

Rigidity and Vanishing Theorems for Almost Even-Clifford Hermitian Manifolds

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Abstract. We prove the rigidity and vanishing of several indices of “geometrically natural” twisted Dirac operators on almost even-Clifford Hermitian manifolds admitting circle actions by automorphisms.

Key words: almost even-Clifford Hermitian manifolds; index of elliptic operator; twisted Dirac operators; circle action by automorphisms

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1 Introduction

There are two classical vanishing theorems for the \hat{A} -genus (the index of the Dirac operator) on Spin manifolds: the Lichnerowicz vanishing [17] which assumes a metric of positive scalar curvature, and the Atiyah–Hirzebruch vanishing [3] which assumes smooth circle action. These vanishings can be seen and have been used frequently as obstructions to the existence of such metrics or actions. More vanishing theorems for the indices of Spin^c Dirac operators were explored by Hattori [11] on almost complex manifolds and Spin^c manifolds with compatible circle actions, which have parallels on complex manifolds with ample line bundles (a positivity condition for certain curvature) as in the case of the Kodaira vanishing theorem. Vanishing theorems have also been proven for indices of twisted Dirac operators on compact quaternion-Kähler manifolds with positive scalar curvature [16], and for almost quaternion-Hermitian manifolds with isometric circle actions that preserve the almost quaternion-Hermitian structure [12].

The vanishings of such indices on manifolds with isometric circle actions are instances of the rigidity of elliptic operators under such actions, an important property in the context of elliptic genera [6, 8, 13, 15, 18, 22, 23, 24]. In this paper, we prove the rigidity and vanishing of the indices of several “geometrically natural” twisted Dirac operators on almost even-Clifford manifolds admitting circle actions by automorphisms, resembling those studied on almost quaternionic-Hermitian manifolds.

The note is organized as follows. In Section 2, we recall some material on Clifford algebras, Spin groups and representations, maximal tori of classical Lie groups, almost even-Clifford Hermitian manifolds and their structure groups. In Section 3, we examine the weights of the Spin representation in terms of the weights of the aforementioned structure groups and explore which representations to use in the twisted Dirac operators. In Section 4, we prove the vanishing Theorems 4.7, 4.8 and 4.9, using the Atiyah–Singer fixed point theorem.

2 Preliminaries

The material presented in this section can be consulted in [1, 7, 9].

2.1 Clifford algebra, spin group and representation

Let Cl_n denote the 2^n -dimensional real Clifford algebra generated by the orthonormal vectors $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ subject to the relations

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

and $\text{Cl}_n = \text{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. The even Clifford subalgebra Cl_r^0 is defined as the invariant (+1)-subspace of the involution of Cl_r induced by the map $-\text{Id}_{\mathbb{R}^r}$.

There exist algebra isomorphisms

$$\text{Cl}_n \cong \begin{cases} \text{End}(\mathbb{C}^{2^k}) & \text{if } n = 2k, \\ \text{End}(\mathbb{C}^{2^k}) \oplus \text{End}(\mathbb{C}^{2^k}) & \text{if } n = 2k + 1, \end{cases} \quad (2.1)$$

and the space of (complex) spinors is defined to be

$$\Delta_n := \mathbb{C}^{2^k} = \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{k \text{ times}}.$$

The map

$$\kappa: \text{Cl}_n \longrightarrow \text{End}(\mathbb{C}^{2^k})$$

is defined to be either the aforementioned isomorphism for n even, or the isomorphism followed by the projection onto the first summand for n odd. In order to make κ explicit, consider the following matrices

$$\text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In terms of the generators e_1, \dots, e_n of the Clifford algebra, κ can be described explicitly as follows

$$\begin{aligned} e_1 &\mapsto \text{Id} \otimes \text{Id} \otimes \dots \otimes \text{Id} \otimes \text{Id} \otimes g_1, \\ e_2 &\mapsto \text{Id} \otimes \text{Id} \otimes \dots \otimes \text{Id} \otimes \text{Id} \otimes g_2, \\ e_3 &\mapsto \text{Id} \otimes \text{Id} \otimes \dots \otimes \text{Id} \otimes g_1 \otimes T, \\ e_4 &\mapsto \text{Id} \otimes \text{Id} \otimes \dots \otimes \text{Id} \otimes g_2 \otimes T, \\ &\dots \dots \dots \\ e_{2k-1} &\mapsto g_1 \otimes T \otimes \dots \otimes T \otimes T \otimes T, \\ e_{2k} &\mapsto g_2 \otimes T \otimes \dots \otimes T \otimes T \otimes T, \end{aligned} \quad (2.2)$$

and, if $n = 2k + 1$,

$$e_{2k+1} \mapsto iT \otimes T \otimes \dots \otimes T \otimes T \otimes T.$$

The vectors

$$u_{+1} = \frac{1}{\sqrt{2}}(1, -i) \quad \text{and} \quad u_{-1} = \frac{1}{\sqrt{2}}(1, i),$$

form a unitary basis of \mathbb{C}^2 with respect to the standard Hermitian product. Thus,

$$\mathcal{B} = \{u_{\varepsilon_1, \dots, \varepsilon_k} = u_{\varepsilon_1} \otimes \dots \otimes u_{\varepsilon_k} \mid \varepsilon_j = \pm 1, j = 1, \dots, k\},$$

is a unitary basis of $\Delta_n = \mathbb{C}^{2^k}$ with respect to the naturally induced Hermitian product.

The Spin group $\text{Spin}(n) \subset \text{Cl}_n$ is the subset

$$\text{Spin}(n) = \{x_1 x_2 \cdots x_{2l-1} x_{2l} \mid x_j \in \mathbb{R}^n, |x_j| = 1, l \in \mathbb{N}\},$$

endowed with the product of the Clifford algebra. It is a Lie group and its Lie algebra is

$$\mathfrak{spin}(n) = \text{span}\{e_i e_j \mid 1 \leq i < j \leq n\}.$$

The restriction of κ to $\text{Spin}(n)$ defines the Lie group representation

$$\kappa_n := \kappa|_{\text{Spin}(n)}: \text{Spin}(n) \longrightarrow \text{GL}(\Delta_n),$$

which is, in fact, special unitary. We have the corresponding Lie algebra representation

$$\kappa_{n*}: \mathfrak{spin}(n) \longrightarrow \mathfrak{gl}(\Delta_n).$$

Recall that the Spin group $\text{Spin}(n)$ is the universal double cover of $\text{SO}(n)$, $n \geq 3$. For $n = 2$ we consider $\text{Spin}(2)$ to be the connected double cover of $\text{SO}(2)$. The covering map will be denoted by

$$\lambda_n: \text{Spin}(n) \rightarrow \text{SO}(n) \subset \text{GL}(\mathbb{R}^n).$$

Its differential is given by $\lambda_{n*}(e_i e_j) = 2E_{ij}$, where $E_{ij} = e_i^* \otimes e_j - e_j^* \otimes e_i$ is the standard basis of the skew-symmetric matrices, and e^* denotes the metric dual of the vector e . Furthermore, we will abuse the notation and also denote by λ_n the induced representation on the exterior algebra $\bigwedge^* \mathbb{R}^n$.

By means of κ , we have the Clifford multiplication

$$\begin{aligned} \mu_n: \mathbb{R}^n \otimes \Delta_n &\longrightarrow \Delta_n, \\ x \otimes \phi &\longmapsto \mu_n(x \otimes \phi) = x \cdot \phi := \kappa(x)(\phi). \end{aligned}$$

The Clifford multiplication μ_n is skew-symmetric with respect to the Hermitian product

$$\langle x \cdot \phi_1, \phi_2 \rangle = \langle \mu_n(x \otimes \phi_1), \phi_2 \rangle = -\langle \phi_1, \mu_n(x \otimes \phi_2) \rangle = -\langle \phi_1, x \cdot \phi_2 \rangle,$$

is $\text{Spin}(n)$ -equivariant and can be extended to a $\text{Spin}(n)$ -equivariant map

$$\begin{aligned} \mu_n: \bigwedge^* (\mathbb{R}^n) \otimes \Delta_n &\longrightarrow \Delta_n, \\ \omega \otimes \psi &\longmapsto \omega \cdot \psi. \end{aligned}$$

When n is even, we define the following involution

$$\begin{aligned} \Delta_n &\longrightarrow \Delta_n, \\ \psi &\longmapsto (-i)^{\frac{n}{2}} \text{vol}_n \cdot \psi, \end{aligned}$$

where $\text{vol}_n = e_1 \cdots e_n$. The ± 1 eigenspace of this involution is denoted Δ_n^\pm . These spaces have equal dimension and are irreducible representations of $\text{Spin}(n)$. Note that our definition differs from the one given in [9] by a $(-1)^{\frac{n}{2}}$. The reason for this difference is that we want the spinor $u_{1,\dots,1}$ to be always positive. In this case, we will denote the two representations by

$$\kappa_n^\pm: \text{Spin}(n) \longrightarrow \text{GL}(\Delta_n^\pm).$$

For future use, let us recall the effect of vol_n on $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ when n is even:

$n \pmod{8}$	Δ_n^+	Δ_n^-
0	1	-1
2	i	$-i$
4	-1	1
6	$-i$	i

Furthermore, for $n \equiv 0 \pmod{4}$, $n \neq 4$,

$$\ker(\kappa_n^+) = \begin{cases} \{1, \text{vol}_r\} & \text{if } r \equiv 0 \pmod{8}, \\ \{1, -\text{vol}_r\} & \text{if } r \equiv 4 \pmod{8}, \end{cases}$$

and

$$\ker(\kappa_n^-) = \begin{cases} \{1, -\text{vol}_r\} & \text{if } r \equiv 0 \pmod{8}, \\ \{1, \text{vol}_r\} & \text{if } r \equiv 4 \pmod{8}. \end{cases}$$

For r even, let

$$\mathbb{P}\text{SO}(r) := \frac{\text{SO}(r)}{\{\pm \text{Id}_r\}} \cong \frac{\text{Spin}(r)}{\{\pm 1, \pm \text{vol}_r\}},$$

and for $r \equiv 0 \pmod{4}$ let

$$\text{Spin}_\pm(r) \cong \frac{\text{Spin}(r)}{\{1, \pm \text{vol}_r\}}.$$

Note that we will always denote by 1 and Id_r the identity elements of $\text{Spin}(r)$ and $\text{SO}(r)$ respectively.

2.2 Maximal tori

2.2.1 $\text{SO}(n)$

Recall that a maximal torus of $\text{SO}(n)$ is given by

$$\begin{pmatrix} \cos(\eta_1) & -\sin(\eta_1) & & & \\ \sin(\eta_1) & \cos(\eta_1) & & & \\ & & \ddots & & \\ & & & \cos(\eta_{n/2}) & -\sin(\eta_{n/2}) \\ & & & \sin(\eta_{n/2}) & \cos(\eta_{n/2}) \end{pmatrix}$$

if n is even, and

$$\begin{pmatrix} \cos(\eta_1) & -\sin(\eta_1) & & & \\ \sin(\eta_1) & \cos(\eta_1) & & & \\ & & \ddots & & \\ & & & \cos(\eta_{[n/2]}) & -\sin(\eta_{[n/2]}) \\ & & & \sin(\eta_{[n/2]}) & \cos(\eta_{[n/2]}) \\ & & & & & 1 \end{pmatrix}$$

if n is odd.

