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Parallelisms & Lie Connections

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Abstract. The aim of this article is to study rational parallelisms of algebraic varieties by means of the transcendence of their symmetries. The nature of this transcendence is measured by a Galois group built from the Picard–Vessiot theory of principal connections.

Key words: parallelism; isogeny; G-structure; linear connection; principal connection; differential Galois theory

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1 Introduction

The aim of this article is to study rational parallelisms of algebraic varieties by means of the transcendence of their symmetries. Our original motivation was to understand the possible obstructions to the third Lie theorem for algebraic Lie pseudogroups. This article is concerned with the simply transitive case. These obstructions should appear in the Galois group of certain connection associated to a Lie algebroid. However, we have written the article in the language of regular and rational parallelisms of algebraic varieties and their symmetries.

A theorem of P. Deligne says that any Lie algebra can be realized as a parallelism of an algebraic variety. This is a sort of algebraic version of the third Lie theorem. Notwithstanding, there is one main problem: given an algebraic variety with a parallelism, how far is it from being an algebraic group? Is it possible to conjugate this parallelism with the canonical parallelism of invariant vector fields on an algebraic group?

In the analytic context, from the Darboux–Cartan theorem [10, p. 212], a $\mathfrak{g}$-parallelized complex manifold M has a natural (G,G) structure where G is a Lie group with $\mathfrak{lie}(G) = \mathfrak{g}$. The obstruction to be a covering of G, as manifold with a (G,G) structure, is contained in a monodromy group [10, p. 130]. In [12], Wang proved that parallelized compact complex manifolds are biholomorphic to quotients of complex Lie groups by discrete cocompact subgroups. This result has been extended by Winkelmann in [13, 14] for some open complex manifolds.

In this article we address the problem of classification of rational parallelisms on algebraic varieties up to birational transformations. Such a classification seems impossible in the algebraic category but we prove a criterion to ensure that a parallelized algebraic variety is isogenous to an algebraic group. Summarizing, we pursue the following plan: We regard infinitesimal symmetries of a rational parallelism as horizontal sections of a connection that we call the reciprocal Lie connection. This connection has a Galois group which is represented as a group of internal automorphisms of a Lie algebra. The obstruction to the algebraic conjugation to an algebraic group, under some assumptions, appear in the Lie algebra of this Galois group.

In Section 2 we introduce the basic definitions; several examples of parallelisms are given here. In Section 3 we study the properties of connections on the tangent bundle whose local analytic horizontal sections form a sheaf of Lie algebras of vector fields. We call them *Lie connections*. They always come by pairs, and they are characterized by having vanishing curvature and constant torsion (Proposition 3.10). We see that each rational parallelism has an attached pair of Lie connections, one of them with trivial Galois group. We compute the Galois groups of some parallelisms given in examples (Proposition 3.14), and prove that any algebraic subgroup of $\mathrm{PSL}_2(\mathbf{C})$ appears as the differential Galois group of a $\mathfrak{sl}_2(\mathbf{C})$-parallelism (Theorem 3.16). Section 4 is devoted to the construction of the isogeny between a $\mathfrak{g}$-parallelized variety and an algebraic group G whose Lie algebra is $\mathfrak{g}$. In order to do this, we introduce the Darboux–Cartan connection, a G-connection whose horizontal sections are parallelism conjugations. We prove that if $\mathfrak{g}$ is centerless then the Darboux–Cartan connection and the reciprocal Lie connection have isogenous Galois groups. We prove that the only centerless Lie algebras $\mathfrak{g}$ such that there exists a $\mathfrak{g}$-parallelism with a trivial Galois group are algebraic Lie algebras, i.e., Lie algebras of algebraic groups. In particular this allows us to give a criterion for a parallelized variety to be isogenous to an algebraic group. The vanishing of the Lie algebra of the Galois group of the reciprocal connection is a necessary and sufficient condition for a parallelized variety to be isogenous to an algebraic group:

Theorem 4.6. *Let $\mathfrak{g}$ be a centerless Lie algebra. An algebraic variety (M,ω) with a rational parallelism of type $\mathfrak{g}$ is isogenous to an algebraic group if and only if $\mathfrak{gal}(\nabla^{\mathrm{rec}}) = \{0\}$.*

The notion of *isogeny* can be extended beyond the simply-transitive case. Let us consider a complex Lie algebra $\mathfrak{g}$. An *infinitesimally homogeneous variety* of type $\mathfrak{g}$ is a pair $(M,\mathfrak{s})$ consisting of a complex smooth irreducible variety M and a finite-dimensional Lie algebra $\mathfrak{s}\subset\mathfrak{X}(M)$ isomorphic to $\mathfrak{g}$ that spans the tangent bundle of M on the generic point.

We are interested in conjugation by rational or by algebraic maps, so that, whenever necessary, we replace M by a suitable Zariski open subset. In this context, we say that a dominant rational map $f\colon M_1 \dashrightarrow M_2$ between varieties of the same dimension conjugates the infinitesimally homogeneous varieties $(M_1,\mathfrak{s}_1)$ and $(M_2,\mathfrak{s}_2)$ if $f^*(\mathfrak{s}_2)=\mathfrak{s}_1$. We say that $(M_1,\mathfrak{s}_1)$ and $(M_2,\mathfrak{s}_2)$ are *isogenous* if they are conjugated to the same infinitesimally homogeneous space of type $\mathfrak{g}$.

Under some hypothesis on the Lie algebra $\mathfrak{s}\subset\mathfrak{X}(M)$ one can prove that $(M,\mathfrak{s})$ is isogenous to a homogeneous space $(G/H,\mathfrak{lie}(G)^{\mathrm{rec}})$ with the action of right invariant vector fields. These hypothesis are satisfied by transitive actions of $\mathfrak{sl}_{n+1}(\mathbf{C})$ on n-dimensional varieties. As a particular case of Theorem 5.12 one has

Theorem. *Let $(M,\mathfrak{s})$ be an infinitesimally homogeneous variety of complex dimension n such that $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}_{n+1}(\mathbf{C})$. Then there exists a dominant rational map $M \dashrightarrow \mathbf{CP}_n$ conjugating $\mathfrak{s}$ with the Lie algebra $\mathfrak{sl}_{n+1}(\mathbf{C})$ of projective vector fields in $\mathbf{CP}_n$.*

Appendix A is devoted to a geometrical presentation of Picard–Vessiot theory for linear and principal connections. Finally, Appendix B contains a detailed proof of Deligne's theorem of the realization of a regular parallelism modeled over any finite-dimensional Lie algebra. This includes also a computation of the Galois group that turns out to be, for this particular construction, an algebraic torus.

2 Parallelisms

Let M be a smooth connected affine variety over $\mathbf{C}$ of dimension r. We denote by $\mathbf{C}[M]$ its ring of regular functions and by $\mathbf{C}(M)$ its field of rational functions. Analogously, we denote by $\mathfrak{X}[M]$ and $\mathfrak{X}(M)$ respectively the Lie algebras of regular and rational vector fields in M, and so on.

Let $\mathfrak{g}$ be a Lie algebra of dimension r. We fix a basis $A_1, \ldots, A_r$ of $\mathfrak{g}$, and the following notation for the associated structure constants $[A_i, A_j] = \sum_k \lambda_{ij}^k A_k$.

A parallelism of type $\mathfrak{g}$ of M is a realization of the Lie algebra $\mathfrak{g}$ as a Lie algebra of pointwise linearly independent vector fields in M. More precisely:

Definition 2.1. A regular parallelism of type $\mathfrak{g}$ in M is a Lie algebra morphism, $\rho\colon \mathfrak{g} \to \mathfrak{X}[M]$ such that $\rho A_1(x), \ldots, \rho A_r(x)$ form a basis of T_xM for any point x of M.

Example 2.2. Let G be an algebraic group and $\mathfrak{g}$ be its Lie algebra of left invariant vector fields. Then the natural inclusion $\mathfrak{g} \subset \mathfrak{X}[G]$ is a regular parallelism of G. The Lie algebra $\mathfrak{g}^{\mathrm{rec}}$ of right invariant vector fields is another regular parallelism of the same type. Let invariant and right invariant vector fields commute, hence, an algebraic group is naturally endowed with a pair of commuting parallelisms of the same type.

From Example 2.2, it is clear that any *algebraic* Lie algebra is realized as a parallelism of some algebraic variety. On the other hand, Theorem B.1 due to P. Deligne and published in [7], ensures that any Lie algebra is realized as a regular parallelism of an algebraic variety. Analogously, we have the definitions of rational and local analytic parallelism. Note that a rational parallelism in M is a regular parallelism in a Zariski open subset $M^\star \subseteq M$.

There is dual definition, equivalent to that of parallelism. This is more suitable for calculations.

Definition 2.3. A regular parallelism form (or coparallelism) of type $\mathfrak{g}$ in M is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1[M] \otimes_{\mathbb{C}} \mathfrak{g}$ such that:

(1) For any $x \in M$, $\omega_x\colon T_xM \to \mathfrak{g}$ is a linear isomorphism.

(2) If A and B are in $\mathfrak{g}$ and X, Y are vector fields such that $\omega(X) = A$ and $\omega(Y) = B$ then $\omega[X,Y] = [A,B]$.

Analogously, we define local analytic and rational coparallelism of type $\mathfrak{g}$ in M. It is clear that each coparallelism induces a parallelism, and reciprocally, by the relation $\omega(\rho(A)) = A$. Thus, there is a natural equivalence between the notions of parallelism and coparallelism. From now on we fix ρ and ω equivalent parallelism and coparallelism of type $\mathfrak{g}$ on M.

The Lie algebra structure of $\mathfrak{g}$ forces ω to satisfy Maurer–Cartan structure equations

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0.$$

Taking components $\omega = \sum_i \omega_i A_i$ we have

$$\mathrm{d}\omega_i + \sum_{j,k=1}^{r} \frac{1}{2}\lambda_{jk}^i \omega_j \wedge \omega_k = 0.$$

Example 2.4. Let G be an algebraic group and $\mathfrak{g}$ be the Lie algebra of left invariant vector fields in G. Then the structure form ω is the coparallelism corresponding to the parallelism of Example 2.2.

Example 2.5. Let $\mathfrak{g} = \langle A_1, A_2 \rangle$ be the 2-dimensional Lie algebra with commutation relation

$$[A_1, A_2] = A_1.$$

The vector fields

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

define a regular parallelism via $\rho(A_i) = X_i$ of $\mathbf{C}^2$. The associated parallelism form is

$$\omega = A_1 \mathrm{d}x + (A_2 - xA_1)\mathrm{d}y.$$

Example 2.6 (Malgrange)**.** Let $\mathfrak{g} = \langle A_1, A_2, A_3\rangle$ be the 3-dimensional Lie algebra with commutation relations

$$[A_1, A_2] = \alpha A_2, \qquad [A_1, A_3] = \beta A_3, \qquad [A_2, A_3] = 0,$$

with α, β, non zero complex numbers. In particular, if α/β is not rational then $\mathfrak{g}$ is not the Lie algebra of an algebraic group. The vector fields

$$X_1 = \frac{\partial}{\partial x} + \alpha y\frac{\partial}{\partial y} + \beta z\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial z},$$

define a regular parallelism via $\rho(A_i) = X_i$ of $\mathbf{C}^3$. The associated parallelism form is

$$\omega = (A_1 - A_2\alpha y - A_3\beta z)\mathrm{d}x + A_2\mathrm{d}y + A_3\mathrm{d}z.$$

Definition 2.7. Let (M, ω) and (N, θ) be algebraic manifolds with coparallelisms of type $\mathfrak{g}$. We say that they are isogenous if there is an algebraic manifold (P, η) with a coparallelism of type $\mathfrak{g}$ and dominant maps $f\colon P \to M$ and $g\colon P \to N$ such that $f^*(\omega) = g^*(\theta) = \eta$.

Clearly, the notion of isogeny of parallelized varieties extends that of isogeny of algebraic groups.

Example 2.8. Let $f\colon M \dashrightarrow G$ be a dominant rational map with values in an algebraic group with $\dim_{\mathbf{C}} M = \dim_{\mathbf{C}} G$. Then $\theta = f^*(\omega)$ is a rational parallelism form in M.

Example 2.9. Let H be a finite subgroup of the algebraic group G and

$$\pi\colon\ G \to M = H \setminus G = \{Hg\colon g \in G\}$$

be the quotient by the action of H on the left side. The structure form ω in G is left-invariant and then it is projectable by π. Then, $\theta = \pi_*(\omega)$ is a regular parallelism form in M.

Example 2.10. Combining Examples 2.8 and 2.9, let $H \subset G$ be a finite subgroup and $f\colon M \to H\backslash G$ be a dominant rational map between manifolds of the same dimension. Then $\theta = f^*(\pi_*(\omega))$ is a rational parallelism form in M.

Example 2.11. By application of Example 2.10 to the case of the multiplicative group we obtain rational multiples of logarithmic forms in $\mathbf{CP}_1$, $\frac{p}{q}\frac{\mathrm{d}f}{f}$ where $f \in \mathbf{C}(z)$. Thus, rational multiples of logarithmic forms in $\mathbf{CP}_1$ are the rational coparallelisms isogenous to that of the multiplicative group.

Example 2.12. By application of Example 2.10 to the case of the additive group we obtain the exact forms in $\mathbf{CP}_1$, $\mathrm{d}F$ where $F \in \mathbf{C}(z)$. Thus, the exact forms in $\mathbf{CP}_1$ are the rational coparallelisms isogenous to that of the additive group.

Example 2.13. Let H be a subgroup of the algebraic group G, with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Let us assume that $\mathfrak{h}$ admits a supplementary Lie algebra $\mathfrak{h}'$

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}' \qquad \text{(as vector spaces)}.$$

We consider the left quotient $M = H \setminus G$ of G by the action of H and the quotient map $\pi\colon G \to M$. It turns out that $\mathfrak{h}'$ is a Lie algebra of vector fields in G projectable by π, and thus $\pi_*|_{\mathfrak{h}'}\colon \mathfrak{h}' \to \mathfrak{X}[M]$ gives a parallelism of M that is regular in the open subset

$$\{Hg \in M\colon\ \mathrm{Adj}_g(\mathfrak{h}) \cap \mathfrak{h}' = \{0\}\}.$$

It turns out to be regular in M if $H \lhd G$. Examples 2.5 and 2.6 are particular cases where G is $\mathrm{Aff}(2, \mathbf{C})$ and $\mathrm{Aff}(3, \mathbf{C})$ respectively.

Remark 2.14. We can see also Example 2.13 as a coparallelism. Let $\pi'\colon \mathfrak{g} \to \mathfrak{h}'$ be the projection given by the vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$. Since $\pi' \circ \omega$ is left invariant form in G, it is projectable by π. Hence, there is a form ω' in M such that $\pi^*\omega' = \pi' \circ \omega$. This form ω' is the corresponding coparallelism.

