1 , W_2 are functions of the tetraspherical coordinates x_i . Applying a Bôcher contraction $A(\epsilon)$ to the free symmetries we obtain

$$\mathcal{X}(\epsilon) = \epsilon^{\alpha_0} \mathcal{X}' + O(\epsilon), \quad \mathcal{L}_j(\epsilon) = \epsilon^{\alpha_j} \mathcal{L}'_j + O(\epsilon), \quad j = 1, 2,$$

where $\mathcal{X}' \in \mathfrak{so}(4, \mathbb{C})$ and the \mathcal{L}'_j are quadratic in $\mathfrak{so}(4, \mathbb{C})$. By a change of basis $\{L_1, L_2\}$ if necessary, one can verify that \mathcal{X}' , \mathcal{L}'_j , $j = 1, 2$ generate a free conformal quadratic algebra. The action of the Bôcher contraction on the 2-dimensional potential space preserves its dimension and maps it smoothly as a function of ϵ , as follows from an examination of the Bertrand-Darboux equations. Thus we get a 2-dimensional potential space in the limit. To find an explicit basis for the contracted potential $V'(\mathbf{b}, \mathbf{y}) = b_1 V'_1(\mathbf{y})' + b_2 V'_2(\mathbf{y})'$ we put $a_1 = \sum_{k=-\infty}^{\infty} c_k \epsilon^k$, $a_2 = \sum_{k=-\infty}^{\infty} d_k \epsilon^k$, where only a finite number of the coefficients c_k , d_k can be nonzero. Then it is a linear algebra problem to determine the c_k , d_k such that $\lim_{\epsilon \rightarrow 0} V(\mathbf{x}(\epsilon), \mathbf{a}(\epsilon)) = V'(\mathbf{b}, \mathbf{y})$ exists for independent potential functions $V'_1(\mathbf{y})'$, $V'_2(\mathbf{y})'$, where the nonzero c_k , d_k are linear in b_1 , b_2 . The limit is guaranteed to exist and is unique up to a change of basis $\{V'_1(\mathbf{y})', V'_2(\mathbf{y})'\}$ for the target potential. Only the last limit and the linear algebra problem need to be solved to identify the contraction. This work was carried out with the assistance of the symbol manipulation programs Maple and Mathematica. There is one additional complication; the results of the contraction

are not invariant under a permutation of the indices of the hyperspherical coordinates defining the contraction matrix A_{ij} . Thus one Bôcher contraction applied to a source system can yield a multiplicity of target results, and all permutations need to be examined.

The results are rather complicated. Fig. 1 provides a clearer idea of what is happening. There are 4 basic Bôcher contractions of 2D Laplace systems and each one when applied to a Laplace system, and each permutation treated, yields another Laplace superintegrable system. A system in class K_1 can be obtained from a system in class K_2 via contraction provided there is a directed arrow path from K_2 to K_1 . All systems follow from A for degenerate potentials, and A is a restriction of [1111] with increased symmetry. Fig. 1 describes only the existence or nonexistence of contractions, not the multiplicity of distinct contractions.

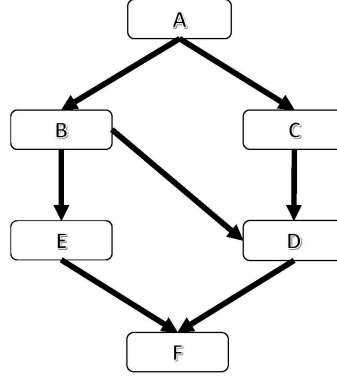


Figure 1. Contraction relations for degenerate Laplace systems.

Our basic interest, however, is in Helmholtz contractions, i.e., contractions of a Helmholtz superintegrable system to another such system. The key is to start with a Laplace system, take a conformal Stäckel transform to a Helmholtz system (which we initially interpret as another Laplace system) and then take a Bôcher contraction of the new system, which as described below gives a new Helmholtz system. The result is the contraction of one Helmholtz system to another. This can be done in such a way the “diagrams commute”, i.e., a Helmholtz contraction is induced by a Bôcher contraction and a Stäckel transform [18]. For example, let \mathcal{H} be the Hamiltonian for class A . In terms of tetraspherical coordinates a general conformal Stäckel transformed potential will take the form

$$V = \frac{\frac{a_3}{x_3^2} + \frac{a_4}{x_4^2}}{\frac{b_3}{x_3^2} + \frac{b_4}{x_4^2}} = \frac{V_A}{F(\mathbf{x}, \mathbf{b})},$$

where $F(\mathbf{x}, \mathbf{b}) = \frac{b_3}{x_3^2} + \frac{b_4}{x_4^2}$, and the transformed Hamiltonian will be $\hat{\mathcal{H}} = \frac{1}{F(\mathbf{x}, \mathbf{b})} \mathcal{H}$, where the transform is determined by the fixed vector (b_3, b_4) . Now we apply the Bôcher contraction $[1, 1, 1, 1] \rightarrow [2, 1, 1]$ to this system. Depending on the permutation of the indices x_j , in the limit as $\epsilon \rightarrow 0$ the potential $V_A \rightarrow V_B$, or $V_A \rightarrow V_C$, and $\mathcal{H} \rightarrow \mathcal{H}'$, the B or C system. Now consider $F(\mathbf{x}(\epsilon), \mathbf{b}) = V'(\mathbf{x}', \mathbf{b})\epsilon^\alpha + O(\epsilon^{\alpha+1})$, where the integer exponent α depends upon our choice of \mathbf{b} . From our theory, the system defined by Hamiltonian $\hat{\mathcal{H}}' = \lim_{\epsilon \rightarrow 0} \epsilon^\alpha \hat{\mathcal{H}}(\epsilon) = \frac{1}{V'(\mathbf{x}', \mathbf{b})} \mathcal{H}'$ is a superintegrable system that arises from the system A by a conformal Stäckel transform induced by the potential $V'(\mathbf{x}', \mathbf{b})$. Thus the Helmholtz superintegrable system with potential $V = V_A/F$ contracts to the Helmholtz superintegrable system with potential V_S/V' , where $S = B$ or $S = C$. The contraction is induced by a generalized Inönü–Wigner Lie algebra contraction of the conformal algebra $\mathfrak{so}(4, \mathbb{C})$. Always the V' can be identified with a specialization of the S potential. Thus a conformal Stäckel transform of A has been contracted to a conformal Stäckel transform of S .

3 Degenerate Helmholtz contractions

The superscript for each targeted Helmholtz system is the value of the exponent α associated with the contraction. In each table below, corresponding to a single Laplace equation equivalence class, the top line is a list of the Helmholtz systems in the class, and the lower lines are the target systems under the Bôcher contraction.

	<i>A</i>	equivalence	class	contractions
contraction	<i>S3</i>	<i>S6</i>	<i>D4D</i>	
[1111] ↓ [211]	<i>E3</i> ² <i>S5</i> ⁰ , <i>E6</i> ²	<i>E18</i> ⁴ <i>S5</i> ⁰	<i>E3</i> ² <i>S5</i> ⁰	
[1111] ↓ [22]	<i>E3</i> ² <i>E14</i> ²	<i>E18</i> ⁴ <i>E12</i> ²	<i>E3</i> ² <i>E12</i> ²	
[1111] ↓ [31]	<i>E5</i> ² <i>E6</i> ² , <i>S5</i> ⁰	<i>E5</i> ² <i>S5</i> ⁰	<i>E5</i> ² , <i>D1D</i> ³ <i>S5</i> ⁰	
[1111] ↓ [4]	<i>E4</i> ⁶	<i>E4</i> ⁶	<i>E4</i> ⁶ , <i>E13</i> ⁸	
	<i>B</i>	equivalence	class	contractions
contraction	<i>S5</i>	<i>E6</i>	<i>D2D</i>	
[1111] ↓ [211]	<i>E14</i> ² <i>S5</i> ⁰	<i>E14</i> ⁰ <i>E6</i> ²	<i>E14</i> ⁰ <i>S5</i> ⁰	
[1111] ↓ [22]	<i>E14</i> ²	<i>E14</i> ²	<i>E12</i> ²	
[1111] ↓ [31]	<i>E5</i> ² <i>S5</i> ⁰	<i>E5</i> ² <i>E6</i> ²	<i>E5</i> ² , <i>D1D</i> ³ <i>S5</i> ⁰ , <i>E6</i> ²	
[1111] ↓ [4]	<i>E4</i> ⁶	<i>E4</i> ⁶	<i>E4</i> ⁶ , <i>E13</i> ⁸	
	<i>C</i>	equivalence	class	contractions
contraction	<i>E3</i>	<i>E18</i>	<i>D3E</i>	
[1111] ↓ [211]	<i>E5</i> ² <i>E3</i> ²	<i>E5</i> ² <i>E18</i> ⁴	<i>E5</i> ^{2,3} <i>E3</i> ²	
[1111] ↓ [22]	<i>E3</i> ²	<i>E18</i> ^{2,4}	<i>E3</i> ² , <i>D3E</i> ²	
[1111] ↓ [31]	<i>E5</i> ²	<i>E5</i> ²	<i>E5</i> ² , <i>D1D</i> ³	
[1111] ↓ [4]	<i>E4</i> ⁶	<i>E4</i> ⁶	<i>E4</i> ⁶ , <i>E13</i> ⁸	
	<i>D</i>	equivalence	class	contractions
contraction	<i>E5</i>	<i>D1D</i>		
[1111] ↓ [211]	<i>E4</i> ⁰ <i>E5</i> ²	<i>E13</i> ⁻¹ , <i>E4</i> ⁰ <i>E5</i> ² , <i>D1D</i> ³		
[1111] ↓ [22]	<i>E4</i> ²	<i>E4</i> ² , <i>E13</i> ²		
[1111] ↓ [31]	<i>E5</i> ²	<i>E5</i> ² , <i>D1D</i> ²		
[1111] ↓ [4]	<i>E4</i> ⁶	<i>E4</i> ⁶ , <i>E13</i> ⁸		
	<i>E</i>	equivalence	class	contractions
contraction	<i>E14</i>	<i>E12</i>		
[1111] ↓ [211]	<i>E4</i> ² <i>E14</i> ⁰	<i>E4</i> ² , <i>E13</i> ³ <i>E14</i> ⁰		
[1111] ↓ [22]	<i>E4</i> ² <i>E14</i> ²	<i>E4</i> ² , <i>E13</i> ⁴ <i>E12</i> ²		
[1111] ↓ [31]	<i>E4</i> ²	<i>E4</i> ² , <i>E13</i> ³		
[1111] ↓ [4]	<i>E4</i> ⁶	<i>E4</i> ⁶ , <i>E13</i> ⁸		

	F	equivalence	class	contractions
contraction	$E4$	$E13$		
$[1111] \downarrow [211]$	$E4^{0,2}$	$E13^{-1,3}$	$E4^2$	
$[1111] \downarrow [22]$	$E4^2$	$E13^2$		
$[1111] \downarrow [31]$	$E4^2$	$E4^2$		
$[1111] \downarrow [4]$	$E4^6$	$E4^6$		

4 Classification of free abstract degenerate classical quadratic algebras

4.1 Abstract quadratic algebras

The special case of a 2D degenerate classical superintegrable system on a constant curvature space or a Darboux space with all parameters equal to zero (no potential) gives rise to a special kind of quadratic algebra which we call a free abstract quadratic algebra. Below is a precise definition.

Definition 4.1. An abstract 2D free degenerate (classical) quadratic algebra is a complex Poisson algebra possessing a linearly independent generating set $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 = \mathcal{H}, \mathcal{X}\}$ that satisfy:

1. The associative product is abelian.
2. \mathcal{A} is graded: $\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}_k$ where each \mathcal{A}_k is a complex vector space, the associative product takes $\mathcal{A}_k \times \mathcal{A}_l$ into \mathcal{A}_{k+l} , and the Poisson brackets $\{--, --\}$ goes from $\mathcal{A}_k \times \mathcal{A}_l$ into \mathcal{A}_{k+l-1} .
3. $\mathcal{A}_0 = \mathbb{C}$.
4. $\mathcal{A}_1 = \mathbb{C}\mathcal{X}$.
5. $\mathcal{A}_2 = \text{span}\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{X}^2\}$.
6. The elements $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4 := \mathcal{X}^2\}$ satisfy a relation given by a homogeneous polynomial of degree 2, $G(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4) = 0$.
7. Any nonzero polynomial F of minimal degree such that $F(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4) = 0$ is a multiple of G .
8. The center of \mathcal{A} contains no elements of order 1 and any element of order two in the center is a multiple of $\mathcal{L}_3 = \mathcal{H}$.
9. The polynomial G depends non-trivially on at least one of the non-central 2nd order generators $\mathcal{L}_1, \mathcal{L}_2$.

We shall simply refer to such an algebra as a *free abstract quadratic algebra*.

Remark 4.2. A free abstract quadratic algebra is a special case of a three-dimensional affine Poisson variety, see, e.g., [4, 20]. Description of some of the properties of such Poisson varieties, along with classification of such Poisson structures on the affine space \mathbb{C}^3 can be found in [20, Section 9.2]. Poisson structures on the manifold \mathbb{R}^3 were classified at [6]. It should be noted that the Poisson algebras considered in this paper, are not quadratic Poisson structures in the sense of [20, Section 8.2]. In addition our classification scheme is different from the above mentioned classifications.