2.2.2 Spin(n)

Each one of the 2×2 rotation blocks is a transformation that can be achieved by using Clifford product. For instance the rotation

$$\begin{pmatrix} \cos(\varphi_1) & -\sin(\varphi_1) & & & & \\ \sin(\varphi_1) & \cos(\varphi_1) & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

can be achieved by using the element

$$e_1(-\cos(\varphi_1/2)e_1 + \sin(\varphi_1/2)e_2) = \cos(\varphi_1/2) + \sin(\varphi_1/2)e_1e_2 \in \text{Spin}(n)$$

as follows

$$\begin{aligned} & (\cos(\varphi_1/2) + \sin(\varphi_1/2)e_1e_2)y(\cos(\varphi_1/2) + \sin(\varphi_1/2)e_2e_1) \\ &= (y_1 \cos(\varphi_1) - y_2 \sin(\varphi_1))e_1 + (y_1 \sin(\varphi_1) + y_2 \cos(\varphi_1))e_2 + y_3e_3 + \cdots + y_n e_n, \end{aligned}$$

for $y = y_1e_1 + \cdots + y_n e_n \in \mathbb{R}^n$. Thus, we see that the corresponding elements in $\text{Spin}(n)$ are

$$\pm(\cos(\varphi_1/2) + \sin(\varphi_1/2)e_1e_2).$$

Furthermore, we see that a maximal torus of $\text{Spin}(n)$ is given by elements of the form

$$t(\varphi_1, \dots, \varphi_k) = \prod_{j=1}^k (\cos(\varphi_j/2) + \sin(\varphi_j/2)e_{2j-1}e_{2j}).$$

By using the explicit description (2.2) of the isomorphisms (2.1), we can check that

$$t(\varphi_1, \dots, \varphi_k) \cdot u_{\varepsilon_1, \dots, \varepsilon_k} = e^{\frac{i}{2} \sum_{j=1}^k \varepsilon_{k+1-j} \varphi_j} \cdot u_{\varepsilon_1, \dots, \varepsilon_k},$$

i.e., the basis vectors $u_{\varepsilon_1, \dots, \varepsilon_k}$ are weight vectors of the spin representation with weight

$$\frac{1}{2} \sum_{j=1}^k \varepsilon_{k+1-j} \varphi_j,$$

which in coordinate vectors with respect to the basis $\{\varphi_j\}$ give the well known expressions

$$\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \dots, \pm \frac{1}{2} \right).$$

Indeed, in terms of the (appropriately ordered) basis \mathcal{B} , the matrix associated to an element $t(\varphi_1, \dots, \varphi_{[\frac{n}{2}]})$ is

$$\begin{pmatrix} e^{\frac{i}{2}(\varphi_1 + \varphi_2 + \cdots + \varphi_{[\frac{n}{2}]})} & & & & & \\ & e^{\frac{i}{2}(-\varphi_1 + \varphi_2 + \cdots + \varphi_{[\frac{n}{2}]})} & & & & \\ & & e^{\frac{i}{2}(\varphi_1 - \varphi_2 + \cdots + \varphi_{[\frac{n}{2}]})} & & & \\ & & & \ddots & & \\ & & & & e^{\frac{i}{2}(-\varphi_1 - \varphi_2 + \cdots + \varphi_{[\frac{n}{2}]})} & \\ & & & & & \ddots \\ & & & & & & e^{\frac{i}{2}(-\varphi_1 - \varphi_2 - \cdots - \varphi_{[\frac{n}{2}]})} \end{pmatrix}.$$

Note that, when n is even, Δ_n^+ is generated by the basis vectors $u_{\varepsilon_1, \dots, \varepsilon_{\frac{n}{2}}}$ with an even number of ε_j equal to -1 , and Δ_n^- is generated by the basis vectors $u_{\varepsilon_1, \dots, \varepsilon_{\frac{n}{2}}}$ with an odd number of ε_j equal to -1 . Therefore, after reordering the basis, the matrix above can be rearranged to have two diagonal blocks of equal size: one block in which the exponents contain an even number of negative signs

$$\begin{pmatrix} e^{\frac{i}{2}(\varphi_1 + \varphi_2 + \dots + \varphi_{\frac{n}{2}})} \\ e^{\frac{i}{2}(-\varphi_1 - \varphi_2 + \dots + \varphi_{\frac{n}{2}})} \\ e^{\frac{i}{2}(-\varphi_1 + \varphi_2 - \dots + \varphi_{\frac{n}{2}})} \\ \dots \end{pmatrix},$$

and another block in which the exponents contain an odd number of negative signs

$$\begin{pmatrix} e^{\frac{i}{2}(-\varphi_1 + \varphi_2 + \dots + \varphi_{\frac{n}{2}})} \\ e^{\frac{i}{2}(\varphi_1 - \varphi_2 + \dots + \varphi_{\frac{n}{2}})} \\ e^{\frac{i}{2}(\varphi_1 + \varphi_2 - \varphi_3 + \dots + \varphi_{\frac{n}{2}})} \\ \dots \end{pmatrix}.$$

2.2.3 $U(m)$

The standard maximal torus of $U(m)$ is

$$\begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \dots & \\ & & & e^{i\theta_m} \end{pmatrix}.$$

2.2.4 $Sp(m)$

The standard maximal torus of $Sp(m)$ is

$$\begin{pmatrix} e^{i\theta_1} & & & & \\ & e^{-i\theta_1} & & & \\ & & \dots & & \\ & & & e^{i\theta_m} & \\ & & & & e^{-i\theta_m} \end{pmatrix}.$$

2.3 Almost even-Clifford Hermitian structures

Definition 2.1. Let $N \in \mathbb{N}$ and (e_1, \dots, e_r) an orthonormal frame of \mathbb{R}^r .

- A *linear even-Clifford structure of rank r* on \mathbb{R}^N is an algebra representation

$$\Phi: Cl_r^0 \longrightarrow \text{End}(\mathbb{R}^N).$$

- A *linear even-Clifford Hermitian structure of rank r* on \mathbb{R}^N (endowed with a positive definite inner product) is a linear even-Clifford structure of rank r such that each bivector $e_i e_j$, $1 \leq i < j \leq r$, is mapped to a skew-symmetric endomorphism $\Phi(e_i e_j) = J_{ij}$.

Remark 2.2.

- Note that $J_{ij}^2 = -\text{Id}_{\mathbb{R}^N}$.
- Given a linear even-Clifford structure of rank r on \mathbb{R}^N , we can average the standard inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^N as follows

$$(X, Y) = \sum_{k=1}^{\lfloor r/2 \rfloor} \left[\sum_{1 \leq i_1 < \dots < i_{2k} < r} \langle \Phi(e_{i_1 \dots i_{2k}})(X), \Phi(e_{i_1 \dots i_{2k}})(Y) \rangle \right],$$

where (e_1, \dots, e_r) is an orthonormal frame of \mathbb{R}^r , so that the linear even-Clifford structure is Hermitian with respect to the averaged inner product.

- Given a linear even-Clifford Hermitian structure of rank r , the subalgebra $\mathfrak{spin}(r)$ is mapped injectively into the skew-symmetric endomorphisms $\text{End}^-(\mathbb{R}^N)$.

Definition 2.3. Let $r \geq 2$.

- A *rank r almost even-Clifford structure* on a smooth manifold M is a smoothly varying choice of a rank r linear even-Clifford structure on each tangent space of M .
- A smooth manifold carrying an almost even-Clifford structure will be called an *almost even-Clifford manifold*.
- A *rank r almost even-Clifford Hermitian structure* on a Riemannian manifold M is a smoothly varying choice of a linear even-Clifford Hermitian structure on each tangent space of M .
- A Riemannian manifold carrying a rank r almost even-Clifford Hermitian structure will be called a *rank r almost even-Clifford Hermitian manifold*, or an *almost- Cl_r^0 -Hermitian manifold* for short.

Remark 2.4. Our definition of almost even-Clifford Hermitian structure does not require the existence of a Riemannian vector bundle of rank r . Therefore, it includes both the notions of even Clifford structure and projective even Clifford structure introduced in [19, Definition 2.2 and Remark 2.5].

2.3.1 Structure groups of almost even-Clifford manifolds

Thanks to [2], we know that the complexification of the tangent space of an almost- Cl_r^0 -Hermitian manifold decomposes as follows

$r \pmod{8}$	$\mathbb{R}^N \otimes \mathbb{C}$	
0	$\mathbb{C}^{m_1} \otimes \Delta_r^+ \oplus \mathbb{C}^{m_2} \otimes \Delta_r^-$	
1, 7	$\mathbb{C}^m \otimes \Delta_r$	
2	$\mathbb{C}^m \otimes \Delta_r^+ \oplus \overline{\mathbb{C}^m} \otimes \Delta_r^-$	(2.3)
6	$\overline{\mathbb{C}^m} \otimes \Delta_r^+ \oplus \mathbb{C}^m \otimes \Delta_r^-$	
3, 5	$\mathbb{C}^{2m} \otimes \Delta_r$	
4	$\mathbb{C}^{2m_2} \otimes \Delta_r^+ \oplus \mathbb{C}^{2m_1} \otimes \Delta_r^-$	

where the different \mathbb{C}^p denote the corresponding standard complex representations of the classical Lie groups $\text{SO}(p)$, $\text{U}(p)$ or $\text{Sp}(p)$. Note that the dimension of an almost even-Clifford Hermitian manifold depends of two or three parameters: the rank r of the even-Clifford structure and the multiplicity m or multiplicities m_1, m_2 .

The structure groups of the aforementioned manifolds, for $r \geq 3$, are given as follows (see [1]):

- For $r \not\equiv 0 \pmod{4}$

$r \pmod{8} \backslash m \pmod{2}$	0	1
1, 7	$\frac{SO(m) \times Spin(r)}{\{\pm(\text{Id}_m, 1)\}}$	$SO(m) \times Spin(r)$
2, 6	$\frac{U(m) \times Spin(r)}{\{\pm(\text{Id}_m, 1), \pm(i\text{Id}_m, -\text{vol}_r)\}}$	
3, 5	$\frac{Sp(m) \times Spin(r)}{\{\pm(\text{Id}_m, 1)\}}$	

- For $r \equiv 0 \pmod{8}$

$m_1 \backslash m_2$	0	0 (mod 2)	1 (mod 2)
0		$\frac{SO(m_2) \times Spin(r)}{\{\pm(\text{Id}_{m_2}, 1), \pm(\text{Id}_{m_2}, -\text{vol}_r)\}}$	$\frac{SO(m_2) \times Spin(r)}{\langle (\text{Id}_{m_2}, -\text{vol}_r) \rangle}$
0 (mod 2)	$\frac{SO(m_1) \times Spin(r)}{\{\pm(\text{Id}_{m_1}, 1), \pm(\text{Id}_{m_1}, \text{vol}_r)\}}$	$\frac{SO(m_1) \times SO(m_2) \times Spin(r)}{\{\pm(\text{Id}_{m_1}, \text{Id}_{m_2}, 1), \pm(\text{Id}_{m_1}, -\text{Id}_{m_2}, \text{vol}_r)\}}$	$\frac{SO(m_1) \times SO(m_2) \times Spin(r)}{\langle (-\text{Id}_{m_1}, \text{Id}_{m_2}, -\text{vol}_r) \rangle}$
1 (mod 2)	$\frac{SO(m_1) \times Spin(r)}{\langle (\text{Id}_{m_1}, \text{vol}_r) \rangle}$	$\frac{SO(m_1) \times SO(m_2) \times Spin(r)}{\langle (\text{Id}_{m_1}, -\text{Id}_{m_2}, \text{vol}_r) \rangle}$	$SO(m_1) \times SO(m_2) \times Spin(r)$

- For $r \equiv 4 \pmod{8}$

	$m_1, m_2 > 0$	$m_1 > 0, m_2 = 0$	$m_1 = 0, m_2 > 0$
$r = 4$	$\frac{Sp(m_1) \times Sp(m_2) \times Spin(r)}{\{\pm(\text{Id}_{2m_1}, \text{Id}_{2m_2}, 1), \pm(\text{Id}_{2m_1}, -\text{Id}_{2m_2}, \text{vol}_r)\}}$	$\frac{Sp(m_1) \times Spin(3)}{\{\pm(\text{Id}_{2m_1}, 1)\}}$	$\frac{Sp(m_2) \times Spin(3)}{\{\pm(\text{Id}_{2m_2}, 1)\}}$
$r > 4$		$\frac{Sp(m_1) \times Spin(r)}{\{\pm(\text{Id}_{2m_1}, 1), \pm(\text{Id}_{2m_1}, \text{vol}_r)\}}$	$\frac{Sp(m_2) \times Spin(r)}{\{\pm(\text{Id}_{2m_2}, 1), \pm(\text{Id}_{2m_2}, -\text{vol}_r)\}}$

Note that for $r = 2$, the structure group is actually $U(m)$.

