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Orthogonality Measure on the Torus for Vector-Valued Jack Polynomials[⋆]

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Abstract. For each irreducible module of the symmetric group on N objects there is a set of parametrized nonsymmetric Jack polynomials in N variables taking values in the module. These polynomials are simultaneous eigenfunctions of a commutative set of operators, self-adjoint with respect to certain Hermitian forms. These polynomials were studied by the author and J.-G. Luque using a Yang–Baxter graph technique. This paper constructs a matrix-valued measure on the N-torus for which the polynomials are mutually orthogonal. The construction uses Fourier analysis techniques. Recursion relations for the Fourier–Stieltjes coefficients of the measure are established, and used to identify parameter values for which the construction fails. It is shown that the absolutely continuous part of the measure satisfies a first-order system of differential equations.

Key words: nonsymmetric Jack polynomials; Fourier–Stieltjes coefficients; matrix-valued measure; symmetric group modules

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1 Introduction

The Jack polynomials form a parametrized basis of symmetric polynomials. A special case of these consists of the Schur polynomials, important in the character theory of the symmetric groups. By means of a commutative algebra of differential-difference operators the theory was extended to nonsymmetric Jack polynomials, again a parametrized basis but now for all polynomials in N variables. These polynomials are orthogonal for several different inner products, and in each case they are simultaneous eigenfunctions of a commutative set of self-adjoint operators. These inner products are invariant under permutations of the coordinates, that is, the symmetric group. One of these inner products is that of $L^2\big(\mathbb{T}^N, K_\kappa(x)\mathrm{d}m(x)\big)$, where

$$\mathbb{T}^N := \big\{x \in \mathbb{C}^N \colon |x_j| = 1,\, 1 \le j \le N\big\},$$
$$\mathrm{d}m(x) = (2\pi)^{-N}\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N, \qquad x_j = \exp(\mathrm{i}\theta_j), \qquad -\pi < \theta_j \le \pi, 1 \le j \le N,$$
$$K_\kappa(x) = \prod_{1\le i<j\le N} |x_i - x_j|^{2\kappa}, \qquad \kappa > -\frac{1}{N};$$

defining the N-torus, the Haar measure on the torus, and the weight function respectively. Beerends and Opdam [1] discovered this orthogonality property of symmetric Jack polynomials. Opdam [12] established orthogonality structures on the torus for trigonometric polynomials associated with Weyl groups; the nonsymmetric Jack polynomials form a special case.

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Details on the derivation of the norm formulae can be found in the treatise by Xu and the author [6, Section 10.6.3]. The weight function K_κ turned out to be the square of the base state for the Calogero–Sutherland quantum mechanical model of N identical particles located at $x_1, x_2, \dots, x_N$ on the circle with a $1/r^2$ potential. This means that the particles repel each other with a force corresponding to a potential $C|x_i - x_j|^{-2}$. See Lapointe and Vinet [10] for the construction of wavefunctions in terms of Jack polynomials for this model. More recently Griffeth [8] constructed vector-valued Jack polynomials for the family $G(n,p,N)$ of complex reflection groups. These are the groups of permutation matrices (exactly one nonzero entry in each row and each column) whose nonzero entries are n^{th} roots of unity and the product of these entries is a $(n/p)^{\mathrm{th}}$ root of unity. The symmetric groups and the hyperoctahedral groups are the special cases $G(1,1,N)$ and $G(2,1,N)$ respectively. The term "vector-valued" means that the polynomials take values in irreducible modules of the underlying group, and the action of the group is on the range as well as the domain of the polynomials. The author [3] together with Luque [5] investigated the symmetric group case more intensively. The results from these two papers are the foundation for the present work.

Since the torus structure is such an important aspect of the theory of Jack polynomials it seemed like an obvious research topic to find the role of the torus in the vector-valued Jack case. Is there a matrix-valued weight function on the torus for which the vector-valued Jack polynomials are mutually orthogonal? Some explorations in the $N = 3$ and $N = 4$ situation showed that the theory is much more complicated than the ordinary (scalar) case. For two-dimensional representations the weight function has hypergeometric function entries (see [4]); this is quite different from the rather natural product $\prod\limits_{1\le i<j\le N} |x_i - x_j|^{2\kappa}$, a power of the discriminant.

In this paper we will produce a matrix-valued measure on the torus for which the vector-valued nonsymmetric polynomials are mutually orthogonal. The result applies to arbitrary irreducible representations of the symmetric groups. In each case there is a permitted range of the parameter. We start with a concise outline of the definitions and construction of the polynomials using the Yang–Baxter graph technique in Section 2, based on [3] and [5]. Section 3 contains the construction of the abstract Hermitian form which is designed to act like an integral over the torus; that is, multiplication by a coordinate function x_j is an isometry. The method is algebraic and based on the Yang–Baxter graph. In Section 4 we use techniques from Fourier analysis to produce the desired measure. The Section begins by using the formulae from the previous sections to define the hypothetical Fourier–Stieltjes coefficients, defined on $\mathbb{Z}^N$ which is the dual group of the torus, a multiplicative group, and then applying a matrix version of a theorem of Bochner about positive-definite functions to get the measure. There is an application of approximate-identity theory using a Cesàro kernel to construct a sequence of positive matrix-valued Laurent polynomials which converges to the orthogonality measure.

Section 5 develops a recurrence relation satisfied by the Fourier–Stieltjes coefficients of the orthogonality measure. The relation allows an inductive calculation for the coefficients (but actual work, even with symbolic computation software, may not be feasible unless the dimensions are reasonably small), and it describes the list of parameter values (certain rational numbers) for which the construction fails.

The scalar weight function on the torus

$$K_\kappa(x) := \prod_{1\le i<j\le N} \left\{ (x_i - x_j)\left(x_i^{-1} - x_j^{-1}\right) \right\}^\kappa$$

satisfies a first-order differential system,

$$x_i \frac{\partial}{\partial x_i} K_\kappa(x) = \kappa K(x) \sum_{j\ne i} \frac{x_i + x_j}{x_i - x_j}, \qquad 1 \le i \le N.$$

In Section 6 we show that there is an analogous matrix differential system which is solved in a distribution sense by the orthogonality measure. We outline a result asserting that the orthogonality measure restricted to the complement of $\bigcup\limits_{1\le i<j\le N}\{x\colon x_i=x_j\}$ is equal to an analytic solution of the differential system times the Haar measure $\mathrm{d}m$. Finally there is Appendix A containing some technical background results.

2 Vector-valued Jack polynomials and the Yang–Baxter graph

This is a summary of the definitions and results from [3] and [5]. For $x=(x_1,\dots,x_N)\in\mathbb{C}^N$ the *monomial* $x^\alpha:=\prod\limits_{i=1}^N x_i^{\alpha_i}$, $\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbb{N}_0^N$, $|\alpha|:=\sum\limits_{i=1}^N\alpha_i$, $\mathbb{N}_0:=\{0,1,2,3,\dots\}$ and α is called a multi-index or a composition of $|\alpha|$. We denote two distinguished elements by $\mathbf{0}=(0,0,\dots,0)$, and $\mathbf{1}=(1,1,\dots,1)$. The *degree* of x^α is $|\alpha|$, and a polynomial is a finite linear combination of monomials. The linear space of all polynomials is denoted by $\mathcal{P}$, and $\mathcal{P}_n=\operatorname{span}\{x^\alpha\colon|\alpha|=n\}$ is the subspace of polynomials homogeneous of degree n. The specific polynomials considered here have coefficients in $\mathbb{Q}(\kappa)$ where κ is transcendental (indeterminate) but which will also take on certain real values. The multi-indices α have an important partial order: let α^+ denote the nonincreasing rearrangement of α, for example if $\alpha=(1,2,1,4)$ then $\alpha^+=(4,2,1,1)$. Let $\mathbb{N}_0^{N,+}$ denote the set of partition multi-indices, that is, $\big\{\lambda\in\mathbb{N}_0^N\colon\lambda_1\ge\lambda_2\ge\dots\ge\lambda_N\big\}$.

Definition 2.1.

$$\alpha\prec\beta\iff\sum_{j=1}^i\alpha_j\le\sum_{j=1}^i\beta_j,\qquad 1\le i\le N,\qquad\alpha\ne\beta,$$
$$\alpha\lhd\beta\iff(|\alpha|=|\beta|)\wedge\big[(\alpha^+\prec\beta^+)\vee(\alpha^+=\beta^+\wedge\alpha\prec\beta)\big].$$

For example $(3,2,1)\lhd(0,2,4)\lhd(4,0,2)$, while $(4,1,1)$, $(3,3,0)$ are not $\lhd$-comparable. The *symmetric group* $\mathcal{S}_N$, the set of permutations of $\{1,2,\dots,N\}$, acts on $\mathbb{C}^N$ by permutation of coordinates. The action is extended to polynomials by $wp(x)=p(xw)$ where $(xw)_i=x_{w(i)}$ (consider x as a row vector and w as a permutation matrix, $[w]_{ij}=\delta_{i,w(j)}$, then $xw=x[w]$). This is a representation of $\mathcal{S}_N$, that is, $w_1(w_2p)(x)=(w_2p)(xw_1)=p(xw_1w_2)=(w_1w_2)p(x)$ for all $w_1,w_2\in\mathcal{S}_N$.

Furthermore $\mathcal{S}_N$ is generated by reflections in the mirrors $\{x\colon x_i-x_j=0\}$ for $1\le i<j\le N$. These are *transpositions,* denoted by (i,j), interchanging x_i and x_j. Define the $\mathcal{S}_N$-action on $\alpha\in\mathbb{N}_0^N$ so that $(xw)^\alpha=x^{w\alpha}$

$$(xw)^\alpha=\prod_{i=1}^N x_{w(i)}^{\alpha_i}=\prod_{j=1}^N x_j^{\alpha_{w^{-1}(j)}},$$

that is $(w\alpha)_i=\alpha_{w^{-1}(i)}$ (take α as column vector, then $w\alpha=[w]\alpha$).