3 Associated Lie connection

3.1 Reciprocal connections

Let ∇ be a linear connection (rational or regular) on TM. The reciprocal connection is defined as

$$\nabla^{\mathrm{rec}}_{\vec{X}}\vec{Y} = \nabla_{\vec{Y}}\vec{X} + \big[\vec{X},\vec{Y}\big].$$

From this definition it is clear that the difference $\nabla - \nabla^{\mathrm{rec}} = \mathrm{Tor}_{\nabla}$ is the torsion tensor, $\mathrm{Tor}_{\nabla} = -\,\mathrm{Tor}_{\nabla^{\mathrm{rec}}}$ and $(\nabla^{\mathrm{rec}})^{\mathrm{rec}} = \nabla$.

3.2 Connections and parallelisms

Let ω be a coparallelism of type $\mathfrak{g}$ in M and ρ its equivalent parallelism. Denote by $\vec{X}_i$ the basis of vector fields in M such that $\omega(\vec{X}_i) = A_i$ is a basis of $\mathfrak{g}$.

Definition 3.1. The connection ∇ associated to the parallelism ω is the only linear connection in M for which ω is a ∇-horizontal form.

Clearly ∇ is a flat connection and the basis $\{\vec{X}_i\}$ is a basis of the space of ∇-horizontal vector fields. In this basis ∇ has vanishing Christoffel symbols

$$\nabla_{\vec{X}_i}\vec{X}_j = 0.$$

Let us compute some infinitesimal symmetries of ω. A vector field $\vec{Y}$ is an infinitesimal symmetry of ω if $\mathrm{Lie}_{\vec{Y}}\omega = 0$, or equivalently, if it commutes with all the vector fields of the parallelism

$$\big[\vec{X}_i,\vec{Y}\big] = 0, \qquad i = 1,\ldots,r.$$

Lemma 3.2. *Let ∇ be the connection associated to the parallelism ω. Then for any vector field $\vec{Y}$ and any $j = 1,\ldots,r$*

$$\big[\vec{X}_j,\vec{Y}\big] = \nabla^{\mathrm{rec}}_{\vec{X}_j}\vec{Y}.$$

Thus, $\vec{Y}$ is an infinitesimal symmetry of ω if and only if it is a horizontal vector field for the reciprocal connection ∇^{rec}.

Proof. A direct computation yields the result. Take $\vec{Y} = \sum\limits_{k=1}^{r} f_k\vec{X}_k$, for each j we have

$$\nabla^{\mathrm{rec}}_{\vec{X}_j}\vec{Y} = \sum_{k=1}^{r}\big((\vec{X}_j f_k)\vec{X}_k + f_k\big[\vec{X}_j,\vec{X}_k\big]\big) = \big[\vec{X}_j,\vec{Y}\big]. \qquad \blacksquare$$

The above considerations also give us the Christoffel symbols for ∇^{rec} in the basis $\{\vec{X}_i\}$

$$\nabla^{\mathrm{rec}}_{\vec{X}_i}\vec{X}_j = \big[\vec{X}_i,\vec{X}_j\big] = \sum_{k=1}^{r}\lambda_{ij}^{k}\vec{X}_k,$$

i.e., the Christoffel symbols of ∇^{rec} are the structure constants of the Lie algebra $\mathfrak{g}$.

Lemma 3.3. *Let ∇ be the connection associated to a coparallelism in M. Then, ∇^{rec} is flat, and the Lie bracket of two ∇^{rec}-horizontal vector fields is a ∇^{rec}-horizontal vector field.*

Proof. The flatness and the preservation of the Lie bracket by ∇^{rec} are direct consequences of the Jacobi identity. Let us compute the curvature

$$\begin{aligned} R(\vec{X}_i, \vec{X}_j, \vec{X}_k) &= \nabla^{\mathrm{rec}}_{\vec{X}_i}(\nabla^{\mathrm{rec}}_{\vec{X}_j} X_k) - \nabla^{\mathrm{rec}}_{\vec{X}_j}(\nabla^{\mathrm{rec}}_{\vec{X}_i} \vec{X}_k) - \nabla^{\mathrm{rec}}_{[\vec{X}_i,\vec{X}_j]}\vec{X}_k \\ &= \rho([A_i,[A_j,A_k]] - [A_j,[A_i,A_k]] - [[A_i,A_j],A_k]) = 0. \end{aligned}$$

Let us compute the Lie bracket for $\vec{Y}$ and $\vec{Z}$ ∇^{rec}-horizontal vector fields

$$\nabla^{\mathrm{rec}}_{\vec{X}_i}[\vec{Y},\vec{Z}] = [\vec{X}_i,[\vec{Y},\vec{Z}]] = [[\vec{X}_i,\vec{Y}],\vec{Z}] + [\vec{Y},[\vec{X}_i,\vec{Z}]] = [\nabla^{\mathrm{rec}}_{\vec{X}_i}\vec{Y},\vec{Z}] + [\vec{Y},\nabla^{\mathrm{rec}}_{\vec{X}_i}\vec{Z}] = 0. \quad \blacksquare$$

Lemma 3.4. *Let $x \in M$ be a regular point of the parallelism form ω. The space of germs at x of horizontal vector fields for ∇^{rec} is a Lie algebra isomorphic to $\mathfrak{g}$. Moreover, let $\vec{Y}_1, \dots, \vec{Y}_r$ be horizontal vector fields with initial conditions $\vec{Y}_i(x) = \vec{X}_i(x)$, then $[\vec{Y}_i, \vec{Y}_j] = -\sum_{k=1}^{r} \lambda^k_{ij}\vec{Y}_k$, where the $\lambda_{i,j}$ are the structure constants of the Lie algebra generated by the $\vec{X}_i$.*

Proof. We can write the vector fields $\vec{Y}_i$ as linear combinations of the vector fields $\vec{X}_i$: $\vec{Y}_i = \sum_{j=1}^{r} a_{ji}\vec{X}_j$. The matrix (a_{ij}) satisfies the differential equation

$$\vec{X}_k a_{ij} = -\sum_{\alpha=1}^{r} \lambda^i_{k\alpha} a_{\alpha j}, \qquad a_{ij}(x) = \delta_{ij}.$$

On the other hand, we have $[\vec{Y}_i, \vec{Y}_j](x) = \sum_{k=1}^{r} \hat{\lambda}^k_{ij}\vec{Y}_k(x)$, for certain unknown structure constants $\hat{\lambda}^k_{ij}$. Let us check that $\hat{\lambda}^k_{ij} = \lambda^k_{ji} = -\lambda^k_{ij}$,

$$\begin{aligned} [\vec{Y}_i, \vec{Y}_j] &= \left[\sum_{\alpha=1}^{r} a_{\alpha i}\vec{X}_\alpha, \sum_{\beta=1}^{r} a_{\beta j}\vec{X}_\beta\right] \\ &= \sum_{\alpha,\beta,\gamma=1}^{r} -a_{\alpha i}\lambda^\beta_{\alpha\gamma}a_{\gamma j}\vec{X}_\beta + \sum_{\alpha,\beta,\gamma=1}^{r} a_{\beta j}\lambda^\alpha_{\beta\gamma}a_{\gamma i}\vec{X}_\alpha + \sum_{\alpha,\beta,\gamma=1}^{r} a_{\beta j}a_{\alpha i}\lambda^\gamma_{\alpha\beta}\vec{X}_\gamma. \end{aligned}$$

Taking values at x, we obtain

$$[\vec{Y}_i, \vec{Y}_j](x) = \sum_{\beta=1}^{r} -\lambda^\beta_{ij}\vec{Y}_\beta(x) + \sum_{\alpha=1}^{r} \lambda^\alpha_{ji}\vec{Y}_\alpha(x) + \sum_{\gamma=1}^{r} \lambda^\gamma_{ij}\vec{Y}_\gamma(x) = \sum_{\alpha=1}^{r} \lambda^k_{ji}\vec{Y}_k(x). \quad \blacksquare$$

Example 3.5. Let G be an algebraic group with Lie algebra $\mathfrak{g}$. As seen in Example 2.4 the Maurer–Cartan structure form ω is a coparallelism in G. Let ∇ be the connection associated to this coparallelism. There is another canonical coparallelism, the right invariant Maurer–Cartan structure form ω_{rec}, let us consider $\mathbf{i}\colon G \to G$ the inversion map,

$$\omega_{\mathrm{rec}} = -\mathbf{i}^*(\omega).$$

As may be expected, the connection associated to the coparallelism ω_{rec} is ∇^{rec}. Right invariant vector fields in G are infinitesimal symmetries of left invariant vector fields and vice versa. In this case, the horizontal vector fields of ∇ and ∇^{rec} are regular vector fields.

As shown in the next three examples, symmetries of a rational parallelism are not in general rational vector fields.

Example 3.6. Let us consider the Lie algebra $\mathfrak{g}$ and the coparallelism $\omega = A_1\mathrm{d}x + (A_2 - xA_1)\mathrm{d}y$, of Example 2.5. Let ∇ be its associated connection. In cartesian coordinates, the only non-vanishing Christoffel symbol of the reciprocal connection is $\Gamma_{21}^1 = -1$. A basis of ∇^{rec}-horizontal vector fields is

$$\vec{Y}_1 = e^y \frac{\partial}{\partial x}, \qquad \vec{Y}_2 = \frac{\partial}{\partial y}.$$

Note that they coincide with $\vec{X}_1$, $\vec{X}_2$ at the origin point and $\left[\vec{Y}_1, \vec{Y}_2\right] = -Y_1$.

Example 3.7. Let us consider the Lie algebra $\mathfrak{g}$ and the coparallelism $\omega = (A_1 - \alpha y A_2 - \beta z A_3)\mathrm{d}x + A_2\mathrm{d}y + A_3\mathrm{d}z$ of Example 2.6. Let ∇ be its associated connection. In cartesian coordinates, the only non-vanishing Christoffel symbols of the reciprocal connection are

$$\Gamma_{11}^2 = -\alpha, \qquad \Gamma_{11}^3 = -\beta.$$

A basis of ∇^{rec}-horizontal vector fields is

$$\vec{Y}_1 = \frac{\partial}{\partial x}, \qquad \vec{Y}_2 = e^{\alpha x}\frac{\partial}{\partial y}, \qquad \vec{Y}_3 = e^{\beta x}\frac{\partial}{\partial z}.$$

Note that they coincide with $\vec{X}_1$, $\vec{X}_2$, $\vec{X}_3$ at the origin point and

$$\left[\vec{Y}_1, \vec{Y}_2\right] = -\alpha Y_2, \qquad \left[\vec{Y}_1, \vec{Y}_3\right] = -\beta \vec{Y}_3.$$

Example 3.8. Let us consider the Lie algebra $\mathfrak{g}$ of Example 2.6 and the coparallelism

$$\omega = (A_1 - \alpha y A_2 - \beta z A_3)\frac{\mathrm{d}x}{x} + A_2\mathrm{d}y + A_3\mathrm{d}z.$$

Let ∇ be its associated connection. In cartesian coordinates, the only non-vanishing Christoffel symbols of the reciprocal connection are

$$\Gamma_{11}^2 = -\alpha, \qquad \Gamma_{11}^3 = -\beta.$$

A basis of ∇^{rec}-horizontal vector fields on a simply connected open subspace $U \subset \mathbb{C}^* \times \mathbb{C}^2$ is

$$\vec{Y}_1 = x\frac{\partial}{\partial x}, \qquad \vec{Y}_2 = x^{\alpha}\frac{\partial}{\partial y}, \qquad \vec{Y}_3 = x^{\beta}\frac{\partial}{\partial z}.$$

3.3 Lie connections

The connections ∇ and ∇^{rec} associated to a coparallelism ω of type $\mathfrak{g}$ are particular cases of the following definition.

Definition 3.9. A Lie connection (regular or rational) in M is a flat connection ∇ in TM such that the Lie bracket of any two horizontal vector fields is a horizontal vector field.

Given a Lie connection ∇ in M, there is a r-dimensional Lie algebra $\mathfrak{g}$ such that the space of germs of horizontal vector fields at a regular point x is a Lie algebra isomorphic to $\mathfrak{g}$. We will say that ∇ is a Lie connection of type $\mathfrak{g}$. The following result gives several algebraic characterizations of Lie connections:

Proposition 3.10. *Let ∇ be a linear connection in TM, the following statements are equivalent:*

(1) ∇ *is a Lie connection;*

(2) ∇^{rec} *is a Lie connection;*

(3) ∇ *is flat and has constant torsion,* $\nabla \operatorname{Tor}_\nabla = 0$*;*

(4) ∇ *and* ∇^{rec} *are flat.*

Proof. Let us first see (1)⇔(2). Let ∇ be a Lie connection. Around each point of the domain of ∇ there is a parallelism, by possibly transcendental vector fields, such that ∇ is its associated connection. Then, Lemma 3.3 states (1)⇒(2). Taking into account that $(\nabla^{\mathrm{rec}})^{\mathrm{rec}} = \nabla$ we have the desired equivalence.

Let us see now that (1)⇔(3). Let us assume that ∇ is a flat connection. For any three vector fields X, Y, Z in M we have

$$(\nabla_X \operatorname{Tor}_\nabla)(Y,Z) = -\operatorname{Tor}_\nabla(\nabla_X Y, Z) - \operatorname{Tor}_\nabla(Y, \nabla_X Z) + \nabla_X \operatorname{Tor}_\nabla(Y,Z).$$

Let us assume that Y and Z are ∇-horizontal vector fields. Then, we have

$$\operatorname{Tor}_\nabla(Y,Z) = \nabla_Y Z - \nabla_Z Y - [Y,Z] = -[Y,Z]$$

and the previous equality yields

$$(\nabla_X \operatorname{Tor}_\nabla)(Y,Z) = -\nabla_X[Y,Z].$$

Thus, we have that $\nabla \operatorname{Tor}_\nabla$ vanishes if and only if the Lie bracket of any two ∇-horizontal vector fields is also ∇-horizontal. This proves (1)⇔(3).