Since all of these groups are quotients of products $G \times Spin(r)$, where G is a (product of) classical Lie group(s), it will be useful to know if they can be mapped to either $Spin(r)$, or $SO(r)$ or $\mathbb{P}SO(r)$. It is easy to see that they map as follows

- For $r \not\equiv 0 \pmod{4}$

$r \pmod{8} \backslash m \pmod{2}$	0	1
1, 7	$SO(r)$	$Spin(r)$
2, 6	$\mathbb{P}SO(r)$	
3, 5	$SO(r)$	

- For $r \equiv 0 \pmod{8}$

$m_1 \backslash m_2$	0 (mod 2)	1 (mod 2)
0 (mod 2)	$\mathbb{P}SO(r)$	$Spin_-(r)$
1 (mod 2)	$Spin_+(r)$	$Spin(r)$

- For $r \equiv 4 \pmod{8}$

$r \equiv 4 \pmod{8} \backslash m_1, m_2$	$m_1, m_2 > 0$	$m_1 > 0, m_2 = 0$	$m_1 = 0, m_2 > 0$
$r = 4$	$\mathbb{P}SO(r)$	$SO(3)$	$SO(3)$
$r > 4$		$\mathbb{P}SO(r)$	$\mathbb{P}SO(r)$

This can be summarized roughly as follows: the structure group of an almost- Cl_r^0 -Hermitian manifold of rank r maps to $\text{SO}(r)$ if r is odd, and maps to $\mathbb{P}\text{SO}(r)$ if r is even.

For future use, we will establish the notation for the decomposition of the complexified tangent bundle of an almost- Cl_r^0 -Hermitian manifold:

$r \pmod{8}$	$TM \otimes \mathbb{C}$	
0	$E_1 \otimes \Delta_r^+ \oplus E_2 \otimes \Delta_r^-$	
1, 7	$E \otimes \Delta_r$	
2	$E \otimes \Delta_r^+ \oplus \bar{E} \otimes \Delta_r^-$	(2.4)
6	$\bar{E} \otimes \Delta_r^+ \oplus E \otimes \Delta_r^-$	
3, 5	$E \otimes \Delta_r$	
4	$E_2 \otimes \Delta_r^+ \oplus E_1 \otimes \Delta_r^-$	

where E , E_1 , E_2 are locally defined vector bundles with fibre \mathbb{C}^p which correspond to the standard complex representation of the different Lie groups mentioned in (2.3).

2.4 A useful lemma

Lemma 2.5. *Let $x \in \mathbb{C}$ and $k, m \in \mathbb{Z}/2$ such that $k + m \in \mathbb{Z}$. If $|k| < |m|$, then*

$$G(z) = \frac{z^k}{z^{-m}e^x - z^m e^{-x}}$$

is a rational function on \mathbb{C} and

$$\lim_{z \rightarrow 0} G(z) = 0 = \lim_{z \rightarrow \infty} G(z).$$

3 Twisted spinor bundles on almost- Cl_r^0 -Hermitian manifolds

In this subsection, we present some calculations relevant to the global definition of twisted spinor bundles.

When the structure group of an oriented N -dimensional Riemannian manifold reduces to a proper subgroup $G \subset \text{SO}(N)$, one can associate vector bundles to the corresponding G -principal bundle P_G by means of the representations of G . If the manifold is Spin, one can ask if there exists a lifting map \tilde{i} making the following diagram commute

$$\begin{array}{ccc} & \text{Spin}(N) & \\ & \nearrow \tilde{i} & \downarrow 2:1 \\ G & \xrightarrow{i} & \text{SO}(N) \end{array}$$

in which case, the Spin representation Δ_N may decompose according to G .

Even when such map \tilde{i} does not exist (necessarily $\pi_1(G) \neq \{1\}$), there may be a finite covering space G' of $G = G'/\Gamma$ for which it does, and one can then decompose the Spin representation according to G' . We can now check how the elements of the finite subgroup Γ act on Δ_N , and at least some of them will act non-trivially, thus confirming that there cannot be a map \tilde{i} . By observing this action, we can then consider tensoring Δ_N with another representation V of G' such that Γ now acts trivially on $\Delta_N \otimes V$.

In the context of almost- Cl_r^0 -Hermitian manifolds, the structure group embeds into the relevant Spin group [1, Theorem 4.1], with the exception of four cases which we will analyze. More precisely, we found that

- $\frac{\mathrm{Sp}(m) \times \mathrm{Spin}(3)}{\{\pm(\mathrm{Id}_{2m}, 1)\}}$ does not embed into $\mathrm{Spin}(4m)$ if m is odd;
- $\frac{\mathrm{Sp}(m_1) \times \mathrm{Sp}(m_2) \times \mathrm{Spin}(4)}{\{\pm(\mathrm{Id}_{2m_1}, \mathrm{Id}_{2m_2}, 1), \pm(\mathrm{Id}_{2m_1}, -\mathrm{Id}_{2m_2}, \mathrm{vol}_4)\}}$ does not embed into $\mathrm{Spin}(4(m_1 + m_2))$ if either m_1 or m_2 (or both) are odd;
- $\frac{\mathrm{U}(m) \times \mathrm{Spin}(6)}{\{\pm(\mathrm{Id}_m, 1), (i\mathrm{Id}_m, -\mathrm{vol}_6)\}}$ does not embed into $\mathrm{Spin}(8m)$ if m is odd;
- $\frac{\mathrm{SO}(m_1) \times \mathrm{SO}(m_2) \times \mathrm{Spin}(8)}{\{(\mathrm{Id}_{m_1}, \mathrm{Id}_{m_2}, 1), (\mathrm{Id}_{m_1}, -\mathrm{Id}_{m_2}, \mathrm{vol}_8)\}}$ if $m_1 + 1 \equiv m_2 \equiv 0 \pmod{2}$, $\frac{\mathrm{SO}(m_1) \times \mathrm{SO}(m_2) \times \mathrm{Spin}(8)}{\{(\mathrm{Id}_{m_1}, \mathrm{Id}_{m_2}, 1), (-\mathrm{Id}_{m_1}, \mathrm{Id}_{m_2}, -\mathrm{vol}_8)\}}$ if $m_1 \equiv m_2 + 1 \equiv 0 \pmod{2}$ do not embed into $\mathrm{Spin}(8(m_1 + m_2))$.

However, by the same calculations in [1] we know that there are homomorphisms

- $\mathrm{Sp}(m) \times \mathrm{Spin}(3) \longrightarrow \mathrm{Spin}(4m)$;
- $\mathrm{Sp}(m_1) \times \mathrm{Sp}(m_2) \times \mathrm{Spin}(4) \longrightarrow \mathrm{Spin}(4(m_1 + m_2))$;
- $\mathrm{U}(m) \times \mathrm{Spin}(6) \longrightarrow \mathrm{Spin}(8m)$;
- $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2) \times \mathrm{Spin}(8) \longrightarrow \mathrm{Spin}(8(m_1 + m_2))$.

In order to analyze this situation and the appropriate twisting bundles for almost- Cl_r^0 -Hermitian manifolds in general, we need to set up some notation regarding weights of Lie groups.

3.1 Weights of $\mathrm{SO}(N)$ with respect to the structure subgroups

We need to rewrite the weights of $\mathrm{SO}(N)$ in terms of the maximal torus of the relevant structure group. Let $(\eta_1, \dots, \eta_{N/2})$ denote the coordinates of a maximal torus of $\mathrm{SO}(N)$, and $(\varphi_1, \dots, \varphi_{\lfloor \frac{r}{2} \rfloor})$ denote the coordinates of a maximal torus of $\mathrm{SO}(r)$.

For r odd, let $\lambda_1, \dots, \lambda_{2\lfloor \frac{r}{2} \rfloor}$ denote the weights of Δ_r

$$\pm \frac{1}{2} \varphi_1 \pm \dots \pm \frac{1}{2} \varphi_{\lfloor \frac{r}{2} \rfloor},$$

listed in some order such that the first half of weights have an even number of negative signs, and the second half of weights have an odd number of negative signs.

For r even, let $\lambda_1^\pm, \dots, \lambda_{\frac{r}{2}-1}^\pm$ denote the weights of Δ_r^\pm

$$\pm \frac{1}{2} \varphi_1 \pm \dots \pm \frac{1}{2} \varphi_{\frac{r}{2}},$$

which have an even and odd number of negative signs respectively. If $r \equiv 0 \pmod{4}$ we will be considering $\lambda_1^\pm, \dots, \lambda_{\frac{r}{2}-1}^\pm$ to be listed in some order so that the first and second halves are interchanged by reflection (changing all the signs),

3.1.1 $r \equiv 1, 7 \pmod{8}$

Let $(\theta_1, \dots, \theta_{\lfloor \frac{m}{2} \rfloor})$ denote the coordinates of maximal tori of $\mathrm{SO}(m)$. Since

$$\mathbb{C}^N = \mathbb{C}^m \otimes \Delta_r,$$

we can set

$$\eta_{(j-1)2\lfloor \frac{r}{2} \rfloor + k} = \theta_j + \lambda_k$$

if m is even, and

$$\eta_{(j-1)2\lfloor \frac{r}{2} \rfloor + k} = \theta_j + \lambda_k, \quad \eta_{\lfloor \frac{m}{2} \rfloor 2\lfloor \frac{r}{2} \rfloor + l} = \lambda_l$$

if m is odd, where $1 \leq j \leq \lfloor \frac{m}{2} \rfloor$, $1 \leq k \leq 2\lfloor \frac{r}{2} \rfloor$ and $1 \leq l \leq 2\lfloor \frac{r}{2} \rfloor - 1$ in both cases.

3.1.2 $r \equiv 2, 6 \pmod{8}$

Let $(\theta_1, \dots, \theta_m)$ denote the coordinates of maximal tori of $U(m)$. Since

$$\mathbb{C}^N = \begin{cases} \mathbb{C}^m \otimes \Delta_r^+ \oplus \overline{\mathbb{C}^m} \otimes \Delta_r^- & \text{if } r \equiv 2 \pmod{8}, \\ \overline{\mathbb{C}^m} \otimes \Delta_r^+ \oplus \mathbb{C}^m \otimes \Delta_r^- & \text{if } r \equiv 6 \pmod{8}, \end{cases}$$

we can set

$$\eta_{(j-1)2^{\frac{r}{2}-1}+k} = \begin{cases} \theta_j + \lambda_k^+ & \text{if } r \equiv 2 \pmod{8}, \\ \theta_j + \lambda_k^- & \text{if } r \equiv 6 \pmod{8}, \end{cases} \quad (3.1)$$

where $1 \leq j \leq m$ and $1 \leq k \leq 2^{\frac{r}{2}-1}$.

3.1.3 $r \equiv 3, 5 \pmod{8}$

Let $(\theta_1, \dots, \theta_m)$ denote the coordinates of maximal tori of $Sp(m)$. Since

$$\mathbb{C}^N = \mathbb{C}^{2m} \otimes \Delta_r,$$

we can set

$$\eta_{(j-1)2^{\lfloor \frac{r}{2} \rfloor}+k} = \theta_j + \lambda_k, \quad (3.2)$$

where $1 \leq j \leq m$ and $1 \leq k \leq 2^{\lfloor \frac{r}{2} \rfloor}$.

3.1.4 $r \equiv 4 \pmod{8}$

Let $(\theta_1, \dots, \theta_{m_1})$ and $(\theta'_1, \dots, \theta'_{m_2})$ denote the coordinates of maximal tori of $Sp(m_1)$ and $Sp(m_2)$ respectively. Since

$$\mathbb{C}^N = \mathbb{C}^{2m_1} \otimes \Delta_r^+ \oplus \mathbb{C}^{2m_2} \otimes \Delta_r^-,$$

we can set

$$\eta_{(j_1-1)2^{\frac{r}{2}-1}+k} = \theta_{j_1} + \lambda_k^+, \quad \eta_{m_1 2^{\frac{r}{2}-1} + (j_2-1)2^{\frac{r}{2}-1}+k} = \theta'_{j_2} + \lambda_k^-, \quad (3.3)$$

where $1 \leq j_1 \leq m_1$, $1 \leq j_2 \leq m_2$ and $1 \leq k \leq 2^{\frac{r}{2}-1}$.