The *simple reflections* $s_i:=(i,i+1)$, $1\le i\le N-1$, generate $\mathcal{S}_N$. They are the key devices for applying inductive methods, and satisfy the *braid* relations:

$$s_is_j=s_js_i,\qquad|i-j|\ge 2;$$
$$s_is_{i+1}s_i=s_{i+1}s_is_{i+1}.$$

We consider the situation where the group $\mathcal{S}_N$ acts on the range as well as on the domain of the polynomials. We use vector spaces (called $\mathcal{S}_N$-modules) on which $\mathcal{S}_N$ has an irreducible unitary (orthogonal) representation: $\tau\colon\mathcal{S}_N\to O_m(\mathbb{R})$ $(\tau(w)^{-1}=\tau(w^{-1})=\tau(w)^T)$. See James and Kerber [9] for representation theory, including a modern discussion of Young's methods. We will specify an orthogonal basis and the images $\tau(s_i)$ for each i, which suffices for our purposes. Identify τ with a partition of N: $(\tau_1, \tau_2, \dots) \in \mathbb{N}_0^{N,+}$ such that $|\tau| = N$. The length of τ is $\ell(\tau) = \max\{i\colon \tau_i > 0\}$. There is a Ferrers diagram of shape τ (this diagram is given the same name), with boxes at points (i,j) with $1 \le i \le \ell(\tau)$ and $1 \le j \le \tau_i$. A *tableau* of shape τ is a filling of the boxes with numbers, and a *reverse standard Young tableau* (RSYT) is a filling with the numbers $\{1, 2, \dots, N\}$ so that the entries decrease in each row and each column. We exclude the one-dimensional representations corresponding to one-row (N) or one-column $(1, 1, \dots, 1)$ partitions, that is, we require $\dim V_\tau \ge 2$. The *hook-length* of the node $(i,j) \in \tau$ is defined to be

$$\operatorname{hook}(\tau; i, j) := \tau_i - j + \#\{k\colon i < k \le \ell(\tau) \wedge j \le \tau_k\} + 1.$$

We will need the key quantity $h_\tau := \operatorname{hook}(\tau; 1, 1) = \tau_1 + \ell(\tau) - 1$, the maximum hook-length of the diagram.

Example 2.2. Here are the Ferrers diagram, a (column-strict) tableau, and an RSYT, all of shape $(5, 3, 2)$

$$\begin{matrix} \square & \square & \square & \square & \square \\ \square & \square & \square & & \\ \square & \square & & & \end{matrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 2 & 3 \\ 1 & 2 & 2 & & \\ 2 & 4 & & & \end{bmatrix}, \quad \begin{bmatrix} 10 & 7 & 4 & 2 & 1 \\ 9 & 6 & 3 & & \\ 8 & 5 & & & \end{bmatrix}.$$

Denote the set of RSYT's of shape τ by $\mathcal{Y}(\tau)$ and let $V_\tau := \operatorname{span}\{T\colon T \in \mathcal{Y}(\tau)\}$ (the field is $\mathbb{C}(\kappa)$) with orthogonal basis $\mathcal{Y}(\tau)$. Furthermore $\dim V_\tau = \#\mathcal{Y}(\tau) = N!/ \prod\limits_{(i,j)\in\tau} \operatorname{hook}(\tau; i, j)$. For $1 \le i \le N$ and $T \in \mathcal{Y}(\tau)$ the entry i is at coordinates $(\operatorname{rw}(i,T), \operatorname{cm}(i,T))$ and the *content* is $c(i,T) := \operatorname{cm}(i,T) - \operatorname{rw}(i,T)$. Each $T \in \mathcal{Y}(\tau)$ is uniquely determined by its *content vector* $[c(i,T)]_{i=1}^N$. For the example $\tau = (3, 1)$

$$\begin{bmatrix} 4 & 2 & 1 \\ 3 & & \end{bmatrix}, \quad \begin{bmatrix} 4 & 3 & 1 \\ 2 & & \end{bmatrix}, \quad \begin{bmatrix} 4 & 3 & 2 \\ 1 & & \end{bmatrix}$$

the list of content vectors is $[2, 1, -1, 0]$, $[2, -1, 1, 0]$, $[-1, 2, 1, 0]$. To recover T from its content vector fill in the entries starting with N, then $N-1$ ($c(N-1, T) = \pm 1$) has two possibilities and so on.

Example 2.3. The list of $\mathcal{Y}(\tau)$ for $\tau = (3, 1, 1)$, $N = 5$

$$\begin{bmatrix} 5 & 2 & 1 \\ 4 & & \\ 3 & & \end{bmatrix}, \quad \begin{bmatrix} 5 & 3 & 1 \\ 4 & & \\ 2 & & \end{bmatrix}, \quad \begin{bmatrix} 5 & 3 & 2 \\ 4 & & \\ 1 & & \end{bmatrix}, \quad \begin{bmatrix} 5 & 4 & 1 \\ 3 & & \\ 2 & & \end{bmatrix}, \quad \begin{bmatrix} 5 & 4 & 2 \\ 3 & & \\ 1 & & \end{bmatrix}, \quad \begin{bmatrix} 5 & 4 & 3 \\ 2 & & \\ 1 & & \end{bmatrix}.$$

The corresponding list of content vectors is $[2, 1, -2, -1, 0]$, $[2, -2, 1, -1, 0]$, $[-2, 2, 1, -1, 0]$, $[2, -2, -1, 1, 0]$, $[-2, 2, -1, 1, 0]$, $[-2, -1, 2, 1, 0]$.

The representation theory can be developed using the content vectors in place of tableaux; this is due to Okounkov and Vershik [14].

2.1 Description of the representation τ

The formulae for the action of $\tau(s_i)$ on the basis $\mathcal{Y}(\tau)$ are from Murphy [11, Theorem 3.12]. Define $b_i(T) := 1/(c(i,T) - c(i+1,T))$. Note that $c(i,T) - c(i+1,T) = 0$ is impossible for RSYT's. If $|c(i,T) - c(i+1,T)| \ge 2$ let $T^{(i)} \in \mathcal{Y}(\tau)$ denote T with i, $i+1$ interchanged. The following describes the action of $\tau(s_i)$ (in each case there is an informal subrectangle description of the relative positions of i and $i+1$ in T; in cases (3) and (4) i and $i+1$ are not necessarily in adjacent rows or columns)

1. If $\operatorname{rw}(i,T)=\operatorname{rw}(i+1,T)$ then $\tau(s_i)T=T$; position is $[i+1,i]$, $b_i(T)=1$.
2. If $\operatorname{cm}(i,T)=\operatorname{cm}(i+1,T)$ then $\tau(s_i)T=-T$; position is $\begin{bmatrix} i+1 \\ i \end{bmatrix}$, $b_i(T)=-1$.
3. if $\operatorname{rw}(i,T)<\operatorname{rw}(i+1,T)$ (then $\operatorname{cm}(i,T)>\operatorname{cm}(i+1,T)$), position $\begin{bmatrix} * & i \\ i+1 & * \end{bmatrix}$, $c(i,T)\geq(\operatorname{cm}(i+1,T)+1)-(\operatorname{rw}(i+1,T)-1)\geq c(i+1,T)+2$, $0<b_i(T)\leq\frac{1}{2}$ then
$$\tau(s_i)T=T^{(i)}+b_i(T)T,$$
$$\tau(s_i)T^{(i)}=\big(1-b_i(T)^2\big)T-b_i(T)T^{(i)}.$$
4. if $\operatorname{rw}(i,T)>\operatorname{rw}(i+1,T)$ (and $\operatorname{cm}(i,T)<\operatorname{cm}(i+1,T)$), position $\begin{bmatrix} * & i+1 \\ i & * \end{bmatrix}$; the formula is found in case (3) interchanging T and $T^{(i)}$, and using $b_i(T)=-b_i(T^{(i)})$.

To eliminate extra parentheses we will write $\tau(i,j)$ for $\tau((i,j))$; where (i,j) is a transposition.

There is a (unique up to constant multiple) positive Hermitian form on V_τ for which τ is unitary (real orthogonal), that is $\langle\tau(w)S_1,S_2\rangle_0=\langle S_1,\tau(w)^{-1}S_2\rangle_0=\langle S_1,\tau(w)^*S_2\rangle_0$, ($S_1,S_2\in V_\tau$, $w\in\mathcal{S}_N$):

Definition 2.4.
$$\langle T,T'\rangle_0:=\delta_{T,T'}\times\prod_{\substack{1\leq i<j\leq N,\\ c(i,T)\leq c(j,T)-2}}\left(1-\frac{1}{(c(i,T)-c(j,T))^2}\right),\qquad T,T'\in\mathcal{Y}(\tau).$$

The verification of the unitary property is based on the relation
$$\big\langle T^{(i)},T^{(i)}\big\rangle_0=\big(1-b_i(T)^2\big)\langle T,T\rangle_0$$
when $0<b_i(T)\leq\frac{1}{2}$. Each $\tau(w)$ is an orthogonal matrix with respect to the orthonormal basis $\big\{\langle T,T\rangle_0^{-1/2}T\colon T\in\mathcal{Y}(\tau)\big\}$. The basis vectors T are simultaneous eigenvectors of the (reverse) *Jucys–Murphy elements* $\omega_i:=\sum\limits_{j=i+1}^{N}(i,j)$ (with $\omega_N=0$), which commute pairwise and $\tau(\omega_i)T=c(i,T)T$, for $1\leq i\leq N$ (see [11, Lemma 3.6]); as usual, τ is extended to a homomorphism of the group algebra $\mathbb{C}\mathcal{S}_N$ by $\tau\left(\sum_w b_w w\right)=\sum_w b_w\tau(w)$.

2.2 Vector-valued nonsymmetric Jack polynomials

The main concern of this paper is $\mathcal{P}_\tau=\mathcal{P}\otimes V_\tau$, the space of V_τ valued polynomials in x, which is equipped with the $\mathcal{S}_N$ action:
$$w\big(x^\alpha\otimes T\big)=(xw)^\alpha\otimes\tau(w)T,\qquad\alpha\in\mathbb{N}_0^N,\qquad T\in\mathcal{Y}(\tau),$$
extended by linearity to
$$wp(x)=\tau(w)p(xw),\qquad p\in\mathcal{P}_\tau.$$

Definition 2.5. The *Dunkl* and *Cherednik–Dunkl* operators are ($1\leq i\leq N$, $p\in\mathcal{P}_\tau$)
$$\mathcal{D}_ip(x):=\partial_ip(x)+\kappa\sum_{j\neq i}\tau(i,j)\frac{p(x)-p(x(i,j))}{x_i-x_j},$$
$$\mathcal{U}_ip(x):=\mathcal{D}_i(x_ip(x))-\kappa\sum_{j=1}^{i-1}\tau(i,j)p(x(i,j)).$$

The commutation relations analogous to the scalar case hold, that is,

$$\begin{aligned}
&\mathcal{D}_i\mathcal{D}_j=\mathcal{D}_j\mathcal{D}_i, \qquad \mathcal{U}_i\mathcal{U}_j=\mathcal{U}_j\mathcal{U}_i, \qquad 1\le i,j\le N,\\
&w\mathcal{D}_i=\mathcal{D}_{w(i)}w, \qquad \forall\, w\in\mathcal{S}_N, \qquad s_j\mathcal{U}_i=\mathcal{U}_is_j, \qquad j\neq i-1,i,\\
&s_i\mathcal{U}_is_i=\mathcal{U}_{i+1}+\kappa s_i, \qquad \mathcal{U}_is_i=s_i\mathcal{U}_{i+1}+\kappa, \qquad \mathcal{U}_{i+1}s_i=s_i\mathcal{U}_i-\kappa.
\end{aligned}$$

The simultaneous eigenfunctions of $\{\mathcal{U}_i\}$ are called (vector-valued) nonsymmetric Jack polynomials (NSJP). For generic κ these eigenfunctions form a basis of $\mathcal{P}_\tau$ (we will specify the excluded rational values in the sequel). They have a triangularity property with respect to the partial order $\rhd$. However the structure does not merely rely on leading terms of the type $x^\alpha\otimes T$. We need the rank function:

Definition 2.6. For $\alpha\in\mathbb{N}_0^N$, $1\le i\le N$

$$r_\alpha(i):=\#\{j\colon \alpha_j>\alpha_i\}+\#\{j\colon 1\le j\le i,\ \alpha_j=\alpha_i\},$$

then $r_\alpha\in\mathcal{S}_N$.