Theorem 4.3. *Keeping the same notations as of Definition 4.1, let \mathcal{A} be a free abstract quadratic algebra. Then there exists a non zero $K \in \mathbb{C}$ such that:*

$$\{\mathcal{L}_1, \mathcal{L}_2\} = K \frac{\partial G}{\partial \mathcal{X}}, \quad \{\mathcal{X}, \mathcal{L}_1\} = K \frac{\partial G}{\partial \mathcal{L}_2}, \quad \{\mathcal{X}, \mathcal{L}_2\} = -K \frac{\partial G}{\partial \mathcal{L}_1}. \quad (4.1)$$

(We shall refer to equations (4.1) as the structure equations of \mathcal{A}).

Proof. Note that $G(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4) = 0$ implies that

$$0 = \{\mathcal{X}, G\} = \frac{\partial G}{\partial \mathcal{L}_1} \{\mathcal{X}, \mathcal{L}_1\} + \frac{\partial G}{\partial \mathcal{L}_2} \{\mathcal{X}, \mathcal{L}_2\}.$$

The above equation is a polynomial expression in $\chi^2, \mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$. Any one of the terms $\frac{\partial G}{\partial \mathcal{L}_1}, \{\mathcal{X}, \mathcal{L}_1\}, \frac{\partial G}{\partial \mathcal{L}_2}, \{\mathcal{X}, \mathcal{L}_2\}$ is either zero or a polynomial of degree one in the variables $\chi^2, \mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$. Since at least one of the terms $\frac{\partial G}{\partial \mathcal{L}_1}, \frac{\partial G}{\partial \mathcal{L}_2}$ is non-zero then we must have

$$\{\mathcal{X}, \mathcal{L}_1\} = K \frac{\partial G}{\partial \mathcal{L}_2}, \quad \{\mathcal{X}, \mathcal{L}_2\} = -K \frac{\partial G}{\partial \mathcal{L}_1},$$

for some K . If $K = 0$ then the first order element \mathcal{X} must be in the center of \mathcal{A} which is impossible. Hence $K \neq 0$. Similarly $G(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4) = 0$ implies

$$0 = \{\mathcal{L}_1, G\} = \frac{\partial G}{\partial \mathcal{X}} \{\mathcal{L}_1, \mathcal{X}\} + \frac{\partial G}{\partial \mathcal{L}_2} \{\mathcal{L}_1, \mathcal{L}_2\},$$

$$0 = \{\mathcal{L}_2, G\} = \frac{\partial G}{\partial \mathcal{X}} \{\mathcal{L}_2, \mathcal{X}\} + \frac{\partial G}{\partial \mathcal{L}_1} \{\mathcal{L}_2, \mathcal{L}_1\},$$

and along with our assumption that at least one of the terms $\frac{\partial G}{\partial \mathcal{L}_1}, \frac{\partial G}{\partial \mathcal{L}_2}$ is non-zero we see that $\{\mathcal{L}_1, \mathcal{L}_2\} = K \frac{\partial G}{\partial \mathcal{X}}$. ■

Corollary 4.4. *Keeping the same notations as of Definition 4.1, let \mathcal{A} be a free abstract quadratic algebra. Then $G(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4)$ depends non-trivially in at least two of the three generators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_4$.*

Proof. It easily follows from the structure equations (4.1) that if $G(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4)$ depends on at most one of the generators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_4$ then the center of \mathcal{A} contains a second order generator which is not a multiple of \mathcal{H} . ■

Free abstract quadratic algebras appear as symmetry algebras of 2D degenerate free superintegrable systems. That is the main motivation for their study. For a given G there is a 1-parameter family of quadratic algebras, parametrized by K . However, these algebras are isomorphic. It follows that the particular choice of nonzero K in the classification theory to follow is immaterial.

Note 4.5. If phase space generators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}, \mathcal{X}$ satisfy a 4th order (in the momentum variables) relation $G = 0$ but the closure relations for $\{\mathcal{X}, \mathcal{L}_1\}, \{\mathcal{X}, \mathcal{L}_2\}, \{\mathcal{L}_1, \mathcal{L}_2\}$, are not satisfied, then K is rational in the generators.

For quantum degenerate systems, knowledge of the Casimir relation $G(L_1, L_2, H, X, \alpha) = 0$ is sufficient to determine the quadratic algebra, subject to the same conditions as the classical case. Details can be found in [7].

4.2 The symmetry group of a free abstract quadratic algebra

In this section we determine the symmetry group of a free abstract quadratic algebra. Note that once we fixed a generating set $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{X}\}$, the Casimir G is given by a symmetric 4×4 matrix $B(G)$ defined by

$$G(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4) = \sum_{i,j=1}^4 B(G)_{ij} \mathcal{L}_i \mathcal{L}_j,$$

The pair $(B(G), K)$ (consisting of the Casimir and the multiplicative constant in the structure equations) is defined only up to a constant. That is, the pair $(zB(G), z^{-1}K)$ for any nonzero z can also serve as a Casimir and a structure constant for the same quadratic algebra and the same basis. A free abstract quadratic algebra is completely determined by a generating set $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{X}\}$ (satisfying the conditions of Definition 4.1) together with the pair $(B(G), K)$. As noted previously, the constant K can always be normalized to 1. Given two sets of generators $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{X}\}$, $\{\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2, \tilde{\mathcal{L}}_3, \tilde{\mathcal{X}}\}$ of the same algebra \mathcal{A} such that:

1. $2 = \deg(\mathcal{L}_1) = \deg(\tilde{\mathcal{L}}_1) = \deg(\mathcal{L}_2) = \deg(\tilde{\mathcal{L}}_2) = \deg(\mathcal{L}_3) = \deg(\tilde{\mathcal{L}}_3)$.
2. $1 = \deg(\mathcal{X}) = \deg(\tilde{\mathcal{X}})$.
3. \mathcal{L}_3 and $\tilde{\mathcal{L}}_3$ are in the center of \mathcal{A} .

Then there is a 'change of basis matrix' A of the form

$$A := \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & 0 \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & 0 \\ 0 & 0 & A_{3,3} & 0 & 0 \\ 0 & 0 & 0 & A_{4,4} & 0 \\ 0 & 0 & 0 & 0 & A_{5,5} \end{pmatrix} \in \text{GL}(5, \mathbb{C}), \quad (4.2)$$

with $A_{4,4} = A_{5,5}^2$ such that for $\mathcal{L} := (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{X})^t$, $\tilde{\mathcal{L}} := (\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2, \tilde{\mathcal{L}}_3, \tilde{\mathcal{L}}_4, \tilde{\mathcal{X}})^t$

$$\mathcal{L} = A\tilde{\mathcal{L}}. \quad (4.3)$$

For A as above we define

$$\hat{A} = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ 0 & 0 & A_{3,3} & 0 \\ 0 & 0 & 0 & A_{4,4} \end{pmatrix}.$$

Besides linear change of basis we can also rescale the invariants $(B(G), K)$. The full symmetry group of a free abstract quadratic algebra is given by the collection of pairs (A, z) , A as in equation (4.2) and $z \in \mathbb{C}^*$. We shall denote this group by G_{degn} . The action of such $(A, z) \in G_{\text{degn}}$ on the generators is given by equation (4.3) and on (G, K) by

$$\begin{aligned} B(G') &= z\hat{A}^t B(G) \hat{A}, \\ K' &= z^{-1} K (A_{1,1} A_{2,2} - A_{1,2} A_{2,1})^{-1} A_{5,5}^{-1}, \end{aligned}$$

where $B(G)$ is the symmetric matrix representing G with respect to $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4\}$, and $B(G')$ is the symmetric matrix representing $G' := (A, z) \cdot G$ with respect to $\{\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2, \tilde{\mathcal{L}}_3, \tilde{\mathcal{L}}_4\}$. We shall call the subgroup of G_{degn} of all pairs of the form $(A, 1)$ as the group of linear change of bases, we denote it by $G_{\text{degn}}^{\text{lin}}$. In the following we shall determine a canonical form for the pair

$(B(G), K)$ of each abstract quadratic algebra. This means that we shall decide upon a unique representative from each orbit of G_{degn} in the space of all possible pairs $(B(G), K)$ that arise from an abstract quadratic algebra. Without loss of generality we can assume that $K = 1$ and determine representative from each orbit of the subgroup of G_{degn} that fixes the value of K . Explicitly this group is given by

$$G_{\text{degn}}^K := \{(A, z) \in G_{\text{degn}} \mid z = (A_{1,1}A_{2,2} - A_{1,2}A_{2,1})^{-1}A_{5,5}^{-1}\}.$$

4.3 The canonical form

In this section after we identify some invariants of orbits of $(B(G), K)$ under the action of G_{degn}^K we consider the reduction to a canonical form. That is, for each orbit we choose a representative which we call the canonical form $(B(G), K)$. For a given Casimir $(B(G), K)$ of a free abstract quadratic algebra \mathcal{A} and a given $(A, z) \in G_{\text{degn}}^K$ we introduce the following notations:

$$\widehat{A} = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ 0 & 0 & A_{3,3} & 0 \\ 0 & 0 & 0 & A_{4,4} \end{pmatrix} = \begin{pmatrix} r & s \\ 0 & t \end{pmatrix},$$

$$B(G) = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{1,2} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{1,3} & b_{2,3} & b_{3,3} & b_{3,4} \\ b_{1,4} & b_{2,4} & b_{3,4} & b_{4,4} \end{pmatrix} = \begin{pmatrix} b & c \\ c^t & d \end{pmatrix}.$$

Direct calculation shows that

$$\widehat{A}^t B(G) \widehat{A} = \begin{pmatrix} r^t b r & r^t (b s + c t) \\ s^t b r + t c^t r & s^t (b s + c t) + t (c^t s + d t) \end{pmatrix}.$$

Hence under the action G_{degn}^K the rank of $B(G)$ and the rank of its upper left 2 by 2 block (which we denote by b) are preserved. We use the theory of symmetric bilinear forms and find $(A, z) \in G_{\text{degn}}^K$ that cast $B(G)$ into exactly one of the following forms depending on $\text{rank}(b)$.

1. $B(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b_{3,3} & b_{3,4} \\ 0 & 0 & b_{3,4} & b_{4,4} \end{pmatrix}$ if $\text{rank}(b) = 2$,
2. $B(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & b_{2,3} & b_{2,4} \\ 0 & b_{2,3} & b_{3,3} & b_{3,4} \\ 0 & b_{2,4} & b_{3,4} & b_{4,4} \end{pmatrix}$ if $\text{rank}(b) = 1$,
3. $B(G) = \begin{pmatrix} 0 & 0 & b_{1,3} & b_{1,4} \\ 0 & 0 & b_{2,3} & b_{2,4} \\ b_{1,3} & b_{2,3} & b_{3,3} & b_{3,4} \\ b_{1,4} & b_{2,4} & b_{3,4} & b_{4,4} \end{pmatrix}$ if $\text{rank}(b) = 0$.

Below we analyze the different cases according to $\text{rank}(b)$ we summarize the results of all cases in Table 3.

4.4 The rank 2 case

If

$$B(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b_{3,3} & b_{3,4} \\ 0 & 0 & b_{4,3} & b_{4,4} \end{pmatrix} \quad \text{and} \quad (A, z) \cdot B(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \tilde{b}_{3,3} & \tilde{b}_{3,4} \\ 0 & 0 & \tilde{b}_{4,3} & \tilde{b}_{4,4} \end{pmatrix},$$

for some $(A, z) \in G_{\text{degn}}^{\text{K}}$ then

$$\hat{A} = \begin{pmatrix} A_{1,1} & A_{1,2} & 0 & 0 \\ A_{2,1} & A_{2,2} & 0 & 0 \\ 0 & 0 & A_{3,3} & 0 \\ 0 & 0 & 0 & A_{4,4} \end{pmatrix}, \quad z = (A_{1,1}A_{2,2} - A_{1,2}A_{2,1})^{-1}A_{5,5}^{-1},$$

with

$$\sqrt{z} \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \in \text{O}_2(\mathbb{C}), \quad A_{5,5}^{-1} = \pm 1, \quad \begin{pmatrix} \tilde{b}_{3,3} & \tilde{b}_{3,4} \\ \tilde{b}_{3,4} & \tilde{b}_{4,4} \end{pmatrix} = \begin{pmatrix} zA_{3,3}^2b_{3,3} & zA_{3,3}b_{3,4} \\ zA_{3,3}b_{3,4} & zb_{4,4} \end{pmatrix}.$$

So if $b_{3,3} \neq 0$ we define the canonical form of $B(G)$ to be

$$B(G)^{21}(b_{3,4}, b_{4,4}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_{3,4} \\ 0 & 0 & b_{3,4} & b_{4,4} \end{pmatrix},$$

with $b_{3,4} \in \{0, 1\}$, $b_{4,4} \in \mathcal{C}$. If $b_{3,3} = 0$ we define the canonical form of $B(G)$ to be

$$B(G)^{22}(b_{3,4}, b_{4,4}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b_{3,4} \\ 0 & 0 & b_{3,4} & b_{4,4} \end{pmatrix},$$

with $b_{3,4}, b_{4,4} \in \{0, 1\}$.