3.1.5 $r \equiv 0 \pmod{8}$

Let $(\theta_1, \dots, \theta_{\lfloor \frac{m_1}{2} \rfloor})$ and $(\theta'_1, \dots, \theta'_{\lfloor \frac{m_2}{2} \rfloor})$ denote the coordinates of maximal tori of $SO(m_1)$ and $SO(m_2)$ respectively. Since

$$\mathbb{C}^N = \mathbb{C}^{m_1} \otimes \Delta_r^+ \oplus \mathbb{C}^{m_2} \otimes \Delta_r^-,$$

we can set

- if m_1, m_2 are even,

$$\eta_{(j_1-1)2^{\frac{r}{2}-1}+k} = \theta_{j_1} + \lambda_k^+, \quad \eta_{m_1 2^{\frac{r}{2}-1} + (j_2-1)2^{\frac{r}{2}-1}+k} = \theta'_{j_2} + \lambda_k^-,$$

where $1 \leq j_1 \leq \frac{m_1}{2}$, $1 \leq j_2 \leq \frac{m_2}{2}$ and $1 \leq k \leq 2^{\frac{r}{2}-1}$;

- if m_1 is even and m_2 is odd,

$$\begin{aligned}\eta_{(j_1-1)2^{\frac{r}{2}-1}+k} &= \theta_{j_1} + \lambda_k^+, & \eta_{m_1 2^{\frac{r}{2}-1} + (j_2-1)2^{\frac{r}{2}-1}+k} &= \theta'_{j_2} + \lambda_k^-, \\ \eta_{m_1 2^{\frac{r}{2}-1} + [\frac{m_2}{2}]2^{\frac{r}{2}-1}+l} &= \lambda_l^-, \end{aligned}\tag{3.4}$$

where $1 \leq j_1 \leq \frac{m_1}{2}$, $1 \leq j_2 \leq [\frac{m_2}{2}]$, $1 \leq k \leq 2^{\frac{r}{2}-1}$ and $1 \leq l \leq 2^{\frac{r}{2}-2}$;

- if m_1 is odd and m_2 is even,

$$\begin{aligned}\eta_{(j_1-1)2^{\frac{r}{2}-1}+k} &= \theta_{j_1} + \lambda_k^+, & \eta_{[\frac{m_1}{2}]2^{\frac{r}{2}-1}+l} &= \lambda_l^+, \\ \eta_{[\frac{m_1}{2}]2^{\frac{r}{2}-1} + 2^{\frac{r}{2}-2} + (j_2-1)2^{\frac{r}{2}-1}+k} &= \theta'_{j_2} + \lambda_k^-, \end{aligned}\tag{3.5}$$

where $1 \leq j_1 \leq [\frac{m_1}{2}]$, $1 \leq j_2 \leq \frac{m_2}{2}$, $1 \leq k \leq 2^{\frac{r}{2}-1}$ and $1 \leq l \leq 2^{\frac{r}{2}-2}$;

- if m_1, m_2 are odd,

$$\begin{aligned}\eta_{(j_1-1)2^{\frac{r}{2}-1}+k} &= \theta_{j_1} + \lambda_k^+, & \eta_{[\frac{m_1}{2}]2^{\frac{r}{2}-1}+l} &= \lambda_l^+, \\ \eta_{[\frac{m_1}{2}]2^{\frac{r}{2}-1} + 2^{\frac{r}{2}-2} + (j_2-1)2^{\frac{r}{2}-1}+k} &= \theta'_{j_2} + \lambda_k^-, & \eta_{[\frac{m_1}{2}]2^{\frac{r}{2}-1} + 2^{\frac{r}{2}-2} + [\frac{m_2}{2}]2^{\frac{r}{2}-1}+l} &= \lambda_l^-, \end{aligned}$$

where $1 \leq j_1 \leq [\frac{m_1}{2}]$, $1 \leq j_2 \leq [\frac{m_2}{2}]$, $1 \leq k \leq 2^{\frac{r}{2}-1}$ and $1 \leq l \leq 2^{\frac{r}{2}-2}$.

3.2 The Spin representation when $r = 3, 4, 6, 8$

The elements of the finite subgroups involved in the structure groups of almost- Cl_r^0 -Hermitian manifolds actually belong to maximal tori. Thus we can calculate their effect on representations in terms of the weights we just described. In this subsection, we examine the cases when the structure group does not embed into $\text{Spin}(N)$.

3.2.1 $r = 3$

Recall (3.2), which in this case is

$$\eta_{2j-1} = \theta_j + \frac{\varphi_1}{2}, \quad \eta_{2j} = \theta_j - \frac{\varphi_1}{2},$$

so that the weights of the spin representation are

$$\pm \frac{\eta_1}{2} \pm \dots \pm \frac{\eta_{2m}}{2} = \sum_{j \in I_1} \frac{\theta_j + \frac{\varphi_1}{2}}{2} - \sum_{j \in \bar{I}_1} \frac{\theta_j + \frac{\varphi_1}{2}}{2} + \sum_{j \in I_2} \frac{\theta_j - \frac{\varphi_1}{2}}{2} - \sum_{j \in \bar{I}_2} \frac{\theta_j - \frac{\varphi_1}{2}}{2},$$

where $I_1, I_2 \subseteq \{1, \dots, m\}$, and $\bar{I}_j = \{1, \dots, m\} - I_j$ denote their complements, $j = 1, 2$. The element $(-\text{Id}_{2m}, -1) \in \text{Sp}(m) \times \text{Spin}(3)$ corresponds to the parameters

$$\theta_j = \pi, \quad \varphi_1 = 2\pi,$$

for $1 \leq j \leq m$, so that such a sum is equal to

$$2|I_1|\pi - m\pi$$

and the effect of $(-\text{Id}_{2m}, -1)$ on each weight line is

$$e^{-im\pi} = (-1)^m.$$

Thus, $(-\text{Id}_{2m}, -1) \in \text{Sp}(m) \times \text{Spin}(3)$ acts trivially on Δ_{4m} if m is even and as multiplication by (-1) if m is odd.

Thus, in order to have a twisted Spin representation

$$\Delta_{4m} \otimes \bigwedge^u \mathbb{C}^{2m} \otimes (\Delta_3)^{\otimes s}$$

of $\frac{\text{Sp}(m) \times \text{Spin}(3)}{\{\pm(\text{Id}_{2m}, 1)\}}$, the exponents must satisfy $m + u + s \equiv 0 \pmod{2}$, which is a well known fact for almost quaternion-Hermitian manifolds [21].

3.2.2 $r = 4$

By (3.3), the weights of the spin representation are

$$\begin{aligned} & \pm \frac{\eta_1}{2} \pm \dots \pm \frac{\eta_{2m_1+2m_2}}{2} \\ &= \sum_{j_1 \in I_1} \frac{\theta_{j_1} + \frac{\varphi_1 + \varphi_2}{2}}{2} - \sum_{j_1 \in \bar{I}_1} \frac{\theta_{j_1} + \frac{\varphi_1 + \varphi_2}{2}}{2} + \sum_{j_1 \in I_2} \frac{\theta_{j_1} + \frac{-\varphi_1 - \varphi_2}{2}}{2} - \sum_{j_1 \in \bar{I}_2} \frac{\theta_{j_1} + \frac{-\varphi_1 - \varphi_2}{2}}{2} \\ &+ \sum_{j_2 \in I'_1} \frac{\theta'_{j_2} + \frac{-\varphi_1 + \varphi_2}{2}}{2} - \sum_{j_2 \in \bar{I}'_1} \frac{\theta'_{j_2} + \frac{-\varphi_1 + \varphi_2}{2}}{2} + \sum_{j_2 \in I'_2} \frac{\theta'_{j_2} + \frac{\varphi_1 - \varphi_2}{2}}{2} - \sum_{j_2 \in \bar{I}'_2} \frac{\theta'_{j_2} + \frac{\varphi_1 - \varphi_2}{2}}{2}, \end{aligned}$$

where $I_1, I_2 \subseteq \{1, \dots, m_1\}$ and $I'_1, I'_2 \subseteq \{1, \dots, m_2\}$.

- The element $(-\text{Id}_{2m_1}, -\text{Id}_{2m_2}, -1) \in \text{Sp}(m_1) \times \text{Sp}(m_2) \times \text{Spin}(4)$ corresponds to the parameters

$$\theta_{j_1} = \pi, \quad \theta'_{j_2} = \pi, \quad \varphi_1 = 2\pi, \quad \varphi_2 = 0,$$

so that such a sum is equal to

$$\pi(2|I_1| - m_1 + 2|I'_2| - m_2)$$

and the effect of $(-\text{Id}_{2m_1}, -\text{Id}_{2m_2}, -1)$ on each weight line is

$$e^{-i\pi(m_1+m_2)} = (-1)^{m_1+m_2}.$$

- The element $(\text{Id}_{2m_1}, -\text{Id}_{2m_2}, \text{vol}_4) \in \text{Sp}(m_1) \times \text{Sp}(m_2) \times \text{Spin}(4)$ corresponds to the parameters

$$\theta_{j_1} = 0, \quad \theta'_{j_2} = \pi, \quad \varphi_1 = \pi, \quad \varphi_2 = \pi,$$

so that the effect of $(\text{Id}_{2m_1}, -\text{Id}_{2m_2}, \text{vol}_4)$ on each weight line is

$$e^{-i\pi m_2} = (-1)^{m_2}.$$

- The element $(-\text{Id}_{2m_1}, \text{Id}_{2m_2}, -\text{vol}_4) \in \text{Sp}(m_1) \times \text{Sp}(m_2) \times \text{Spin}(4)$ corresponds to the parameters

$$\theta_{j_1} = \pi, \quad \theta'_{j_2} = 0, \quad \varphi_1 = \pi, \quad \varphi_2 = -\pi,$$

so that the effect of $(-\text{Id}_{2m_1}, \text{Id}_{2m_2}, -\text{vol}_4)$ on each weight line is

$$e^{-i\pi m_1} = (-1)^{m_1}.$$

Thus, in order to have a twisted Spin representation

$$\Delta_{4(m_1+m_2)} \otimes \wedge^{u_1} \mathbb{C}^{2m_1} \otimes \wedge^{u_2} \mathbb{C}^{2m_2} \otimes (\Delta_4^+)^{\otimes s} \otimes (\Delta_4^-)^{\otimes t}$$

of

$$\frac{\text{Sp}(m_1) \times \text{Sp}(m_2) \times \text{Spin}(4)}{\{\pm(\text{Id}_{2m_1}, \text{Id}_{2m_2}, 1), \pm(\text{Id}_{2m_1}, -\text{Id}_{2m_2}, \text{vol}_4)\}},$$

the exponents must satisfy

$$m_1 + u_1 + t \equiv 0 \pmod{2}, \quad m_2 + u_2 + s \equiv 0 \pmod{2}.$$

3.2.3 $r = 6$

By (3.1), the weights of the spin representation are

$$\begin{aligned} & \pm \frac{\eta_1}{2} \pm \cdots \pm \frac{\eta_{8m}}{2} \\ &= \sum_{j \in I_1} \frac{\theta_j + \frac{-\varphi_1 + \varphi_2 + \varphi_3}{2}}{2} - \sum_{j \in \bar{I}_1} \frac{\theta_j + \frac{-\varphi_1 + \varphi_2 + \varphi_3}{2}}{2} + \sum_{j \in I_2} \frac{\theta_j + \frac{\varphi_1 - \varphi_2 + \varphi_3}{2}}{2} - \sum_{j \in \bar{I}_2} \frac{\theta_j + \frac{\varphi_1 - \varphi_2 + \varphi_3}{2}}{2} \\ &+ \sum_{j \in I_3} \frac{\theta_j + \frac{\varphi_1 + \varphi_2 - \varphi_3}{2}}{2} - \sum_{j \in \bar{I}_3} \frac{\theta_j + \frac{\varphi_1 + \varphi_2 - \varphi_3}{2}}{2} + \sum_{j \in I_4} \frac{\theta_j + \frac{-\varphi_1 - \varphi_2 - \varphi_3}{2}}{2} - \sum_{j \in \bar{I}_4} \frac{\theta_j + \frac{-\varphi_1 - \varphi_2 - \varphi_3}{2}}{2}, \end{aligned}$$

where $I_1, I_2, I_3, I_4 \subseteq \{1, \dots, m_1\}$.