Finally, let us see (1)⇔(4). It is clear that (1) implies (4) so we only need to see (4)⇒(1). Assume ∇ and ∇^{rec} are flat. Then, locally, there exist a basis $\{\vec{X}_i\}$ of ∇-horizontal vector fields and a basis $\{\vec{Y}_i\}$ of ∇^{rec}-horizontal vector fields. By the definition of the reciprocal connection, we have that a vector field $\vec{X}$ is ∇-horizontal if and only if it satisfies $[\vec{X}, \vec{Y}_i] = 0$ for $i = 1, \dots, r$. By the Jacobi identity we have

$$\big[[\vec{X}_i, \vec{X}_j], \vec{Y}_k\big] = 0.$$

The Lie brackets $\big[\vec{X}_i, \vec{X}_j\big]$ are also ∇-horizontal and ∇ is a Lie connection. ■

Lemma 3.11. *Let* ∇ *be a Lie connection on* M*. Let* x *be a regular point and* $\vec{X}_1, \dots, \vec{X}_r$ *and* $\vec{Y}_1, \dots, \vec{Y}_r$ *be basis of horizontal vector field germs on* M *for* ∇ *and* ∇^{rec} *respectively with same initial conditions* $\vec{X}_i(x) = \vec{Y}_i(x)$*. Then*

$$\big[\vec{X}_i, \vec{X}_j\big](x) = -\big[\vec{Y}_i, \vec{Y}_j\big](x).$$

It follows that ∇ *and* ∇^{rec} *are of the same type* $\mathfrak{g}$*.*

Proof. By definition ∇ is the connection associated to the local analytic parallelism given by the basis $\{\vec{X}_i\}$ of horizontal vector fields. Then we apply Lemma 3.4 in order to obtain the desired conclusion. ■

3.4 Some results on Lie connections by means of Picard–Vessiot theory

Definitions and general results concerning the Picard–Vessiot theory of connections are given in Appendix A.

Proposition 3.12. *Let* ∇ *be a rational Lie connection in* TM*. The* ∇*-horizontal vector fields are the symmetries of a rational parallelism of* M *if and only if* $\operatorname{Gal}(\nabla^{\mathrm{rec}}) = \{1\}$*.*

Proof. We will use the notations of Section A.6: $R^1(TM)$ is the $\mathrm{GL}_n(\mathbb{C})$-principal bundle associated to TM and $\mathcal{F}'$ is the $\mathrm{GL}_n(\mathbb{C})$-invariant foliation on $R^1(TM)$ given by graphs of local basis of ∇-horizontal sections. The Galois group $\mathrm{Gal}(\nabla^{\mathrm{rec}})$ can be computed as soon as we know the Zariski closure $\overline{\mathscr{L}}$ of a leaf $\mathscr{L}$ of the induced foliation $\mathcal{F}'$ on $R^1(TM)$. $\mathrm{Gal}(\nabla^{\mathrm{rec}})$ is finite is and only if $\overline{\mathscr{L}} = \mathscr{L}$ and is $\{1\}$ if and only if $\mathscr{L}$ is the graph of a rational section $M \to R^1(TM)$. This means that there exists a basis of rational ∇^{rec}-horizontal sections. These sections give the desired parallelism. ■

Proposition 3.13. *For any Lie connection* ∇*,* $\mathrm{Gal}(\nabla) \subseteq \mathrm{Aut}(\mathfrak{g})$*.*

Proof. Let us choose a point $x \in M$ regular for ∇ and a basis $A_1, \dots, A_r$ of $\mathfrak{g}$, i.e., a basis $Y_1, \dots, Y_r$ of local ∇-horizontal section of TM at x.

Using notation of Section A.6, we will identify $R^1(T_xM)$ with the set of isomorphisms of linear spaces $\sigma\colon \mathfrak{g} \to T_xM$; now $\mathrm{Gal}(\nabla) \subseteq \mathrm{GL}(\mathfrak{g})$. Because of the construction of $\mathfrak{g}$, we have a canonical point in $R^1(TM)$ corresponding to the identity $\sigma_o\colon \mathfrak{g} \to T_xM$.

For $m \in M$, if σ is an isomorphism from $\mathfrak{g}$ to T_mM then one defines $H^k_{i,j}(\sigma)$ to be

$$\left.\frac{[X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_k} \wedge \cdots \wedge X_r}{X_k \wedge X_1 \wedge \cdots \wedge \widehat{X_k} \wedge \cdots \wedge X_r}\right|_m,$$

where X_i is the horizontal section such that $X_i(m) = \sigma A_i$. These functions are regular functions on $R^1(TM)$. Moreover they are constant and equal to the constant structures on the Zariski closure of the leaf passing through σ_o. The Galois group is the stabilizer of this leaf then the functions $H^k_{i,j}$ are invariant under the action of the Galois group, i.e., the Galois group preserves the Lie bracket. ■

Proposition 3.14. *Let* $\mathfrak{h}'$ *be a Lie sub-algebra of the Lie algebra of some algebraic group and let* G *be the smallest algebraic subgroup such that* $\mathfrak{lie}(G) = \mathfrak{g} \supset \mathfrak{h}'$*. Assume the existence of an algebraic subgroup* H *of* G *whose Lie algebra* $\mathfrak{h}$ *is supplementary to* $\mathfrak{h}'$ *in* $\mathfrak{g}$*,* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$*. Let us consider the following objects:*

- (*a*) *the quotient map* $\pi\colon G \to M$ *where* M *is the variety of cosets* $H \setminus G$*, and* ∇ *the Lie connection associated to the parallelism* $\pi_*\colon \mathfrak{h}' \to \mathfrak{X}[M]$ *in* M (*as given in Example 2.13*);
- (*b*) *its reciprocal Lie connection* ∇^{rec} *on* M*;*
- (*c*) *the Lie algebras of right invariant vector fields*

  $$\mathfrak{g}^{\mathrm{rec}} = \mathbf{i}_*(\mathfrak{g}), \qquad \mathfrak{h}'^{\mathrm{rec}} = \mathbf{i}_*(\mathfrak{h}'),$$

  where $\mathbf{i}$ *is the inverse map on* G*.*

Then, the following statements are true:

- (*i*) $\mathfrak{h}'$ *is an ideal of* $\mathfrak{g}$ (*equivalently* $\mathfrak{h}'^{\mathrm{rec}}$ *is an ideal of* $\mathfrak{g}^{\mathrm{rec}}$)*;*
- (*ii*) $\mathfrak{h}$ *is commutative* (*equivalently* H *is virtually abelian*)*;*
- (*iii*) *the adjoint action of* G *on* $\mathfrak{g}^{\mathrm{rec}}$ *preserves* $\mathfrak{h}'^{\mathrm{rec}}$ *and thus gives, by restriction, a morphism* $\overline{\mathrm{Adj}}\colon G \to \mathrm{Aut}(\mathfrak{h}'^{\mathrm{rec}})$*;*
- (*iv*) *The Galois group of the connection* ∇^{rec} *is* $\overline{\mathrm{Adj}}(H) \subseteq \mathrm{Aut}(\mathfrak{h}'^{\mathrm{rec}})$ *and thus virtually abelian.*

Proof. We have that $\mathfrak{g}$ is the algebraic hull of $\mathfrak{h}'$. From Lemma B.3 in Appendix B we obtain $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}'$. Statement (i) follows straightforwardly. Let us consider A and B in $\mathfrak{h}$. Then $[A, B]$ is in $\mathfrak{h}$ and also in $\mathfrak{h}'$ by the previous argument. Thus, $[A, B] = 0$ and this finishes the proof of statement (ii). Let us denote by H' the subgroup of G spanned by the image of $\mathfrak{h}'$ by the exponential map. For each element $h \in H'$, the adjoint action of h preserves the Lie algebra $\mathfrak{h}'$. By continuity of the adjoint action in the Zariski topology, we have that $\mathfrak{h}'$ is preserved by the adjoint action of all elements of G. This proves statement (iii). In order to prove the last statement in the proposition we have to construct a Picard–Vessiot extension for the connection ∇^{rec}. Let us consider a basis $\{A_1, \dots, A_m\}$ of $\mathfrak{h}'$ and let $\bar{A}_i$ be the projection $\pi_*(A_i)$. We have an extension of differential fields

$$(\mathbf{C}(M), \bar{\mathcal{D}}) \subseteq (\mathbf{C}(G), \mathcal{D}),$$

where $\bar{\mathcal{D}}$ stands for the $\mathbf{C}(M)$-vector space of derivations spanned by $\bar{A}_1, \dots, \bar{A}_m$ and $\mathcal{D}$ stands for the $\mathbf{C}(G)$-vector space of derivations spanned by $A_1, \dots, A_m$ (see Appendix A for our conventions on differential fields).

The projection π is a principal H-bundle. Any rational first integral of $\{A_1, \dots, A_m\}$ is constant along H' and thus it is necessarily a complex number. Thus, the above extension has no new constants and it is strongly normal in the sense of Kolchin, with Galois group H. Note that the differential field automorphism corresponding to an element $h \in H$ is the pullback of functions by the left translation L_h^{-1}, that is, $(hf)(g) = f\big(h^{-1}g\big)$.

The horizontal sections for the connection ∇^{rec} are characterized by the differential equations

$$[\bar{A}_i, X] = 0. \tag{3.1}$$

Let us consider $\{B_1, \dots, B_m\}$ a basis of $\mathfrak{h}'^{\mathrm{rec}}$. From the Zariski closedness of H in G it follows that there are regular functions $f_{ij} \in \mathbf{C}[G]$ such that $B_i = \sum\limits_{j=1}^{m} f_{ij} A_j$. Thus let us define $\bar{B}_i = \sum\limits_{j=1}^{m} f_{ij} \bar{A}_j$. Those objects are vector fields in M with coefficients in $\mathbf{C}[G]$, and clearly satisfy equation (3.1). Thus, the Picard–Vessiot extension of ∇^{rec} is spanned by the functions f_{ij} and it is embedded, as a differential field, in $\mathbf{C}(G)$. Let us denote such extension by $\mathbf{L}$. We have a chain of extensions

$$\mathbf{C}(M) \subseteq \mathbf{L} \subseteq \mathbf{C}(G).$$

By Galois correspondence, the Galois group of ∇^{rec} is a quotient H/K where K is the subgroup of elements of H that fix, by left translation, the functions f_{ij}. In order to prove statement (iv) we need to check that this group K is the kernel of the morphism $\overline{\mathrm{Adj}}$.

Let us note that the image under the adjoint action by $g \in G$ of an element $B \in \mathfrak{h}'^{\mathrm{rec}}$ is given by the left translation, $\overline{\mathrm{Adj}}(g)(B) = L_{g*}(B)$. This transformation makes sense for any derivation of $\mathbf{C}[G]$, and thus we have an action of G on $\mathfrak{X}(G)$. Let us take h in the kernel of $\overline{\mathrm{Adj}}$, thus, $\overline{\mathrm{Adj}}(h)(B_j) = B_j$ for any index j. Applying the transformation L_{h*} to the expression of B_i as linear combination of the left invariant vector fields A_j we obtain $B_i = \sum\limits_{j=1}^{m} L_{h*}(f_{ij}A_j) = \sum\limits_{j=1}^{m} h(f_{ij})A_j$. The coefficients of B_i as linear combination of the A_j are unique, and thus, $h(f_{ij}) = f_{ij}$ we conclude that h is an automorphism fixing $\mathbf{L}$. On the other hand, let us take $h \in H$ fixing $\mathbf{L}$. Then $L_{h*}\big(\sum f_{ij}A_j\big) = \sum f_{ij}A_j$ thus $\overline{\mathrm{Adj}}(h)(B_i) = B_i$ and then h is in the kernel of $\overline{\mathrm{Adj}}$. ∎

3.5 Some examples of $\mathfrak{sl}_2$-parallelisms

We will construct some parallelized varieties as subvarieties of the arc space of the affine line $\mathbb{A}^1_{\mathbf{C}}$. This family of examples show how to realize every subgroup of $\mathrm{PSL}_2(\mathbf{C})$ as the Galois group of the reciprocal Lie connection.

3.5.1 The arc space of the affine line and its Cartan 1-form

In our special case, the arc space of the affine line $\mathbb{A}^1_{\mathbf{C}}$ with affine coordinate z, is the space of all formal power series $\widehat{z} = \sum z^{(i)} \frac{x^i}{i!}$. It will be denoted by $\mathscr{L}$, its ring of regular functions is $\mathbf{C}[\mathscr{L}] = \mathbf{C}\big[z^{(0)}, z^{(1)}, z^{(2)}, \dots\big]$. For an open subset $U \subset \mathbf{C}$ one denotes by $\mathscr{L}U$ the set of power series $\widehat{z}$ with $z^{(0)} \in U$.

A biholomorphism $f\colon U \to V$ between open sets of $\mathbf{C}$ can be lift to a biholomorphism $\mathscr{L}f\colon \mathscr{L}U \to \mathscr{L}V$ by composition $\widehat{z} \to f \circ \widehat{z}$.

Let $\widehat{\mathfrak{X}}$ be the Lie algebra of formal vector fields $\mathbf{C}[[x]]\frac{\partial}{\partial x}$. One can build a rational form $\sigma\colon T\mathscr{L} \to \widehat{\mathfrak{X}}$ in following way (see [5, Section 2]). Let $v = \sum a_i \frac{\partial}{\partial z^{(i)}}$ be a tangent vector at the formal coordinate $\widehat{p}$, i.e., an arc in the Zariski open subset $\{z^{(1)} \neq 0\}$. The local coordinate $\widehat{p}$ can be used to have formal coordinates $p_0, p_1, p_2, \dots$, on $\mathscr{L}$ and v can be written $v = \sum b_i \frac{\partial}{\partial p_i}$. The form σ is defined by $\sigma(v) = \sum b_i \frac{x^i}{i!}\frac{\partial}{\partial x}$. This form is rational and is an isomorphism between $T_p\mathscr{L}$ and $\widehat{\mathfrak{X}}$ satisfying $d\sigma = -\frac{1}{2}[\sigma,\sigma]$ and $(\mathscr{L}f)^*\sigma = \sigma$ for any biholomorphism f.

This means that σ provides an action of $\widehat{\mathfrak{X}}$ commuting with the lift of biholomorphisms. This form seems to be a coparallelism but it is not compatible with the natural structure of pro-variety of $\mathscr{L}$ and $\widehat{\mathfrak{X}}$: $\sigma^{-1}\big(\frac{\partial}{\partial x}\big) = \sum\limits_{i\geq 0} z^{(i+1)}\frac{\partial}{\partial z^{(i)}}$ is a derivation of degree $+1$ with respect to the pro-variety structure of $\mathscr{L}$. The total derivation above will be denoted by E_{-1}. This gives a differential structure to the ring $\mathbf{C}[\mathscr{L}]$.

3.5.2 The parallelized varieties

Let $\nu \in \mathbf{C}(z)$ be a rational function, f be the rational function on the arc space given by the Schwarzian derivative

$$f\big(z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}\big) = \frac{z^{(3)}}{z^{(1)}} - \frac{3}{2}\left(\frac{z^{(2)}}{z^{(1)}}\right)^2 + \nu\big(z^{(0)}\big)\big(z^{(1)}\big)^2,$$

and $I \subset \mathbf{C}[\mathscr{L}]$ be the E_{-1}-invariant ideal generated by $p\big(z^{(0)}\big){z^{(1)}}^2 f\big(z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}\big)$ where p is a minimal denominator of ν.