4.5 The rank 1 case

If

$$B(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & b_{2,3} & b_{2,4} \\ 0 & b_{2,3} & b_{3,3} & b_{3,4} \\ 0 & b_{2,4} & b_{3,4} & b_{4,4} \end{pmatrix} \quad \text{and} \quad (A, z) \cdot B(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \tilde{b}_{2,3} & \tilde{b}_{2,4} \\ 0 & \tilde{b}_{2,3} & \tilde{b}_{3,3} & \tilde{b}_{3,4} \\ 0 & \tilde{b}_{2,4} & \tilde{b}_{3,4} & \tilde{b}_{4,4} \end{pmatrix},$$

for some $(A, z) \in G_{\text{degn}}^{\text{K}}$ then

$$\hat{A} = \begin{pmatrix} \pm z^{-1/2} & 0 & \mp z^{1/2}b_{2,3}A_{3,3}A_{2,1} & \mp z^{-1/2}b_{2,4}A_{4,4}A_{2,1} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ 0 & 0 & A_{3,3} & 0 \\ 0 & 0 & 0 & A_{4,4} \end{pmatrix}, \quad z = A_{2,2}^{-2}A_{4,4}^{-1},$$

and

$$\tilde{b}_{2,3} = b_{2,3}A_{2,2}^{-1}A_{3,3}A_{4,4}^{-1}, \quad (4.4)$$

$$\tilde{b}_{2,4} = b_{2,4}A_{2,2}^{-1}, \quad (4.5)$$

$$\tilde{b}_{3,3} = b_{2,3}^2A_{2,1}^2A_{2,2}^{-4}A_{3,3}^2A_{4,4}^{-2} + 2b_{2,3}A_{2,3}A_{2,2}^{-2}A_{3,3}A_{4,4}^{-1} + b_{3,3}A_{2,2}^{-2}A_{3,3}^2A_{4,4}^{-1},$$

$$\tilde{b}_{3,4} = b_{2,3}b_{2,4}A_{2,1}^2A_{2,2}^{-4}A_{3,3}A_{4,4}^{-1} + b_{2,3}A_{2,4}A_{2,2}^{-2}A_{3,3}A_{4,4}^{-1} + b_{2,4}A_{2,3}A_{2,2}^{-2} + b_{3,4}A_{2,2}^{-2}A_{3,3},$$

$$\tilde{b}_{4,4} = b_{2,4}^2A_{2,1}^2A_{2,2}^{-4} + 2b_{2,4}A_{2,4}A_{2,2}^{-2} + b_{4,4}A_{2,2}^{-2}A_{4,4}.$$

From (4.4) we see that $\tilde{b}_{2,3} = 0$ if and only if $b_{2,3} = 0$, similarly from (4.5) we see that $\tilde{b}_{2,4} = 0$ if and only if $b_{2,4} = 0$. Hence we continue our analysis according to the vanishing of $b_{2,3}$ and $b_{2,4}$.

4.5.1 $\text{rank}(b) = 1$, $b_{2,3} = 0$ and $b_{2,4} = 0$

If

$$B(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_{3,3} & b_{3,4} \\ 0 & 0 & b_{3,4} & b_{4,4} \end{pmatrix} \quad \text{and} \quad (A, z) \cdot B(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{b}_{3,3} & \tilde{b}_{3,4} \\ 0 & 0 & \tilde{b}_{3,4} & \tilde{b}_{4,4} \end{pmatrix},$$

for some $(A, z) \in G_{\text{degn}}^{\text{K}}$ then

$$\hat{A} = \begin{pmatrix} \pm z^{-1/2} & 0 & 0 & 0 \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ 0 & 0 & A_{3,3} & 0 \\ 0 & 0 & 0 & A_{4,4} \end{pmatrix}, \quad z = A_{2,2}^{-2}A_{4,4}^{-1},$$

and

$$\begin{pmatrix} \tilde{b}_{3,3} & \tilde{b}_{3,4} \\ \tilde{b}_{3,4} & \tilde{b}_{4,4} \end{pmatrix} = \begin{pmatrix} A_{2,2}^{-2}A_{4,4}^{-1}A_{3,3}^2b_{3,3} & A_{2,2}^{-2}A_{4,4}^{-1}A_{3,3}A_{4,4}b_{3,4} \\ A_{2,2}^{-2}A_{4,4}^{-1}A_{3,3}A_{4,4}b_{3,4} & A_{2,2}^{-2}A_{4,4}b_{4,4} \end{pmatrix}.$$

We define the canonical form to be

$$B(G)^{11}(b_{3,3}, b_{3,4}, b_{4,4}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_{3,3} & b_{3,4} \\ 0 & 0 & b_{3,4} & b_{4,4} \end{pmatrix},$$

with $b_{3,3}, b_{3,4}, b_{4,4} \in \{0, 1\}$.

4.5.2 $\text{rank}(b) = 1$, $b_{2,3} \neq 0$ and $b_{2,4} \neq 0$

In this case we can act with $(A, z) \in G_{\text{degn}}^{\text{K}}$ to obtain

$$B(G)^{15}(b_{3,4}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & b_{3,4} \\ 0 & 1 & b_{3,4} & 0 \end{pmatrix},$$

for a unique $b_{3,4} \in \{0, 1\}$. We define this form to be the canonical form in this case.

4.5.3 $\text{rank}(b) = 1$, $b_{2,3} \neq 0$ and $b_{2,4} = 0$

In this case we can act with $(A, z) \in G_{\text{degn}}^{\text{K}}$ to obtain

$$B(G)^{16}(b_{4,4}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b_{4,4} \end{pmatrix},$$

with unique $b_{4,4} \in \{0, 1\}$ which is defined to be the canonical form in this case.

4.5.4 $\text{rank}(b) = 1$, $b_{2,3} = 0$ and $b_{2,4} \neq 0$

In this case we can act with $(A, z) \in G_{\text{degn}}^{\text{K}}$ to obtain

$$B(G)^{17}(b_{3,3}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & b_{3,3} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

with unique $b_{3,3} \in \{0, 1\}$ which is defined to be the canonical form in this case.

4.6 The rank 0 case

If

$$B(G) = \begin{pmatrix} 0 & 0 & b_{1,3} & b_{1,4} \\ 0 & 0 & b_{2,3} & b_{2,4} \\ b_{1,3} & b_{2,3} & b_{3,3} & b_{3,4} \\ b_{1,4} & b_{2,4} & b_{3,4} & b_{4,4} \end{pmatrix} \quad \text{and} \quad (A, z) \cdot B(G) = \begin{pmatrix} 0 & 0 & \tilde{b}_{1,3} & \tilde{b}_{1,4} \\ 0 & 0 & \tilde{b}_{2,3} & \tilde{b}_{2,4} \\ \tilde{b}_{1,3} & \tilde{b}_{2,3} & \tilde{b}_{3,3} & \tilde{b}_{3,4} \\ \tilde{b}_{1,4} & \tilde{b}_{2,4} & \tilde{b}_{3,4} & \tilde{b}_{4,4} \end{pmatrix},$$

for some $(A, z) \in G_{\text{degn}}^{\text{K}}$ then

$$\begin{pmatrix} \tilde{b}_{1,3} & \tilde{b}_{1,4} \\ \tilde{b}_{2,4} & \tilde{b}_{2,4} \end{pmatrix} = z \begin{pmatrix} A_{1,1} & A_{2,1} \\ A_{1,2} & A_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,3} & b_{1,4} \\ b_{2,3} & b_{2,4} \end{pmatrix} \begin{pmatrix} A_{3,3} & 0 \\ 0 & A_{4,4} \end{pmatrix},$$

and

$$\begin{aligned} \begin{pmatrix} \tilde{b}_{3,3} & \tilde{b}_{3,4} \\ \tilde{b}_{4,3} & \tilde{b}_{4,4} \end{pmatrix} &= z \begin{pmatrix} A_{1,3} & A_{2,3} \\ A_{1,4} & A_{2,4} \end{pmatrix} \begin{pmatrix} b_{1,3} & b_{1,4} \\ b_{2,3} & b_{2,4} \end{pmatrix} \begin{pmatrix} A_{3,3} & 0 \\ 0 & A_{4,4} \end{pmatrix} \\ &+ z \begin{pmatrix} A_{3,3} & 0 \\ 0 & A_{4,4} \end{pmatrix} \begin{pmatrix} b_{1,3} & b_{2,3} \\ b_{1,4} & b_{2,4} \end{pmatrix} \begin{pmatrix} A_{1,3} & A_{1,4} \\ A_{2,3} & A_{2,4} \end{pmatrix} \\ &+ z \begin{pmatrix} A_{3,3} & 0 \\ 0 & A_{4,4} \end{pmatrix} \begin{pmatrix} b_{3,3} & b_{3,4} \\ b_{3,4} & b_{4,4} \end{pmatrix} \begin{pmatrix} A_{3,3} & 0 \\ 0 & A_{4,4} \end{pmatrix}. \end{aligned}$$

Hence in the case of $b = 0$, the rank of $c = \begin{pmatrix} b_{1,3} & b_{1,4} \\ b_{2,3} & b_{2,4} \end{pmatrix}$ is invariant under the action of $G_{\text{degn}}^{\text{K}}$. We continue our analysis according to the value of this rank. By Corollary 4.4 $c \neq 0$.

4.6.1 $\text{rank}(b) = 0, \text{rank}(c) = 1$

In this case we have the following forms

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & b_{3,3} & b_{3,4} \\ 0 & 1 & b_{3,4} & b_{4,4} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & b_{3,3} & b_{3,4} \\ 0 & 0 & b_{3,4} & b_{4,4} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & b_{3,3} & b_{3,4} \\ 0 & 1 & b_{3,4} & b_{4,4} \end{pmatrix}.$$

The first case can be reduced to the canonical form

$$B(G)^{05} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & b_{3,4} \\ 0 & 1 & b_{3,4} & 0 \end{pmatrix},$$

for a unique $b_{3,4} \in \{0, 1\}$. The second case can be reduced to the canonical form

$$B(G)^{06} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b_{4,4} \end{pmatrix},$$

for a unique $b_{4,4} \in \{0, 1\}$. The case with $b_{4,4} = 0$ is not possible due to Corollary 4.4. The third case can be reduced to the following canonical form

$$B(G)^{07}(b_{3,3}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & b_{3,3} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

for a unique $b_{3,3} \in \{0, 1\}$.

4.6.2 $\text{rank}(b) = 0, \text{rank}(c) = 2$

In this case we only have the form

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & b_{3,3} & b_{3,4} \\ 0 & 1 & b_{3,4} & b_{4,4} \end{pmatrix}, \tag{4.6}$$

which can be reduced to the canonical form

$$B(G)^{08} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

4.7 Comparison of geometric and abstract free degenerate quadratic algebras

We examine the entries in Table 3 to determine which can be realized as a classical 2D degenerate superintegrable system or a classical Lie algebra with linearly independent generators. There is a close relationship between the canonical forms of free abstract degenerate quadratic algebras

degenerate quadratic algebras			
#	rank(b)	invariant form	canonical form
1	2	$B(G)^{21}(b_{34}, b_{44})$	$\mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{H}^2 + b_{44}\mathcal{X}^4 + 2b_{34}\mathcal{H}\mathcal{X}^2, b_{34} \in \{0, 1\}, b_{44} \in \mathbb{C}$
2	2	$B(G)^{22}(b_{34}, b_{44})$	$\mathcal{L}_1^2 + \mathcal{L}_2^2 + b_{44}\mathcal{X}^4 + 2b_{34}\mathcal{H}\mathcal{X}^2, b_{3,4}, b_{44} \in \{0, 1\}$
3	1	$B(G)^{11}(b_{33}, b_{34}, b_{44})$	$\mathcal{L}_1^2 + b_{33}\mathcal{H}^2 + b_{44}\mathcal{X}^4 + 2b_{34}\mathcal{H}\mathcal{X}^2, b_{33}, b_{34}, b_{44} \in \{0, 1\}, b_{34} + b_{44} \neq 0$
4	1	$B(G)^{15}(b_{34})$	$\mathcal{L}_1^2 + 2\mathcal{L}_2\mathcal{H} + 2\mathcal{L}_2\mathcal{X}^2 + 2b_{34}\mathcal{H}\mathcal{X}^2, b_{34} \in \{0, 1\}$
5	1	$B(G)^{16}(b_{44})$	$\mathcal{L}_1^2 + 2\mathcal{L}_2\mathcal{H} + b_{44}\mathcal{X}^4, b_{44} \in \{0, 1\}$
6	1	$B(G)^{17}(b_{33})$	$\mathcal{L}_1^2 + 2\mathcal{L}_2\mathcal{X}^2 + b_{33}\mathcal{H}^2, b_{33} \in \{0, 1\}$
7	0	$B(G)^{05}(b_{34})$	$2\mathcal{L}_2\mathcal{H} + 2\mathcal{L}_2\mathcal{X}^2 + 2b_{34}\mathcal{H}\mathcal{X}^2, b_{34} \in \{0, 1\}$
8	0	$B(G)^{06}$	$2\mathcal{L}_2\mathcal{H} + \mathcal{X}^4$
9	0	$B(G)^{07}(b_{33})$	$2\mathcal{L}_2\mathcal{X}^2 + b_{33}\mathcal{H}^2, b_{33} \in \{0, 1\}$
10	0	$B(G)^{08}$	$2\mathcal{L}_1\mathcal{H} + 2\mathcal{L}_2\mathcal{X}^2$

Table 3. List of canonical forms of free abstract degenerate quadratic algebras. The canonical forms $B(G)^{ab}$ are given explicitly in Sections 4.4, 4.5, 4.6 above.

and Stäckel equivalence classes of degenerate superintegrable systems. We demonstrate this by treating one example in detail. The superintegrable system $S3$, with degenerate potential, can be defined by

$$\mathcal{G} = \mathcal{L}_1^2 + \mathcal{L}_2^2 - \mathcal{L}_1\mathcal{H} + \mathcal{L}_1\mathcal{X}^2 + a_1\mathcal{X}^2 + (a_1 + a_2)\mathcal{L}_1 = 0,$$

where the a_j are the parameters in the potential. To perform a general Stäckel transform of this system with nonsingular transform matrix $C = (c_{jk})$: 1) we set $a_j = \sum_{k=1}^2 c_{jk}b_k, k = 1, 2$ where the b_k are the new parameters, 2) we make the replacements $\mathcal{H} \rightarrow -b_2, b_2 \rightarrow -\mathcal{H}$ and 3) we then set all parameters $b_j = 0$ to determine the free degenerate quadratic algebra. The result is

$$[A]: \quad \mathcal{G} = \mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{L}_1\mathcal{X}^2 - c_{12}\mathcal{H}\mathcal{X}^2 - (c_{12} + c_{22})\mathcal{H}\mathcal{L}_1 = 0,$$

where $|c_{12}| + |c_{22}| > 0$. The canonical forms in Table 3 associated with the equivalence class $[A]$ are 1: $b_{44} = 1$ and 2: $b_{44} = 1$.