- The element $(-\text{Id}_m, -1) \in \text{U}(m) \times \text{Spin}(6)$ corresponds to the parameters

$$\theta_j = \pi, \quad \varphi_1 = 2\pi, \quad \varphi_2 = 0, \quad \varphi_3 = 0,$$

so that its effect on each weight line is

$$e^{-2i\pi m} = 1.$$

- The element $(i\text{Id}_m, -\text{vol}_6) \in \text{U}(m) \times \text{Spin}(6)$ corresponds to the parameters

$$\theta_j = \frac{\pi}{2}, \quad \varphi_1 = -\pi, \quad \varphi_2 = \pi, \quad \varphi_3 = \pi,$$

so that its effect on each weight line is

$$e^{-i\pi m} = (-1)^m.$$

Thus, in order to have a twisted Spin representation

$$\Delta_{8m} \otimes \wedge^{u_1} \mathbb{C}^m \otimes \wedge^{u_2} \overline{\mathbb{C}^m} \otimes (\Delta_6^+)^{\otimes s} \otimes (\Delta_6^-)^{\otimes t}$$

of $\frac{\text{U}(m) \times \text{Spin}(6)}{\{\pm(\text{Id}_m, 1), \pm(i\text{Id}_m, -\text{vol}_6)\}}$, the exponents must satisfy

$$\begin{aligned} u_1 + u_2 + s + t &\equiv 0 \pmod{2}, & 2m + u_1 + 3u_2 + s + 3t &\equiv 0 \pmod{4}, \\ 2m + 3u_1 + u_2 + 3s + t &\equiv 0 \pmod{4}. \end{aligned}$$

3.2.4 $r = 8$

By (3.5), if $m_1 + 1 \equiv m_2 \equiv 0 \pmod{2}$, the element $(\text{Id}_{m_1}, -\text{Id}_{m_2}, \text{vol}_8) \in \text{SO}(m_1) \times \text{SO}(m_2) \times \text{Spin}(8)$ corresponds to the parameters

$$\theta_{j_1} = 0, \quad \theta'_{j_2} = \pi, \quad \varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \pi,$$

and its effect on each weight line is multiplication by -1 . Thus, we can have twisted Spin representations

$$\Delta_{8(m_1+m_2)} \otimes \wedge^{u_1} \mathbb{C}^{m_1} \otimes \wedge^{u_2} \mathbb{C}^{m_2} \otimes (\Delta_8^+)^{\otimes s} \otimes (\Delta_8^-)^{\otimes t}$$

of

$$\frac{\text{SO}(m_1) \times \text{SO}(m_2) \times \text{Spin}(8)}{\{(\text{Id}_{m_1}, \text{Id}_{m_2}, 1), (\text{Id}_{m_1}, -\text{Id}_{m_2}, \text{vol}_8)\}}$$

if $u_2 + t \equiv 1 \pmod{2}$ and $u_1, s \in \mathbb{N}$.

Similarly, by (3.4), if $m_1 \equiv m_2 + 1 \equiv 0 \pmod{2}$, we can have twisted Spin representations

$$\Delta_{8(m_1+m_2)} \otimes \wedge^{u_1} \mathbb{C}^{m_1} \otimes \wedge^{u_2} \mathbb{C}^{m_2} \otimes (\Delta_8^+)^{\otimes s} \otimes (\Delta_8^-)^{\otimes t}$$

of

$$\frac{\mathrm{SO}(m_1) \times \mathrm{SO}(m_2) \times \mathrm{Spin}(8)}{\{(\mathrm{Id}_{m_1}, \mathrm{Id}_{m_2}, 1), (-\mathrm{Id}_{m_1}, \mathrm{Id}_{m_2}, -\mathrm{vol}_8)\}}$$

if $u_1 + s \equiv 1 \pmod{2}$ and $u_2, t \in \mathbb{N}$.

3.3 Twisting representations

For most r , almost- Cl_r^0 -Hermitian manifolds are Spin [1, Theorem 4.1]. In particular, this is the case when $r \geq 5$ and $r \neq 6, 8$. Thus, we only need to choose suitable representations of the structure group G to twist the spinor bundle:

- For $r \equiv 1, 7 \pmod{8}$ our candidates are

$$\wedge^u \mathbb{C}^m \otimes \Delta_r^{\otimes s}, \quad S^u \mathbb{C}^m \otimes \Delta_r^{\otimes s}.$$

They are representations of the structure group when

$$\begin{aligned} u + s &\equiv 0 \pmod{2} && \text{if } m \text{ is even,} \\ u, s &\in \mathbb{N} && \text{if } m \text{ is odd.} \end{aligned}$$

- For $r \equiv 3, 5 \pmod{8}$ our candidates are

$$\wedge^u \mathbb{C}^{2m} \otimes \Delta_r^{\otimes s}, \quad S^u \mathbb{C}^{2m} \otimes \Delta_r^{\otimes s}.$$

They are representations of the structure group when

$$u + s \equiv 0 \pmod{2}.$$

- For $r \equiv 4 \pmod{8}$ our candidates are

$$\begin{aligned} \wedge^{u_1} \mathbb{C}^{2m_1} \otimes \wedge^{u_2} \mathbb{C}^{2m_2} \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t}, \\ S^{u_1} \mathbb{C}^{2m_1} \otimes S^{u_2} \mathbb{C}^{2m_2} \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t}. \end{aligned}$$

They are representations of the structure group when

$$u_2 + s \equiv 0 \pmod{2}, \quad u_1 + t \equiv 0 \pmod{2}.$$

- For $r \equiv 2 \pmod{8}$, $r \neq 2$, our candidates are

$$\wedge^{u_1} \mathbb{C}^m \otimes \wedge^{u_2} \overline{\mathbb{C}^m} \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t}, \quad S^{u_1} \mathbb{C}^m \otimes S^{u_2} \overline{\mathbb{C}^m} \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t}.$$

They are representations of the structure group when

$$\begin{aligned} u_1 + u_2 + s + t &\equiv 0 \pmod{2}, && u_1 + 3u_2 + 3s + t \equiv 0 \pmod{4}, \\ 3u_1 + u_2 + s + 3t &\equiv 0 \pmod{4}. \end{aligned}$$

- For $r \equiv 6 \pmod{8}$ our candidates are

$$\bigwedge^{u_1} \mathbb{C}^m \otimes \bigwedge^{u_2} \overline{\mathbb{C}^m} \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t}, \quad S^{u_1} \mathbb{C}^m \otimes S^{u_2} \overline{\mathbb{C}^m} \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t}.$$

They are representations of the structure group when

$$\begin{aligned} u_1 + u_2 + s + t &\equiv 0 \pmod{2}, & u_1 + 3u_2 + s + 3t &\equiv 0 \pmod{4}, \\ 3u_1 + u_2 + 3s + t &\equiv 0 \pmod{4}. \end{aligned}$$

- For $r \equiv 0 \pmod{8}$ our candidates are

$$\begin{aligned} \bigwedge^{u_1} \mathbb{C}^{m_1} \otimes \bigwedge^{u_2} \mathbb{C}^{m_2} \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t}, \\ S^{u_1} \mathbb{C}^{m_1} \otimes S^{u_2} \mathbb{C}^{m_2} \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t}. \end{aligned}$$

They are representations of the structure group when

$$\begin{aligned} \begin{cases} u_2 + t \equiv 0 \pmod{2} \\ u_1 + s \equiv 0 \pmod{2} \end{cases} & \text{if } m_1 \equiv m_2 \equiv 0 \pmod{2}, \\ \begin{cases} u_2 + t \equiv 0 \pmod{2} \\ u_1, s \in \mathbb{N} \end{cases} & \text{if } m_1 + 1 \equiv m_2 \equiv 0 \pmod{2}, \\ \begin{cases} u_2, t \in \mathbb{N} \\ u_1 + s \equiv 0 \pmod{2} \end{cases} & \text{if } m_1 \equiv m_2 + 1 \equiv 0 \pmod{2}, \\ u_1, u_2, s, t \in \mathbb{N} & \text{if } m_1 \equiv m_2 \equiv 0 \pmod{2}. \end{aligned}$$

4 Index calculations

In this section, we recall the definition of twisted Dirac operators, how to apply the Atiyah–Singer fixed point formula [4], (infinitesimal) automorphisms of almost- Cl_r^0 -Hermitian manifolds and prove the vanishing Theorems 4.7, 4.8 and 4.9.

4.1 Rigidity of elliptic operators

Definition 4.1. Let $D: \Gamma(E) \rightarrow \Gamma(F)$ be an elliptic operator acting on sections of the vector bundles E and F over a compact manifold M . The index of D is the virtual vector space $\text{ind}(D) = \ker(D) - \text{coker}(D)$. If M admits a circle action preserving D , i.e., such that S^1 acts on E and F , and commutes with D , $\text{ind}(D)$ admits a Fourier decomposition into complex 1-dimensional irreducible representations of S^1 $\text{ind}(D) = \sum a_m L^m$, where $a_m \in \mathbb{Z}$ and L^m is the representation of S^1 on \mathbb{C} given by $\lambda \mapsto \lambda^m$. The elliptic operator D is called *rigid* if $a_m = 0$ for all $m \neq 0$, i.e., $\text{ind}(D)$ consists only of the trivial representation with multiplicity a_0 .

Let us recall three examples.

Example 4.2. The deRham complex

$$d + d^*: \Omega^{\text{even}} \rightarrow \Omega^{\text{odd}}$$

from even-dimensional forms to odd-dimensional ones, where d^* denotes the adjoint of the exterior derivative d , is *rigid* for any circle action on M by isometries since by Hodge theory the kernel and the cokernel of this operator consist of harmonic forms, which by homotopy invariance stay fixed under the circle action.

Example 4.3. The signature operator on an oriented manifold

$$d_s: \Omega_c^+ \longrightarrow \Omega_c^-$$

from even to odd complex forms under the Hodge $*$ operator is *rigid* for any circle action on M by isometries since the kernel and cokernel of this operator consist of harmonic forms.

Example 4.4. The Dirac operator on a Spin manifold is *rigid* for any circle action by isometries [3].

4.2 Twisted Dirac operators

In this subsection, let M be a $4n$ -dimensional oriented Riemannian manifold. M is Spin if its orthonormal frame bundle $P_{\text{SO}(4n)}$ admits a double cover by a principal bundle $P_{\text{Spin}(4n)}$ with structure group $\text{Spin}(4n)$, which gives rise to the spinor bundle

$$P_{\text{Spin}(4n)} \times_{\kappa} \Delta_{4n}.$$

The Levi-Civita connection on $P_{\text{SO}(4n)}$ can be lifted to $P_{\text{Spin}(4n)}$ to define a covariant differentiation ∇ on Δ

$$\nabla: \Gamma(\Delta) \longrightarrow \Gamma(T^* \otimes \Delta),$$

and the (elliptic and self-adjoint) Dirac operator

$$\not{D}(\psi) = \sum_{i=1}^{4n} e^i \cdot \nabla_{e_i} \psi$$

for $\psi \in \Gamma(\Delta)$, where (e_1, \dots, e_{4n}) is a local orthonormal frame. Since the spin representation decomposes, the Dirac operator can be split into two parts

$$\not{D}: \Gamma(\Delta_+) \longrightarrow \Gamma(\Delta_-), \quad \not{D}^*: \Gamma(\Delta_-) \longrightarrow \Gamma(\Delta_+).$$

We are interested in Dirac operators with coefficients in auxiliary vector bundles F equipped with a covariant derivative $\nabla^F: \Gamma(F) \longrightarrow \Gamma(T^* \otimes F)$. The Dirac operator twisted by F (or with coefficients in F)

$$(\not{D} \otimes F): \Gamma(\Delta_+ \otimes F) \longrightarrow \Gamma(\Delta_- \otimes F)$$

is defined by

$$(\not{D} \otimes F)(\psi \otimes f) = \left(\sum_{i=1}^{4n} e^i \cdot \nabla_{e_i} \psi \right) \otimes f + \sum_{i=1}^{4n} \mu(e^i \otimes \psi) \otimes \nabla_{e_i}^F f,$$

where $\psi \in \Gamma(\Delta)$, $f \in \Gamma(F)$.