Lemma 3.15. *The zero set V of I is a dimension 3 subvariety of $\mathscr{L}$ and $\omega(TV) = \mathfrak{sl}_2(\mathbf{C}) \subset \widehat{\mathfrak{X}}$. This provides a $\mathfrak{sl}_2$-parallelism on V.*

Proof. One can compute explicitly this parallelism using $z^{(0)}$, $z^{(1)}$ and $z^{(2)}$ as étale coordinates on a Zariski open subset of V. Let us first compute the $\mathfrak{sl}_2$ action on $\mathscr{L}$. The standard inclusion of $\mathfrak{sl}_2$ in $\widehat{\mathfrak{X}}$ is given by $E_{-1} = \frac{\partial}{\partial x}$, $E_0 = x\frac{\partial}{\partial x}$ and $E_1 = x^2\frac{\partial}{\partial x}$. Their actions on $\mathscr{L}$ are given by $E_{-1} = \sum z^{(i+1)}\frac{\partial}{\partial z^{(i)}}$, $E_0 = \sum iz^{(i)}\frac{\partial}{\partial z^{(i)}}$ and $E_1 = \sum i(i-1)z^{(i-1)}\frac{\partial}{\partial z^{(i)}}$. The ideal I is generated by the functions $E_{-1}^n \cdot f$. By definition $E_{-1}\cdot f \in I$, a direct computation gives that $E_0 \cdot f = 2f \in I$, $E_1 \cdot f = 0 \in I$. The relations in $\mathfrak{sl}_2$ give that $E_{-1}\cdot I \subset I$, $E_0 \cdot I \subset I$ and $E_1 \cdot I \subset I$, i.e., the vector fields E_{-1}, E_0 and E_1 are tangent to V. ■

Now parameterizing V by $z^{(0)}$, $z^{(1)}$ and $z^{(2)}$ one gets

$$E_{-1}|_{\mathbf{C}^3} = z^{(1)}\frac{\partial}{\partial z^{(0)}} + z^{(2)}\frac{\partial}{\partial z^{(1)}} + \left(-\nu\big(z^{(0)}\big)\big(z^{(1)}\big)^3 + \frac{3}{2}\frac{(z^{(2)})^2}{z^{(1)}}\right)\frac{\partial}{\partial z^{(2)}},$$

$$E_0|_{\mathbf{C}^3} = z^{(1)}\frac{\partial}{\partial z^{(1)}} + 2z^{(2)}\frac{\partial}{\partial z^{(2)}}, \qquad E_1|_{\mathbf{C}^3} = 2z^{(1)}\frac{\partial}{\partial z^{(2)}}.$$

They form a rational $\mathfrak{sl}_2$-parallelism on $\mathbf{C}^3$ depending on the choice of a rational function in one variable.

3.5.3 Symmetries and the Galois group of the reciprocal connection

Theorem 3.16. *Any algebraic subgroup of* $\mathrm{PSL}_2(\mathbf{C})$ *can be realized as the Galois group of the reciprocal connection of a parallelism of* $\mathbf{C}^3$.

Proof. A direct computation shows that $z \mapsto \varphi(z)$ is an holomorphic function satisfying

$$\frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 + \nu(\varphi)(\varphi')^2 = \nu(z),$$

if and only if its prolongation $\mathscr{L}\varphi\colon \widehat{z} \mapsto \varphi(\widehat{z})$ on the space $\mathscr{L}$ preserves V and preserves each of the vector fields E_{-1}, E_0 and E_1.

Taking infinitesimal generators of this pseudogroup, one gets for any local analytic solution of the linear equation

$$a''' + 2\nu a' + \nu' a = 0, \tag{3.2}$$

a vector field $X = a(z)\frac{\partial}{\partial z}$ whose prolongation on $\mathscr{L}$ is

$$\mathscr{L}X = a\big(z^{(0)}\big)\frac{\partial}{\partial z^{(0)}} + a'\big(z^{(0)}\big)z^{(1)}\frac{\partial}{\partial z^{(1)}} + \big(a''\big(z^{(0)}\big)\big(z^{(1)}\big)^2 + a'\big(z^{(0)}\big)z^{(2)}\big)\frac{\partial}{\partial z^{(2)}} + \cdots.$$

The equation (3.2) ensures that $\mathscr{L}X$ is tangent to V. The invariance of σ, $(\mathscr{L}X)_*\sigma = 0$, ensures that $\mathscr{L}X$ commutes with the $\mathfrak{sl}_2$-parallelism given above. This means that for any solution a of (3.2) the vector field

$$a\big(z^{(0)}\big)\frac{\partial}{\partial z^{(0)}} + a'\big(z^{(0)}\big)z^{(1)}\frac{\partial}{\partial z^{(1)}} + \big(a''\big(z^{(0)}\big)\big(z^{(1)}\big)^2 + a'\big(z^{(0)}\big)z^{(2)}\big)\frac{\partial}{\partial z^{(2)}},$$

commutes with $E_{-1}|_{\mathbf{C}^3}$, $E_0|_{\mathbf{C}^3}$ and $E_1|_{\mathbf{C}^3}$.

Then the linear differential system of flat section for the reciprocal connection reduces to the linear equation (3.2). This equation is the second symmetric power of $y'' = \nu(z)y$. If $G \subset \mathrm{SL}_2(\mathbf{C})$ is the Galois group of $y'' = \nu(z)y$ then the image of its second symmetric power representation $s^2\colon G \to \mathrm{Sym}^2(\mathbf{C}^2)$ is the Galois group of (3.2). The kernel of this representation is $\{\mathrm{Id}, -\mathrm{Id}\}$ then the Galois group of (3.2) is an algebraic subgroup of $\mathrm{PSL}_2(\mathbf{C})$.

Let us remark that, as it follows from its definition, the Galois group of an equation contains the monodromy group. Moreover one can determine the monodromy group of classical differential equations. Hypergeometric equations depend on three complex numbers (a, b, c)

$$z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0,$$

and is equivalent to

$$y'' = \nu(\ell, n, m; z)y,$$

with

$$\nu(\ell, m, n; z) = \frac{\big(1-\ell^2\big)}{4z^2} + \frac{1-m^2}{4(1-z)^2} + \frac{1-\ell^2-m^2+n^2}{4z(1-z)},$$

and

$$F = z^{-c/2}(1-z)^{(c-a-b-1)/2}y, \qquad \ell = 1-c, \qquad m = c-a-b, \qquad n = a-b.$$

These two equations have the same projectivized Galois group in $\mathrm{PGL}_2(\mathbf{C})$. Any algebraic subgroup of $\mathrm{PGL}_2(\mathbf{C})$ will be realized by an appropriate choice of (a, b, c).

3.5.4 The whole group

For $a = b = 1/2$, $c = 1$, the hypergeometric equation is the Picard–Fuchs equation of Legendre family. Its monodromy group is $\Gamma(2) \subset \mathrm{SL}_2(\mathbf{Z})$ and is Zariski dense in $\mathrm{SL}_2(\mathbf{C})$.

3.5.5 The triangular subgroups

For $b = 0$ and $a = -1$ one can compute a basis of solutions of the equation: 1 and $\int \left(\frac{1-z}{z}\right)^c \mathrm{d}z$. If c is not rational, the Galois group is the group of invertible matrices $\left[\begin{smallmatrix} u & v \\ 0 & 1 \end{smallmatrix}\right]$. When c is rational then u must be a root of the unity of order the denominators of c. When $c \in \mathbf{Z}$, the Galois group is the group of matrices $\left[\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right]$.

For $b = 0$ and $c = a + 1$ a basis of solution is given by z^{-a} and 1. Its Galois group is a subgroup of the group of matrices $\left[\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right]$. The parameter a is rational if and only if it is a finite subgroup.

3.5.6 The dihedral subgroups

For $c = 1/2$ and $a + b = 0$, a basis of solution is given by $\left(\sqrt{z} + \sqrt{1-z}\right)^a$ and $\left(\sqrt{z} - \sqrt{1-z}\right)^a$. The monodromy group is a dihedral group in $\mathrm{GL}_2(\mathbf{C})$ whose quotients give dihedral subgroups of $\mathrm{PGL}_2(\mathbf{C})$.

3.5.7 The tetrahedral subgroup

This group is the monodromy group of hypergeometric equation for $\ell = 1/3$, $m = 1/2$ and $n = 1/3$. A basis of solution is given by

$$(z-1)^{-1/12}\left(\sqrt{3}\big(z^{1/3}+1\big) \pm 2\sqrt{z^{2/3}+z^{1/3}+1}\right)^{1/4}.$$

3.5.8 The octahedral subgroup

This group is the monodromy group of hypergeometric equation for $\ell = 1/2$, $m = 1/3$ and $n = 1/4$. A basis of solution is given by

$$\begin{aligned}(z-1)^{-1/24}\Big[&\sqrt{3}\Big(\big(\sqrt{z}-1\big)^{1/3} + \big(\sqrt{z}+1\big)^{1/3}\Big)^{1/3} \\ &\pm 2\sqrt{\big(\sqrt{z}-1\big)^{2/3} + (z-1)^{1/3} + \big(\sqrt{z}+1\big)^{2/3}}\,\Big]^{1/4}.\end{aligned}$$

3.5.9 The icosaedral subgroup

This group is the monodromy group of hypergeometric equation for $\ell = 1/2$, $m = 1/3$ and $n = 1/5$. As icosahedral group is not solvable, the solution space is not described using formulas as simple as in preceding examples. ■

4 Darboux–Cartan connections

4.1 Connection of parallelism conjugations

Let ω be a rational coparallelism on M of type $\mathfrak{g}$ and G an algebraic group with Lie algebra of left invariant vector fields $\mathfrak{g}$ and Maurer–Cartan form θ. Denote by $M^\star$ the open subset of M in wich ω is regular. We will study the contruction of conjugating maps between the parallelisms (M, ω) and (G, θ).

Let us consider the trivial principal bundle $\pi\colon P = G\times M \to M$. In this bundle we consider the action of G by right translations $(g,x)*g' = (gg',x)$. Let Θ be the $\mathfrak{g}$-valued form $\Theta = \theta - \omega$ in P.

Definition 4.1. The kernel of Θ is a rational flat invariant connection on the principal bundle $\pi\colon P\to M$. We call it the Darboux–Cartan connection of parallelism conjugations from (M,ω) to (G,θ).

The equation $\Theta = 0$ defines a foliation on P transversal to the fibers at regular points of ω. The leaves of the foliation are the graphs of analytic parallelism conjugations from (M,ω) to (G,θ). By means of differential Galois theory the Darboux–Cartan connection has a Galois group $\mathrm{Gal}(\Theta)$ with Lie algebra $\mathfrak{gal}(\Theta)$. The following facts are direct consequences of the definition of the Galois group:

(a) there is a regular covering map $c\colon (M^\star,\omega)\to(U,\theta)$ with U an open subset of G, and $c^*(\theta)=\omega$ if and only if $\mathrm{Gal}(\Theta)=\{1\}$;

(b) there is a regular covering map $c\colon (M^\star,\omega)\to(U,q_*\theta)$ with U an open subset of G/H, H a group of finite index, and $c^*(q_*\theta|_U)=\omega$ if and only if $\mathfrak{gal}(\Theta)=\{0\}$.

In any case, the necessary and sufficient condition for (M,ω) and (G,θ) to be isogenous parallelized varieties is that $\mathfrak{gal}(\Theta)=\{0\}$.

4.2 Darboux–Cartan connection and Picard–Vessiot

Note that the coparallelism ω gives a rational trivialization of TM as the trivial bundle of fiber $\mathfrak{g}$. In TM we have defined the connection ∇^{rec} whose horizontal vector fields are the symmetries of the parallelism. On the other hand, G acts in $\mathfrak{g}$ by means of the adjoint action. The Cartan-Darboux connection induces then a connection ∇^{adj} in the associated trivial bundle $\mathfrak{g}\times M$ of fiber $\mathfrak{g}$.

Proposition 4.2. *The map*

$$\tilde\omega\colon\ \left(TM,\nabla^{\mathrm{rec}}\right)\to\left(\mathfrak{g}\times M,\nabla^{\mathrm{adj}}\right),\qquad X_x\mapsto(\omega_x(X_x),x)$$

is a birational conjugation of the linear connections ∇^{rec} and ∇^{adj}.

Proof. It is clear that the map $\tilde\omega$ is birational. Let us consider $\{A_1,\dots,A_m\}$ a basis of $\mathfrak{g}$. Let $\rho\colon\mathfrak{g}\to\mathfrak{X}(M)$ be the parallelism associated to the parallelism ω and let us define $X_i=\rho(A_i)$. Then $\{X_1,\dots,X_n\}$ is a rational frame in M and the map $\tilde\omega$ conjugates the vector field X_i with the constant section A_i of the trivial bundle of fiber $\mathfrak{g}$. By definition of the reciprocal connection

$$\nabla^{\mathrm{rec}}_{X_i}X_j=[X_i,X_j].$$

On the other hand, by definition of the adjoint action and application of the covariant derivative as in equation (A.1) of Appendix A.7 we obtain

$$\nabla^{\mathrm{adj}}_{X_i}A_j=[A_i,A_j].$$

Therefore we have that $\tilde\omega$ is a rational morphism of linear connections that conjugates ∇^{rec} with ∇^{adj}. ∎

The following facts follow directly from Proposition 4.2, and basic properties of the Galois group.

Corollary 4.3. *Let us consider the adjoint action* $\mathrm{Adj}\colon \mathrm{G} \to \mathrm{GL}(\mathfrak{g})$ *and its derivative* $\mathrm{adj}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$*. The following facts hold:*

(*a*) $\mathrm{Gal}(\nabla^{\mathrm{rec}}) = \mathrm{Adj}(\mathrm{Gal}(\Theta))$*;*

(*b*) $\mathfrak{gal}(\nabla^{\mathrm{rec}}) = \mathrm{adj}(\mathfrak{gal}(\Theta))$*;*

(*c*) *if* $\mathfrak{g}$ *is centerless then* $\mathfrak{gal}(\nabla^{\mathrm{rec}})$ *is isomorphic to* $\mathfrak{gal}(\Theta)$*;*

(*d*) *assume* $\mathfrak{g}$ *is centerless, then the necessary and sufficient condition for* (M,ω) *and* (G,θ) *to be isogenous is that* $\mathfrak{gal}(\nabla^{\mathrm{rec}}) = \{0\}$*.*

Proof. (a) and (b). First, by Proposition 4.2 we have that $\mathrm{Gal}(\nabla^{\mathrm{rec}}) = \mathrm{Gal}(\nabla^{\mathrm{adj}})$ and so $\mathfrak{gal}(\nabla^{\mathrm{rec}}) = \mathfrak{gal}(\nabla^{\mathrm{adj}})$. By definition ∇^{adj} is the associated connection induced by Θ in the associated bundle $\mathfrak{g} \times M$. This trivial bundle is the associated bundle induced by the adjoint representation $\mathrm{Adj}\colon G \to \mathrm{End}(\mathfrak{g})$. Then, by Theorem A.6, we have $\mathrm{Gal}(\nabla^{\mathrm{rec}}) = \mathrm{Adj}(\mathrm{Gal}(\Theta))$ and $\mathrm{Gal}(\nabla^{\mathrm{rec}}) = \mathrm{Adj}(\mathrm{Gal}(\Theta))$.