A consequence is that $r_\alpha\alpha=\alpha^+$, the nonincreasing rearrangement of α, for any $\alpha\in\mathbb{N}_0^N$. For example if $\alpha=(1,2,1,4)$ then $r_\alpha=[3,2,4,1]$ and $r_\alpha\alpha=\alpha^+=(4,2,1,1)$ (recall $w\alpha_i=\alpha_{w^{-1}(i)}$). Also $r_\alpha=I$ if and only if α is a partition ($\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_N$).

For each $\alpha\in\mathbb{N}_0^N$ and $T\in\mathcal{Y}(\tau)$ there is a NSJP $\zeta_{\alpha,T}$ with leading term $x^\alpha\otimes\tau\big(r_\alpha^{-1}\big)T$, that is,

$$\begin{aligned}
&\zeta_{\alpha,T}=x^\alpha\otimes\tau\big(r_\alpha^{-1}\big)T+\sum_{\alpha\rhd\beta}x^\beta\otimes t_{\alpha\beta}(\kappa), \qquad t_{\alpha\beta}(\kappa)\in V_\tau,\\
&\mathcal{U}_i\zeta_{\alpha,T}=\big(\alpha_i+1+\kappa c(r_\alpha(i),T)\big)\zeta_{\alpha,T}, \qquad 1\le i\le N.
\end{aligned}$$

2.3 The Yang–Baxter graph

The NSJP's can be constructed by means of a Yang–Baxter graph. The details are in [5]; this paper has several figures illustrating some typical graphs.

A node consists of

$$(\alpha,T,\xi_{\alpha.T},r_\alpha,\zeta_{\alpha,T}),$$

where $\alpha\in\mathbb{N}_0^N$, $\xi_{\alpha,T}$ is the spectral vector $\xi_{\alpha,T}(i)=\alpha_i+1+\kappa c(r_\alpha(i),T)$, $1\le i\le N$. The root is $\big(\mathbf{0},T_0,[1+\kappa c(i,T_0)]_{i=1}^N,I,1\otimes T_0\big)$ where T_0 is formed by entering $N,N-1,\dots,1$ column-by-column in the Ferrers diagram, for example $\tau=(3,3,1)$

$$T_0=\begin{bmatrix}7&4&2\\6&3&1\\5&&\end{bmatrix}, \qquad c(\cdot,T_0)=[1,2,0,1,-2,-1,0].$$

There is an adjacency relation in $\mathcal{Y}(\tau)$ based on the positions of the pairs $\{i,i+1\}$ and an inversion counter.

Definition 2.7. For $T\in\mathcal{Y}(\tau)$ set

$$\operatorname{inv}(T):=\#\big\{(i,j)\colon i<j,\ c(i,T)-c(j,T)\le-2\big\}.$$

Recall from Section 2.1 that there are four types of positions of a given pair $\{i,i+1\}$ in T, and in case (3) it is straightforward to check that $\operatorname{inv}(T^{(i)})=\operatorname{inv}(T)+1$.

If $\alpha_i \neq \alpha_{i+1}$ then $r_{s_i\alpha} = r_\alpha s_i$. The cycle $w_0 := (123\dots N)$ and the affine transformation

$$\Phi(a_1, a_2, \dots, a_N) := (a_2, a_3, \dots, a_N, a_1 + 1)$$

are fundamental parts of the construction; and $r_{\Phi\alpha} = r_\alpha w_0$ for any α, that is,

$$r_\alpha w_0(i) = r_\alpha(w_0(i)) = r_\alpha(i+1) = r_{\Phi\alpha}(i), \qquad 1 \leq i < N,$$
$$r_\alpha w_0(N) = r_\alpha(w_0(N)) = r_\alpha(1) = r_{\Phi\alpha}(N).$$

The *jumps* in the graph, which raise the degree by one, are

$$(\alpha, T, \xi_{\alpha,T}, r_\alpha, \zeta_{\alpha,T}) \xrightarrow{\Phi} \big(\Phi\alpha, T, \Phi\xi_{\alpha,T}, r_\alpha w_0, x_N w_0^{-1}\zeta_{\alpha,T}\big), \qquad (2.1)$$
$$\zeta_{\Phi\alpha,T} = x_N w_0^{-1}\zeta_{\alpha,T}$$

the leading term is $x^{\Phi\alpha} \otimes \tau\big(w_0^{-1} r_\alpha^{-1}\big)T$ and $w_0^{-1} r_\alpha^{-1} = (r_\alpha w_0)^{-1}$. For example: $\alpha = (0, 3, 5, 0)$, $r_\alpha = [3, 2, 1, 4]$, $\Phi\alpha = (3, 5, 0, 1)$, $r_{\Phi\alpha} = [2, 1, 4, 3]$.

There are two types of *steps*, labeled by s_i:

1. If $\alpha_i < \alpha_{i+1}$, then

$$(\alpha, T, \xi_{\alpha,T}, r_\alpha, \zeta_{\alpha,T}) \xrightarrow{s_i} (s_i\alpha, T, s_i\xi_{\alpha,T}, r_\alpha s_i, \zeta_{s_i\alpha,T}),$$
$$\zeta_{s_i\alpha,T} = s_i\zeta_{\alpha,T} - \frac{\kappa}{\xi_{\alpha,T}(i) - \xi_{\alpha,T}(i+1)}\zeta_{\alpha,T}.$$

Observe that this construction is valid provided $\xi_{\alpha,T}(i) \neq \xi_{\alpha,T}(i+1)$, that is, $\alpha_{i+1} - \alpha_i \neq \kappa(c(r_\alpha(i), T) - c(r_\alpha(i+1), T))$. The extreme values of $c(\cdot, T)$ are $\tau_1 - 1$ and $1 - \ell(\tau)$, thus $|c(r_\alpha(i), T) - c(r_{\alpha+1}(i), T)| \leq h_\tau - 1$. Furthermore $\alpha_{i+1} - \alpha_i \geq 1$ and the step is valid provided $\kappa m \notin \{1, 2, 3, \dots\}$ for $m = 1 - h_\tau, 2 - h_\tau, \dots, h_\tau - 1$. The bound $-1/(h_\tau - 1) < \kappa < 1/(h_\tau - 1)$ is sufficient.

2. If $\alpha_i = \alpha_{i+1}$, and the positions of $j := r_\alpha(i)$, $j+1$ in T are of type (3), that is, $c(j, T) - c(j+1, T) \geq 2$ (the definition of r_α implies $r_\alpha(i+1) = j+1$ and $s_i r_\alpha^{-1} = r_\alpha^{-1} s_j$). Set

$$b' = \frac{1}{c(j, T) - c(j+1, T)} = \frac{\kappa}{\xi_{\alpha,T}(i) - \xi_{\alpha,T}(i+1)};$$

thus $0 < b' \leq \frac{1}{2}$, there is a step

$$(\alpha, T, \xi_{\alpha,T}, r_\alpha, \zeta_{\alpha,T}) \xrightarrow{s_i} \big(\alpha, T^{(j)}, s_i\xi_{\alpha,T}, r_\alpha, \zeta_{\alpha,T^{(j)}}\big),$$
$$\zeta_{\alpha,T^{(j)}} = s_i\zeta_{\alpha,T} - b'\zeta_{\alpha,T},$$

($T^{(j)}$ is the result of interchanging j and $j+1$ in T). The leading term is transformed $s_i\big(x^\alpha \otimes \tau\big(r_\alpha^{-1}\big)T\big) = (xs_i)^\alpha \otimes \tau\big(s_i r_\alpha^{-1}\big)T = x^\alpha \otimes \tau\big(r_\alpha^{-1}\big)\tau(s_j)T$ and $\tau(s_j)T = T^{(j)} + b'T$.

There are two other possibilities corresponding to (1) and (2) for the action of s_i on $\zeta_{\alpha,T}$ when $\alpha_i = \alpha_{i+1}$ (note $r_\alpha(i+1) = r_\alpha(i) + 1$): (1) $\mathrm{rw}(r_\alpha(i), T) = \mathrm{rw}(r_\alpha(i) + 1, T)$, then $s_i\zeta_{\alpha,T} = \zeta_{\alpha,T}$; (2) $\mathrm{cm}(r_\alpha(i), T) = \mathrm{cm}(r_\alpha(i) + 1, T)$, then $s_i\zeta_{\alpha,T} = -\zeta_{\alpha,T}$.