Remark 4.5. If the manifold is not Spin, there may exist well defined twisted spinor bundles (as above), as it happens when the structure group of M reduces to a subgroup of $\text{SO}(4n)$ and $\Delta_{4n} \otimes F$ is a representation of such subgroup.

4.3 Index formula and localization

Let M be an compact $4n$ -dimensional oriented Riemannian manifold. Let us assume that the bundle

$$\Delta_{4n} \otimes F$$

is well defined, where we will use the same symbol to denote the representation and the associated vector bundle, where the $\dim_{\mathbb{C}}(F) = p$. Since $\Delta_{4n} \otimes F$ is a Clifford bundle, by the Atiyah–Singer index theorem [5, 20], the index of the twisted Dirac operators can be computed as

$$\text{ind}(\not{\partial} \otimes F) = \langle \widehat{A}(M) \text{ch}(F), [M] \rangle,$$

where $\text{ch}(\cdot)$ denotes the Chern character, $\widehat{A}(M)$ denotes the \widehat{A} -genus, and $[M]$ denotes the fundamental cycle of M . In terms of formal roots,

$$\begin{aligned} c(TM \otimes \mathbb{C}) &= (1 + \eta_1)(1 - \eta_1) \cdots (1 + \eta_{2n})(1 - \eta_{2n}), \\ p(TM) &= (1 + \eta_1^2) \cdots (1 + \eta_{2n}^2), \\ c(F) &= (1 + \nu_1) \cdots (1 + \nu_p), \\ \text{ch}(F) &= \sum_{l=1} e^{\nu_l}, \\ \text{ind}(\not{\partial} \otimes F) &= \left\langle \sum_{l=1} e^{\nu_l} \cdot \prod_{i=1}^{2n} \frac{\eta_i}{e^{\frac{\eta_i}{2}} - e^{-\frac{\eta_i}{2}}}, [M] \right\rangle. \end{aligned}$$

If M admits a non-trivial S^1 action that lifts to $\Delta_{4n} \otimes F$, the equivariant version of the index can be written in terms of the local data of the S^1 -fixed point set M^{S^1} . More precisely, let $z \in S^1$ be a generic element of S^1 . By the Atiyah–Singer fixed point theorem [4, 5]

$$\text{ind}(\not{\partial} \otimes F)_z = \sum_{P \subset M^{S^1}} \mu(P, z),$$

where $\mu(P, z)$ is the local contribution of the oriented fixed point submanifold $P \subset M^{S^1}$, which can be computed as follows. The S^1 action on M induces a decomposition of TM over P ,

$$TM|_P = \sum_k N_k, \tag{4.1}$$

where N_k is a bundle over P whose fibers are representations of S^1 on which $z \in S^1$ acts as an automorphism with multiple eigenvalue z^k , $k \in \mathbb{Z}$. Note that P inherits an orientation since M is oriented and the bundles N_k for $k \neq 0$ are naturally oriented. Formally, by means of the splitting principle, we can write

$$TM|_P = \mathcal{L}^{q_1} + \cdots + \mathcal{L}^{q_{2n}}, \tag{4.2}$$

where \mathcal{L} corresponds to the standard representation of S^1 on \mathbb{C} , so that $z \in S^1$ acts by multiplication by z^{q_i} on \mathcal{L}^{q_i} . The integers $q_i = q_i(P) \in \mathbb{Z}$ are the exponents of the action at P , which correspond to the aforementioned numbers k . Thus, following [14, p. 67],

$$\mu(P, z) = \left\langle \sum z^{-n_k} e^{\nu_k} \prod_{q_i=0} \frac{\eta_i}{e^{\frac{\eta_i}{2}} - e^{-\frac{\eta_i}{2}}} \prod_{q_j \neq 0} \frac{1}{z^{-\frac{q_j}{2}} e^{\frac{\eta_j}{2}} - z^{\frac{q_j}{2}} e^{-\frac{\eta_j}{2}}}, [P] \right\rangle,$$

where $n_k = n_k(P)$ are the exponents of the action on F restricted to P . The function $\mu(P, z)$ is a rational function of the complex variable z with zeroes at 0 and ∞ as long as

$$|n_k| < \frac{1}{2} (|q_1(P)| + \cdots + |q_{2n}(P)|) \quad (4.3)$$

for all $1 \leq k \leq p$. If such a condition is fulfilled for all $P \subset M^{S^1}$, then $\text{ind}(\not\partial \otimes F)_z$ is a rational function of z with zeroes at 0 and ∞ . Notice that $\text{ind}(\not\partial \otimes F)_z$ also belongs to the representation ring $R(S^1)$ of S^1 , which can be identified with the Laurent polynomial ring $\mathbb{Z}[z, z^{-1}]$. Hence, by Lemma 2.5,

$$\text{ind}(\not\partial \otimes F) = \text{ind}(\not\partial \otimes F)_1 = 0,$$

i.e.,

$$\langle \widehat{A}(M) \text{ch}(F), [M] \rangle = 0.$$

4.4 \widehat{A} -genus of almost- Cl_r^0 -Hermitian manifolds

Given a $4n$ -dimensional Riemannian manifold, according to the splitting principle with respect to the maximal torus of $\text{SO}(4n)$, its complexified tangent bundle splits formally as follows

$$TM \otimes \mathbb{C} = L_1 \oplus L_1^{-1} \oplus \cdots \oplus L_{2n} \oplus L_{2n}^{-1}$$

and, therefore,

$$c(TM \otimes \mathbb{C}) = (1 + x_1)(1 - x_1) \cdots (1 + x_{2n})(1 - x_{2n}) = (1 - x_1^2) \cdots (1 - x_{2n}^2)$$

and its Pontrjagin class is

$$p(TM) = (1 + x_1^2) \cdots (1 + x_{2n}^2),$$

and the \widehat{A} -genus is given by

$$\widehat{A}(M) = \prod_{j=1}^{2n} \frac{x_j/2}{\sinh(x_j/2)} = \prod_{j=1}^{2n} \frac{x_j}{e^{x_j/2} - e^{-x_j/2}}.$$

In the following, we will set $x_i = \eta_i$ from Section 3.1.

4.4.1 $r \equiv 1, 7 \pmod{8}$

The \widehat{A} -genus is given by

$$\widehat{A}(M) = \prod_{j=1}^{\lfloor \frac{m}{2} \rfloor} \prod_{k=1}^{2^{\lfloor \frac{r}{2} \rfloor}} \frac{\theta_j + \lambda_k}{e^{\frac{\theta_j + \lambda_k}{2}} - e^{-\frac{\theta_j + \lambda_k}{2}}}$$

if m is even, and

$$\widehat{A}(M) = \prod_{j=1}^{\lfloor \frac{m}{2} \rfloor} \prod_{k=1}^{2^{\lfloor \frac{r}{2} \rfloor}} \frac{\theta_j + \lambda_k}{e^{\frac{\theta_j + \lambda_k}{2}} - e^{-\frac{\theta_j + \lambda_k}{2}}} \prod_{l=1}^{2^{\lfloor \frac{r}{2} \rfloor - 1}} \frac{\lambda_l}{e^{\frac{\lambda_l}{2}} - e^{-\frac{\lambda_l}{2}}}$$

if m is odd.

4.4.2 $r \equiv 2, 6 \pmod{8}$

The \widehat{A} -genus is given by

$$\widehat{A}(M) = \begin{cases} \prod_{j=1}^m \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta_j + \lambda_k^+}{e^{\frac{\theta_j + \lambda_k^+}{2}} - e^{-\frac{\theta_j + \lambda_k^+}{2}}} & \text{if } r \equiv 2 \pmod{8}, \\ \prod_{j=1}^m \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta_j + \lambda_k^-}{e^{\frac{\theta_j + \lambda_k^-}{2}} - e^{-\frac{\theta_j + \lambda_k^-}{2}}} & \text{if } r \equiv 6 \pmod{8}. \end{cases}$$

4.4.3 $r \equiv 3, 5 \pmod{8}$

The \widehat{A} -genus is given by

$$\widehat{A}(M) = \prod_{j=1}^m \prod_{k=1}^{2^{\lfloor \frac{r}{2} \rfloor}} \frac{\theta_j + \lambda_k}{e^{\frac{\theta_j + \lambda_k}{2}} - e^{-\frac{\theta_j + \lambda_k}{2}}}.$$

4.4.4 $r \equiv 4 \pmod{8}$

The \widehat{A} -genus is given by

$$\widehat{A}(M) = \prod_{j_1=1}^{m_1} \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta_{j_1} + \lambda_k^+}{e^{\frac{\theta_{j_1} + \lambda_k^+}{2}} - e^{-\frac{\theta_{j_1} + \lambda_k^+}{2}}} \prod_{j_2=1}^{m_2} \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta'_{j_2} + \lambda_k^-}{e^{\frac{\theta'_{j_2} + \lambda_k^-}{2}} - e^{-\frac{\theta'_{j_2} + \lambda_k^-}{2}}}.$$

4.4.5 $r \equiv 0 \pmod{8}$

We can set

- if m_1, m_2 are even,

$$\widehat{A}(M) = \prod_{j_1=1}^{\frac{m_1}{2}} \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta_{j_1} + \lambda_k^+}{e^{\frac{\theta_{j_1} + \lambda_k^+}{2}} - e^{-\frac{\theta_{j_1} + \lambda_k^+}{2}}} \prod_{j_2=0}^{\frac{m_2}{2}} \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta'_{j_2} + \lambda_k^-}{e^{\frac{\theta'_{j_2} + \lambda_k^-}{2}} - e^{-\frac{\theta'_{j_2} + \lambda_k^-}{2}}};$$

- if m_1 is even and m_2 is odd,

$$\widehat{A}(M) = \prod_{j_1=1}^{\frac{m_1}{2}} \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta_{j_1} + \lambda_k^+}{e^{\frac{\theta_{j_1} + \lambda_k^+}{2}} - e^{-\frac{\theta_{j_1} + \lambda_k^+}{2}}} \prod_{j_2=1}^{\lceil \frac{m_2}{2} \rceil} \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta'_{j_2} + \lambda_k^-}{e^{\frac{\theta'_{j_2} + \lambda_k^-}{2}} - e^{-\frac{\theta'_{j_2} + \lambda_k^-}{2}}} \prod_{l=1}^{2^{\frac{r}{2}-2}} \frac{\lambda_l^-}{e^{\frac{\lambda_l^-}{2}} - e^{-\frac{\lambda_l^-}{2}}};$$

- if m_1 is odd and m_2 is even,

$$\widehat{A}(M) = \prod_{j_1=1}^{\lfloor \frac{m_1}{2} \rfloor} \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta_{j_1} + \lambda_k^+}{e^{\frac{\theta_{j_1} + \lambda_k^+}{2}} - e^{-\frac{\theta_{j_1} + \lambda_k^+}{2}}} \prod_{l=1}^{2^{\frac{r}{2}-2}} \frac{\lambda_l^+}{e^{\frac{\lambda_l^+}{2}} - e^{-\frac{\lambda_l^+}{2}}} \prod_{j_2=1}^{\frac{m_2}{2}} \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta'_{j_2} + \lambda_k^-}{e^{\frac{\theta'_{j_2} + \lambda_k^-}{2}} - e^{-\frac{\theta'_{j_2} + \lambda_k^-}{2}}};$$

- if m_1, m_2 are odd,

$$\begin{aligned} \widehat{A}(M) &= \prod_{j_1=1}^{\lfloor \frac{m_1}{2} \rfloor} \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta_{j_1} + \lambda_k^+}{e^{\frac{\theta_{j_1} + \lambda_k^+}{2}} - e^{-\frac{\theta_{j_1} + \lambda_k^+}{2}}} \prod_{l=1}^{2^{\frac{r}{2}-2}} \frac{\lambda_l^+}{e^{\frac{\lambda_l^+}{2}} - e^{-\frac{\lambda_l^+}{2}}} \\ &\times \prod_{j_2=1}^{\lceil \frac{m_2}{2} \rceil} \prod_{k=1}^{2^{\frac{r}{2}-1}} \frac{\theta'_{j_2} + \lambda_k^-}{e^{\frac{\theta'_{j_2} + \lambda_k^-}{2}} - e^{-\frac{\theta'_{j_2} + \lambda_k^-}{2}}} \prod_{l=1}^{2^{\frac{r}{2}-2}} \frac{\lambda_l^-}{e^{\frac{\lambda_l^-}{2}} - e^{-\frac{\lambda_l^-}{2}}}. \end{aligned}$$