The superintegrable system $E6$, with degenerate potential, can be defined by

$$\mathcal{G} = \mathcal{L}_1^2 - \mathcal{L}_2\mathcal{H} + \mathcal{L}_2\mathcal{X}^2 + a_1\mathcal{X}^2 + a_2\mathcal{L}_2 = 0.$$

Going through the same procedure as above, we obtain the equivalence class

$$[B]: \quad \mathcal{G} = \mathcal{L}_1^2 + \mathcal{L}_2\mathcal{X}^2 - c_{12}\mathcal{H}\mathcal{X}^2 - c_{22}\mathcal{H}\mathcal{L}_2 = 0,$$

where $|c_{12}| + |c_{22}| > 0$. The canonical form associated with this equivalence class is 4: all cases.

The superintegrable system $E3$, with degenerate potential, can be defined by

$$\mathcal{G} = \mathcal{L}_1^2 + \mathcal{L}_2^2 - \mathcal{L}_1\mathcal{H} + a_2(\mathcal{X}^2 + a_2)\mathcal{L}_1.$$

The equivalence class is

$$[C]: \quad \mathcal{G} = \mathcal{L}_1^2 + \mathcal{L}_2^2 - c_{12}\mathcal{H}\mathcal{X}^2 - c_{22}\mathcal{H}\mathcal{L}_1 = 0,$$

rank	canonical form	
2	1: all cases	2: all cases
1	3: all cases except	4: all cases
	$(b_{44}, b_{34} = 1, b_{33} = 0, 1), (b_{44} = 0, b_{34}, b_{33} = 1)$	
1	5: all cases	6: all cases
0	7: no	8: missing $2\mathcal{L}_2\mathcal{H} + \mathcal{X}^4 = 0$
0	9: missing $\mathcal{L}_2\mathcal{X}^2 = 0$	10: all cases

Table 4. Matching of geometric with abstract quadratic algebras.

where $|c_{12}| + |c_{22}| > 0$. The canonical forms associated with this equivalence class are 1: $b_{44} = 0$, and 2: $b_{44} = 0$.

The superintegrable system $E5$ can be defined by

$$\mathcal{G} = \mathcal{L}_1^2 + \mathcal{X}^4 - \mathcal{H}\mathcal{X}^2 + a_1\mathcal{L}_2 + a_2\mathcal{X}^2 = 0.$$

The equivalence class is

$$[D]: \quad \mathcal{G} = \mathcal{L}_1^2 + \mathcal{X}^4 - c_{22}\mathcal{H}\mathcal{X}^2 - c_{12}\mathcal{H}\mathcal{L}_2 = 0,$$

where $|c_{12}| + |c_{22}| > 0$. The canonical forms associated with this equivalence class are 3: $b_{44} = 1$, $b_{34} = 0$ and 5: $b_{44} = 1$.

The superintegrable system $E14$ can be defined by

$$\mathcal{G} = -\mathcal{L}_1^2 - \mathcal{L}_2\mathcal{X}^2 + a_1\mathcal{H} - a_1a_2 = 0.$$

The equivalence class is

$$[E]: \quad \mathcal{G}^2 = -\mathcal{L}_1^2 - \mathcal{L}_2\mathcal{X}^2 - c_{12}c_{22}\mathcal{H}^2 = 0,$$

where $|c_{12}| + |c_{22}| > 0$. The canonical forms associated with $[E]$ are 6: all cases.

The superintegrable system $E4$ can be defined by

$$\mathcal{G} = \mathcal{H}^2 + \mathcal{X}^4 + 2\mathcal{H}\mathcal{X}^2 - 4\mathcal{L}_2\mathcal{X}^2 - 4ia_1\mathcal{L}_1 - 2a_2\mathcal{X}^2 - 2a_2\mathcal{H} + a_2^2 = 0.$$

The equivalence class is

$$[F]: \quad \mathcal{G} = \mathcal{X}^4 - 4\mathcal{L}_2\mathcal{X}^2 - 4ic_{12}\mathcal{H}\mathcal{L}_1 + 2c_{22}\mathcal{H}\mathcal{X}^2 + c_{22}^2\mathcal{H}^2 = 0,$$

where $|c_{12}| + |c_{22}| > 0$. The canonical forms associated with $[F]$ are 9: $b_{33} = 1$ and 10: all cases.

Heisenberg systems: In addition there are systems that can be obtained from the degenerate geometric systems above by contractions from $\mathfrak{so}(4, \mathbb{C})$ to $\mathfrak{e}(3, \mathbb{C})$. These are not Bôcher contractions and the contracted systems are not superintegrable, because the Hamiltonians become singular. However, they do form quadratic algebras and several have the interpretation of time-dependent Schrödinger equations in 2D spacetime, so we also consider them geometrical. Some of these were classified in [16] where they were called Heisenberg systems since they appeared in quadratic algebras formed from 2nd order elements in the Heisenberg algebra with generators $\mathcal{M}_1 = p_x$, $\mathcal{M}_2 = xp_y$, $\mathcal{E} = p_y$, where $\mathcal{E}^2 = \mathcal{H}$. The systems are all of type 4. We will devote a future paper to their study. The possible canonical forms are 3: $b_{33} = b_{44} = 0$, $b_{34} = 1$ and 5: $b_{44} = 0$.

All these results relating geometric systems to abstract systems are summarized in Table 4. We see that every abstract quadratic algebra is isomorphic to a quadratic algebra corresponding to a superintegrable system, with just 5 exceptions.

Theorem 4.6. *Every free quadratic algebra realizable by functions on 4-dimensional phase space with grading the order of polynomials in the momenta is isomorphic to a free quadratic algebra of a superintegrable system.*

Proof: We show that the 5 exceptional free quadratic algebras cannot be realized in phase space. We assume in each case that the algebra is realizable in terms of functions on phase space and obtain a contradiction.

1. Case 3: $\mathcal{L}_1^2 + (\mathcal{H} + \mathcal{X}^2)^2 = 0$. We can factor \mathcal{G} as $(\mathcal{H} + \mathcal{X}^2 + i\mathcal{L}_1)(\mathcal{H} + \mathcal{X}^2 - i\mathcal{L}_1) = 0$. This is possible only if one of the factors vanishes; hence the generators are linearly dependent. Impossible!
2. Case 3: $\mathcal{L}_1^2 + 2\mathcal{H}\mathcal{X}^2 + \mathcal{X}^4 = 0$. Here, $\mathcal{L}_1^2 = -\mathcal{X}^2(2\mathcal{H} + \mathcal{X}^2)$ so $\mathcal{L}_1 = \mathcal{X}\mathcal{Y}$ where \mathcal{Y} is a 1st order constant of the motion. Thus \mathcal{Y} must be a multiple of \mathcal{X} and the generators are linearly dependent. Impossible!
3. Case 3: $\mathcal{L}_1^2 + \mathcal{H}^2 + 2\mathcal{H}\mathcal{X}^2 = 0$. Here, $\mathcal{L}_1^2 = -\mathcal{H}(\mathcal{H} + 2\mathcal{X}^2)$. If \mathcal{L}_1 doesn't factor then it must be a multiple of \mathcal{H} and of $\mathcal{H} + 2\mathcal{X}^2$ simultaneously. Impossible! Suppose then that $\mathcal{L}_1 = \mathcal{Y}\mathcal{Z}$, a product of two 1st order factors. If \mathcal{Z} is a multiple of \mathcal{Y} then \mathcal{Y} is a constant of the motion, hence proportional to \mathcal{X} . Impossible! Thus \mathcal{Y}, \mathcal{Z} must be distinct linear factors. If \mathcal{H} is divisible by each factor then $\mathcal{L}_1, \mathcal{H}$ are linearly dependent. Impossible! So \mathcal{H} must be divisible by the square of a single factor. By renormalizing \mathcal{Y} and \mathcal{Z} we can assume $\mathcal{H} = \mathcal{Y}^2$. Thus \mathcal{Y} is a 1st order constant of the motion, necessarily proportional to \mathcal{X} . We conclude that $\mathcal{H} \sim \mathcal{X}^2$. Impossible!
4. Case 8: $2\mathcal{L}_2\mathcal{H} + \mathcal{X}^4 = 0$. Since \mathcal{X} is 1st order, and $2\mathcal{L}_2\mathcal{H} = -\mathcal{X}^4$ both \mathcal{H} and \mathcal{L}_2 must be divisible by \mathcal{X}^2 . Thus each is a perfect square of a 1st order symmetry, necessarily a scalar multiple of \mathcal{X} . Hence the 2nd order generators are linearly dependent. Impossible!
5. Case 9: $\mathcal{L}_2\mathcal{X} = 0$. Here at least one of the generators must vanish. Impossible!

Thus we have shown that the only free degenerate quadratic algebras that can be constructed in phase space are those that arise from superintegrability. The remaining 5 abstract systems must lie in different graded Poisson algebras.

5 Classification of contractions of free abstract quadratic degenerate 2D superintegrable systems on constant curvature spaces and Darboux spaces

In this section we define contractions between free abstract quadratic algebras. Then we list the canonical forms of the Casimirs of free abstract quadratic algebras that arises as the symmetry algebras of degenerate 2D free superintegrable systems on constant curvature spaces or Darboux spaces. Finally using the canonical forms, we classify all possible contractions relating free abstract quadratic algebras of degenerate 2D superintegrable systems on constant curvature spaces or Darboux spaces.

5.1 Contraction of a free abstract quadratic algebra

Definition 5.1. Let \mathcal{A}, \mathcal{B} be free abstract quadratic algebras with Casimirs $(B(G_{\mathcal{A}}), K_{\mathcal{A}})$ and $(B(G_{\mathcal{B}}), K_{\mathcal{B}})$, respectively. Suppose that there is a continuous map $\epsilon \mapsto (A_{\epsilon}, z_{\epsilon})$ from some punctured neighborhood of 0 in \mathbb{C}^* , the nonzero complex numbers, into G_{degn} and such that $\lim_{\epsilon \rightarrow 0} (A_{\epsilon}, z_{\epsilon}) \cdot (B(G_{\mathcal{A}}), K_{\mathcal{A}}) = (B(G_{\mathcal{B}}), K_{\mathcal{B}})$. Then we say that \mathcal{B} is a contraction of \mathcal{A} .

The meaning of the definition is that in any generating set $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4\}$ of \mathcal{A} (satisfying the same assumptions as before) the corresponding matrix $B(G_{\mathcal{A}})$ and $K_{\mathcal{A}}$ satisfy

$$\lim_{\epsilon \rightarrow 0} z_{\epsilon} \widehat{A}_{\epsilon}^t B(G_{\mathcal{A}}) \widehat{A}_{\epsilon} = B(G_{\mathcal{B}}), \quad (5.1)$$

$$\lim_{\epsilon \rightarrow 0} z_{\epsilon}^{-1} K_{\mathcal{A}} ((A_{\epsilon})_{1,1}(A_{\epsilon})_{2,2} - (A_{\epsilon})_{1,2}(A_{\epsilon})_{2,1})^{-1} (A_{\epsilon})_{5,5}^{-1} = K_{\mathcal{B}}, \quad (5.2)$$

where $(B(G_{\mathcal{B}}), K_{\mathcal{B}})$ is a realization of the Casimir of \mathcal{B} in some generating set. In the classification below we are using a more refined class of contractions. We shall call these contractions algebraic. By definition an *algebraic contraction* is a contraction that can be realized via a map $\epsilon \mapsto (A_{\epsilon}, z_{\epsilon})$ from some punctured neighborhood of 0 in \mathbb{C}^* into G_{degn} such that z_{ϵ} as well as the entries of A_{ϵ} are rational functions in ϵ .