The proofs that these formulae are mutually compatible for different paths in the graph from the root $(\mathbf{0}, T_0)$ to a given node (α, T), use inductive arguments based on the fact that these paths have the same length. The number of jumps is clearly $|\alpha|$ and the number of steps is $S(\alpha) + \mathrm{inv}(T) - \mathrm{inv}(T_0)$, where

$$S(\alpha) := \frac{1}{2} \sum_{1 \leq i < j \leq N} (|\alpha_i - \alpha_j| + |\alpha_i - \alpha_j + 1| - 1).$$

3 Hermitian forms

For a complex vector space V a Hermitian form is a mapping $\langle\cdot,\cdot\rangle\colon V\otimes V\to\mathbb{C}$ such that $\langle u,cv\rangle=c\langle u,v\rangle$, $\langle u,v_1+v_2\rangle=\langle u,v_1\rangle+\langle u,v_2\rangle$ and $\langle u,v\rangle=\overline{\langle v,u\rangle}$ for $u,v_1,v_2\in V$, $c\in\mathbb{C}$. The form is positive semidefinite if $\langle u,u\rangle\geq 0$ for all $u\in V$. The concern of this paper is with a particular Hermitian form on $\mathcal{P}_\tau$ which has the properties (for all $f,g\in\mathcal{P}_\tau$):

$$\begin{aligned}&\left\langle 1\otimes T,1\otimes T'\right\rangle=\left\langle T,T'\right\rangle_0, \qquad T,T'\in\mathcal{Y}(\tau),\\ &\langle wf,wg\rangle=\langle f,g\rangle, \qquad w\in\mathcal{S}_N,\\ &\langle x_i\mathcal{D}_if,g\rangle=\langle f,x_i\mathcal{D}_ig\rangle, \qquad 1\leq i\leq N.\end{aligned}\tag{3.1}$$

The commutation $\mathcal{U}_i=\mathcal{D}_ix_i-\kappa\sum\limits_{j<i}(i,j)=x_i\mathcal{D}_i+1+\kappa\sum\limits_{j>i}(i,j)$ together with $\langle(i,j)f,g\rangle=\langle f,(i,j)g\rangle$ show that $\langle\mathcal{U}_if,g\rangle=\langle f,\mathcal{U}_ig\rangle$ for all i. Thus the uniqueness of the spectral vectors discussed above implies that $\langle\zeta_{\alpha,T},\zeta_{\beta,T'}\rangle=0$ whenever $(\alpha,T)\neq(\beta,T')$. In particular polynomials homogeneous of different degrees are mutually orthogonal, by the basis property of $\{\zeta_{\alpha,T}\}$. We can deduce contiguity relations corresponding to the steps described above and implied by the properties of the form. Consider step type (1) with

$$s_i\zeta_{\alpha,T}=\zeta_{s_i\alpha,T}+b'\zeta_{\alpha,T}, \qquad b'=\frac{\kappa}{\xi_{\alpha,T}(i)-\xi_{\alpha,T}(i+1)}.$$

The conditions $\langle s_i\zeta_{\alpha,T},s_i\zeta_{\alpha,T}\rangle=\langle\zeta_{\alpha,T},\zeta_{\alpha,T}\rangle$ and $\langle\zeta_{\alpha,T},\zeta_{s_i\alpha,T}\rangle=0$ imply

$$\begin{aligned}&\langle\zeta_{\alpha,T},\zeta_{\alpha,T}\rangle=\left\langle\zeta_{s_i\alpha,T}+b'\zeta_{\alpha,T},\zeta_{s_i\alpha,T}+b'\zeta_{\alpha,T}\right\rangle=\langle\zeta_{s_i\alpha,T},\zeta_{s_i\alpha,T}\rangle+b'^2\langle\zeta_{\alpha,T},\zeta_{\alpha,T}\rangle,\\ &\langle\zeta_{s_i\alpha,T},\zeta_{s_i\alpha,T}\rangle=\left(1-b'^2\right)\langle\zeta_{\alpha,T},\zeta_{\alpha,T}\rangle.\end{aligned}$$

A necessary condition that the form be positive-definite ($f\neq 0$ implies $\langle f,f\rangle>0$) is that $-1<b'<1$ in each of the possible steps. Since (with $j=r_\alpha(i)$ and $\ell=r_\alpha(i+1)$)

$$1-b'^2=\frac{[\alpha_{i+1}-\alpha_i+(c(\ell,T)-c(j,T)+1)\kappa][\alpha_{i+1}-\alpha_i+(c(\ell,T)-c(j,T)-1)\kappa]}{[\alpha_{i+1}-\alpha_i+(c(\ell,T)-c(j,T))\kappa]^2},$$

the extreme values of $(c(\ell,T)-c(j,T)\pm 1)$ are $\pm h_\tau$, and $\alpha_{i+1}-\alpha_i\geq 1$, it follows that $-1/h_\tau<\kappa<1/h_\tau$ implies $1-b'^2>0$. Since steps of type (1) link any (α,T) to (α^+,T) one can obtain (with $\varepsilon=\pm 1$)

$$\begin{aligned}&\mathcal{E}_\varepsilon(\alpha,T):=\prod_{\substack{1\leq i<j\leq N\\ \alpha_i<\alpha_j}}\left(1+\frac{\varepsilon\kappa}{\alpha_j-\alpha_i+\kappa(c(r_\alpha(j),T)-c(r_\alpha(i),T))}\right),\\ &\langle\zeta_{\alpha,T},\zeta_{\alpha,T}\rangle=\left(\mathcal{E}_1(\alpha,T)\mathcal{E}_{-1}(\alpha,T)\right)^{-1}\langle\zeta_{\alpha^+,T},\zeta_{\alpha^+,T}\rangle.\end{aligned}\tag{3.2}$$

Similarly the steps of type (2) (with $\alpha_i=\alpha_{i+1}$ and $j=r_\alpha(i)$, $b'=\frac{1}{c(j,T)-c(j+1,T)}$) imply the relation

$$\left\langle\zeta_{\alpha,T^{(j)}},\zeta_{\alpha,T^{(j)}}\right\rangle=\left(1-b'^2\right)\langle\zeta_{\alpha,T},\zeta_{\alpha,T}\rangle.$$

It was shown in [3] (this is a special case of a result of Griffeth [8, Theorem 6.1]) that the definition for $\lambda\in\mathbb{N}_0^{N,+}$

$$\langle\zeta_{\lambda,T},\zeta_{\lambda,T}\rangle=\langle T,T\rangle_0\prod_{i=1}^N(1+\kappa c(i,T))_{\lambda_i}\prod_{1\leq i<j\leq N}\prod_{\ell=1}^{\lambda_i-\lambda_j}\left(1-\left(\frac{\kappa}{\ell+\kappa(c(i,T)-c(j,T))}\right)^2\right).$$

together with formula (3.2) produce a Hermitian form (called the covariant form) satisfying (3.1) and the additional property $\langle x_i f, g\rangle = \langle f, \mathcal{D}_i g\rangle$ for all $f, g \in \mathcal{P}_\tau$ and $1 \le i \le N$ (the bound $-1/h_\tau < \kappa < 1/h_\tau$ for positivity of this form was found by Etingof and Stoica [7]).

Here we want a Hermitian form for which multiplication by any x_i is an isometry, that is, $\langle x_i f, x_i g\rangle = \langle f, g\rangle$ for all $f, g \in \mathcal{P}_\tau$ and $1 \le i \le N$. Heuristically this should involve an integral over the N-torus. The isometry postulate, and the equations (3.1) determine the form uniquely, as will be shown. The postulate $\langle x_N f, g\rangle = \langle f, \mathcal{D}_N g\rangle$ in the covariant form is used to compute the effect of a jump $\zeta_{\Phi\alpha,T} = x_N w_0^{-1}\zeta_{\alpha,T}$ (see (2.1)), that is, to evaluate $\langle \zeta_{\Phi\alpha,T}, \zeta_{\Phi\alpha,T}\rangle / \langle \zeta_{\alpha,T}, \zeta_{\alpha,T}\rangle$. From the proofs in [3, Appendix, Corollary 5, Theorem 10] and [5] we see that the factor $\prod_{i=1}^{N}(1+\kappa c(i,T))_{\lambda_i}$ arises from ratios of this type. This aspect (here we need the ratio to be 1) motivates the following:

Definition 3.1. For $\lambda \in \mathbb{N}_0^{N,+}$, $\alpha, \beta \in \mathbb{N}_0^N$, and $T, T' \in \mathcal{Y}(\tau)$ the Hermitian form $\langle\cdot,\cdot\rangle_{\mathbb{T}}$ on $\mathcal{P}_\tau$ is specified by

$$(\alpha, T) \ne (\beta, T') \Longrightarrow \langle \zeta_{\alpha,T}, \zeta_{\beta,T}\rangle_{\mathbb{T}} = 0,$$
$$\langle \zeta_{\lambda,T}, \zeta_{\lambda,T}\rangle_{\mathbb{T}} = \langle T, T\rangle_0 \prod_{1\le i<j\le N} \prod_{\ell=1}^{\lambda_i-\lambda_j} \left(1 - \left(\frac{\kappa}{\ell + \kappa(c(i,T) - c(j,T))}\right)^2\right),$$
$$\langle \zeta_{\alpha,T}, \zeta_{\alpha,T}\rangle_{\mathbb{T}} = \big(\mathcal{E}_1(\alpha,T)\mathcal{E}_{-1}(\alpha,T)\big)^{-1} \langle \zeta_{\alpha^+,T}, \zeta_{\alpha^+,T}\rangle_{\mathbb{T}};$$

the form is extended to all of $\mathcal{P}_\tau$ by linearity in the second variable and Hermitian symmetry, that is, $\langle f, c_1 g + c_2 h\rangle_{\mathbb{T}} = c_1\langle f, g\rangle_{\mathbb{T}} + c_2\langle f, h\rangle_{\mathbb{T}}$ and $\langle f, g\rangle_{\mathbb{T}} = \overline{\langle g, f\rangle_{\mathbb{T}}}$, for $f, g, h \in \mathcal{P}_\tau$ and $c_1, c_2 \in \mathbb{C}$.

Observe that the formula is invariant when λ is replaced by $\lambda + m\,\mathbf{1} = (\lambda_1 + m, \lambda_2 + m, \ldots, \lambda_N + m)$ for any $m \in \mathbb{N}$. This follows easily from the commutation (where $e_N := x_1 x_2 \cdots x_N$)

$$\mathcal{U}_i\big(e_N^m f\big) = m e_N^m f + e_N^m \mathcal{U}_i f, \qquad 1 \le i \le N, \qquad m = 1, 2, \ldots,$$

thus $\mathcal{U}_i(e_N^m \zeta_{\alpha,T}) = (m + \alpha_i + \kappa c(r_\alpha(i), T)) e_N^m \zeta_{\alpha,T}$, and $e_N^m \zeta_{\alpha,T}$ is a simultaneous eigenfunction of $\{\mathcal{U}_i\}$ with the same eigenvalues and the same leading term as $\zeta_{\alpha+m\mathbf{1},T}$. Hence $\zeta_{\alpha+m\mathbf{1},T} = e_N^m \zeta_{\alpha,T}$. We now extend the structure of NSJP's to V_τ-valued Laurent polynomials, thereby producing a basis:

Definition 3.2. Suppose $\alpha \in \mathbb{Z}^N$ then set $\zeta_{\alpha,T} = e_N^{-m}\zeta_{\alpha+m\mathbf{1},T}$ where $m \in \mathbb{N}_0$ and satisfies $m \ge -\min_j \alpha_i$. This is valid since $\alpha + m\mathbf{1} \in \mathbb{N}_0^N$ and by the relation $\zeta_{\beta+k\mathbf{1},T} = e_N^k \zeta_{\beta,T}$ for $\beta \in \mathbb{N}_0^N$ and $k \in \mathbb{N}_0$.

The proof that the form satisfies the properties (3.1) with respect to steps is the same as the one in [3, Propositions 8 and 9], and it suffices to verify the effect of a jump.