4.5 Infinitesimal automorphisms

An *automorphism* of an almost- Cl_r^0 -Hermitian manifold M is an isometry which preserves the almost even-Clifford Hermitian structure. A vector field X on M is an *infinitesimal automorphism* if it is a Killing vector field that preserves the structure, i.e., locally

$$\mathcal{L}_X J_{ij} = \sum_{k < l} \alpha_{kl}^{(ij)} J_{kl},$$

for some (local) functions $\alpha_{kl}^{(ij)}$, where \mathcal{L}_X denotes the Lie derivative in the direction of X . Consider

$$\mathcal{L}_X(J_{ij}(Y)) = (\mathcal{L}_X J_{ij})(Y) + J_{ij}(\mathcal{L}_X Y),$$

which can be written in terms of the Levi-Civita connection ∇ as follows

$$\nabla_X(J_{ij}(Y)) - \nabla_{J_{ij}(Y)}X = \sum_{k < l} \alpha_{kl}^{(ij)} J_{kl}(Y) + J_{ij}(\nabla_X Y - \nabla_Y X).$$

Now, if $p \in M$ is such that $X_p = 0$,

$$-\nabla_{J_{ij}(Y)}X = \sum_{k < l} \alpha_{kl}^{(ij)} J_{kl}(Y) - J_{ij}(\nabla_Y X),$$

i.e.,

$$[J_{ij}, \nabla X](Y) = \sum_{k < l} \alpha_{kl}^{(ij)} J_{kl}(Y).$$

Hence, $(\nabla X)_p$ is a skew-symmetric endomorphism such that

$$[J_{ij}, \nabla X] = \sum_{k < l} \alpha_{kl}^{(ij)} J_{kl}.$$

i.e., $(\nabla X)_p$ belongs to Lie algebra of the structure group of M [1, 2].

We will say that a smooth circle action on an almost- Cl_r^0 -Hermitian manifold is an *action by automorphisms* if the corresponding Killing vector field is an infinitesimal automorphism.

Example 4.6. The 16-dimensional symmetric space

$$\frac{F_4}{\text{Spin}(9)}$$

has an almost- Cl_0^0 -Hermitian structure admitting S^1 actions by automorphisms [10].

4.6 Exponents of the S^1 action

In this section, let M be a compact, rank r almost even-Clifford Hermitian manifold with a non-trivial (effective) S^1 action by automorphisms. Let $P \subset M^{S^1}$ be an S^1 -fixed submanifold. The corresponding infinitesimal isometry X is such that $(\nabla X)_p \in \mathfrak{so}(N)$ at any fixed point $p \in P$. This corresponds to the induced action of S^1 to $T_p M$, and such a circle lies in a maximal torus. The tangent space at p decomposes as in Section 4.3. In fact, we can now be more precise about these exponents. By Section 4.5, $(\nabla X)_p$ belongs to a Cartan subalgebra of the Lie algebra of the structure group, and we can assume that the decomposition (4.1) is compatible with a decomposition such as (4.2) into complex lines with respect to a maximal torus of such

a group. Hence, we can read off the exponents of the action with respect to the weights given in Section 3.1:

$r \pmod{8}$	$\pm q_i$		
0	$\frac{t_{j_1} \pm h_k^+}{2}, \frac{t'_{j_2} \pm h_k^-}{2}$ $(\frac{h_l}{2}, \frac{h}{2} \frac{r-1+l}{2})$	$1 \leq j_1 \leq [\frac{m_1}{2}],$ $1 \leq j_2 \leq [\frac{m_2}{2}]$	$1 \leq k \leq 2^{\lceil \frac{r}{2} \rceil - 2}$ $(1 \leq l \leq 2^{\lceil \frac{r}{2} \rceil - 3})$
1, 7	$\frac{t_j \pm h_k}{2} (\frac{h_l}{2})$	$1 \leq j \leq [\frac{m}{2}]$	$1 \leq k \leq 2^{\lceil \frac{r}{2} \rceil - 1}$ $(1 \leq l \leq 2^{\lceil \frac{r}{2} \rceil - 2})$
2, 6	$\frac{t_j \pm h_k^+}{2}, \frac{-t_j \pm h_k^-}{2}$	$1 \leq j \leq m$	$1 \leq k \leq 2^{\lceil \frac{r}{2} \rceil - 2}$
3, 5	$\frac{t_j \pm h_k}{2}$	$1 \leq j \leq m$	$1 \leq k \leq 2^{\lceil \frac{r}{2} \rceil - 1}$
4	$\frac{t_{j_1} \pm h_k^+}{2}, \frac{t'_{j_2} \pm h_k^-}{2}$	$1 \leq j_1 \leq m_1, 1 \leq j_2 \leq m_2$	$1 \leq k \leq 2^{\lceil \frac{r}{2} \rceil - 2}$

where m, m_1, m_2 denote the corresponding multiplicities. Here, the numbers $\frac{t_j}{2}, \frac{t'_j}{2}$ are the exponents corresponding to the complex representations E, E_1, E_2 described in Section 2.3.1, f_j are the exponents for the $\text{SO}(r)$ representation for r odd or $\mathbb{P}\text{SO}(r)$ representation for r even, h_k denote the numbers

$$\pm f_1 \pm \cdots \pm f_{\lceil \frac{r}{2} \rceil}$$

in some order for r odd, and h_k^\pm denote the numbers

$$\pm f_1 \pm \cdots \pm f_{\frac{r}{2}}$$

with an even or odd number of negative signs respectively, listed in some order for r even.

4.7 Vanishing theorems

In this section, we give the main details of the proofs of the vanishing theorems.

Theorem 4.7. *Let M be a compact N -dimensional almost- Cl_r^0 -Hermitian admitting a smooth circle action by automorphisms, $r \geq 3$. Let E, E_1, E_2 be the (locally defined) bundles described in (2.4), m, m_1, m_2 the corresponding multiplicities and u, u_1, u_2, s, t be non-negative integers satisfying the conditions given in Sections 3.2 and 3.3. Then,*

- for $r \equiv 1, 7 \pmod{8}$, if $0 \leq u + s < [\frac{m}{2}]$,

$$\langle \text{ch}(\wedge^u E) \text{ch}(\Delta_r)^s \widehat{A}(M), [M] \rangle = 0;$$

- for $r \equiv 3, 5 \pmod{8}$, if $0 \leq u + s < m$,

$$\langle \text{ch}(\wedge^u E) \text{ch}(\Delta_r)^s \widehat{A}(M), [M] \rangle = 0;$$

- for $r \equiv 0 \pmod{8}$, if $0 \leq u_1 + s < [\frac{m_1}{2}], 0 \leq u_2 + t < [\frac{m_2}{2}]$,

$$\langle \text{ch}(\wedge^{u_1} E_1) \text{ch}(\wedge^{u_2} E_2) \text{ch}(\Delta_r^+)^s \text{ch}(\Delta_r^-)^t \widehat{A}(M), [M] \rangle = 0;$$

- for $r \equiv 2, 6 \pmod{8}$, if $0 \leq u_1 + s < m$ and $0 \leq u_2 + t < m$, or if $0 \leq u_1 + t < m$ and $0 \leq u_2 + s < m$,

$$\langle \text{ch}(\wedge^{u_1} E) \text{ch}(\wedge^{u_2} \overline{E}) \text{ch}(\Delta_r^+)^s \text{ch}(\Delta_r^-)^t \widehat{A}(M), [M] \rangle = 0;$$

- for $r \equiv 4 \pmod{8}$, if $0 \leq u_1 + s < m_1$ and $0 \leq u_2 + t < m_2$,

$$\langle \text{ch}(\wedge^{u_1} E_1) \text{ch}(\wedge^{u_2} E_2) \text{ch}(\Delta_r^+)^s \text{ch}(\Delta_r^-)^t \widehat{A}(M), [M] \rangle = 0.$$

If the inequalities are not strict, the indices are rigid.

Proof. Since the S^1 action is by automorphisms of the almost even-Clifford Hermitian structure, the action lifts to the bundles associated to the structure group, such as the twisted spin bundles we are considering. Given that the arguments are similar in all cases, we will only describe the calculation for $r \equiv 1, 7 \pmod{8}$ and m even.

Let $P \subset M^{S^1}$ be an S^1 -fixed submanifold. By Section 4.5, over P the circle group of automorphisms maps non-trivially to the structure group $\text{SO}(m)\text{Spin}(r)$, so that the fibers of the bundles $\Delta_N \otimes \wedge^u E \otimes \Delta_r^{\otimes s}$ over points of P decompose as sums of representations of S^1 . Recall that

$$\text{ch}(\wedge^u E) = \sum_{1 \leq i_1 < \dots < i_u \leq 2[\frac{m}{2}]} e^{\vartheta_{i_1} + \dots + \vartheta_{i_u}},$$

where

$$\vartheta_j = \theta_j, \quad \vartheta_{[m/2]+j} = -\theta_j, \quad j = 1, \dots, [m/2].$$

Thus, the exponents of the twist will be of the form

$$\frac{1}{2} \left(\sum_{a=1}^c (-1)^{\varepsilon_a} t_{i_a} + \sum_{b=1}^s (-1)^{\delta_b} h_{l_b} \right),$$

where $0 \leq c \leq u$, $\varepsilon_a, \delta_b \in \{0, 1\}$. There are two points to verify in the proof: firstly, that the contributions $\mu(P, z)$ are rational functions and, secondly, that the exponents of the twisting bundles and the tangent space satisfy the inequality (4.3).

The first one follows from the fact that the fibers of the bundles $\Delta_N \otimes \wedge^u E \otimes \Delta_r^{\otimes s}$ over P decompose as sums of representations of S^1 . Formally, according to the splitting principle, if

$$TM_c = L_1 \oplus L_1^{-1} \oplus \dots \oplus L_{N/2} \oplus L_{N/2}^{-1},$$

then

$$\begin{aligned} \Delta_N &= (L_1^{1/2} \oplus L_1^{-1/2}) \otimes \dots \otimes (L_{N/2}^{1/2} \oplus L_{N/2}^{-1/2}) \\ &= L_1^{1/2} \otimes \dots \otimes L_{N/2}^{1/2} \oplus \dots \oplus L_1^{-1/2} \otimes \dots \otimes L_{N/2}^{-1/2}, \end{aligned}$$

so that the S^1 -exponents on these lines will be of the form

$$\sum \left(\pm \frac{t_j \pm h_k}{4} \right) + \sum \left(\pm \frac{h_l}{4} \right).$$

The bundle $\Delta_N \otimes \wedge^u E \otimes \Delta_r^{\otimes s}$ will have integer exponents over P of the form

$$\begin{aligned} &\frac{1}{2} \left(\sum_{a=1}^c (-1)^{\varepsilon_a} t_{i_a} + \sum_{b=1}^s (-1)^{\delta_b} h_{l_b} \right) + \sum_{q_j \neq 0} \left((-1)^{\gamma_j} \frac{t_j + h_k}{4} \right) \\ &+ \sum_{q_{j'} \neq 0} \left((-1)^{\gamma_{j'}} \frac{t_{j'} - h_k}{4} \right) + \sum_{q_l \neq 0} \left((-1)^{\zeta_l} \frac{h_l}{4} \right). \end{aligned}$$

Thus, the powers of z in each summand of $\mu(P, z)$ can be rearranged in order to show that such a summand is a product of rational functions such as the one described in Lemma 2.5.