(c) It is a direct consequence of (b). The kernel of $\mathrm{adj}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the center of $\mathfrak{g}$.

(d) It follows from the definition of Darboux–Cartan connection (see remarks after Definition 4.1) that the necessary and sufficient condition for (M,ω) and (G,θ) to be isogenous is that $\mathfrak{gal}(\nabla^{\mathrm{rec}}) = \{0\}$. By point (b) we conclude. ■

4.3 Algebraic Lie algebras

Let us consider (M,ω) a rational coparallelism of type $\mathfrak{g}$ with $\mathfrak{g}$ a centerless Lie algebra. We do not assume *a priori* that $\mathfrak{g}$ is an algebraic Lie algebra. The connection ∇^{rec} is, as said in Proposition 4.2, conjugated to the connection in $\mathfrak{g} \times M$ induced by the adjoint action. Note that, in order to define this connection we do not need the group operation but just the Lie bracket in $\mathfrak{g}$. We have an exact sequence

$$0 \to \mathfrak{g}' \to \mathfrak{g} \to \mathfrak{g}^{ab} \to 0,$$

where $\mathfrak{g}'$ is the derived algebra $[\mathfrak{g},\mathfrak{g}]$. Since the Galois group acts by adjoint action, we have that $\mathfrak{g}' \times M$ is stabilized by the connection ∇^{rec} and thus we have an exact sequence of connections

$$0 \to (\mathfrak{g}' \times M, \nabla') \to \left(\mathfrak{g} \times M, \nabla^{\mathrm{rec}}\right) \to \left(\mathfrak{g}^{ab} \times M, \nabla^{ab}\right) \to 0.$$

Lemma 4.4. *The Galois group of* ∇^{ab} *is the identity, therefore* ∇^{ab} *has a basis of rational horizontal sections.*

Proof. By definition, the action of $\mathfrak{g}$ in $\mathfrak{g}^{ab}$ vanishes. Thus, the constant functions $M \to \mathfrak{g}^{ab}$ are rational horizontal sections. ■

Lemma 4.5. *Let* ω *be a rational coparallelism of* M *of type* $\mathfrak{g}$ *with* $\mathfrak{g}$ *a centerless Lie algebra. If* $\mathfrak{gal}(\nabla^{\mathrm{rec}}) = \{0\}$ *then* $\mathfrak{g}$ *is an algebraic Lie algebra.*

Proof. Assume $\mathfrak{g}$ is a linear Lie algebra and et E be the smallest algebraic subgroup such that $\mathrm{Lie}(E) = \mathfrak{e} \supset \mathfrak{g}$. We may assume that E is also centerless. Let $A_1,\ldots,A_r$ be a basis of $\mathfrak{g}$, for $i = 1,\ldots,r$, $X_i = \omega^{-1}(A_i)$. Complete with $B_1,\ldots,B_p$ in such way that $A_1,\ldots,A_r,B_1,\ldots,B_p$ is a basis of $\mathfrak{e}$. We consider in $E \times M$ the distribution spanned by the vector fields $A_i + X_i$. This is a E-principal connection called ∇.

Let $\overline{\nabla}$ be the induced connection via the adjoint representation on $\mathfrak{e} \times M$ then

1) $\overline{\nabla}$ preserves $\mathfrak{g}$ and $\overline{\nabla}|_{\mathfrak{g}} = \nabla^{\mathrm{rec}}$, by hypothesis $\mathfrak{gal}(\overline{\nabla}|_{\mathfrak{g}}) = \{0\}$;

2) if $\widetilde{\nabla}$ is the quotient connection on $\mathfrak{e}/\mathfrak{g}$ then $\mathfrak{gal}(\widetilde{\nabla}) = \{0\}$.

If $\varphi \in \mathfrak{gal}(\overline{\nabla})$ then for any $X \in \mathfrak{g}$, $[X, B_i] \in \mathfrak{g}$ thus $0 = \varphi[X, B_i] = [X, \varphi B_i]$ and φB_i commute with $\mathfrak{g}$. From the second point above $\varphi B \in \mathfrak{g}$. By hypothesis $\varphi B_i = 0$ and $\mathfrak{gal}(\overline{\nabla}) = \{0\}$. The projection on E of an algebraic leaf of ∇ gives an algebraic leaf for the foliation of E by the left translation by $\mathfrak{g}$. This proves the lemma. ■

Theorem 4.6. *Let $\mathfrak{g}$ be a centerless Lie algebra. An algebraic variety (M, ω) with a rational parallelism of type $\mathfrak{g}$ is isogenous to an algebraic group if and only if $\mathfrak{gal}(\nabla^{\mathrm{rec}}) = \{0\}$.*

Proof. It follows directly from Lemma 4.5 and Corollary 4.3. ■

Corollary 4.7. *Let $\mathfrak{g}$ be a centerless Lie algebra. Any algebraic variety endowed with a pair of commuting rational parallelisms of type $\mathfrak{g}$ is isogenous to an algebraic group endowed with its two canonical parallelisms of left and right invariant vector fields.*

Proof. Just note that to have a pair of commuting parallelism is a more restrictive condition than having a parallelism with vanishing Lie algebra of the Galois group of its reciprocal connection. ■

This result can be seen as an algebraic version of Wang result in [12]. It gives the classification of algebraic varieties endowed with pairs of commuting parallelisms. Assuming that the Lie algebra is centerless is not a superfluous hypothesis, note that the result clearly does not hold for abelian Lie algebras. There are rational 1-forms in $\mathbf{CP}_1$ that are not exact (isogenous to $(\mathbf{C}, \mathrm{d}z)$) nor logarithmic (isogenous to $(\mathbf{C}^*, \mathrm{d}\log(z))$). In these examples, the pair of commuting parallelisms is given by twice the same parallelism.

Remark 4.8. Let (M, ω, ω') be a manifold endowed with a pair of commuting parallelism forms of type $\mathfrak{g}$, a centerless Lie algebra. From Lemma 4.5 we have that $\mathfrak{g}$ is an algebraic Lie algebra. We can construct the algebraic group enveloping $\mathfrak{g}$ as follows. We consider the adjoint action

$$\mathrm{adj}\colon\ \mathfrak{g} \hookrightarrow \mathrm{End}(\mathfrak{g}).$$

The algebraic group enveloping $\mathfrak{g}$ is identified with the algebraic subgroup G of $\mathrm{Aut}(\mathfrak{g})$ whose Lie algebra is $\mathrm{adj}(\mathfrak{g})$. From Corollary 4.3(a), we have that $\mathrm{Gal}(\Theta) = \{e\}$. Thus, there is a rational map $f\colon M \to G$ such that $f^*(\theta) = \omega$, where θ is the Maurer–Cartan form of G. We can express explicitly this map in terms of the commuting parallelism forms. For each $x \in M$ in the domain of regularity of the parallelisms, $\omega(x)$ and $\omega'(x)$ are isomorphisms of T_xM with $\mathfrak{g}$. We define

$$f(x) = -\omega(x) \circ \omega'(x)^{-1}.$$

Remark 4.9. In virtue of Corollary 4.7, if $\mathfrak{g}$ is a non-algebraic centerless Lie algebra, there is no algebraic variety endowed with a pair of regular commuting parallelisms of type $\mathfrak{g}$. This limits the possible generalizations of Theorem B.1.

Remark 4.10. B. Malgrange has given in [8] another criterion: If (M, ω) is a parallelized variety and $\mathcal{F}$ is the foliation on $M \times M$ given by $\mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega = 0$. Then (M, ω) is birational to an algebraic group if and only if leaves of $\mathcal{F}$ are graphs of rational maps. The relations with Theorem 4.6 and Corollary 4.7 are the following. One can identify TM with the vertical tangent (i.e., the kernel of $\mathrm{d}\,\mathrm{pr}_2$) along the diagonal in $M \times M$. The diagonal is a leaf of $\mathcal{F}$ and the linearization of $\mathcal{F}$ along the diagonal defines a connection $\nabla_{\mathcal{F}}$ on TM. By construction:

- $\nabla_{\mathcal{F}}$-horizontal sections commute with the parallelism, it is the reciprocal Lie connection;
- if leaves of $\mathcal{F}$ are algebraic then $\nabla_{\mathcal{F}}$-horizontal section are algebraic.

5 Some homogeneous varieties

The notion of *isogeny* can be extended beyond the simply-transitive case. Let us consider a complex Lie algebra $\mathfrak{g}$. An *infinitesimally homogeneous variety* of type $\mathfrak{g}$ is a pair $(M, \mathfrak{s})$ consisting of a complex smooth irreducible variety M and a finite-dimensional Lie algebra $\mathfrak{s} \subset \mathfrak{X}(M)$ isomorphic to $\mathfrak{g}$.

As before, we are interested in conjugation by rational and algebraic maps so that, whenever necessary, we replace M by a suitable Zariski open subset. In this context, we say that a dominant rational map $f\colon M_1 \dashrightarrow M_2$ between varieties of the same dimension conjugates the infinitesimally homogeneous varieties $(M_1, \mathfrak{s}_1)$ and $(M_2, \mathfrak{s}_2)$ if $f^*(\mathfrak{s}_2) = \mathfrak{s}_1$. We say that $(M_1, \mathfrak{s}_1)$ and $(M_2, \mathfrak{s}_2)$ are *isogenous* if they are conjugated to the same infinitesimally homogeneous space of type $\mathfrak{g}$.

Let G be an algebraic group over $\mathbf{C}$, K an algebraic subgroup, $\mathfrak{lie}(G)$ its Lie algebra of left invariant vector fields and $\mathfrak{lie}(G)^{\mathrm{rec}}$ its Lie algebra of right invariant vector fields. A natural example of infinitesimally homogeneous space are the homogeneous spaces G/H endowed with the induced action of the Lie algebra $\mathfrak{lie}(G)^{\mathrm{rec}}$. We want to recognize when a infinitesimally homogeneous space is isogenous to an homogeneous space. We prove that if $\mathfrak{s} \subset \mathfrak{X}(M)$ is a *normal* Lie algebra of vector fields then $(M, \mathfrak{s})$ is isogenous to a homogeneous space. In particular, we prove that any n-dimensional infinitesimally homogeneous space of type $\mathfrak{sl}_{n+1}(\mathbf{C})$ is isogenous to the projective space. Our answer is based on a generalization of the computations done in Section 3.5.

5.1 The $\mathfrak{sl}_2$ case

Theorem 5.1 (Loray–Pereira–Touzet (private communication))**.** *Let $\mathscr{C}$ be a curve with X, Y, H three rational vector fields such that $[X,Y] = H$, $[H,X] = -X$ and $[H,Y] = Y$. Then there exists a rational dominant map $h\colon \mathscr{C} \dashrightarrow \mathbf{CP}_1$ such that $X = h^*\big(\frac{\partial}{\partial z}\big)$, $H = h^*\big(z\frac{\partial}{\partial z}\big)$ and $Y = h^*\big(z^2\frac{\partial}{\partial z}\big)$.*

Their proof is elementary. We outline here a more sophisticated proof in the case $\mathscr{C} = A^1_{\mathbf{C}}$ that will be generalized in the next section.

Proof. Notations are the ones introduced in Section 3.5. $\mathscr{L}$ is the space of parameterized arcs $\widehat{z} = \sum_i z^{(i)} \frac{x^i}{i!}$ on $\mathscr{C}$. The vector space $\mathbf{C}X + \mathbf{C}H + \mathbf{C}Y$ is denoted by $\mathfrak{g}$. Let $r_o\colon (\mathbf{C}, 0) \to A^1_{\mathbf{C}}$ be an arc with $r_o'(0) \neq 0$ and consider $V \subset \mathscr{L}$ defined by

$$V = \{\widehat{z} \in \mathscr{L} \mid \widehat{z}^*\mathfrak{g} = r_o^*\mathfrak{g}\}.$$

Claim 5.2. *This is a 3-dimensional algebraic variety.*

Claim 5.3. *The prolongations $\mathscr{L}X$, $\mathscr{L}Y$ and $\mathscr{L}H$ define a $\mathfrak{sl}_2$-parallelism on V.*

Let us describe the canonical structure of $\mathscr{L}$ (see [5, pp. 11–12] or next section for a different presentation). For k an integer greater or equal to -1, let us consider the vector field on $\mathscr{L}$

$$E_k = \sum_{i \geq k} \frac{i!}{(i-k-1)!} z^{(i-k)} \frac{\partial}{\partial z^{(i)}}.$$

We define a morphism of Lie algebra $\rho\colon \widehat{\mathfrak{X}} \to \mathfrak{X}(\mathscr{L})$ by $x^{k+1}\frac{\partial}{\partial x} \mapsto E_k$ and the adic continuity.

Claim 5.4. *The Cartan form σ (as defined in Section 3.5.1) restricted to V takes values in the Lie algebra $r_0^*(\mathfrak{g})$. It is the parallelism form reciprocal to the parallelism $\mathscr{L}X$, $\mathscr{L}H$ and $\mathscr{L}Y$ of V.*

Using Corollary 4.7, V is isogeneous to $\mathrm{PSL}_2(\mathbf{C})$ as defined in Definition 2.7. For $p \in M$, $V_p = \{\widehat{z} \in V \mid \widehat{z}(0) = p\}$ are homogeneous spaces for the action of $\widetilde{K} = \{\varphi\colon (\mathbf{C}, 0) \to (\mathbf{C}, 0) \,|\, r_o \circ \varphi \in V\}$, i.e., $\mathscr{C} = V/\widetilde{K}$. Let K be the subgroup of $\mathrm{PSL}_2(\mathbf{C})$ of upper triangular matrices.