Theorem 3.3. *Suppose $\alpha \in \mathbb{N}_0^{N,+}$ then $\langle \zeta_{\Phi\alpha,T}, \zeta_{\Phi\alpha,T}\rangle_{\mathbb{T}} = \langle \zeta_{\alpha,T}, \zeta_{\alpha,T}\rangle_{\mathbb{T}}$.*

Proof. We use (2.1) to relate $\langle \zeta_{\Phi\alpha,T}, \zeta_{\Phi\alpha,T}\rangle_{\mathbb{T}}$ to $\langle \zeta_{\beta,T}, \zeta_{\beta,T}\rangle_{\mathbb{T}}$ where $\beta = (\Phi\alpha)^+ = (\alpha_1 + 1, \alpha_2, \ldots, \alpha_N)$. The product $\mathcal{E}_\varepsilon(\Phi\alpha, T)$ is over the pairs $\alpha_i = (\Phi\alpha)_{i-1} < (\Phi\alpha)_N = \alpha_1 + 1$ for $2 \le i \le N$, thus

$$\mathcal{E}_\varepsilon(\Phi\alpha, T) = \prod_{i=2}^{N}\left(1 - \frac{\varepsilon\kappa}{\alpha_1 + 1 - \alpha_i + \kappa(c(r_{\Phi\alpha}(N), T) - c(r_{\Phi\alpha}(i-1), T))}\right)$$
$$= \prod_{i=2}^{N}\left(1 - \frac{\varepsilon\kappa}{\alpha_1 + 1 - \alpha_i + \kappa(c(1,T) - c(i,T))}\right).$$

By definition

$$\begin{aligned}\langle\zeta_{\Phi\alpha,T},\zeta_{\Phi\alpha,T}\rangle_{\mathbb{T}} &= \big(\mathcal{E}_1(\Phi a,T)\mathcal{E}_{-1}(\Phi\alpha,T)\big)^{-1}\langle\zeta_{\beta,T},\zeta_{\beta,T}\rangle_{\mathbb{T}}\\ &= \prod_{i=2}^{N}\left(1-\left(\frac{\kappa}{\alpha_1+1-\alpha_i+\kappa(c(1,T)-c(i,T))}\right)^2\right)^{-1}\\ &\quad\times\langle T,T\rangle_0\prod_{1\le i<j\le N}\prod_{\ell=1}^{\beta_i-\beta_j}\left(1-\left(\frac{\kappa}{\ell+\kappa(c(i,T)-c(j,T))}\right)^2\right)\\ &= \langle T,T\rangle_0\prod_{2\le i<j\le N}\prod_{\ell=1}^{\alpha_i-\alpha_j}\left(1-\left(\frac{\kappa}{\ell+\kappa(c(i,T)-c(j,T))}\right)^2\right)\\ &\quad\times\prod_{j=2}^{N}\prod_{\ell=1}^{\alpha_1-\alpha_j}\left(1-\left(\frac{\kappa}{\ell+\kappa(c(1,T)-c(j,T))}\right)^2\right)=\langle\zeta_{\alpha,T},\zeta_{\alpha,T}\rangle_{\mathbb{T}}.\end{aligned}$$

The terms in the product for $i=1$ and $2\le j\le N$, $\ell=\beta_1-\beta_j=\alpha_1+1-\alpha_j$ are canceled out. ■

We summarize the key results. We say κ is *generic* if $(\alpha,T)\ne(\beta,T')$ implies the spectral vectors $\xi_{\alpha,T}\ne\xi_{\beta,T'}$.

Proposition 3.4. *For generic κ the Hermitian form $\langle\cdot,\cdot\rangle_{\mathbb{T}}$ satisfies*

1) *if f, g are homogeneous and $\deg f\ne\deg g$ then $\langle f,g\rangle_{\mathbb{T}}=0$,*
2) $\langle wf,wg\rangle_{\mathbb{T}}=\langle f,g\rangle_{\mathbb{T}}$, $f,g\in\mathcal{P}_\tau$, $w\in\mathcal{S}_N$,
3) $\langle x_i\mathcal{D}_if,g\rangle_{\mathbb{T}}=\langle f,x_i\mathcal{D}_ig\rangle_{\mathbb{T}}$ *for* $f,g\in\mathcal{P}_\tau$ *and* $1\le i\le N$,
4) $\langle x_if,x_ig\rangle_{\mathbb{T}}=\langle f,g\rangle_{\mathbb{T}}$ *for* $f,g\in\mathcal{P}_\tau$ *and* $1\le i\le N$.

Proof. For generic κ the NSJP's $\zeta_{\alpha,T}$ with $|\alpha|=n$ form a basis for $\mathcal{P}_{\tau,n}$; this immediately implies (1). For (2) the fact that $\langle s_i\zeta_{\alpha,T},s_i\zeta_{\beta,T'}\rangle_{\mathbb{T}}=\langle\zeta_{\alpha,T},\zeta_{\beta,T'}\rangle_{\mathbb{T}}$ for $1\le i<N$ follows from the corresponding results in [3, Propositions 8 and 9, Corollary 3] when $|\alpha|=|\beta|$, otherwise from Definition 3.1. This suffices for (2) since $\{s_i\}$ generates $\mathcal{S}_N$. The definition of NSJP's implies trivially that $\langle\mathcal{U}_i\zeta_{\alpha,T},\zeta_{\beta,T'}\rangle_{\mathbb{T}}=\langle\zeta_{\alpha,T},\mathcal{U}_i\zeta_{\beta,T'}\rangle_{\mathbb{T}}$ for all i and (α,T), (β,T') because both sides vanish if $(\alpha,T)\ne(\beta,T')$, otherwise equal $\xi_{\alpha,T}(i)\langle\zeta_{\alpha,T},\zeta_{\alpha,T}\rangle_{\mathbb{T}}$. The commutation $\mathcal{U}_i=\mathcal{D}_ix_i-\kappa\sum\limits_{j<i}(i,j)=x_i\mathcal{D}_i+1+\kappa\sum\limits_{j>i}(i,j)$ together with $\langle(i,j)f,g\rangle_{\mathbb{T}}=\langle f,(i,j)g\rangle_{\mathbb{T}}$ from (2) show that $\langle x_i\mathcal{D}_if,g\rangle_{\mathbb{T}}=\langle f,x_i\mathcal{D}_ig\rangle_{\mathbb{T}}$ for $1\le i\le N$. For part (4) (recall $w_0=(123\dots N)$) $\zeta_{\Phi\alpha,T}=x_Nw_0^{-1}\zeta_{\alpha,T}$ and $\zeta_{\Phi\beta,T'}=x_Nw_0^{-1}\zeta_{\beta,T'}$. By Theorem 3.3 $\langle\zeta_{\Phi\alpha,T},\zeta_{\Phi\beta,T'}\rangle_{\mathbb{T}}=\langle\zeta_{\alpha,T},\zeta_{\beta,T'}\rangle_{\mathbb{T}}$ (if $(\alpha,T)\ne(\beta,T')$ then $(\Phi\alpha,T)\ne(\Phi\beta,T')$). Thus for each (α,T), (β,T')

$$\big\langle x_N\big(w_0^{-1}\zeta_{\alpha,T}\big),x_N\big(w_0^{-1}\zeta_{\beta,T'}\big)\big\rangle_{\mathbb{T}}=\langle\zeta_{\alpha,T},\zeta_{\beta,T'}\rangle_{\mathbb{T}}=\big\langle w_0^{-1}\zeta_{\alpha,T},w_0^{-1}\zeta_{\beta,T'}\big\rangle_{\mathbb{T}}.$$

The set $\big\{w_0^{-1}\zeta_{\alpha,T}:(\alpha,T)\big\}$ is a basis for $\mathcal{P}_\tau$ thus $\langle x_Nf,x_Ng\rangle_{\mathbb{T}}=\langle f,g\rangle_{\mathbb{T}}$ for all $f,g\in\mathcal{P}_\tau$. For any i

$$\begin{aligned}\langle x_if,x_ig\rangle_{\mathbb{T}}&=\langle(i,N)x_if,(i,N)x_ig\rangle_{\mathbb{T}}=\langle x_N(i,N)f,x_N(i,N)g\rangle_{\mathbb{T}}\\ &=\langle(i,N)f,(i,N)g\rangle_{\mathbb{T}}=\langle f,g\rangle_{\mathbb{T}};\end{aligned}$$

and this completes the proof. ■

This lays the abstract foundation for the next developments.

4 Fourier–Stieltjes coefficients on the torus

The torus $\mathbb{T}^N := \{x \in \mathbb{C}^N : |x_i| = 1,\ 1 \le i \le N\}$ is a multiplicative compact abelian group with dual group $\mathbb{Z}^N$. We will use this property to find the measure of orthogonality for the NSJP's on the torus. First we produce the Fourier–Stieltjes coefficients of the hypothetical measure and then use a matrix version of a theorem of Bochner to deduce the existence of the measure.

When κ is generic the NSJP's form a basis for $\mathcal{P}_\tau$ and it is possible to make the definition

$$\widetilde{A}(\alpha,\beta,T,T') := \big(\langle T,T\rangle_0 \langle T',T'\rangle_0\big)^{-1/2} \langle x^\alpha \otimes T, x^\beta \otimes T'\rangle_{\mathbb{T}}$$

for $\alpha,\beta \in \mathbb{N}_0^N$ and $T,T' \in \mathcal{Y}(\tau)$. In effect this uses the orthonormal basis of V_τ. By the symmetry of the form $\widetilde{A}(\alpha,\beta,T,T') = \widetilde{A}(\beta,\alpha,T',T)$. By Proposition 3.4 $|\alpha| \ne |\beta|$ implies $\widetilde{A}(\alpha,\beta,T,T') = 0$. Another consequence is $\widetilde{A}(\mathbf{0},\mathbf{0},T,T') = \delta_{T,T'}$.

Definition 4.1. For each $\gamma \in \mathbb{Z}^N$ with $\sum_{i=1}^N \gamma_i = 0$ let $\gamma_i^\pi = \max(\gamma_i,0)$ and $\gamma_i^\nu = -\min(\gamma_i,0)$ for $1 \le i \le N$; then $\gamma = \gamma^\pi - \gamma^\nu$ and $\gamma^\pi,\gamma^\nu \in \mathbb{N}_0^N$. Furthermore $|\gamma^\pi| = |\gamma^\nu|$ and $\sum_i |\gamma_i| = |\gamma^\pi| + |\gamma^\nu|$ is even.

Introduce the index set $\boldsymbol{Z}_N$ and its graded components by

$$\boldsymbol{Z}_N := \left\{\alpha \in \mathbb{Z}^N : \sum_{i=1}^N \alpha_i = 0\right\},$$

$$\boldsymbol{Z}_{N,n} := \left\{\alpha \in \boldsymbol{Z}_N : \sum_{i=1}^N |\alpha_i| = 2n\right\}, \qquad n = 0,1,2,\dots.$$

Formula (A.1) for $\#\boldsymbol{Z}_{N,n}$ is in Appendix A.