For the second point, it is sufficient to consider the exponents of the form

$$\frac{1}{2} \left(\sum_{a=1}^u (-1)^{\varepsilon_a} t_{i_a} \pm s h_k \right).$$

Since $u + s < \lfloor \frac{m}{2} \rfloor$, there exists an s -tuple of indices $j_1 < \dots < j_s$ such that $\{j_1, \dots, j_s\} \subset \{1, \dots, \lfloor \frac{m}{2} \rfloor\} - \{i_1, \dots, i_u\}$. Thus,

$$\begin{aligned} \left| \sum_{a=1}^u (-1)^{\varepsilon_a} t_{i_a} \pm s h_k \right| &= \left| \sum_{a=1}^u (-1)^{\varepsilon_a} \left(\frac{t_{i_a} + h_k}{2} + \frac{t_{i_a} - h_k}{2} \right) \pm \sum_{b=1}^s \left(\frac{h_k + t_{j_b}}{2} + \frac{h_k - t_{j_b}}{2} \right) \right| \\ &\leq \sum_{a=1}^u \left(\left| \frac{t_{i_a} + h_k}{2} \right| + \left| \frac{t_{i_a} - h_k}{2} \right| \right) + \sum_{b=1}^s \left(\left| \frac{h_k + t_{j_b}}{2} \right| + \left| \frac{h_k - t_{j_b}}{2} \right| \right) \\ &\leq \sum_{i=1}^m \left| \frac{t_i + h_k}{2} \right| + \left| \frac{t_i - h_k}{2} \right| < \sum_{l=1}^{2^{\lfloor r/2 \rfloor - 1}} \sum_{i=1}^m \left| \frac{t_i + h_l}{2} \right| + \left| \frac{t_i - h_l}{2} \right| \\ &\leq \sum_{c=1}^{N/2} |q_c|, \end{aligned}$$

which is the corresponding version of the inequality (4.3) in Section 4.3 ■

Theorem 4.8. *Let M be a compact N -dimensional almost- Cl_r^0 -Hermitian admitting a smooth circle action by automorphisms, $r \geq 3$. Let E, E_1, E_2 be the (locally defined) bundles described in (2.4), m, m_1, m_2 the corresponding multiplicities and u, u_1, u_2, s, t be non-negative integers satisfying the conditions given in Sections 3.2 and 3.3. Then,*

- for $r \equiv 1, 7 \pmod{8}$, if $0 \leq u + s < \lfloor \frac{m}{2} \rfloor$ and $u \leq 2^{\lfloor \frac{r}{2} \rfloor - 1}$,

$$\langle \text{ch}(S^u E) \text{ch}(\Delta_r)^s \widehat{A}(M), [M] \rangle = 0;$$

- for $r \equiv 3, 5 \pmod{8}$, if $0 \leq u + s < m$ and $u \leq 2^{\lfloor \frac{r}{2} \rfloor - 1}$,

$$\langle \text{ch}(S^u E) \text{ch}(\Delta_r)^s \widehat{A}(M), [M] \rangle = 0;$$

- for $r \equiv 0 \pmod{8}$, if $0 \leq u_1 + s < \lfloor \frac{m_1}{2} \rfloor$, $0 \leq u_2 + t < \lfloor \frac{m_2}{2} \rfloor$ and $u_1, u_2 \leq 2^{\lfloor \frac{r}{2} \rfloor - 2}$,

$$\langle \text{ch}(S^{u_1} E_1) \text{ch}(S^{u_2} E_2) \text{ch}(\Delta_r^+)^s \text{ch}(\Delta_r^-)^t \widehat{A}(M), [M] \rangle = 0;$$

- for $r \equiv 2, 6 \pmod{8}$, if $u_1, u_2 \leq 2^{\lfloor \frac{r}{2} \rfloor - 2}$ and one of $0 \leq u_1 + s < m$, $0 \leq u_2 + t < m$ or $0 \leq u_1 + t < m$, $0 \leq u_2 + s < m$,

$$\langle \text{ch}(S^{u_1} E) \text{ch}(S^{u_2} \overline{E}) \text{ch}(\Delta_r^+)^s \text{ch}(\Delta_r^-)^t \widehat{A}(M), [M] \rangle = 0;$$

- for $r \equiv 4 \pmod{8}$, if $0 \leq u_1 + s < m_1$, $0 \leq u_2 + t < m_2$ and $u_1, u_2 \leq 2^{\lfloor \frac{r}{2} \rfloor - 2}$,

$$\langle \text{ch}(S^{u_1} E_1) \text{ch}(S^{u_2} E_2) \text{ch}(\Delta_r^+)^s \text{ch}(\Delta_r^-)^t \widehat{A}(M), [M] \rangle = 0.$$

If the inequalities are not strict, the indices are rigid.

Proof. We will only describe the relevant changes to the calculation for $r \equiv 1, 7 \pmod{8}$ and m even. Let $P \subset M^{S^1}$ be an S^1 -fixed submanifold. Recall that

$$\text{ch}(S^u E) = \sum_{1 \leq i_1 \leq \dots \leq i_u \leq 2 \lfloor \frac{m}{2} \rfloor} e^{\vartheta_{i_1} + \dots + \vartheta_{i_u}},$$

where

$$\vartheta_j = \theta_j, \quad \vartheta_{\lfloor m/2 \rfloor + j} = -\theta_j, \quad j = 1, \dots, \lfloor m/2 \rfloor.$$

Thus, the exponents of the twist will be of the form

$$\frac{1}{2} \left(\sum_{a=1}^u (-1)^{\varepsilon_a} t_{i_a} + \sum_{b=1}^s (-1)^{\delta_b} h_{l_b} \right),$$

where $\varepsilon_a, \delta_b \in \{0, 1\}$. It is sufficient to consider the exponents of the form

$$\frac{1}{2} \left(\sum_{a=1}^u (-1)^{\varepsilon_a} t_{i_a} \pm s h_k \right).$$

Among them, there are two extreme types, namely the ones equal to exponents of the exterior powers which we already know how to deal with, and the ones such as ut_1 . For such an exponent, consider

$$\begin{aligned} |ut_1| &= \left| \sum_{l=1}^u \frac{t_1 + h_l}{2} + \frac{t_1 - h_l}{2} \right| \leq \sum_{l=1}^u \left| \frac{t_1 + h_l}{2} \right| + \left| \frac{t_1 - h_l}{2} \right| \leq \sum_{l=1}^{2 \lfloor \frac{m}{2} \rfloor - 1} \left| \frac{t_1 + h_l}{2} \right| + \left| \frac{t_1 - h_l}{2} \right| \\ &< \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{l=1}^{2 \lfloor \frac{m}{2} \rfloor - 1} \left| \frac{t_1 + h_l}{2} \right| + \left| \frac{t_1 - h_l}{2} \right| \leq \sum_{c=1}^{N/2} |q_c|, \end{aligned}$$

if $u < 2^{\lfloor r/2 \rfloor - 1}$. ■

Theorem 4.9. *Let M be a compact N -dimensional almost- Cl_r^0 -Hermitian admitting a smooth circle action by automorphisms, $r \geq 3$. Let E, E_1, E_2 be the (locally defined) bundles described in (2.4), m, m_1, m_2 the corresponding multiplicities and $u_i, v_i, u'_i, v'_i, s, t$ be non-negative integers satisfying analogous conditions to those given in Sections 3.2 and 3.3. Then,*

- for $r \equiv 1, 7 \pmod{8}$, if

$$0 \leq \sum_{i=1}^b u_i + \sum_{j=1}^b v_j + s < \left\lfloor \frac{m}{2} \right\rfloor$$

and

$$a + \sum_{i=1}^b v_i \leq 2^{\lfloor \frac{r}{2} \rfloor - 1},$$

$$\left\langle \text{ch} \left(\bigotimes_{i=1}^a \wedge^{u_i} E \otimes \bigotimes_{j=1}^b S^{v_j} E \otimes (\Delta_r)^{\otimes s} \right) \widehat{A}(M), [M] \right\rangle = 0;$$

- for $r \equiv 3, 5 \pmod{8}$, if

$$0 \leq \sum_{i=1}^b u_i + \sum_{j=1}^b v_j + s < m$$

and

$$a + \sum_{i=1}^b v_i \leq 2^{\lfloor \frac{r}{2} \rfloor - 1},$$

$$\left\langle \text{ch} \left(\bigotimes_{i=1}^a \wedge^{u_i} E \otimes \bigotimes_{j=1}^b S^{v_j} E \otimes (\Delta_r)^{\otimes s} \right) \widehat{A}(M), [M] \right\rangle = 0;$$

- for $r \equiv 0 \pmod{8}$, if

$$0 \leq \sum_{i=1}^b u_i + \sum_{j=1}^b v_j + s < \left\lfloor \frac{m_1}{2} \right\rfloor, \quad 0 \leq \sum_{i=1}^c u'_i + \sum_{j=1}^d v'_j + t < \left\lfloor \frac{m_2}{2} \right\rfloor$$

and

$$a + \sum_{i=1}^b v_i, c + \sum_{i=1}^d v'_i \leq 2^{\lfloor \frac{r}{2} \rfloor - 2},$$

$$\left\langle \text{ch} \left(\bigotimes_{i=1}^a \wedge^{u_i} E_1 \otimes \bigotimes_{j=1}^b S^{v_j} E_1 \otimes \bigotimes_{k=1}^c \wedge^{u'_k} E_2 \otimes \bigotimes_{l=1}^d S^{v'_l} E_2 \right. \right. \\ \left. \left. \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t} \right) \widehat{A}(M), [M] \right\rangle = 0;$$

- for $r \equiv 2 \pmod{8}$, if

$$a + \sum_{i=1}^b v_i, \quad c + \sum_{i=1}^d v'_i \leq 2^{\lfloor \frac{r}{2} \rfloor - 2},$$

and

$$0 \leq \sum_{i=1}^b u_i + \sum_{j=1}^b v_j + s < m, \quad 0 \leq \sum_{i=1}^c u'_i + \sum_{j=1}^d v'_j + t < m$$

or

$$0 \leq \sum_{i=1}^b u_i + \sum_{j=1}^b v_j + t < m, \quad 0 \leq \sum_{i=1}^c u'_i + \sum_{j=1}^d v'_j + s < m,$$

$$\left\langle \text{ch} \left(\bigotimes_{i=1}^a \wedge^{u_i} E \otimes \bigotimes_{j=1}^b S^{v_j} E \otimes \bigotimes_{k=1}^c \wedge^{u'_k} \bar{E} \otimes \bigotimes_{l=1}^d S^{v'_l} \bar{E} \right. \right. \\ \left. \left. \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t} \right) \widehat{A}(M), [M] \right\rangle = 0;$$

- for $r \equiv 4 \pmod{8}$, if

$$0 \leq \sum_{i=1}^b u_i + \sum_{j=1}^b v_j + s < m_1, \quad 0 \leq \sum_{i=1}^c u'_i + \sum_{j=1}^d v'_j + t < m_2$$

and

$$a + \sum_{i=1}^b v_i, \quad c + \sum_{i=1}^d v'_i \leq 2^{\lfloor \frac{r}{2} \rfloor - 2},$$

$$\left\langle \text{ch} \left(\bigotimes_{i=1}^a \wedge^{u_i} E_1 \otimes \bigotimes_{j=1}^b S^{v_j} E_1 \otimes \bigotimes_{k=1}^c \wedge^{u'_k} E_2 \otimes \bigotimes_{l=1}^d S^{v'_l} E_2 \right. \right.$$

$$\left. \left. \otimes (\Delta_r^+)^{\otimes s} \otimes (\Delta_r^-)^{\otimes t} \right) \widehat{A}(M), [M] \right\rangle = 0.$$

If the inequalities are not strict, the indices are rigid.

Remark 4.10. When $r = 3$, Theorems 4.7 and 4.8 return the vanishings for almost quaternion-Hermitian manifolds proved in [12].

Remark 4.11. Theorems 4.7, 4.8 and 4.9 do not restrict to the well known vanishings for almost Hermitian manifolds proved in [11], which require a divisibility condition on $c_1(M)$. This is due to the fact that the structure group of a $2m$ -dimensional almost Hermitian manifold is $U(m)$ instead of

$$\frac{U(m) \times \text{Spin}(2)}{\{\pm(\text{Id}_2, 1), \pm(i\text{Id}_2, -\text{vol}_2)\}}.$$

Remark 4.12. For $r \neq 3, 4, 6, 8$, an almost- Cl_r^0 -Hermitian manifold is Spin (see [1, Theorem 4.1]). Thus, for $u = u_1 = u_2 = s = t = 0$, the vanishings in the theorems restrict to Atiyah–Hirzebruch’s vanishing.

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