Claim 5.5. *The actions of $\widetilde{K}$ on V and the right action of K on $\mathrm{PSL}_2(\mathbf{C})$ are conjugated by the isogeny.*

This induces an isogeny between $\mathscr{C}$ and $\mathbf{CP}_1$. Let π_1 and π_2 be the two maps of the isogeny. A local transformation φ such that $\pi_1 \circ \varphi = \pi_1$ satisfies $\varphi^*\pi_1^*(X, H, Y) = \pi_1^*(X, H, Y)$ and the same is true for the push-forward $(\pi_2)_*\varphi$ of φ on $\mathbf{CP}_1$. Then $(\pi_2)_*\varphi$ preserves $\frac{\partial}{\partial z}$ and $z\frac{\partial}{\partial z}$. It is the identity. This finishes the proof. ■

5.2 Some jet spaces

Let M be a n-dimensional affine variety. The space of parameterized subspaces of M is the set of formal maps: $M^{[n]} = \{r\colon (\mathbf{C}^n, 0) \to M\}$. Like the arc space, it has a natural structure of pro-algebraic variety. We will give the construction of its coordinate ring following [1, Section 2.3.2, p. 80]. Let $\mathbf{C}[\partial_1, \dots, \partial_n]$ be the $\mathbf{C}$-vector space of linear partial differential operators with constant coefficients. The coordinate ring of $M^{[n]}$ is $\mathrm{Sym}(\mathbf{C}[M] \otimes \mathbf{C}[\partial_1, \dots, \partial_n])/\mathcal{L}$ where

- the tensor product is a tensor product of $\mathbf{C}$-vector spaces;
- $\mathrm{Sym}(V)$ is the $\mathbf{C}$-algebra generated by the vector space V;
- $\mathbf{C}[M] \otimes \mathbf{C}[\partial_1, \dots, \partial_n]$ has a structure of $\mathbf{C}[\partial_1, \dots, \partial_n]$-module *via* the right composition of differential operators;
- $\mathrm{Sym}(\mathbf{C}[M] \otimes \mathbf{C}[\partial_1, \dots, \partial_n])$ has the induced structure of $\mathbf{C}[\partial_1, \dots, \partial_n]$-algebra;
- the Leibniz ideal $\mathcal{L}$ is the $\mathbf{C}[\partial_1, \dots, \partial_n]$-ideal generated by $fg \otimes 1 - (f \otimes 1)(g \otimes 1)$ for all $(f, g) \in \mathbf{C}[M]^2$ and by $1 - 1 \otimes 1$.

Local coordinates $(z_1, \dots, z_n)$ on M induce local coordinates on $M^{[n]}$ *via* the Taylor expansion of maps r at 0

$$r(x_1 \dots, x_n) = \left(\sum_{\alpha \in \mathbf{N}^n} r_1^\alpha \frac{x^\alpha}{\alpha!}, \dots, \sum_{\alpha \in \mathbf{N}^n} r_n^\alpha \frac{x^\alpha}{\alpha!} \right).$$

One denotes by $z_i^\alpha\colon M^{[n]} \to \mathbf{C}$ the function defined by $z_i^\alpha(r) = r_i^\alpha$. This function is the element $z_i \otimes \partial^\alpha$ in $\mathbf{C}[M^{[n]}]$.

5.2.1 Prolongation of vector fields

Any derivation Y of $\mathbf{C}[M]$ can be trivially extended to a derivation of $\mathrm{Sym}(\mathbf{C}[M] \otimes \mathbf{C}[\partial_1, \dots, \partial_n])$. It preserves the ideal generated by $fg \otimes 1 - (f \otimes 1)(g \otimes 1)$ for all $(f, g) \in \mathbf{C}[M]^2$ and by $1 - 1 \otimes 1$ and commutes with the action of $\mathbf{C}[\partial_1, \dots, \partial_n]$ then it preserves the Leibniz ideal and defines a derivation of $\mathbf{C}[M^{[n]}]$. This derivation is called the prolongation of Y, and it is denoted by $Y^{[n]}$.

The same procedure can be used to define the prolongation of analytic or formal vector fields on M to $M^{[n]}$.

5.2.2 The canonical structure

The jet space $M^{[n]}$ is endowed with a differential structure on its coordinate ring and with a group action by "reparameterizations". The compatibility condition between these two structures is well-known (see [5, pp. 11–23]) and is easily obtained using the construction above.

The action of $\partial_j \colon \mathbf{C}[M^{[n]}] \to \mathbf{C}[M^{[n]}]$ can be written in local coordinates and gives the total derivative operator $\sum_{i,\alpha} z_i^{\alpha+1_j} \frac{\partial}{\partial z_i^\alpha}$. It is the differential structure of the jet space. The pro-algebraic group

$$\Gamma = \left\{\gamma \colon (\mathbf{C}^n, 0) \xrightarrow{\sim} (\mathbf{C}^n, 0);\ \text{formal invertible}\right\}$$

acts on $M^{[n]}$.This action is denoted by $S\gamma(r) = r \circ \gamma$.

These two actions arise from the action of the Lie algebra $\widehat{\mathfrak{X}} = \bigoplus \mathbf{C}[[x_1, \dots, x_n]]\partial_i$ on $M^{[n]}$. This action is described on the coordinate ring in the following way. For $\xi \in \widehat{\mathfrak{X}}$, $f \in \mathbf{C}[M]$ and $P \in \mathbf{C}[\partial_1, \dots, \partial_n]$, we define $\xi \cdot (f \otimes P) = f \otimes (P \circ \xi)|_0$ where the composition is evaluated in 0 in order to get an element of $\mathbf{C}[\partial_1, \dots, \partial_n]$. The action of $\bigoplus \mathbf{C}\partial_i$ is the differential structure. The action of $\widehat{\mathfrak{X}}^0 = \mathfrak{lie}(\Gamma)$, the Lie subalgebra of vector fields vanishing at 0 is the infinitesimal part of the action of Γ.

Theorem 5.6 ([5]). *Let $M^{[n]*}$ be the open subset of submersions. The action above gives a canonical form $\sigma \colon TM^{[n]*} \to \widehat{\mathfrak{X}}$ satisfying:*

- *for any $r \in M^{[n]*}$, σ is a isomorphism from $T_rM^{[n]*}$ to $\widehat{\mathfrak{X}}$;*
- *for any $\gamma \in \Gamma$, $(S\gamma)^*\sigma = \gamma^* \circ \sigma$;*
- *$d\sigma = -\frac{1}{2}[\sigma, \sigma]$.*

These equalities are *not* compatible with the projective systems.

5.3 Normal Lie algebras of vectors fields

Without lost of generality, we should

1) identify $\mathfrak{g}$ with its image in $\mathfrak{X}(M)$;
2) replace M by a Zariski open subvariety on which $\mathfrak{g}$ is defined and of maximal rank at any point.

If $p \in M$ one can identify $\mathfrak{g}$ with a Lie subalgebra of $\widehat{\mathfrak{X}}(M, p)$, the Lie algebra of formal vector fields on M at p.

Definition 5.7. For a Lie subalgebra $\mathfrak{g} \subset \mathfrak{X}[M]$, its normalizer at $p \in M$ is

$$\widehat{N}(\mathfrak{g}, p) = \left\{Y \in \widehat{\mathfrak{X}}(M, p) \,|\, Y, \mathfrak{g}] \subset \mathfrak{g}\right\}.$$

Definition 5.8. A Lie subalgebra $\mathfrak{g} \subset \mathfrak{X}[M]$ is said to be normal if for generic $p \in M$ on has $\widehat{N}(\mathfrak{g}, p) = \mathfrak{g}$.

Lemma 5.9. *If $\mathfrak{g}$ is transitive then the Lie algebra $\widehat{N}(\mathfrak{g}, p)$ is finite-dimensional.*

Proof. Let k be an integer large enough so that the only element of $\mathfrak{g}$ vanishing at order k at p is 0. If $\widehat{N}(\mathfrak{g}, p)$ is not finite-dimensional then there exists a non-zero $Y \in \widehat{N}(\mathfrak{g}, p)$ vanishing at order $k+1$ at p. For $X \in \mathfrak{g}$, the Lie bracket $[Y, X]$ is an element of $\mathfrak{g}$ vanishing at order k at p. It is zero meaning that Y is invariant under the flows of vector fields in $\mathfrak{g}$. The transitivity hypothesis together with $Y(p) = 0$ proves the lemma. ■

Lemma 5.10. *If there exists a point $p \in M$ such that $\mathfrak{g}$ is maximal among finite-dimensional Lie subalgebra of $\widehat{\mathfrak{X}}(M, p)$ then $\mathfrak{g}$ is normal.*

Proof. Because of the preceding lemma, if such a point exists then $\mathfrak{g} = \widehat{N}(\mathfrak{g}, p)$ in $\widehat{\mathfrak{X}}(M, p)$. By transitivity, for any couple of points $(p_1, p_2) \in M^2$ there is a composition of flows of elements of $\mathfrak{g}$ sending p_1 on p_2. These flows preserve $\mathfrak{g}$ thus the equality holds at any p. ■

Example 5.11. Let M be n-dimensional and $\mathfrak{g}$ be a transitive Lie subalgebra of rational vector fields isomorphic to $\mathfrak{sl}_{n+1}(\mathbf{C})$. Then $\mathfrak{g}$ is normal (see [3]).

5.4 Centerless, transitive and normal $\Rightarrow$ isogenous to a homogeneous space

Theorem 5.12. *Let M be a smooth irreducible algebraic variety over $\mathbf{C}$ and $\mathfrak{g}$ be a transitive, centerless, normal, finite-dimensional Lie subalgebra of $\mathfrak{X}(M)$. Then there exists an algebraic group G, an algebraic subgroup $H \subset G$ and an isogeny between $(M, \mathfrak{g})$ and $(G/H, \mathfrak{lie}(G))$. Moreover, if $N_G(\mathfrak{lie}(H)) = H$ then the isogeny is a dominant rational map $M \dashrightarrow G/H$.*

Because of the finiteness and the transitivity, there exists an integer k such that at any $p \in M$ and for any $Y \in \widehat{N}(\mathfrak{g}, p)$, $j_k(Y)(p) \neq 0$, unless $Y = 0$.

Let $r_o \colon (\mathbf{C}^n, 0) \to M$ be an invertible formal map with $r_o(0) = p$ a regular point. Let us consider the subspace of $M^{[n]}$ defined by

$$V = \{r \colon (\mathbf{C}^n, 0) \to M \,|\, r^*\mathfrak{g} = r_o^*\mathfrak{g}\}.$$

Lemma 5.13. *V is finite-dimensional.*

Proof. If $r_o^{-1} \circ r$ is tangent to the identity at order k then the induced automorphism of $\mathfrak{g}$ is the identity. The map $r_o^{-1} \circ r$ fixes p, thus it is the identity. This proves the lemma. ■

Using r_o one can identify the Lie algebra $\widehat{N}(\mathfrak{g}, p)$ with a Lie subalgebra of $\widehat{\mathfrak{X}}$. The latter acts on $M^{[n]}$ as described in Section 5.2.2. As an application of the Theorem 5.6, one gets:

Lemma 5.14. *The restriction of the canonical structure of $M^{[n]}$ gives an parallelism*

$$TV = r_o^*(\widehat{N}(\mathfrak{g}, p)) \times V,$$

called the canonical parallelism.

Lemma 5.15. *The horizontal sections of the reciprocal Lie connection of the canonical parallelism are $Y^{[n]}$ for $Y \in \widehat{N}(\mathfrak{g}, q)$ for $q \in M$.*

Lemma 5.16. *Under the hypothesis of normality of $\mathfrak{g}$, V has two commuting parallelisms of type $\mathfrak{g}$.*

Using Corollary 4.7, $\mathfrak{g}$ is the Lie algebra of an algebraic group G isogeneous to V. V is foliated by the orbits of the subgroup K of Γ stabilizing V. This group is algebraic with Lie algebra $\mathfrak{k} = r_o^*(\mathfrak{g}) \cap \widehat{\mathfrak{X}}^0$. Let $\mathfrak{h} \subset \mathfrak{lie}(G)$ be the Lie algebra corresponding to $\mathfrak{k}$ by the isogeny. Then the orbits of $\mathfrak{h}$ are algebraic. This means that $\mathfrak{h}$ is the Lie algebra of an algebraic subgroup H of G, and that V/K and G/H are isogenous.

Assume that $N_G(\mathfrak{lie}(H)) = H$. If W is the isogeny between V and G. The push-forward of a local analytic deck transformation of $W \to V$ is a transformation of G preserving each element of $\mathfrak{g}$, it is a right translation. A deck transformation preserves the orbits of the pull-back of $\mathfrak{k}$ on W. Its push-forward preserves the orbits of a group containing H with the same Lie algebra. By hypothesis the push-forward is in H and then the isogony obtained by taking the quotient under K and H is the graph of a dominant rational map.

A Picard–Vessiot theory of a principal connection

In the previous reasoning we have used the concept of differential Galois group of a connection. Here, we present a dictionary between invariant connection and strongly normal differential field extension (in the sense of Kolchin). In our setting a differential field is a pair $(\mathcal{K}, \mathcal{D})$ where $\mathcal{K}$ is a finitely generated field over $\mathbf{C}$ and $\mathcal{D}$ is a $\mathcal{K}$ vector space of derivations of $\mathcal{K}$ stable by Lie bracket. The dimension of $\mathcal{D}$ is called the rank of the differential field. Note that we can adapt this notion easily to that of a finite number of commuting derivations by taking a suitable basis of $\mathcal{D}$. However we prefer to consider the whole space of derivations. With our definition a differential field extension $(\mathcal{K},\mathcal{D}) \to (\mathcal{K}',\mathcal{D}')$ is a field extension $\mathcal{K}\subset\mathcal{K}'$ such that each element of $\mathcal{D}$ extends to a unique element of $\mathcal{D}'$ and such extensions span the space $\mathcal{D}'$ as $\mathcal{K}'$-vector space.

A.1 Differential field extensions and foliated varieties

First, let us see that there is a natural dictionary between finitely generated differential fields over $\mathbf{C}$ and irreducible foliated varieties over $\mathbf{C}$ modulo birational equivalence. Let $(M,\mathcal{F})$ be an irreducible foliated variety of dimension n. The distribution $T\mathcal{F}\subset TM$ is of rank $r\leq n$. We denote by $\mathfrak{X}_{\mathcal{F}}$ the space of rational vector fields in $T\mathcal{F}$; it is a $\mathbf{C}(M)$-Lie algebra of dimension r. Hence, the pair $(\mathbf{C}(M),\mathfrak{X}_{\mathcal{F}})$ is a differential field. The field of constants is the field $\mathbf{C}(M)^{\mathcal{F}}$ of rational first integrals of the foliation.

Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliated varieties. A regular (rational) map $\phi\colon (M',\mathcal{F}') \dashrightarrow (M,\mathcal{F})$ is a regular (rational) morphism of foliated varieties if $\mathrm{d}\phi$ induces an isomorphism between $T_x\mathcal{F}'$ and $T_{\phi(x)}\mathcal{F}$ for (generic values of) $x\in M'$. It is clear that $\mathcal{F}'$ and $\mathcal{F}$ have the same rank.

A differential field extension, correspond here to a dominant rational map of irreducible foliated varieties $\phi\colon (M',\mathcal{F}')\dashrightarrow (M,\mathcal{F})$. It induces the extension $\phi^*\colon (\mathbf{C}(M),\mathfrak{X}_{\mathcal{F}})\to(\mathbf{C}(M'),\mathfrak{X}_{\mathcal{F}'})$ by composition with ϕ.

Example A.1. Let $\mathcal{F}$ the foliation of $\mathbf{C}^2$ defined by $\{\mathrm{d}y - y\mathrm{d}x = 0\}$. It corresponds to the differential field $\left(\mathbf{C}(x,e^x),\langle\frac{\mathrm{d}}{\mathrm{d}x}\rangle\right)$.

Remark A.2. Throughout this appendix "connection" means "flat connection".

A.2 Invariant $\mathcal{F}$-connections

Let us consider from now a foliated manifold of dimension n and rank r without rational first integrals $(M,\mathcal{F})$, an algebraic group G and a principal irreducible G-bundle $\pi\colon P\to M$. A G-invariant connection in the direction of $\mathcal{F}$ is a foliation $\mathcal{F}'$ of rank r in P such that:

(a) $\pi\colon (P,\mathcal{F}')\to(M,\mathcal{F})$ is a dominant regular map of foliated varieties;

(b) The foliation $\mathcal{F}'$ is invariant by the action of G in P.

With this definition $(\mathbf{C}(M),\mathfrak{X}_{\mathcal{F}})\to(\mathbf{C}(P),\mathfrak{X}_{\mathcal{F}'})$ is a differential field extension. Also, each element $g\in G$ induces a differential field automorphism of $(\mathbf{C}(P),\mathfrak{X}_{F'})$ that fixes $(\mathbf{C}(M),\mathfrak{X}_F)$ by setting $(g\cdot f)(x) = f(x\cdot g)$.

Let $\mathfrak{g}$ be the Lie algebra of G. There is a way of defining a G-equivariant form $\Theta_{\mathcal{F}'}$ with values in $\mathfrak{g}$, and defined in $\mathrm{d}\pi^{-1}(T\mathcal{F})$ in such way that $T\mathcal{F}'$ is the kernel of $\Theta_{\mathcal{F}'}$. First, there is a canonical form Θ_0 defined in $\ker(d\pi)$ that sends each vertical vector $X_p\in\ker d_p\pi\subset T_pP$ to the element $\mathfrak{g}$ that verifies,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} p\cdot\exp\varepsilon A = X_p.$$

This form is G-equivariant in the sense that $R_g^*(\Theta_0) = \mathrm{Adj}_{g^{-1}}\circ\omega$. We have a decomposition of the vector bundle $\mathrm{d}\pi^{-1}(T\mathcal{F}) = \ker(\mathrm{d}\pi)\oplus T\mathcal{F}'$. This decomposition allows to extend Θ_0 to a form $\Theta_{\mathcal{F}'}$ defined for vectors in $\mathrm{d}\pi^{-1}(T\mathcal{F})$ whose kernel is precisely $T\mathcal{F}$. We call *horizontal frames* to those sections s of π such that $s^*(\Theta_{\mathcal{F}}) = 0$.

A.3 Picard–Vessiot bundle

We say that the principal G-bundle with invariant $\mathcal{F}$-connection $\pi\colon (P, \mathcal{F}') \to (M, \mathcal{F})$ is a *Picard–Vessiot* bundle if there are no rational first integrals of $\mathcal{F}'$. The notion of Picard–Vessiot bundle corresponds exactly to that of primitive extension of Kolchin. In such case G is the group of differential field automorphisms of $(\mathbf{C}(P), \mathfrak{X}_{F'})$ that fix $(\mathbf{C}(M), \mathfrak{X}_F)$ and $(\mathbf{C}(M), \mathfrak{X}_F) \to (\mathbf{C}(P), \mathfrak{X}_{F'})$ is a strongly normal extension. Moreover, any strongly normal extension with constant field $\mathbf{C}$ can be constructed in this way (see [6, Chapter VI, Section 10, Theorem 9]).

One of the most remarkable properties of strongly normal extensions is the Galois correspondence (from [6, Chapter VI, Section 4]).

Theorem A.3 (Galois correspondence)**.** *Assume that $(\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}}) \to (\mathbf{C}(P), \mathfrak{X}_{\mathcal{F}'})$ is strongly normal with group of automorphisms G. Then, there is a bijection between the set of intermediate differential field extensions and algebraic subgroups of G. To each intermediate differential field extension, it corresponds the group of automorphisms that fix such an extension point-wise. To each subgroup of automorphisms it corresponds its subfield of fixed elements.*

A.4 The Picard–Vessiot bundle of an invariant $\mathcal{F}$-connection

Let us consider an irreducible principal G-bundle $\pi\colon (P, \mathcal{F}') \to (M, \mathcal{F})$ endowed with an invariant $\mathcal{F}$-connection $\mathcal{F}'$. We assume that $\mathcal{F}$ has no rational first integrals. A result of Bonnet (see [2, Theorem 1.1]) ensures that for a very generic point in M the leaf passing through such point is Zariski dense in M. Let us consider such a Zariski-dense leaf $\mathcal{L}$ of $\mathcal{F}$ in M. Let us consider any leaf $\mathcal{L}'$ of $\mathcal{F}'$ in P that projects by π onto $\mathcal{L}$. Its Zariski closure is unique in the following sense:

Theorem A.4. *Let $\mathcal{L}'$ and $\mathcal{L}''$ two leaves of $\mathcal{F}'$ whose projections by π are Zariski dense in M. Then, there exist an element $g \in G$ such that $\overline{\mathcal{L}'} \cdot g = \overline{\mathcal{L}''}$.*

Proof. By construction, there is some $x \in \pi(\mathcal{L}') \cap \pi(\overline{\mathcal{L}''})$. Let us consider $p \in \pi^{-1}(\{x\}) \cap \mathcal{L}'$ and $q \in \pi^{-1}(\{x\}) \cap \overline{\mathcal{L}''}$. Since p and q are in the same fiber, there is a unique element $g \in G$ such that $p \cdot g = q$. By the G-invariance of the connection $\mathcal{L}' \cdot g$ is the leaf of $\mathcal{F}'$ that passes through q. The set $\overline{\mathcal{L}''}$ is, by construction, union of leaves of $\mathcal{F}'$ and contains the point q. Thus, $\overline{\mathcal{L}' \cdot g} \subseteq \overline{\mathcal{L}''}$, and $\overline{\mathcal{L}'} \cdot g \subseteq \overline{\mathcal{L}''}$. Now, by exchanging the roles of $\mathcal{L}'$ and $\mathcal{L}''$, we prove that there is an element h such that $\overline{\mathcal{L}''} \cdot h \subseteq \overline{\mathcal{L}'}$. It follows $h = g^{-1}$. This finishes the proof. ■

Let L be the Zariski closure of $\mathcal{L}'$. Let us consider the algebraic subgroup

$$H = \{g \in G\colon L \cdot g = L\}$$

stabilizing L. The projection π restricted to L is dominant, thus there is a Zariski open subset $M^\star$ such that $\pi^\star\colon L^\star \to M^\star$ is surjective. Let us call $\mathcal{F}^\star$ the restriction of $\mathcal{F}'$ to $L^\star$. It follows that the bundle: $\pi^\star\colon (L^\star, \mathcal{F}^\star) \to (M^\star, \mathcal{F}|_{M^\star})$ is a principal bundle of structure group H called Picard–Vessiot bundle. The differential field extension $(\mathbf{C}(M), \mathfrak{X}_{\mathcal{F}}) \to (\mathbf{C}(L^\star), \mathfrak{X}_{\mathcal{F}^\star})$ is the so-called Picard–Vessiot extension associated to the connection. The algebraic group H is the differential Galois group of the connection.

A.5 Split of a connection

Let us consider a pair of morphisms of foliated varieties

$$\phi_j\colon\ (M_j, \mathcal{F}_j) \to (M, \mathcal{F}), \qquad \text{for} \quad j = 1, 2.$$

Then, we can define in $M_1 \times_M M_2$ a foliation $\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ in the following way. A vector $X = (X_1, X_2)$ is in $T(\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2)$ if and only if $\mathrm{d}\phi_1(X_1) = \mathrm{d}\phi_2(X_2) \in T\mathcal{F}$. Let us consider $(P, \mathcal{F}')$ a principal $\mathcal{F}$ connection. Note that the projection

$$\pi_1\colon\ (M_1 \times_M P, \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}') \to (M_1, \mathcal{F}_1)$$

is a principal G-bundle endowed of a $\mathcal{F}_1$-connection. We call this bundle the pullback of $(P, \mathcal{F}')$ by ϕ_1.

We also may consider the trivial G-invariant connection $\mathcal{F}_0$ in the trivial principal G-bundle

$$\pi_0\colon\ (M \times G, \mathcal{F}_0) \to (M, \mathcal{F}),$$

for what the leaves of $\mathcal{F}_0$ are of the form $(\mathcal{L}, g)$ where $\mathcal{L}$ is a leaf of $\mathcal{F}$ and g a fixed element of G. We say that the G-invariant connection $(P, \mathcal{F}')$ is rationally trivial if there is a birational G-equivariant isomorphism of foliated manifolds between $(P, \mathcal{F})$ and $(M \times G, \mathcal{F}_0)$.

Invariant connections are always trivialized after pullback; there is a universal G-equivariant isomorphism defined over P

$$(P \times G, \mathcal{F}' \times_{\mathcal{F}} \mathcal{F}_0) \to (P \times_M P, \mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'), \qquad (p, g) \mapsto (p, p \cdot g),$$

that trivializes any G-invariant connection. However, the differential field $(\mathbf{C}(P), \mathfrak{X}_{\mathcal{F}'})$ may have new constant elements. To avoid this, we replace the pullback to P by a pullback to the Picard–Vessiot bundle $L^\star$

$$(L^\star \times G, \mathcal{F}^\star \times_{\mathcal{F}} \mathcal{F}_0) \to (L^\star \times_M P, \mathcal{F}^\star \times_{\mathcal{F}} \mathcal{F}'), \qquad (p, g) \mapsto (p, p \cdot g).$$

The Picard–Vessiot bundle has some minimality property. It is the smallest bundle on M that trivializes the connection. We have the following result.

Theorem A.5. *Let us consider $\pi\colon (P, \mathcal{F}') \to (M, \mathcal{F})$ be as above, $\pi^\star\colon (L^\star, \mathcal{F}^\star) \to (M, \mathcal{F})$ the Picard–Vessiot bundle, and and $\phi\colon \big(\tilde{M}, \tilde{\mathcal{F}}\big) \to (M, \mathcal{F})$ any dominant rational map of foliated varieties such that:*

- (a) *$\tilde{\mathcal{F}}$ has no rational first integrals in $\tilde{M}$;*
- (b) *the pullback $\big(\tilde{M} \times_M P, \tilde{F} \times_{\mathcal{F}} \mathcal{F}'\big) \to \big(\tilde{M}, \tilde{\mathcal{F}}\big)$ is rationally trivial.*

There is a dominant rational map of foliated varieties $\psi\colon \tilde{M} \dashrightarrow L^\star$ such that $\pi^\star \circ \psi = \phi$ in their common domain.

Proof. Let us take $\tau\colon \tilde{M} \times G \dashrightarrow \tilde{M} \times_M P$ a birational trivialization, $\pi_2\colon \tilde{M} \times_M P \to P$ be the projection in the second factor, and $\iota\colon \tilde{M} \to \tilde{M} \times G$ the inclusion $p \mapsto (p, e)$. Then, $\tilde{\psi} = \pi_2 \circ \tau \circ \iota$ is a rational map from $\tilde{M}$ to P whose differential sends $T\tilde{\mathcal{F}}$ to $T\mathcal{F}$. By Bonnet theorem, $\tilde{M}$ is the Zariski closure of a leaf of $\tilde{\mathcal{F}}$ that projects by ϕ into a Zariski dense leaf of $\mathcal{F}$. From this, $\tilde{\psi}$ contains a dense leaf of $\mathcal{F}'$ in P. By applying a suitable right translation in P and the uniqueness Theorem A.4, we obtain the desired conclusion. ■

A.6 Linear connections

Let $(M, \mathcal{F})$ be as above, of dimension n and rank r. Let $\xi\colon E \to M$ be a vector bundle of rank k. A linear integrable $\mathcal{F}$-connection is a foliation $\mathcal{F}_E$ of rank r which is compatible with the structure of vector bundle in the following sense: the point-wise addition of two leaves of any dilation of a leaf is also a leaf. This can also be stated in terms of a covariant derivative operator ∇ wich is defined only in the direction of $\mathcal{F}$. First, the kernel of $\mathrm{d}\xi$ is naturally projected onto E itself

$$\mathrm{vert}_0\colon\ \ker(\mathrm{d}\xi) \to E, \qquad X_v \mapsto w,$$

where $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} v + \varepsilon w = X_v$. Then, the decomposition of $\mathrm{d}\xi^{-1}(T\mathcal{F})$ as $\ker(\mathrm{d}\xi) \oplus T\mathcal{F}_E$ allows us to extend vert_0 to a projection

$$\mathrm{vert}\colon\ \mathrm{d}\xi^{-1}(T\mathcal{F}) \to E.$$

Thus, we define for each section s its covariant derivative $\nabla s = s^*(\mathrm{vert} \circ \mathrm{d}s|_{T\mathcal{F}})$. This is a 1-form on M defined only for vectors in $T\mathcal{F}$. This covariant derivative has the desired properties, it is additive and satisfies the Leibniz formula

$$\nabla(fs) = \mathrm{d}f|_{T\mathcal{F}} \otimes s + f\nabla s.$$

In general, we write for X a vector in $T\mathcal{F}$, $\nabla_X s$ for the contraction of ∇s with the vector X. It is an element of E over the same base point in M that the vector X. We call *horizontal sections* to those sections s of ξ such that $\nabla s = 0$.