Proposition 5.2. *\mathcal{B} is a contraction of \mathcal{A} if and only if there is a continuous map $\epsilon \mapsto (A_{\epsilon}, z_{\epsilon})$ from some punctured neighborhood of 0 in \mathbb{C}^* into G_{degn} such that*

$$\lim_{\epsilon \rightarrow 0} z_{\epsilon} \widehat{A}_{\epsilon}^t B(G_{\mathcal{A}})_{\text{can}} \widehat{A}_{\epsilon} = B(G_{\mathcal{B}})_{\text{can}}, \quad (5.3)$$

$$\lim_{\epsilon \rightarrow 0} z_{\epsilon}^{-1} ((A_{\epsilon})_{1,1}(A_{\epsilon})_{2,2} - (A_{\epsilon})_{1,2}(A_{\epsilon})_{2,1})^{-1} (A_{\epsilon})_{5,5}^{-1} = 1, \quad (5.4)$$

where $(B(G_{\mathcal{A}})_{\text{can}}, 1)$ and $(B(G_{\mathcal{B}})_{\text{can}}, 1)$ are the canonical forms of the Casimirs of \mathcal{A} and \mathcal{B} respectively.

Proof. Obviously if equations (5.3)–(5.4) hold then \mathcal{B} is a contraction of \mathcal{A} . For the other direction assume that equations (5.1)–(5.2) hold. We can further assume that $(B(G_{\mathcal{A}}), K_{\mathcal{A}})$ is in its canonical form $(B(G_{\mathcal{A}})_{\text{can}}, 1)$. Let $(A, z) \in G_{\text{degn}}$ such that $(B(G_{\mathcal{B}}), K_{\mathcal{B}}) = (A, z) \cdot (B(G_{\mathcal{B}})_{\text{can}}, 1)$, then the continuity of the action of G_{degn} implies that $\lim_{\epsilon \rightarrow 0} ((A, z)^{-1}(A_{\epsilon}, z_{\epsilon})) \cdot (B(G_{\mathcal{A}})_{\text{can}}, 1) = (B(G_{\mathcal{B}})_{\text{can}}, 1)$. \blacksquare

Proposition 5.3. *Let \mathcal{A}, \mathcal{B} be free abstract quadratic algebras and $\epsilon \mapsto (A_{\epsilon}, 1)$ a continuous map from \mathbb{C}^* into G_{degn} such that*

$$\lim_{\epsilon \rightarrow 0} \widehat{A}_{\epsilon}^t B(G_{\mathcal{A}})_{\text{can}} \widehat{A}_{\epsilon} = B(G_{\mathcal{B}})_{\text{can}}.$$

Then there is another continuous map $\epsilon \mapsto (C_{\epsilon}, z_{\epsilon})$ from \mathbb{C}^ into G_{degn} such that $\lim_{\epsilon \rightarrow 0} (C_{\epsilon}, z_{\epsilon}) \cdot (B(G_{\mathcal{A}})_{\text{can}}, 1) = (B(G_{\mathcal{B}})_{\text{can}}, 1)$.*

Proof. Define

$$z_{\epsilon} := ((A_{\epsilon})_{1,1}(A_{\epsilon})_{2,2} - (A_{\epsilon})_{1,2}(A_{\epsilon})_{2,1})^4 (A_{\epsilon})_{5,5}^4,$$

$$C_{\epsilon} = ((A_{\epsilon})_{1,1}(A_{\epsilon})_{2,2} - (A_{\epsilon})_{1,2}(A_{\epsilon})_{2,1})^{-1} (A_{\epsilon})_{5,5}^{-1} A_{\epsilon}. \quad \blacksquare$$

Note that Propositions 5.2 and 5.3 also hold for algebraic contractions. The conclusion from the last two proposition is that for the purpose of classifying contractions of free abstract quadratic algebras it is enough to take the Casimirs $B(G)_{\mathcal{A}}$ and $B(G)_{\mathcal{B}}$ in their canonical forms and to consider only the action of the group of invertible matrices of the form

$$\widehat{A} = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ 0 & 0 & A_{3,3} & 0 \\ 0 & 0 & 0 & A_{4,4} \end{pmatrix}, \quad (5.5)$$

on symmetric matrices B by $B \mapsto \widehat{A}^t B \widehat{A}$. We shall denote the space of 4 by 4 complex symmetric matrices by $\text{Sym}(4, \mathbb{C})$ and the group of matrices of the form (5.5) by $\widehat{G}_{\text{degn}}$. The group $\widehat{G}_{\text{degn}}$ is a complex algebraic group and the space $\text{Sym}(4, \mathbb{C})$ is a complex algebraic variety on which $\widehat{G}_{\text{degn}}$ acts algebraically. As was explained in [5, Section 7.1.2] this implies that if \mathcal{B} is a contraction of a quadratic algebra \mathcal{A} then \mathcal{A} is not a contraction of \mathcal{B} (unless \mathcal{A} and \mathcal{B} are isomorphic). In addition if \mathcal{B} is a contraction of \mathcal{A} then the rank of any matrix that represents $B(G)_{\mathcal{B}}$ and the rank of its upper left 2 by 2 block can not exceed the corresponding ranks for any matrix that represents $B(G)_{\mathcal{A}}$. Hence there is certain hierarchy for contractions that is governed by the rank. By a rank of a free abstract quadratic algebra we mean $(\text{rank}(B), \text{rank}(b))$ of the corresponding matrices B and b in any bases. We shall make this more precise below.

5.2 Classification of abstract algebraic contractions of superintegrable systems

Organized according to their rank, the canonical forms of the free abstract quadratic algebras of free triplets of 2D constant curvature spaces and Darboux spaces are given in Table 5 below. We shall classify all possible algebraic contractions between any two free abstract quadratic

canonical forms for the Casimirs of free degenerate 2D 2nd order superintegrable systems on constant curvature and Darboux spaces				
#	system	rank(B)	rank(b)	canonical form
1	$S6$	4	2	$B^{22}(1, 1), \mathcal{L}_1^2 + \mathcal{L}_2^2 + 2\mathcal{H}\mathcal{X}^2 + \mathcal{X}^4$
2	$E18$	4	2	$B^{22}(0, 1), \mathcal{L}_1^2 + \mathcal{L}_2^2 + 2\mathcal{H}\mathcal{X}^2$
3	$D3E$	4	2	$B^{21}(1, 0), \mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{H}^2 + 2\mathcal{H}\mathcal{X}^2$
4	$D4(b)D$	4	2	$B^{21}(1, -2), \mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{H}^2 + 2\mathcal{H}\mathcal{X}^2 - 2\mathcal{X}^4$
5	$S3$	4	2	$B^{21}(\sqrt{2}e^{i\frac{3\pi}{4}}, -2i), \mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{H}^2 + 2\sqrt{2}e^{i\frac{3\pi}{4}}\mathcal{H}\mathcal{X}^2 - 2i\mathcal{X}^4$
6	$E3$	3	2	$B^{21}(0, 0), \mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{H}^2$
7	$E12$	4	1	$B^{17}(1), \mathcal{L}_1^2 + 2\mathcal{L}_2\mathcal{X}^2 + \mathcal{H}^2$
8	$D1D$	4	1	$B^{16}(1), \mathcal{L}_1^2 + 2\mathcal{L}_2\mathcal{H} + \mathcal{H}^2$
9	$D2D$	4	1	$B^{15}(1), \mathcal{L}_1^2 + 2\mathcal{L}_2\mathcal{H} + 2\mathcal{L}_2\mathcal{X}^2 + 2\mathcal{H}\mathcal{X}^2$
10	$E6$	3	1	$B^{15}(0), \mathcal{L}_1^2 + 2\mathcal{L}_2\mathcal{H} + 2\mathcal{L}_2\mathcal{X}^2$
11	$E5$	3	1	$B^{11}(0, 0, 1), \mathcal{L}_1^2 + 2\mathcal{H}\mathcal{X}^2 + \mathcal{X}^4$
12	$E14$	3	1	$B^{17}(0), \mathcal{L}_1^2 + 2\mathcal{L}_2\mathcal{X}^2$
13	$S5$	3	1	$B^{17}(0), \mathcal{L}_1^2 + 2\mathcal{L}_2\mathcal{X}^2$
14	$E13$	4	0	$B^{08}, 2\mathcal{L}_1\mathcal{H} + 2\mathcal{L}_2\mathcal{X}^2$
15	$E4$	3	0	$B^{07}(1), \mathcal{H}^2 + 2\mathcal{L}_2\mathcal{X}^2$

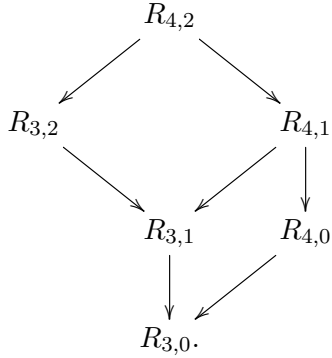
Table 5. List of canonical forms of free abstract degenerate quadratic algebras.

algebras that appear in Table 5. More precisely for any two such algebras we determine if such a contraction is possible or not. If it does we give one realization of it, unless it is a contraction from a free abstract quadratic algebra to itself. We shall start with some general observations. We note that the quadratic algebras of $E14$ and $S5$ coincide so we only keep $E14$ in our notation. We divide the free abstract quadratic algebras of the second order free degenerate superintegrable systems on 2D constant curvature spaces and 2D Darboux spaces according to their ranks: $R_{4,2} := \{S6, E18, D3E, D4(b)D, S3\}$, $R_{3,2} := \{E3\}$, $R_{4,1} := \{E12, D1D, D2D\}$,

	$S6$	$E18$	$D3E$	$D4(b)D$	$S3$	$E3$	$E12$	$D1D$	$D2D$	$E6$	$E5$	$E14$	$E13$	$E4$
$S6$	+	+	-	-	-	-	+	-	-	-	+	+	+	+
$E18$	-	+	-	-	-	-	-	-	-	-	+	-	+	+
$D3E$	-	+	+	-	-	+	-	+	-	-	+	-	+	+
$D4(b)D$	-	-	-	+	-	+	+	+	-	-	+	+	+	+
$S3$	-	-	-	-	+	+	-	+	-	+	+	+	+	+
$E3$	-	-	-	-	-	+	-	-	-	-	+	-	-	+
$E12$	-	-	-	-	-	-	+	-	-	-	+	+	+	+
$D1D$	-	-	-	-	-	-	-	+	-	-	+	-	+	+
$D2D$	-	-	-	-	-	-	+	+	+	+	+	+	+	+
$E6$	-	-	-	-	-	-	-	-	-	+	+	+	-	+
$E5$	-	-	-	-	-	-	-	-	-	-	+	-	-	+
$E14$	-	-	-	-	-	-	-	-	-	-	+	+	-	+
$E13$	-	-	-	-	-	-	-	-	-	-	-	-	+	+
$E4$	-	-	-	-	-	-	-	-	-	-	-	-	-	+

Table 6. List of algebraic contractions between free abstract quadratic algebras of 2D free degenerate superintegrable systems on constant curvature spaces and Darboux spaces. A plus in the rubric placed in the i -th row and j -th column indicates that there is a contraction from the system listed in the i -th row of the first column to the system listed in the j -th column of the first row. A minus indicates that there is no such contraction.

$R_{3,1} := \{E6, E5, E14\}$, $R_{4,0} := \{E13\}$, $R_{3,0} := \{E4\}$. By the discussion above, besides contractions between two algebras in the same class $R_{i,j}$ we can potentially find a contraction only according to the following diagram:



The classification is summarized in Table 6. Detailed analysis is presented below. Representatives for all possible contractions are given in Section 5.3.7.

5.2.1 Contractions between rank two free abstract quadratic algebras

Note that for any of the B_{can} matrices for the systems $R_{4,2} \cup R_{3,2} = \{S6, E18, D3E, D4(b)D, S3\} \cup \{E3\}$ we have

$$\widehat{A}^t B \widehat{A} = \begin{pmatrix} r^t & 0 \\ s^t & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} = \begin{pmatrix} r^t r & r^t s \\ s^t r & s^t s + t d t \end{pmatrix}. \quad (5.6)$$

Consider $A = A(\epsilon)$ in equation (5.6) such that

$$\lim_{\epsilon \rightarrow 0} \widehat{A(\epsilon)}^t B \widehat{A(\epsilon)} = B^0, \quad (5.7)$$

where $B^0 \in R_{4,2} \cup R_{3,2}$. Below we prove that without loss of generality we can assume that $r_\epsilon^t r_\epsilon$ is diagonal and s_ϵ is the zero two by two matrix.