Definition 4.2. For $\gamma \in \mathbb{Z}^N$ the matrix A_γ (of size $\#\mathcal{Y}(\tau) \times \#\mathcal{Y}(\tau)$) is given by

$$(A_\gamma)_{T,T'} = \widetilde{A}(\gamma^\pi,\gamma^\nu,T,T'), \qquad T,T' \in \mathcal{Y}(\tau), \qquad \gamma \in \boldsymbol{Z}_N,$$
$$A_\gamma = 0, \qquad \gamma \notin \boldsymbol{Z}_N.$$

Proposition 4.3. *Suppose $\alpha,\beta \in \mathbb{N}_0^N$ and $T,T' \in \mathcal{Y}(\tau)$ then $\widetilde{A}(\alpha,\beta,T,T') = (A_{\alpha-\beta})_{T,T'}$.*

Proof. If $|\alpha| \ne |\beta|$ then $\sum\limits_{i=1}^N (\alpha_i - \beta_i) \ne 0$, $\widetilde{A}(\alpha,\beta,T,T') = 0$ and $A_{\alpha-\beta} = 0$ by definition. If $|\alpha| = |\beta|$ let $\zeta_i = \min(\alpha_i,\beta_i)$ for $1 \le i \le N$ then x^ζ is a factor of both x^α and x^β; by Proposition 3.4 $\langle x^\alpha \otimes T, x^\beta \otimes T'\rangle_{\mathbb{T}} = \langle x^{\alpha-\zeta} \otimes T, x^{\beta-\zeta} \otimes T'\rangle_{\mathbb{T}}$. By construction $(\alpha-\beta)^\pi = \alpha-\zeta$, $(\alpha-\beta)^\nu = \beta-\zeta$. It follows that

$$\langle x^\alpha \otimes T, x^\beta \otimes T'\rangle = T^* A_{\alpha-\beta} T' = \big(\langle T,T\rangle_0 \langle T',T'\rangle_0\big)^{1/2} (A_{\alpha-\beta})_{T,T'}. \qquad \blacksquare$$

For a formal Laurent series $h(x) = \sum\limits_{\alpha \in \mathbb{Z}^N} c_\alpha x^\alpha$ let $\mathrm{CT}(h(x)) = c_{\mathbf{0}}$, the constant term. Then

$$\langle x^\alpha \otimes T, x^\beta \otimes T'\rangle = \big(\langle T,T\rangle_0 \langle T',T'\rangle_0\big)^{1/2} \mathrm{CT}\left(x^{-\alpha} \sum_{\gamma \in \mathbb{Z}^N} (A_\gamma)_{T,T'} x^\gamma x^\beta\right).$$

In the next section we investigate analytical properties of the formal series, but first we consider algebraic properties, that is, those not needing any convergence results.

Theorem 4.4. *Suppose $\gamma \in \boldsymbol{Z}_N$ and $w \in \mathcal{S}_N$ then $A_{-\gamma} = A_\gamma^*$ and $A_{w\gamma} = \tau(w) A_\gamma \tau(w^{-1})$.*

Proof. The relation $\widetilde{A}(\alpha,\beta,T,T')=\widetilde{A}(\beta,\alpha,T',T)$ shows $(A_{\alpha-\beta})_{T,T'}=(A_{\beta-\alpha})_{T',T}$. By definition

$$\begin{aligned}\langle w(x^{\alpha}\otimes T),w(x^{\beta}\otimes T')\rangle_{\mathbb{T}}&=\langle x^{w\alpha}\otimes\tau(w)T,x^{w\beta}\otimes\tau(w)T'\rangle_{\mathbb{T}}\\&=\left(\langle T,T\rangle_0\langle T',T'\rangle_0\right)^{1/2}T^*\tau(w)^*A_{w\alpha-w\beta}\tau(w)T'\\&=\langle x^{\alpha}\otimes T,x^{\beta}\otimes T'\rangle_{\mathbb{T}}=\left(\langle T,T\rangle_0\langle T',T'\rangle_0\right)^{1/2}T^*A_{\alpha-\beta}T'\end{aligned}$$

and thus $A_\gamma=\tau(w)^{-1}A_{w\gamma}\tau(w)$ (recall τ is real-orthogonal so $\tau(w)^*=\tau(w^{-1})$). ∎

Summing over the graded components $\boldsymbol{Z}_{N,n}$ produces Laurent polynomials with good properties, such as analyticity in $(\mathbb{C}\backslash\{0\})^N$. The maps $a\mapsto w\alpha$ $(w\in\mathcal{S}_N)$ and $\alpha\mapsto-\alpha$ act as permutations on each $\boldsymbol{Z}_{N,n}$.

Definition 4.5. For $n=0,1,2,\dots$ let

$$H_n(x):=\sum_{\alpha\in\boldsymbol{Z}_{N,n}}A_\alpha x^\alpha,$$

a Laurent polynomial with matrix coefficients.

For complex Laurent polynomials $f(x)=\sum_{\alpha\in\mathbb{Z}^N}c_\alpha x^\alpha$ (finite sum) define $f(x)^*=\sum_{\alpha\in\mathbb{Z}^N}\overline{c_\alpha}x^{-\alpha}$; if the coefficients $\{c_\alpha\}$ are matrices then $f(x)^*=\sum_{\alpha\in\mathbb{Z}^N}c_\alpha^*x^{-\alpha}$. There is a slight abuse of notation here: if $x\in\mathbb{T}^N$ then $\overline{(x^\alpha)}=x^{-\alpha}$ and $f(x)^*$ agrees with the adjoint of the matrix $f(x)$.

Proposition 4.6. *Suppose $n=0,1,2,\dots$ and $w\in\mathcal{S}_N$ then $H_n(xw)=\tau(w)^{-1}H_n(x)\tau(w)$ and $H_n(x)^*=H_n(x)$.*

Proof. Compute

$$\begin{aligned}H_n(xw)&=\sum_{\alpha\in\boldsymbol{Z}_{N,n}}A_\alpha(wx)^\alpha=\sum_{\alpha\in\boldsymbol{Z}_{N,n}}A_\alpha x^{w\alpha}=\sum_{\beta\in\boldsymbol{Z}_{N,n}}A_{w^{-1}\beta}x^\beta\\&=\tau(w^{-1})\sum_{\beta\in\boldsymbol{Z}_{N,n}}A_\beta x^\beta\tau(w)=\tau(w)^{-1}H_n(x)\tau(w).\end{aligned}$$

Also $H_n(x)^*=\sum_{\alpha\in\boldsymbol{Z}_{N,n}}A_\alpha^*x^{-\alpha}=\sum_{\alpha\in\boldsymbol{Z}_{N,n}}A_{-\alpha}x^{-\alpha}=H_n(x)$. ∎

As a consequence we find an important commutation satisfied by a particular point value of $H_n(x)$ (recall the N-cycle $w_0=(1,2,\dots,N)$).

Corollary 4.7. *Suppose $n=1,2,3,\dots$ then $\tau(w_0)^{-1}H_n(x_0)\tau(w_0)=H_n(x_0)$, where $x_0=(1,\omega,\dots,\omega^{N-1})$, $\omega=\exp\frac{2\pi\mathrm{i}}{N}$.*

Proof. By definition $x_0w_0=(\omega,\dots,\omega^{N-1},1)=\omega x_0$. Each monomial x^α for $\alpha\in\boldsymbol{Z}_N$ is homogeneous of degree zero (suppose $c\in\mathbb{C}\backslash\{0\}$ then $(cx)^\alpha=c^{\alpha_1+\cdots+\alpha_N}x^\alpha=x^\alpha$) thus $H_n(x_0w_0)=H_n(\omega x_0)=H_n(x_0)$ and $H_n(x_0w_0)=\tau(w_0)^{-1}H_n(x_0)\tau(w_0)$. ∎

We turn to the harmonic analysis significance of the matrices $\{A_\alpha\}$. For an integrable function f on $\mathbb{T}^N$ the Fourier transform (coefficient) is

$$\widehat{f}(\alpha)=\int_{\mathbb{T}^N}f(x)x^{-\alpha}\mathrm{d}m(x),\qquad \alpha\in\mathbb{Z}^N,$$

where $x := (\exp(\mathrm{i}\theta_1), \dots, \exp(\mathrm{i}\theta_N))$ and $\mathrm{d}m(x) = (2\pi)^{-N}\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N$; and for a Baire measure μ on $\mathbb{T}^N$ the Fourier–Stieltjes transform is

$$\widehat{\mu}(\alpha) = \int_{\mathbb{T}^N} x^{-\alpha}\mathrm{d}\mu(x), \qquad \alpha \in \mathbb{Z}^N.$$

We will show that there is a matrix-valued measure μ, positive in a certain sense, such that $\widehat{\mu}(\alpha) = A_\alpha$ for all $\alpha \in \mathbb{Z}^N$ provided that $-1/h_\tau < \kappa < 1/h_\tau$. There is a version of a theorem of Bochner about positive-definite functions on a locally compact abelian group which proves this claim. The details of the proof and some consequences are in Appendix A.

Let $n = \dim V_\tau$ and identify V_τ with $\mathbb{C}^n$ whose elements are considered as column vectors (in effect we use indices $1 \le i \le n$ instead of $\{T \in \mathcal{Y}(\tau)\}$). The inner product on $\mathbb{C}^n$ is $\langle u, v\rangle := \sum_{i=1}^n \overline{u_i}v_i$, and the norm is $|v| = \sqrt{\langle v, v\rangle}$. Note that $\langle u, Av\rangle = u^*Av$. A positive-definite matrix P satisfies $\langle u, Pu\rangle \ge 0$ for all $u \in \mathbb{C}^n$ (this implies $P^* = P$).

Definition 4.8. A function $F\colon \mathbb{Z}^N \to M_n(\mathbb{C})$ is positive-definite if

$$\sum_{\alpha,\beta\in\mathbb{Z}^N} f(\alpha)^*F(\alpha-\beta)f(\beta) \ge 0$$

for any finitely supported $\mathbb{C}^n$-valued function f on $\mathbb{Z}^N$.

Theorem 4.9. *Suppose F is positive-definite then there exist Baire measures $\{\mu_{jk}\colon 1 \le j,k \le n\}$ on $\mathbb{T}^N$ such that*

$$\int_{\mathbb{T}^N} x^{-\alpha}\mathrm{d}\mu_{jk}(x) = F(\alpha)_{jk}, \qquad \alpha \in \mathbb{Z}^N, \qquad 1 \le j,k \le n.$$

Furthermore each μ_{jj} is positive and

$$\langle f, g\rangle_F := \sum_{i,j=1}^n \int_{\mathbb{T}^N} \overline{f(x)_i}g(x)_j\mathrm{d}\mu_{ij}(x)$$

defines a positive-semidefinite Hermitian form on $C(\mathbb{T}^N;\mathbb{C}^n)$ (continuous $\mathbb{C}^n$-valued functions on $\mathbb{T}^N$) satisfying $|\langle f, g\rangle_F| \le B\|f\|_\infty\|g\|_\infty$ for $f, g \in C(\mathbb{T}^N;\mathbb{C}^n)$ with $B < \infty$.