Let $\pi\colon R^1(E) \to M$ be the bundle of linear frames in E. It is a principal linear $\mathrm{GL}_k(\mathbf{C})$-bundle. The foliation $\mathcal{F}_E$ induces a foliation $\mathcal{F}'$ in $R^1(E)$ that is a G-invariant $\mathcal{F}$-connection. Let us consider the Picard–Vessiot bundle, $(L^\star, \mathcal{F}^\star)$. The uniqueness Theorem A.5 on the Picard–Vessiot bundle, can be rephrased algebraically in the following way. The Picard–Vessiot extension $(\mathbf{C}(M), \mathfrak{X}_\mathcal{F}) \to (\mathbf{C}(L^\star), \mathfrak{X}_{\mathcal{F}^\star})$ is characterized by the following properties (cf. [11, Section 1.3]):

(a) there are no new constants, $\mathbf{C}(L^\star) = \mathbf{C}$;

(b) it is spanned, as a field extension of $\mathbf{C}(M)$, by the coefficients of a fundamental matrix of solutions of the differential equation of the horizontal sections.

A.7 Associated connections

Let $\pi\colon (P, \mathcal{F}') \to (M, \mathcal{F})$ be a G-invariant connection, as before, where $\mathcal{F}$ is a foliation in M without rational first integrals. Let us consider $\phi\colon G \to \mathrm{GL}(V)$ a finite-dimensional linear representation of G. It is well known that the associated bundle $\pi_P\colon V_P \to M$,

$$V_P = P \times_G V = (P \times V)/G, \qquad (p \cdot g, v) \sim (p, g \cdot v),$$

is a vector bundle with fiber V. Here we represent the action of G in V by the same operation symbol than before. The G-invariant connection $\mathcal{F}'$ rises to a foliation in $P \times G$ and then it is projected to a foliation $\mathcal{F}_V$ in V_P. In this way, the projection

$$\pi_P\colon\ (V_P, \mathcal{F}_V) \to (M, \mathcal{F}),$$

turns out to be a linear $\mathcal{F}$-connection. It is called the *Lie–Vessiot* connection induced in the associated bundle. The Galois group of the principal and the associated Lie–Vessiot connection are linked in the following way.

Theorem A.6. *Let $H \subset G$ be the Galois group of the principal connection $\mathcal{F}'$. Then, the Galois group of the associated Lie–Vessiot connection $\mathcal{F}_V$ is $\phi(H) \subseteq \mathrm{GL}(V)$.*

Proof. Let us consider the bundle of frames $R^1(V_P)$, with its induced invariant connection $\mathcal{F}''$. Let us fix a basis $\{v_1, \ldots, v_r\}$ of V. Then, we have a map

$$\tilde{\pi}\colon\ P \to R^1(V_P), \qquad p \mapsto ([p, v_1], \ldots, [p, v_r]),$$

where the pair $[p, v]$ represents the class of the pair $(p, v) \in P \times V$. By construction, $\tilde{\pi}$ sends $T\mathcal{F}'$ to $T\mathcal{F}''$. It implies that, if $\mathcal{L}^\star$ is a Picard–Vessiot bundle for $\mathcal{F}'$ then $\tilde{\pi}(L^\star)$ is a Picard–Vessiot bundle for $\mathcal{F}''$. Second, if $\mathcal{L}^\star$ is a principal H bundle, then $\tilde{\pi}(L^\star)$ is a principal H/K bundle where K is the subgroup of H that stabilizes the basis $\{v_1, \ldots, v_r\}$. ■

Let us discuss how the covariant derivative operator in ∇ is defined in terms of $\Theta_{\mathcal{F}'}$ and the action of G in V. Let us denote by $\phi'\colon \mathfrak{g} \to \mathfrak{gl}(V)$ the induced Lie algebra morphism. Let s be a local section of ξ. Let us consider the canonical projection $\bar{\pi}\colon P \times V \to V(P)$. This turns out to be also a principal bundle, here the action on pairs is $(p, v) \cdot g = \big(p \cdot g, g^{-1} \cdot v\big)$. Now we can take any section r of this bundle, and define $\tilde{s} = r \circ s$. As r takes values in a cartesian product, we obtain $\tilde{s} = (s_1, s_2)$ where s_1 is a section of π and s_2 is a function with values in V. Finally we obtain

$$\nabla s = \mathrm{d}s_2|_{T\mathcal{F}} - \phi'(s_1^*(\Theta_{\mathcal{F}'}))(s_2). \tag{A.1}$$

A calculation shows that it does not depend of the choice of r and it is the covariant derivative operator associated to $\mathcal{F}_V$. In particular, if s_2 is already an horizontal frame, then the covariant differential is given by the first term $\mathrm{d}s_s|_{T\mathcal{F}}$.

B Deligne's realization of Lie algebra

The proof of the existence of a regular parallelism for any complex Lie algebra $\mathfrak{g}$ is written in a set of two letters from P. Deligne to B. Malgrange (dated from November of 2005 and February of 2010 respectively) that are published verbatim in [7]. We reproduce here the proof with some extra details.

Theorem B.1 (Deligne)**.** *Given any complex Lie algebra $\mathfrak{g}$ there exist an algebraic variety endowed with a regular parallelism of type $\mathfrak{g}$.*

Lemma B.2. *Let T be an algebraic torus acting regularly by automophisms in some algebraic group H and let $\mathfrak{t}$ be the Lie algebra of T. Let us consider the semidirect product*

$$\mathfrak{t} \ltimes H, \qquad (t, h)(t', h') = (t + t', (\exp(t') \cdot h)h')$$

as an algebraic variety and analytic Lie group. Its left invariant vector fields form a regular parallelism of $\mathfrak{t} \ltimes H$. The Galois group of this parallelism is a torus.

Proof. Let us denote by α the action of T in H and $\alpha'\colon \mathfrak{t} \mapsto \mathfrak{X}[H]$ the Lie algebra isomorphism given by the infinitesimal generators

$$(\alpha' X)_h = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \alpha_{\exp(\varepsilon t)}(h).$$

Let X be an invariant vector field in $\mathfrak{t}$. Let us compute the left invariant vector field in $t \ltimes H$ whose value at the identity is $(X_0, 0)$. In order to perform the computation we write the vector as an infinitesimally near point to $(0, e)$.

$$L_{(t,h)}(0 + \varepsilon X_0, e) = (t + \varepsilon X_t, \alpha_{\exp(\varepsilon X)}(h)) = (t + \varepsilon X_t, h + \varepsilon(\alpha' X)_h).$$

And thus $dL_{(t,h)}(X_0,0) = (X_t,(\alpha'X)_h)$. We conclude that $(X,\alpha'X) \in \mathfrak{X}[\mathfrak{t} \ltimes H]$ is the left invariant vector field whose value at $(0,e)$ is $(X_0,0)$. Let us consider now Y a left invariant vector field in H. Let us compute, as before, the left invariant vector field whose value at (t,h) is $(0,Y_h)$

$$L_{(t,h)}(0,e+\varepsilon Y_e) = (t, L_h(e+\varepsilon Y_e)) = (t, h+\varepsilon Y_h).$$

And thus $(0,Y)$ is the left invariant vector field whose value at $(0,e)$ is $(0,Y_e)$. These vector fields of the form $(X,\alpha'X)$ and $(0,Y)$ are regular and span the Lie algebra of left invariant vector fields in $\mathfrak{t} \ltimes H$. Hence, they form a regular parallelism.

In order to compute the Galois group of the parallelism, let us compute its reciprocal parallelism. It is formed by the right invariant vector fields in the analytic Lie group $\mathfrak{t} \ltimes H$. A similar computation proves that if X is an invariant vector field in $\mathfrak{t}$ then $(X,0)$ is right invariant in $\mathfrak{t} \ltimes H$. For each element $\tau \in T$, α_τ is a group automorphism of H. Thus, it induces a derived automorphism $\alpha_{\tau *}$ of the Lie algebra of regular vector fields in H. Let Y be now a right invariant vector field in H. Let us compute the right invariant vector field Z in $\mathfrak{t} \ltimes H$ whose value at $(0,e)$ is $(0,Y_e)$:

$$R_{(t,h)}(0,e+\varepsilon Y_e) = (t, \alpha_{\exp(t)}(e+\varepsilon Y_e)h) = (t, h+\varepsilon(\alpha_{\exp(t)*}Y)_h)$$

and $Z_{t,h} = (0,(\alpha_{\exp(t)*}Y)_h)$. Those analytic vector fields depend on the exponential function in a torus thus we can conclude, by a standard argument of differential Galois theory, that the associated differential Galois group is a torus. ∎

Let us consider $\mathfrak{g}$ an arbitrary, non algebraic, finite-dimensional complex Lie algebra. We consider an embedding of $\mathfrak{g}$ in the Lie algebra of general linear group and E the smallest algebraic subgroup whose Lie algebra $\mathfrak{e}$ contains $\mathfrak{g}$. E is a connected linear algebraic group.

Lemma B.3 (also in [4, Proposition 1])**.** *With the above definitions and notation* $[\mathfrak{e},\mathfrak{e}] = [\mathfrak{g},\mathfrak{g}]$.

Proof. Let H be the group of matrices that stabilizes $\mathfrak{g}$ and acts trivially on $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Its Lie algebra $\mathfrak{h}$ contains $\mathfrak{g}$ and thus $H \supseteq E$ and $\mathfrak{h} \supseteq \mathfrak{e}$. By definition of H we have $[\mathfrak{h},\mathfrak{g}] = [\mathfrak{g},\mathfrak{g}]$, therefore $[\mathfrak{e},\mathfrak{g}] \subseteq [\mathfrak{g},\mathfrak{g}]$. Let us now consider the group H_1 that stabilizes $\mathfrak{e}$ and $\mathfrak{g}$ and that acts trivially in $\mathfrak{e}/[\mathfrak{g},\mathfrak{g}]$. This is again an algebraic group containing E, and its Lie algebra $\mathfrak{h}_1$ satisfies $[\mathfrak{h}_1,\mathfrak{e}] \subseteq [\mathfrak{g},\mathfrak{g}]$. Taking into account $\mathfrak{e} \subseteq \mathfrak{h}_1$ we have $[\mathfrak{e},\mathfrak{e}] \subseteq [\mathfrak{g},\mathfrak{g}]$. The other inclusion is trivial. ∎

Because of Lemma B.3, the abelianized Lie algebra $\mathfrak{g}^{ab} = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is a subspace of $\mathfrak{e}^{ab} = \mathfrak{e}/[\mathfrak{e},\mathfrak{e}]$. Moreover, if we consider the quotient map, $\pi\colon \mathfrak{e} \to \mathfrak{e}^{ab}$, then $\mathfrak{g} = \pi^{-1}(\mathfrak{g}^{ab})$.

Let us consider an algebraic Levy decomposition $E \simeq L \ltimes U$ (see [9, Chapter 6]). Here, L is reductive and U is the unipotent radical, consisting in all the unipotent elements of E. The semidirect product structure is produced by an action of L in U, so that, $(l_1,u_1)(l_2,u_2) = (l_1l_2,(l_2\cdot u_1)u_2)$.

Since L is reductive, its commutator subgroup L' is semisimple. Let T be the center of L, which is a torus, the map

$$\varphi\colon\ T \times L' \to L, \qquad (t,l) \mapsto tl,$$

is an isogeny. The isogeny defines an action of $T \times L'$ in U by $(t,l)\cdot u = tl\cdot u$. We have found an isogeny

$$(T \times L') \ltimes U \to E.$$

The Lie algebra $\mathfrak{u}$ of U is a nilpotent Lie algebra, so that the exponential map $\exp\colon \mathfrak{u} \to U$ is regular and bijective. In general, if V is an abelian quotient of U with Lie algebra $\mathfrak{v}$ then the exponential map conjugates the addition law in $\mathfrak{v}$ with the group law in V.

Lemma B.4. *With the above definitions and notation, let $\bar{\mathfrak{u}}$ be the biggest quotient of $\mathfrak{u}^{ab}$ in which L acts by the identity. We have a Lie algebra isomorphism $\mathfrak{e}^{ab} \simeq \mathfrak{t} \times \bar{\mathfrak{u}}$.*

Proof. Let us compute $\mathfrak{e}^{ab}$. We compute the commutators $\mathfrak{e}$ by means of the isomorphism $\mathfrak{e} \simeq (\mathfrak{t} \times \mathfrak{l}') \ltimes \mathfrak{u}$. We obtain

$$[(t_1, l_1, u_1), (t_2, l_2, u_2)] = (0, [l_1, l_2], a(t_2, l_2)u_1 + [u_1, u_2]),$$

where a represents the derivative at (e, e) of the action of L in U. From this we obtain that $[\mathfrak{e}, \mathfrak{e}]$ is spanned by $(\{0\} \times \mathfrak{l}') \ltimes \{0\}$, $\{0\} \ltimes [\mathfrak{u}, \mathfrak{u}]$ and $\{0\} \times \langle a(\mathfrak{l})\mathfrak{u}\rangle$. Taking into account that $\bar{\mathfrak{u}}/(\langle a(\mathfrak{l})\mathfrak{u}\rangle + [\mathfrak{u}, \mathfrak{u}])$ is the biggest quotient of $\mathfrak{u}^{ab}$ in which L acts trivially, we obtain the result of the lemma. ∎

Let $\mathfrak{t}$ be the Lie algebra of T. Its exponential map is an analytic group morphism and thus we may consider the analytic action of $\mathfrak{t} \times L'$ in U given by $(t, l) \cdot u = (\exp(t)l) \cdot u$. Let $\tilde{E}$ be the algebraic variety and analytic Lie group $(\mathfrak{t} \times L') \ltimes U$. By application of Lemma B.2, and taking into account that $\tilde{E} \simeq \mathfrak{t} \ltimes H$, where H is the group $L' \cdot U$, we have that the left invariant vector fields in $\tilde{E}$ are regular. Let us consider the projection

$$\pi_1 \colon\ \tilde{E} \to \mathfrak{e}^{ab} = \mathfrak{t} \times \bar{\mathfrak{u}}, \qquad (t, l, u) \mapsto (t, [\log(u)]),$$

this projection is algebraic by construction, and also a morphism of Lie groups. By Lemmas B.3 and B.4, $\mathfrak{g}^{ab}$ is a vector subspace of the image. Then, let us take $\tilde{G}$ the fiber $\pi_1^{-1}(\mathfrak{g}^{ab})$. It is an algebraic submanifold of $\tilde{E}$ and an analytic Lie group. The derivative at the identity of π_1 is precisely the abelianization π and it follows that the Lie algebra of $\tilde{G}$ is precisely $\mathfrak{g}$. Finally $\tilde{G}$ is an algebraic variety with a regular $\mathfrak{g}$-parallelism. This finishes the proof of Theorem B.1.

Remark B.5. The right invariant vector fields in $\tilde{G}$ are constructed as in Lemma B.2 by means of the exponential function in the torus. Hence, Galois groups of the parallelisms obtained via this construction are always tori.

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