Proposition 5.4. *Suppose that $\{B_\epsilon\}_{\epsilon \in \mathbb{R}^+}$ is a family of 4×4 symmetric matrices with entries in $\mathbb{C}[\epsilon]$ and such that*

$$\lim_{\epsilon \rightarrow 0^+} \begin{pmatrix} (B_\epsilon)_{11} & (B_\epsilon)_{12} & (B_\epsilon)_{13} & (B_\epsilon)_{14} \\ (B_\epsilon)_{12} & (B_\epsilon)_{22} & (B_\epsilon)_{23} & (B_\epsilon)_{24} \\ (B_\epsilon)_{13} & (B_\epsilon)_{23} & (B_\epsilon)_{33} & (B_\epsilon)_{34} \\ (B_\epsilon)_{14} & (B_\epsilon)_{24} & (B_\epsilon)_{34} & (B_\epsilon)_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & l_{33} & l_{34} \\ 0 & 0 & l_{34} & l_{44} \end{pmatrix}.$$

Then there exists a continuous function $\epsilon \mapsto A_\epsilon$ from \mathbb{C}^* to $\widehat{G}_{\text{degn}}$ with each entry polynomial in ϵ such that $\widehat{A}_\epsilon^t B_\epsilon \widehat{A}_\epsilon$ is of the form

$$\begin{pmatrix} (\widetilde{B}_\epsilon)_{11} & 0 & 0 & 0 \\ 0 & (\widetilde{B}_\epsilon)_{22} & 0 & 0 \\ 0 & 0 & (\widetilde{B}_\epsilon)_{33} & (\widetilde{B}_\epsilon)_{34} \\ 0 & 0 & (\widetilde{B}_\epsilon)_{34} & (\widetilde{B}_\epsilon)_{44} \end{pmatrix},$$

and

$$\lim_{\epsilon \rightarrow 0} \widehat{A}_\epsilon^t B_\epsilon \widehat{A}_\epsilon = \lim_{\epsilon \rightarrow 0} B_\epsilon.$$

Proof. Note that for $\widehat{A}_\epsilon^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -(B_\epsilon)_{12} & (B_\epsilon)_{11} & 0 & 0 \\ -(B_\epsilon)_{13} & 0 & (B_\epsilon)_{11} & 0 \\ -(B_\epsilon)_{14} & 0 & 0 & (B_\epsilon)_{11} \end{pmatrix}$ $\lim_{\epsilon \rightarrow 0} \widehat{A}_\epsilon^t$ is the identity matrix and $\widehat{A}_\epsilon^t B_\epsilon \widehat{A}_\epsilon$ is a symmetric matrix with all entries in the first row and first column besides the $(1, 1)$ entry equal to zero. Similar matrix take care of the second row and second column. ■

Now suppose that equation (5.7) holds. Using the last proposition this means that we can assume that $\widehat{A}(\epsilon) = \begin{pmatrix} r_\epsilon & 0 \\ 0 & t_\epsilon \end{pmatrix}$ with $r_\epsilon \in \text{GL}(2, \mathbb{C})$ and, as always, t_ϵ a diagonal invertible matrix. From this we easily see that the only possible contractions between two algebras in $R_{4,2} \cup R_{3,2}$ are $S6 \rightarrow E18$, $D3E \rightarrow E18$, $D3E \rightarrow E3$, $D4(b)D \rightarrow E3$, $S3 \rightarrow E3$.

5.3 Contractions of rank two free abstract quadratic algebras to rank one algebras

Proposition 5.5. *Suppose that $\{B_\epsilon\}_{\epsilon \in \mathbb{R}^+}$ is a family of 4×4 symmetric matrices with entries in $\mathbb{C}[\epsilon]$ such that*

$$\lim_{\epsilon \rightarrow 0^+} B_\epsilon = L = \begin{pmatrix} 1 & 0 & l_{13} & l_{14} \\ 0 & 0 & l_{23} & l_{24} \\ l_{13} & l_{23} & l_{33} & l_{34} \\ l_{14} & l_{24} & l_{34} & l_{44} \end{pmatrix}.$$

Then there exists a continuous function $\epsilon \mapsto A_\epsilon$ from \mathbb{C}^* to $\widehat{G}_{\text{degn}}$ with each entry polynomial in ϵ such that $\widehat{A}_\epsilon^t B_\epsilon \widehat{A}_\epsilon$ is of the form

$$\begin{pmatrix} (\widetilde{B}_\epsilon)_{11} & 0 & 0 & 0 \\ 0 & (\widetilde{B}_\epsilon)_{22} & (\widetilde{B}_\epsilon)_{23} & (\widetilde{B}_\epsilon)_{24} \\ 0 & (\widetilde{B}_\epsilon)_{23} & (\widetilde{B}_\epsilon)_{33} & (\widetilde{B}_\epsilon)_{34} \\ 0 & (\widetilde{B}_\epsilon)_{24} & (\widetilde{B}_\epsilon)_{34} & (\widetilde{B}_\epsilon)_{44} \end{pmatrix},$$

and

$$\lim_{\epsilon \rightarrow 0} \widehat{A}_\epsilon^t B_\epsilon \widehat{A}_\epsilon = \lim_{\epsilon \rightarrow 0} B_\epsilon.$$

If in addition $l_{13} = 0$ then $(\widetilde{B}_\epsilon)_{13} = 0$ and similarly if $l_{14} = 0$ then $(\widetilde{B}_\epsilon)_{14} = 0$.

The proof is similar to the proof of Proposition 5.4.

Corollary 5.6. *In any contraction from a rank two systems $R_{4,2} \cup R_{3,2} = \{S6, E18, D3E, D4(b)D, S3, E3\}$ to of one of the rank one systems $R_{4,1} \cup R_{3,1} = \{E12, D1D, D2D, E6, E5, E14\}$ we can assume that*

$$\widehat{A(\epsilon)}^t B \widehat{A(\epsilon)} = \begin{pmatrix} (B_\epsilon)_{11} & 0 & 0 & 0 \\ 0 & (B_\epsilon)_{22} & (B_\epsilon)_{23} & (B_\epsilon)_{24} \\ 0 & (B_\epsilon)_{23} & (B_\epsilon)_{33} & (B_\epsilon)_{34} \\ 0 & (B_\epsilon)_{24} & (B_\epsilon)_{34} & (B_\epsilon)_{44} \end{pmatrix}. \quad (5.8)$$

5.3.1 Contractions of $D3E$ to rank one algebras

For $A(\epsilon) \in \widehat{G}_{\text{degn}}$ the matrix $\widehat{A(\epsilon)}^t B(G)^{21}(1,0)\widehat{A(\epsilon)}$ is given by

$$\begin{pmatrix} A_{11}^2 + A_{21}^2 & A_{11}A_{12} + A_{21}A_{22} & A_{11}A_{13} + A_{21}A_{23} & A_{11}A_{14} + A_{21}A_{24} \\ A_{12}A_{11} + A_{22}A_{21} & A_{12}^2 + A_{22}^2 & A_{12}A_{13} + A_{22}A_{23} & A_{12}A_{14} + A_{22}A_{24} \\ A_{11}A_{13} + A_{21}A_{23} & A_{12}A_{13} + A_{22}A_{23} & A_{13}^2 + A_{23}^2 + A_{33}^2 & A_{13}A_{14} + A_{23}A_{24} + A_{33}A_{44} \\ A_{11}A_{14} + A_{21}A_{24} & A_{12}A_{14} + A_{22}A_{24} & A_{13}A_{14} + A_{23}A_{24} + A_{33}A_{44} & A_{14}^2 + A_{24}^2 \end{pmatrix}.$$

Assuming that this matrix is in the form of (5.8) then this implies that exist β, γ, δ complex valued functions of ϵ define on \mathbb{C}^* such that

$$(A_{12}, A_{22}) = \beta(A_{21}, -A_{11}), \quad (A_{13}, A_{23}) = \gamma(A_{21}, -A_{11}), \quad (A_{14}, A_{24}) = \delta(A_{21}, -A_{11}).$$

Hence $\widehat{A(\epsilon)}^t B(G)^{21}(1,0)\widehat{A(\epsilon)}$ takes the form

$$\begin{pmatrix} A_{11}^2 + A_{21}^2 & 0 & 0 & 0 \\ 0 & \beta^2(A_{11}^2 + A_{21}^2) & \beta\gamma(A_{11}^2 + A_{21}^2) & \beta\delta(A_{11}^2 + A_{21}^2) \\ 0 & \beta\gamma(A_{11}^2 + A_{21}^2) & \gamma^2(A_{11}^2 + A_{21}^2) + A_{33}^2 & \gamma\delta(A_{11}^2 + A_{21}^2) + A_{33}A_{44} \\ 0 & \beta\delta(A_{11}^2 + A_{21}^2) & \gamma\delta(A_{11}^2 + A_{21}^2) + A_{33}A_{44} & \delta^2(A_{11}^2 + A_{21}^2) \end{pmatrix}.$$

We are going to consider the limit of ϵ goes to zero of such a matrix and check if it can be equal to one of the matrices of the canonical forms of the systems in $R_{4,1} \cup R_{3,1}$. This means that

$\lim_{\epsilon \rightarrow 0^+} (A_{11}^2 + A_{21}^2) = 1$ and we can replace $\widehat{A(\epsilon)}^t B(G)^{21}(1,0)\widehat{A(\epsilon)}$ by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta^2 & \beta\gamma & \beta\delta \\ 0 & \beta\gamma & \gamma^2 + A_{33}^2 & \gamma\delta + A_{33}A_{44} \\ 0 & \beta\delta & \gamma\delta + A_{33}A_{44} & \delta^2 \end{pmatrix},$$

with $\lim_{\epsilon \rightarrow 0^+} \beta = 0$. From this we can show that $D3E$ can not be contracted to $E6, E12, E14, D2D$.

5.3.2 Contractions of $E18$ to rank one algebras

Following the same steps as in the case of $D3E$ we can replace $\widehat{A(\epsilon)}^t B(G)^{22}(0,1)\widehat{A(\epsilon)}$ by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta^2 & \beta\gamma & \beta\delta \\ 0 & \beta\gamma & \gamma^2 & \gamma\delta + A_{33}A_{44} \\ 0 & \beta\delta & \gamma\delta + A_{33}A_{44} & \delta^2 \end{pmatrix},$$

with $\lim_{\epsilon \rightarrow 0^+} \beta = 0$. From this we can show that $E18$ can not be contracted to $E6, E12, D1D, D2D$.

5.3.3 Contractions of $S6$ to rank one algebras

Following the same steps as in the case of $D3E$ we can replace $\widehat{A(\epsilon)}^t B(G)^{22}(1,1)\widehat{A(\epsilon)}$ by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta^2 & \beta\gamma & \beta\delta \\ 0 & \beta\gamma & \gamma^2 & \gamma\delta + A_{33}A_{44} \\ 0 & \beta\delta & \gamma\delta + A_{33}A_{44} & \delta^2 + A_{44}^2 \end{pmatrix},$$

with $\lim_{\epsilon \rightarrow 0^+} \beta = 0$. From this we can show that $S6$ can not be contracted to $E6$, $D1D$, $D2D$.

5.3.4 Contractions of $D4(b)D$ to rank one algebras

Following the same steps as in the case of $D3E$ we can replace $\widehat{A(\epsilon)}^t B(G)^{21}(1,-2)\widehat{A(\epsilon)}$ by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta^2 & \beta\gamma & \beta\delta \\ 0 & \beta\gamma & \gamma^2 + A_{33}^2 & \gamma\delta + A_{33}A_{44} \\ 0 & \beta\delta & \gamma\delta + A_{33}A_{44} & \delta^2 - 2A_{44}^2 \end{pmatrix},$$

with $\lim_{\epsilon \rightarrow 0^+} \beta = 0$. From this we can show that $D4(b)D$ can not be contracted to $D2D$, $E6$.

5.3.5 Contractions of $S3$ to rank one algebras

Following the same steps as in the case of $D3E$ we can replace $\widehat{A(\epsilon)}^t B(G)^{21}(\sqrt{2}e^{i\frac{3\pi}{4}}, -2i)\widehat{A(\epsilon)}$ by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta^2 & \beta\gamma & \beta\delta \\ 0 & \beta\gamma & \gamma^2 + A_{33}^2 & \gamma\delta + \sqrt{2}e^{i\frac{3\pi}{4}} A_{33}A_{44} \\ 0 & \beta\delta & \gamma\delta + \sqrt{2}e^{i\frac{3\pi}{4}} A_{33}A_{44} & \delta^2 - 2iA_{44}^2 \end{pmatrix},$$

with $\lim_{\epsilon \rightarrow 0^+} \beta = 0$. From this we can show that $S3$ can not be contracted to $D2D$, $E12$.

5.3.6 Contractions of $E3$ to rank one algebras

Following the same steps as in the case of $D3E$ we can replace $\widehat{A(\epsilon)}^t B(G)^{21}(0,0)\widehat{A(\epsilon)}$ by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta^2 & \beta\gamma & \beta\delta \\ 0 & \beta\gamma & \gamma^2 & \gamma\delta \\ 0 & \beta\delta & \gamma\delta & \delta^2 \end{pmatrix},$$

with $\lim_{\epsilon \rightarrow 0^+} \beta = 0$. From this we can show that $E3$ can not be contracted to $E6$ and $E14$.