The proof is in Appendix A.1 and Theorem A.3. (In general the measures μ_{jk} are not real-valued for $j \ne k$.) For notational simplicity we introduce

$$\int_{\mathbb{T}^N} f(x)^*\mathrm{d}\mu(x)g(x) := \sum_{i,j=1}^n \int_{\mathbb{T}^N} \overline{f(x)_i}g(x)_j\mathrm{d}\mu_{ij}(x). \tag{4.1}$$

To show that $\alpha \mapsto A_\alpha$ is positive-definite let f be a finitely supported $\mathbb{C}^n$-valued function f on $\mathbb{Z}^N$ and let $p(x) = \sum_{\alpha,T} \langle T, T\rangle_0^{-1/2} f_T(\alpha)x^\alpha \otimes T$ be the associated Laurent polynomial (now we use the T indices on $\mathbb{C}^n$). Because this is a finite sum there is a nonnegative integer m such that $e_N^m p(x)$ is polynomial (no negative powers). Then for $-1/h_\tau < \kappa < 1/h_\tau$

$$\begin{aligned}0 \le \langle e_N^m p, e_N^m p\rangle_{\mathbb{T}} &= \sum_{\alpha,\beta\in\mathbb{Z}^N}\sum_{T,T'} \left(\langle T, T\rangle_0 \langle T', T'\rangle_0\right)^{-1/2} \overline{f_T(\alpha)} f_{T'}(\beta) \langle x^{\alpha+m\mathbf{1}} \otimes T, x^{\beta+m\mathbf{1}} \otimes T'\rangle \\ &= \sum_{\alpha,\beta\in\mathbb{Z}^N}\sum_{T,T'} \overline{f_T(\alpha)} f_{T'}(\beta)(A_{\alpha-\beta})_{T,T'}.\end{aligned}$$

Let $\mu = [\mu_{T,T'}]$ be the matrix of measures produced by the theorem, that is,

$$(A_\alpha)_{T,T'} = \int_{\mathbb{T}^N} x^{-\alpha}\mathrm{d}\mu_{T,T'}(x), \qquad \alpha \in \mathbb{Z}^N, \qquad T, T' \in \mathcal{Y}(\tau).$$

Theorem 4.10. *For $-1/h_\tau < \kappa < 1/h_\tau$ there exists a matrix of Baire measures $\mu = [\mu_{T,T'}]$ on $\mathbb{T}^N$ such that*

$$\langle f, g\rangle_{\mathbb{T}} = \int_{\mathbb{T}^N} f(x)^* \mathrm{d}\mu(x) g(x)$$

for all Laurent polynomials f, g with coefficients in V_τ, in particular for all NSJP's f, g.

Of course we want more detailed information about these measures. The first step is to apply an approximate identity, a tool from the convolution structure for measures and functions on the torus. We consider Cesàro summation of the series $\sum_\alpha A_\alpha x^\alpha$ based on summing first over each $\boldsymbol{Z}_{N,n}$. Set

$$S_n(x) := \sum_{\alpha\in\boldsymbol{Z}_{N,n}} x^\alpha,$$

a Laurent polynomial, and the corresponding (C,δ)-kernel (for $\delta > 0$) is defined to be (the Pochhammer symbol is $(t)_m = \prod\limits_{i=1}^{m}(t+i-1)$)

$$\sigma_n^\delta(x) := \sum_{k=0}^{n} \frac{(-n)_k}{(-n-\delta)_k} S_k(x).$$

The point is that $\lim\limits_{n\to\infty} \frac{(-n)_k}{(-n-\delta)_k} = 1$ for fixed k. In terms of convolution

$$\begin{aligned}
\sigma_n^\delta * \mu(x) &= \int_{\mathbb{T}^N} \sigma_n^\delta\big(xy^{-1}\big)\mathrm{d}\mu(y),\\
\big(\widehat{\sigma_n^\delta * \mu}\big)(\alpha) &= \int_{\mathbb{T}^N}\int_{\mathbb{T}^N} x^{-\alpha}\sigma_n^\delta\big(xy^{-1}\big)\mathrm{d}\mu(y)\mathrm{d}m(x)\\
&= \int_{\mathbb{T}^N}\int_{\mathbb{T}^N} (xy)^{-\alpha}\sigma_n^\delta(x)\mathrm{d}\mu(y)\mathrm{d}m(x) = A_\alpha\widehat{\sigma_n^\delta}(\alpha),
\end{aligned}$$

and $\widehat{\sigma_n^\delta}(\alpha) = \frac{(-n)_k}{(-n-\delta)_k}$ for $\alpha \in \boldsymbol{Z}_{N,k}$ for $0 \le k \le n$ and $= 0$ for $|\alpha| > 2n$ (or $\alpha \notin \boldsymbol{Z}_N$). Thus $\sigma_n^\delta * \mu(x) = \sum\limits_{k=0}^{n} \frac{(-n)_k}{(-n-\delta)_k} H_k(x)$. In fact $\sigma_n^{N-1}(x) \ge 0$ for all $x \in \mathbb{T}^N$ (Corollary 4.12 below) which implies $\sigma_n^{N-1} * \mu$ converges to μ in a useful sense (weak-$*$, see Theorem 4.17(4)) and $\sigma_n^{N-1} * \mu(x)$ is a Laurent polynomial all of whose point values are positive-semidefinite matrices. Also $\big\|\sigma_n^{N-1}\big\|_1 := \int_{\mathbb{T}^N} \big|\sigma_n^{N-1}\big|\mathrm{d}m = 1$.

The *complete symmetric polynomial* in N variables and degree n is given by

$$h_n(x) := \sum\left\{x^\alpha \colon \alpha\in\mathbb{N}_0^N \colon \sum_{i=1}^{N}\alpha_i = n\right\}.$$

Recall

$$\#\left\{\alpha\in\mathbb{N}_0^N \colon \sum_{i=1}^{N}\alpha_i = m\right\} = \frac{(N)_m}{m!} \qquad \text{for} \quad m = 0,1,2,3,\dots.$$

Theorem 4.11. *For $n \ge 0$*

$$h_n\left(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_N}\right) h_n(x_1,\dots,x_N) = \frac{(N)_n}{n!}\sigma_n^{N-1}(x). \tag{4.2}$$

Proof. The product is a sum of terms $x^{\alpha-\beta}$ with $\alpha,\beta\in\mathbb{N}_0^N$ and $|\alpha|=n=|\beta|$. For example the term $x^0=1$ appears exactly $\frac{(N)_n}{n!}$ times, because the number of terms in h_n is $\frac{(N)_n}{n!}$. Consider a fixed $\gamma\in\boldsymbol{Z}_{N,m}$ for some m with $0\le m\le n$. The term x^γ appears in the product for each pair (α,β) with

$$\alpha=\gamma^\pi+\alpha',\qquad \beta=\gamma^\nu+\alpha',\qquad \gamma=\alpha-\beta,$$

where $\alpha'\in\mathbb{N}_0^N$ and $\sum\limits_{i=1}^N\alpha_i'=n-m$. Recall $\gamma_i^\pi=\max(\gamma_i,0)$ and $\gamma_i^\nu=-\min(0,\gamma_i)=\max(0,-\gamma_i)$; thus $\sum\limits_{i=1}^N\gamma_i^\pi=m$ and $\sum\limits_{i=1}^N\alpha_i=n$. Therefore the coefficient of x^γ is

$$\#\left\{\alpha'\in\mathbb{N}_0^N\colon\ \sum_{i=1}^N\alpha_i'=n-m\right\}=\frac{(N)_{n-m}}{(n-m)!}.$$

Hence

$$h_n\left(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_N}\right)h_n(x_1,\dots,x_N)=\sum_{m=0}^N\frac{(N)_{n-m}}{(n-m)!}S_m(x).$$

To finish the proof multiply this relation by $\frac{n!}{(N)_n}$ and compute

$$\begin{aligned}\frac{n!}{(N)_n}\frac{(N)_{n-m}}{(n-m)!}&=(-1)^m(-n)_m\frac{(N)_{n-m}}{(N)_{n-m}(N+n-m)_m}\\&=(-1)^m\frac{(-n)_m}{(N+n-m)_m}=\frac{(-n)_m}{(1-N-n)_m}.\end{aligned}$$

■

Corollary 4.12. *$\sigma_n^{N-1}(x)\ge0$ for all $x\in\mathbb{T}^N$.*

Proof. $\overline{h_n(x_1,\dots,x_N)}=h_n\left(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_N}\right)$ for $x\in\mathbb{T}^N$. ■

Observe that this kernel applies to the quotient space $\mathbb{T}^N/\mathbb{D}$ where

$$\mathbb{D}:=\{(u,u,\dots,u)\colon u\in\mathbb{C},\ |u|=1\},$$

is the diagonal subgroup. That is, each $S_n(x)$ is homogeneous of degree zero, constant on sets $\{(ux_1,ux_2,\dots,ux_N)\colon|u|=1\}$ for fixed $x\in\mathbb{T}^N$.

Here are approximate identity properties of σ_n^{N-1}; we use $\mathbb{T}^N/\mathbb{D}$ to refer to functions homogeneous of degree zero. There is a standard formula:

Lemma 4.13. *Suppose $g,h\in C(\mathbb{T}^N)$ and ν is a Baire measure on $\mathbb{T}^N$ then for $h^\dagger(x):=h(x^{-1})$*

$$\int_{\mathbb{T}^N}g(x)(h*\nu)(x)\mathrm{d}m(x)=\int_{\mathbb{T}^N}\big(g*h^\dagger\big)(y)\mathrm{d}\nu(y).$$

Proof. The left side equals

$$\int_{\mathbb{T}^N}\int_{\mathbb{T}^N}g(x)h\big(xy^{-1}\big)\mathrm{d}\nu(y)\mathrm{d}m(x)=\int_{\mathbb{T}^N}\int_{\mathbb{T}^N}g(x)h^\dagger\big(yx^{-1}\big)\mathrm{d}m(x)\mathrm{d}\nu(y)$$

(by Fubini's theorem) which equals the right side. ■

The following is a standard result on approximate identities.