5.3.7 Explicit contractions

E13:

$$\begin{matrix} E13 \longrightarrow E4 \\ \begin{pmatrix} \epsilon & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

E14:

$$\begin{array}{cc} E14 \longrightarrow E4 & E14 \longrightarrow E5 \\ \begin{pmatrix} \epsilon & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{array}$$

E5:

$$E5 \longrightarrow E4 \\ \begin{pmatrix} \epsilon^2 & \epsilon & 1 & \epsilon^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i\epsilon^{-1} \end{pmatrix}.$$

D2D:

$$\begin{array}{ccc} D2D \longrightarrow E6 & D2D \longrightarrow E12 & D2D \longrightarrow D1D \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \epsilon^{-1} & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \epsilon^{-1} \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & \frac{\epsilon}{2} & 0 \\ 0 & 0 & \epsilon^{-1} & 0 \\ 0 & 0 & 0 & \epsilon^2 \end{pmatrix}, \\ \\ D2D \longrightarrow E5 & D2D \longrightarrow E14 & D2D \longrightarrow E4 & D2D \longrightarrow E13 \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & \frac{1}{2\epsilon} \\ 0 & 0 & 2\epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, & \begin{pmatrix} \epsilon & 0 & 1 & 1 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, & \begin{pmatrix} 0 & \epsilon & 0 & \epsilon^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i\epsilon^{-1} \end{pmatrix}. \end{array}$$

E6:

$$\begin{array}{ccc} E6 \longrightarrow E14 & E6 \longrightarrow E5 & E6 \longrightarrow E4 \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & \epsilon & \epsilon^{-1} \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \frac{1}{2}\epsilon \end{pmatrix}, & \begin{pmatrix} \epsilon & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{array}$$

E12:

$$\begin{array}{ccc} E12 \longrightarrow E14 & E12 \longrightarrow E5 & E12 \longrightarrow E4 & E12 \longrightarrow E13 \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & \epsilon & 1 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} \epsilon & 0 & \epsilon^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i\epsilon^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{array}$$

The non-contraction of $E12$ to $D1D$, $E6$, $D2D$: Demanding that $\lim_{\epsilon \rightarrow 0} \widehat{A}(\epsilon)^t B^{17}(1) \widehat{A}(\epsilon)$ converge to the Canonical form of one of the systems $D1D$, $E6$, $D2D$, and observing the entries on places (1, 1), (2, 2), (3, 1), and (3, 2) in the matrix equation we obtain the equations:

$$\lim_{\epsilon \rightarrow 0} A_{1,1}^2 = 1, \quad \lim_{\epsilon \rightarrow 0} A_{1,2}^2 = 0, \quad \lim_{\epsilon \rightarrow 0} A_{1,3} A_{1,1} = 0, \quad \lim_{\epsilon \rightarrow 0} A_{1,2} A_{1,3} = 1,$$

which obviously can not hold simultaneously.

D1D:

$$\begin{array}{ccc}
 D1D \longrightarrow E5 & D1D \longrightarrow E13 & D1D \longrightarrow E4 \\
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} \epsilon^2 & \epsilon & 0 & \epsilon^{-1} \\ \epsilon^{-1} & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & i\epsilon^{-1} \end{pmatrix}, & \begin{pmatrix} \epsilon^2 & \epsilon & 0 & \epsilon^{-1} \\ 0 & \epsilon & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i\epsilon^{-1} \end{pmatrix}.
 \end{array}$$

The non-contraction of $D1D$ to $E6$, $E12$, $S5$: Demanding that $\lim_{\epsilon \rightarrow 0} \widehat{A(\epsilon)}^t B^{16}(1) \widehat{A(\epsilon)}$ converge to the canonical form of one of the systems $E6$, $E12$, $S5$, and observing the entries on places $(1, 1)$, $(1, 2)$, $(4, 1)$, and $(4, 2)$ in the matrix equation we obtain the equations:

$$\lim_{\epsilon \rightarrow 0} A_{1,1} = \pm 1, \quad \lim_{\epsilon \rightarrow 0} A_{1,2} = 0, \quad \lim_{\epsilon \rightarrow 0} A_{1,2} A_{1,4} = 1, \quad \lim_{\epsilon \rightarrow 0} A_{1,1} A_{1,4} = 0,$$

which obviously can not hold simultaneously.

E3:

$$\begin{array}{ccc}
 E3 \longrightarrow E5 & E3 \longrightarrow E4 \\
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 1 & 1 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} \epsilon & 0 & 1 & -i\epsilon^{-1} \\ i\epsilon & \epsilon & i & \epsilon^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
 \end{array}$$

D4(b)D:

$$\begin{array}{ccc}
 D4(b)D \longrightarrow E4 & D4(b)D \longrightarrow E14 & D4(b)D \longrightarrow E5 \\
 \begin{pmatrix} \epsilon & 0 & 1 & 0 \\ 0 & \epsilon & 0 & \epsilon^{-1} \\ 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}\epsilon} \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & \epsilon^{-1} \\ 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}\epsilon} \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 1 & 1 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, \\
 \\
 D4(b)D \longrightarrow D1D & D4(b)D \longrightarrow E12 & D4(b)D \longrightarrow E13 \\
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\epsilon & -\epsilon^{-1} & 0 \\ 0 & 0 & i\epsilon^{-1} & 0 \\ 0 & 0 & 0 & \epsilon^2 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\epsilon & \frac{1}{\sqrt{3}} & -\epsilon^{-1} \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}\epsilon} \end{pmatrix}, & \begin{pmatrix} \epsilon & \epsilon & \frac{1}{2}\epsilon^{-1} & \frac{1}{2}\epsilon^{-1} \\ i\epsilon & -i\epsilon & -\frac{i}{2}\epsilon^{-1} & \frac{i}{2}\epsilon^{-1} \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, \\
 \\
 D4(b)D \longrightarrow E3 \\
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}.
 \end{array}$$

S3:

$$\begin{array}{ccc}
 S3 \longrightarrow E5 & S3 \longrightarrow E14 & S3 \longrightarrow E4 \\
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & i & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}e^{-i\frac{3\pi}{4}} \end{pmatrix}, & \begin{pmatrix} 1 & 0 & \epsilon & -\epsilon \\ 0 & \epsilon & \epsilon^3 & \epsilon^{-1} \\ 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & \frac{i}{\epsilon\sqrt{2}}e^{-i\frac{3\pi}{4}} \end{pmatrix}, & \begin{pmatrix} \epsilon & 0 & 1 & 0 \\ 0 & \epsilon & 0 & \epsilon^{-1} \\ 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & i\frac{1}{\epsilon\sqrt{2}}e^{-i\pi/4} \end{pmatrix}, \\
 \\
 S3 \longrightarrow D1D & S3 \longrightarrow E13 & S3 \longrightarrow E6 \\
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\epsilon & -\epsilon^{-1} & 0 \\ 0 & 0 & i\epsilon^{-1} & 0 \\ 0 & 0 & 0 & \epsilon^2 \end{pmatrix}, & \begin{pmatrix} \epsilon & \epsilon & \frac{1}{2}\epsilon^{-1} & \frac{1}{2}\epsilon^{-1} \\ i\epsilon & -i\epsilon & -\frac{i}{2}\epsilon^{-1} & \frac{i}{2}\epsilon^{-1} \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\epsilon & -\epsilon^{-1} & -\epsilon^{-1} \\ 0 & 0 & -i\epsilon^{-1} & 0 \\ 0 & 0 & 0 & \frac{e^{i3\pi/4}}{\sqrt{2}\epsilon} \end{pmatrix},
 \end{array}$$

$$S3 \longrightarrow E3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}.$$

S6:

$$S6 \longrightarrow E12 \quad S6 \longrightarrow E14 \quad S6 \longrightarrow E5$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 1 & \epsilon^{-1} \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i\epsilon^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & \epsilon^{-1} \\ 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & i\epsilon^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S6 \longrightarrow E4 \quad S6 \longrightarrow E13 \quad S6 \longrightarrow E18$$

$$\begin{pmatrix} \epsilon^2 & \epsilon & 1 & \epsilon^{-1} \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i\epsilon^{-1} \end{pmatrix}, \quad \begin{pmatrix} \epsilon & -i\epsilon & \epsilon^{-1} & 0 \\ 0 & \epsilon & i\epsilon^{-1} & \epsilon^{-1} \\ 0 & 0 & -\epsilon^{-1} & 0 \\ 0 & 0 & 0 & i\epsilon^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon^{-1} & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}.$$

E18:

$$E18 \longrightarrow E5 \quad E18 \longrightarrow E13 \quad E18 \longrightarrow E4$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \epsilon & \epsilon & \frac{1}{2}\epsilon^{-1} & \frac{1}{2}\epsilon^{-1} \\ \pm i\epsilon & \mp i\epsilon & \mp \frac{i}{2}\epsilon^{-1} & \pm \frac{i}{2}\epsilon^{-1} \\ 0 & 0 & \frac{i}{\sqrt{2}}\epsilon^{-1} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}}\epsilon^{-1} \end{pmatrix}, \quad \begin{pmatrix} \epsilon^2 & \epsilon & 1 & (1-i)(2\epsilon)^{-1} \\ 0 & \epsilon & 0 & (1+i)(2\epsilon)^{-1} \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & (1+i)(2\epsilon)^{-1} \end{pmatrix}.$$

D3E:

$$D3E \longrightarrow E5 \quad D3E \longrightarrow D1D \quad D3E \longrightarrow E13$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 1 & 1 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\epsilon & -\epsilon^{-1} & -1 \\ 0 & 0 & i\epsilon^{-1} & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad \begin{pmatrix} \epsilon^{-1} & \epsilon^2 & \frac{\epsilon}{1-i} & \frac{1}{\epsilon^2(1+i)} \\ i\epsilon^{-1} & \epsilon^2 & -\frac{\epsilon}{1-i} & \frac{1}{\epsilon^2(1-i)} \\ 0 & 0 & -\frac{\sqrt{2}\epsilon}{1+i} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}\epsilon^2} \end{pmatrix},$$

$$D3E \longrightarrow E4 \quad D3E \longrightarrow E3 \quad D3E \longrightarrow E18$$

$$\begin{pmatrix} \epsilon & 0 & 1 & 0 \\ 0 & \epsilon & 0 & \epsilon^{-1} \\ 0 & 0 & \epsilon^2 & \frac{-1+i}{\sqrt{2}\epsilon} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}\epsilon} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon^{-1} \end{pmatrix}.$$

5.4 Contractions of the degenerate quadratic algebras and the lower half of the Askey scheme

The bottom half of the contraction Askey scheme relating orthogonal polynomials via contractions to degenerate, singular and free superintegrable systems is presented in Fig. 2. The top half of the Scheme, relating to contractions of nondegenerate superintegrable systems can be found in [5] and the full scheme in [17]. On the left side are the orthogonal polynomials that realize finite-dimensional representations of the quadratic algebras via difference or differential operators and on the right those that realize infinite-dimensional bounded below representations. The arrows from the nondegenerate superintegrable system $S9$, $E1$, $E8$ and $E3'$ correspond to restriction/contractions to degenerate systems, i.e., the parameters in the 3-parameter potentials are restricted to the case of only 1 parameter and such that one of the symmetry operators

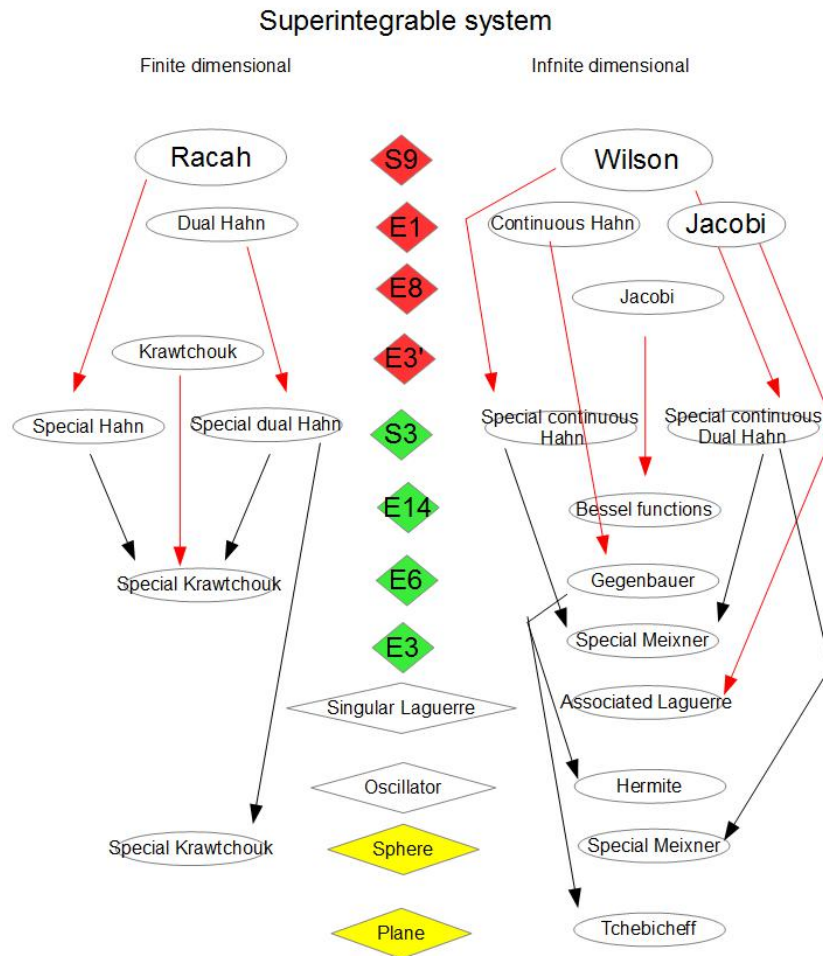


Figure 2. Contractions of degenerate systems and the bottom half of the Askey scheme.

becomes a perfect square. This increases the symmetry algebra of the resulting degenerate system.

Remark 5.7. For reference, the corresponding potentials are

- $E1$: $V = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}$,
- $E3'$: $V = \alpha(x^2 + y^2) + \beta x + \gamma y$,
- $E8$: $V = \frac{\alpha(x-iy)}{(x+iy)^3} + \frac{\beta}{(x+iy)^2} + \gamma(x^2 + y^2)$,

on flat space and

- $S9$: $V = \frac{\alpha}{s_1^2} + \frac{\beta}{s_2^2} + \frac{\gamma}{s_3^2}$, $s_1^2 + s_2^2 + s_3^2 = 1$,

on the 2-sphere.

The arrows from one degenerate superintegrable system to another are the standard contractions studied above. The singular Laguerre and oscillator superintegrable systems have singular Hamiltonians, and for these systems knowledge of the free quadratic algebra does not necessarily determine the full superintegrable system. The singular Laguerre system is a restriction/contraction of $E1$. The resulting quadratic algebra is isomorphic to $\{H\} \oplus \mathfrak{sl}(2, \mathbb{R})$ [17], in the same sense that the quadratic algebra of the 2D Kepler system is said to be $\mathfrak{so}(3, \mathbb{R})$. This is

true only if the system is restricted to an eigenspace of H . The Oscillator system is a contraction of $E6$ and its quadratic algebra is isomorphic to the 4-dimensional oscillator algebra [17]. The plane and sphere systems are restriction/contractions of degenerate superintegrable systems to free superintegrable systems on the plane and the 2-sphere, respectively.

6 Conclusions and discussion

This paper is devoted to the study of the geometric quadratic algebras that correspond to 2D degenerate 2nd order superintegrable systems and general abstract degenerate quadratic algebras. Since the geometric quadratic algebras are uniquely determined by their free restrictions we studied only parameter-free geometric and abstract algebras. The geometric algebras were already known; in this paper we classified all abstract algebras in Table 3. We related the geometric and free quadratic algebras in Table 4. We showed that there were 5 abstract algebras with no geometric counterpart, but that it was impossible to represent them in phase space.

In Section 3 we derived and classified all Bôcher contractions of 2D degenerate 2nd order superintegrable systems. In Fig. 2 we showed the relationship between our results and the bottom half of the Askey scheme. We derived and classified all abstract contractions of the geometric quadratic algebras, presenting the results in Table 6. Comparing the Bôcher and abstract contractions and taking into account the isomorphism of the $E14$ and $S5$ algebras, we see that there is a match except for 6 abstract contractions with no geometric realization:

paren algebra	abstract contracted algebra
$S6$	$D1D$
$S3$	$D1$
$S3$	$E13$
$D3E$	$E18$
$E12$	$E5$
$E14$	$E5$

In two cases the failure of geometric realization is obvious: It is not possible to contract a constant curvature space to a Darboux space [7]. This paper is a partial warm-up for an analogous study of quadratic algebras for 3D superintegrable systems, e.g., [2] and for cubic algebras, e.g., [21].

A Summary of degenerate Laplace and Helmholtz systems

The degenerate superintegrable systems can occur only on the 2-sphere, 2D flat space, or one of the 4 Darboux spaces. The notation for these systems is taken from [14, 15]. (We write the systems in classical form; the quantum analogs have the same potentials and the obvious replacements of classical momenta by quantum derivatives.) We assume all variables to be complex.

Degenerate complex Euclidean systems $H = p_x^2 + p_y^2 + \alpha V(x, y)$:

1. $E18$: $V = \frac{1}{\sqrt{x^2+y^2}}$, Kepler potential,
2. $E3$: $V = x^2 + y^2$, harmonic oscillator,
3. $E6$: $V = \frac{1}{x^2}$, radial potential,
4. $E5$: $V = x$, linear potential,
5. $E12$: $V = \frac{x+iy}{\sqrt{(x+iy)^2+c^2}}$,
6. $E14$: $V = \frac{1}{(x+iy)^2}$,

7. E4: $V = x + iy$,
8. E13: $V = \frac{1}{\sqrt{x+iy}}$.

The last 4 systems are real in Minkowski space.

Degenerate systems on the complex sphere: We use the classical realization for $o(3, \mathcal{C})$ with basis $J_1 = s_2 p_{s_3} - s_3 p_{s_2}$, $J_2 = s_3 p_{s_1} - s_1 p_{s_3}$, $J_3 = s_1 p_{s_2} - s_2 p_{s_1}$, and Hamiltonian $H = J_1^2 + J_2^2 + J_3^2 + \alpha V$. Here $s_1^2 + s_2^2 + s_3^2 = 1$.

1. S6: $V = \frac{s_3}{\sqrt{s_1^2 + s_2^2}}$, Kepler analog,
2. S3: $V = \frac{1}{s_3}$, Higg's oscillator,
3. S5: $V = \frac{1}{(s_1 + is_2)^2}$.

The last system is real on the 2-sheet hyperboloid.

Degenerate systems on Darboux spaces:

1. D1D: $H = \frac{1}{4x}(p_x^2 + p_y^2) + \frac{\alpha}{x}$,
2. D2D: $H = \frac{x^2}{x^2+1}(p_x^2 + p_y^2) + \frac{\alpha}{x^2+1}$,
3. D3E: $H = \frac{1}{2} \frac{e^{2x}}{e^x+1}(p_x^2 + p_y^2) + \frac{\alpha}{e^x+1}$,
4. D4(b)D: $H = -\frac{\sin^2 2x}{2 \cos 2x+b}(p_x^2 + p_y^2) + \frac{\alpha}{2 \cos 2x+b}$.

Remark A.1. Every degenerate system occurs as a “restriction” of at least one nondegenerate system, although the symmetry algebra grows. For example the classical nondegenerate S9 system has the Hamiltonian

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$$

and a basis of symmetries

$$L_1 = J_3^2 + a_1 \frac{s_2^2}{s_1^2} + a_2 \frac{s_1^2}{s_2^2}, \quad L_2 = J_1^2 + a_2 \frac{s_3^2}{s_2^2} + a_3 \frac{s_2^2}{s_3^2}, \quad L_3 = J_2^2 + a_3 \frac{s_1^2}{s_3^2} + a_1 \frac{s_3^2}{s_1^2},$$

where $H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3$. If we let $a_1 \rightarrow 0$, $a_2 \rightarrow 0$ we obtain the Hamiltonian for S3: $H' = J_1^2 + J_2^2 + J_3^2 + \frac{a_3}{s_3^2}$. However, now

$$L_1 \rightarrow L'_1 = J_3^2, \quad L_2 \rightarrow L'_2 = J_1^2 + a_3 \frac{s_2^2}{s_3^2}, \quad L_3 \rightarrow L'_3 = J_2^2 + a_3 \frac{s_1^2}{s_3^2},$$

and the restricted system now admits the 1st order symmetry J_3 as well as a new 2nd order symmetry $\{J_3, L'_2\}$. These 4 symmetries are related by the Casimir. A table with all of the restrictions of nondegenerate systems on constant curvature spaces to degenerate systems can be found in [16] and a table with all restrictions of nondegenerate systems on Darboux spaces can be found in [7].

Acknowledgments

This work was partially supported by a grant from the Simons Foundation (# 208754 to Willard Miller, Jr and by CONACYT grant (# 250881 to M.A. Escobar). The author M.A. Escobar is grateful to ICN UNAM for the kind hospitality during his visit, where a part of the research was done, he was supported in part by DGAPA grant IN108815 (Mexico). We thank a referee for pointing out the relevance of references [4, 6, 20].

References

- [1] Bôcher M., Über die Riehenentwickelungen der Potentialtheory, B.G. Teubner, Leipzig, 1894.
- [2] Capel J.J., Kress J.M., Post S., Invariant classification and limits of maximally superintegrable systems in 3D, *SIGMA* **11** (2015), 038, 17 pages, [arXiv:1501.06601](#).
- [3] Daskaloyannis C., Tanoudis Y., Quantum superintegrable systems with quadratic integrals on a two dimensional manifold, *J. Math. Phys.* **48** (2007), 072108, 22 pages, [math-ph/0607058](#).
- [4] Dufour J.P., Zung N.T., Poisson structures and their normal forms, *Progress in Mathematics*, Vol. 242, Birkhäuser Verlag, Basel, 2005.
- [5] Escobar Ruiz M.A., Kalnins E.G., Miller Jr. W., Subag E., Bôcher and abstract contractions of 2nd order quadratic algebras, *SIGMA* **13** (2017), 013, 38 pages, [arXiv:1611.02560](#).
- [6] Grabowski J., Marmo G., Perelomov A.M., Poisson structures: towards a classification, *Modern Phys. Lett. A* **8** (1993), 1719–1733.
- [7] Heinonen R., Kalnins E.G., Miller Jr. W., Subag E., Structure relations and Darboux contractions for 2D 2nd order superintegrable systems, *SIGMA* **11** (2015), 043, 33 pages, [arXiv:1502.00128](#).
- [8] Izmet'ev A.A., Pogosyan G.S., Sissakian A.N., Winternitz P., Contractions of Lie algebras and separation of variables, *J. Phys. A: Math. Gen.* **29** (1996), 5949–5962.
- [9] Izmet'ev A.A., Pogosyan G.S., Sissakian A.N., Winternitz P., Contractions of Lie algebras and the separation of variables: interbase expansions, *J. Phys. A* **34** (2001), 521–554.
- [10] Kalnins E.G., Kress J.M., Miller Jr. W., Second-order superintegrable systems in conformally flat spaces. I. Two-dimensional classical structure theory, *J. Math. Phys.* **46** (2005), 053509, 28 pages.
- [11] Kalnins E.G., Kress J.M., Miller Jr. W., Second order superintegrable systems in conformally flat spaces. II. The classical two-dimensional Stäckel transform, *J. Math. Phys.* **46** (2005), 053510, 15 pages.
- [12] Kalnins E.G., Kress J.M., Miller Jr. W., Second order superintegrable systems in conformally flat spaces. IV. The classical 3D Stäckel transform and 3D classification theory, *J. Math. Phys.* **47** (2006), 043514, 26 pages.
- [13] Kalnins E.G., Kress J.M., Miller Jr. W., Second-order superintegrable systems in conformally flat spaces. V. Two- and three-dimensional quantum systems, *J. Math. Phys.* **47** (2006), 093501, 25 pages.
- [14] Kalnins E.G., Kress J.M., Miller Jr. W., Winternitz P., Superintegrable systems in Darboux spaces, *J. Math. Phys.* **44** (2003), 5811–5848, [math-ph/0307039](#).
- [15] Kalnins E.G., Kress J.M., Pogosyan G.S., Miller Jr. W., Completeness of superintegrability in two-dimensional constant-curvature spaces, *J. Phys. A: Math. Gen.* **34** (2001), 4705–4720, [math-ph/0102006](#).
- [16] Kalnins E.G., Miller Jr. W., Quadratic algebra contractions and second-order superintegrable systems, *Anal. Appl. (Singap.)* **12** (2014), 583–612, [arXiv:1401.0830](#).
- [17] Kalnins E.G., Miller Jr. W., Post S., Contractions of 2D 2nd order quantum superintegrable systems and the Askey scheme for hypergeometric orthogonal polynomials, *SIGMA* **9** (2013), 057, 28 pages, [arXiv:1212.4766](#).
- [18] Kalnins E.G., Miller Jr. W., Subag E., Bôcher contractions of conformally superintegrable Laplace equations, *SIGMA* **12** (2016), 038, 31 pages, [arXiv:1512.09315](#).
- [19] Koenigs G.X.P., Sur les géodésiques a integrales quadratiques, in *Le cons sur la théorie générale des surfaces*, Vol. 4, Editor J.G. Darboux, Chelsea Publishing, 1972, 368–404.
- [20] Laurent-Gengoux C., Pichereau A., Vanhaecke P., Poisson structures, *Grundlehren der Mathematischen Wissenschaften*, Vol. 347, Springer, Heidelberg, 2013.
- [21] Marquette I., Superintegrability with third order integrals of motion, cubic algebras, and supersymmetric quantum mechanics. II. Painlevé transcendent potentials, *J. Math. Phys.* **50** (2009), 095202, 18 pages, [arXiv:0811.1568](#).
- [22] Miller Jr. W., Post S., Winternitz P., Classical and quantum superintegrability with applications, *J. Phys. A: Math. Theor.* **46** (2013), 423001, 97 pages, [arXiv:1309.2694](#).
- [23] Tempesta P., Winternitz P., Harnad J., Miller Jr. W., Pogosyan G., Rodriguez M. (Editors), Superintegrability in classical and quantum systems, *CRM Proceedings and Lecture Notes*, Vol. 37, Amer. Math. Soc., Providence, RI, 2004.