Proposition 4.14. *Suppose $f \in C\big(\mathbb{T}^N/\mathbb{D}\big)$ then $\big\|f - f * \sigma_n^{N-1}\big\|_\infty \to 0$ as $n \to \infty$.*

Proof. For $\varepsilon > 0$ there exists a Laurent polynomial p on $\mathbb{T}^N/\mathbb{D}$ such that $\|f-p\|_\infty < \varepsilon$. Then

$$f - f * \sigma_n^{N-1} = (f-p) + \big(p - \sigma_n^{N-1} * p\big) + (p-f) * \sigma_n^{N-1},$$

and $\big\|(p-f) * \sigma_n^{N-1}\big\|_\infty \le \|p-f\|_\infty \big\|\sigma_n^{N-1}\big\|_1 < \varepsilon$. Let $p(x) = \sum\limits_{m=0}^{M} \sum\limits_{\alpha \in \boldsymbol{Z}_{N,m}} c_\alpha x^\alpha$ for some coefficients c_α (and finite M); thus

$$\big(p - \sigma_n^{N-1} * p\big)(x) = \sum_{m=0}^{M} \sum_{\alpha \in \boldsymbol{Z}_{N,m}} \left(1 - \frac{(-n)_m}{(1-N-n)_m}\right) c_\alpha x^\alpha,$$

which tends to zero in norm as $n \to \infty$. ■

Corollary 4.15. *Suppose ν is a Baire measure on $\mathbb{T}^N$ and $f \in C\big(\mathbb{T}^N/\mathbb{D}\big)$ then*

$$\lim_{n\to\infty} \int_{\mathbb{T}^N} f(x)\big(\sigma_n^{N-1} * \nu\big)(x)\mathrm{d}m(x) = \int_{\mathbb{T}^N} f(x)\mathrm{d}\nu(x).$$

Proof. By Lemma 4.13

$$\int_{\mathbb{T}^N} f(x)\big(\sigma_n^{N-1} * \nu\big)(x)\mathrm{d}m(x) = \int_{\mathbb{T}^N} \big(f * \sigma_n^{N-1}\big)(x)\mathrm{d}\nu(x),$$

since $\big(\sigma_n^{N-1}\big)^\dagger = \sigma_n^{N-1}$, and $f * \sigma_n^{N-1}$ converges uniformly to f as $n \to \infty$. ■

Definition 4.16. Define the μ-approximating Laurent polynomials

$$K_n(x) := \sigma_n^{N-1} * \mu(x) = \sum_{m=0}^{n} \frac{(-n)_m}{(1-N-n)_m} \sum_{\alpha \in \boldsymbol{Z}_{N,n}} A_\alpha x^\alpha.$$

Note $\frac{(-n)_m}{(1-N-n)_m} = \frac{(n-m+1)_{N-1}}{(n+1)_{N-1}}$ for $0 \le m \le n$; for example with $N=3$, $\frac{(-n)_m}{(-2-n)_m} = \big(1 - \frac{m}{n+1}\big)\big(1 - \frac{m}{n+2}\big)$, and $=0$ for $m > n$.

Theorem 4.17. *For $-1/h_\tau < \kappa < 1/h_\tau$ and $n = 1,2,3,\dots$ the following hold:*

(1) *$K_n(x)$ is positive semi-definite for each $x \in \mathbb{T}^N$,*

(2) *$K_n(xw) = \tau(w)^{-1} K_n(x) \tau(w)$ for each $x \in \mathbb{T}^N, w \in \mathcal{S}_N$,*

(3) *$K_n(x_0)\tau(w_0) = \tau(w_0) K_n(x_0)$,*

(4) *$\lim\limits_{n\to\infty} \int_{\mathbb{T}^N} f(x)^* K_n(x) g(x) \mathrm{d}m(x) = \langle f, g\rangle_{\mathbb{T}}$ for all $f, g \in \mathcal{P}_\tau$; the limit exists for any $f, g \in C\big(\mathbb{T}^N; V_\tau\big)$ and defines a $\|\cdot\|_\infty$-bounded positive Hermitian form.*

Proof. Part (1) is a consequence of Theorem A.4. Parts (2) and (3) follow from the properties of H_m in Proposition 4.6. For part (4) there is an intermediate step of averaging over the diagonal group $\mathbb{D}$. Define the operator ρ: $C\big(\mathbb{T}^N\big) \to C\big(\mathbb{T}^N/\mathbb{D}\big)$ by

$$\rho(p)(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} p\big(e^{\mathrm{i}\theta} x\big)\mathrm{d}\theta, \qquad p \in C\big(\mathbb{T}^N\big).$$

Clearly $\|\rho(p)\|_\infty \le \|p\|_\infty$; in effect ρ is the projection onto Fourier series supported by $\boldsymbol{Z}_N$. Then

$$\int_{\mathbb{T}^N} p(x)\mathrm{d}\mu_{T,T'}(x) = \int_{\mathbb{T}^N} \rho(p)(x)\mathrm{d}\mu_{T,T'}(x),$$
$$\int_{\mathbb{T}^N} p(x)(K_n(x))_{T,T'}\mathrm{d}m(x) = \int_{\mathbb{T}^N} \rho(p)(x)(K_n(x))_{T,T'}\mathrm{d}m(x), \qquad T,T' \in \mathcal{Y}(\tau).$$

To extend this to the form $\langle\cdot,\cdot\rangle_{\mathbb{T}}$ express the typical sum $\sum_{i,j=1}^{n} \overline{f_i}B_{ij}g_j = \operatorname{tr}\big((f\otimes g^*)^*B\big)$ where tr denotes the trace and $(f\otimes g^*)_{ij} = f_i\overline{g_j}$ $(1\le i,j\le n)$. Then (ρ is applied to matrices entry-wise) for $f,g\in C\big(\mathbb{T}^N;V_\tau\big)$

$$\int_{\mathbb{T}^N} f(x)^*\mathrm{d}\mu(x)g(x) = \int_{\mathbb{T}^N} \operatorname{tr}\big[(f(x)\otimes g(x)^*)^*\mathrm{d}\mu(x)\big] = \int_{\mathbb{T}^N} \operatorname{tr}\big[\{\rho(f(x)\otimes g(x)^*)\}^*\mathrm{d}\mu(x)\big],$$
$$\begin{aligned}\int_{\mathbb{T}^N} f(x)^*K_n(x)g(x)\mathrm{d}m(x) &= \int_{\mathbb{T}^N} \operatorname{tr}\big[(f(x)\otimes g(x)^*)^*K_n(x)\big]\mathrm{d}m(x)\\ &= \int_{\mathbb{T}^N} \operatorname{tr}\big[\{\rho(f(x)\otimes g(x)^*)\}^*K_n(x)\big]\mathrm{d}m(x).\end{aligned}$$

The convergence properties of Proposition 4.14 imply part (4). ■

5 Recurrence relations

As a simple illustration consider $\boldsymbol{Z}_{N,1}$ where it suffices to find $A_{1,-1,0\dots,0}$. Introduce the unit basis vectors ε_i for $\mathbb{Z}^N$ (with $(\varepsilon_i)_j = \delta_{ij}$), so that $(1,-1,0,\dots) = \varepsilon_1-\varepsilon_2$. The relation $(\varepsilon_2-\varepsilon_1) = (1,2)(\varepsilon_1-\varepsilon_2)$ implies $A_{\varepsilon_2-\varepsilon_1} = \tau(1,2)A_{\varepsilon_1-\varepsilon_2}\tau(1,2) = A^*_{\varepsilon_1-\varepsilon_2}$. From $\langle x_1\mathcal{D}_1f,g\rangle_{\mathbb{T}} = \langle f,x_1\mathcal{D}_1g\rangle_{\mathbb{T}}$ (Proposition 3.4(3)) we find

$$\begin{aligned}x_1\mathcal{D}_1(x_1\otimes T) &= x_1\otimes T + \kappa x_1\sum_{j=2}^{N}\frac{x_1-(x(1,j))_1}{x_1-x_j}\tau(1,j)T\\ &= x_1\otimes T + \kappa x_1\otimes\tau(\omega_1)T = x_1\otimes(I+\kappa\tau(\omega_1))T,\\ x_1\mathcal{D}_1(x_2\otimes T') &= \kappa x_1\sum_{j=2}^{N}\frac{x_2-(x(2,j))_1}{x_1-x_j}\tau(1,j)T' = -\kappa x_1\tau(1,2)T';\end{aligned}$$

recall the Jucys–Murphy elements $\omega_i := \sum_{j=i+1}^{N}(i,j)$ and the action $\tau(\omega_i)T = c(i,T)T$ for $T\in\mathcal{Y}(\tau)$. Next the equation $\langle x_1\mathcal{D}_1(x_1\otimes T),x_2\otimes T'\rangle_{\mathbb{T}} = \langle x_1\otimes T,x_1\mathcal{D}_1(x_2\otimes T')\rangle_{\mathbb{T}}$ yields

$$T^*(I+\kappa\tau(\omega_1))^*A_{\varepsilon_1-\varepsilon_2}T' = -\kappa T^*A_{\mathbf{0}}\tau(1,2)T',$$

and $A_{\mathbf{0}} = I$. This holds for arbitrary T, T', and $\tau(\omega_1)$ is diagonal with the entry at (T,T) being $c(1,T)$ thus

$$(I+\kappa\tau(\omega_1))A_{\varepsilon_1-\varepsilon_2} = -\kappa\tau(1,2), \qquad A_{\varepsilon_1-\varepsilon_2} = -\kappa(I+\kappa\tau(\omega_1))^{-1}\tau(1,2),$$

provided $\kappa c(1,T)\ne -1$ for all $T\in\mathcal{Y}(\tau)$.

Lemma 5.1. *For $\alpha \in \mathbb{N}_0^N$, $T \in \mathcal{Y}(\tau)$ and $1 \le i \le N$*

$$x_i \mathcal{D}_i\big(x^\alpha \otimes T\big) = \alpha_i x^\alpha \otimes T - \kappa \sum_{\alpha_j > \alpha_i} \sum_{\ell=1}^{\alpha_j - \alpha_i} x^{\alpha + \ell(\varepsilon_i - \varepsilon_j)} \otimes \tau(i,j) T$$
$$+ \kappa \sum_{\alpha_i > \alpha_j} \sum_{\ell=0}^{\alpha_i - \alpha_j - 1} x^{\alpha + \ell(\varepsilon_j - \varepsilon_i)} \otimes \tau(i,j) T.$$

Proof. This follows from performing the division in $\frac{x^\alpha - x^{(i,j)\alpha}}{x_i - x_j}$. ∎

Proposition 5.2. *For $\alpha, \beta \in \mathbb{N}_0^N$ such that $|\alpha| = |\beta|$ and $1 \le i \le N$*

$$(\alpha_i - \beta_i) A_{\alpha - \beta} = \kappa \sum_{\alpha_j > \alpha_i} \sum_{\ell=1}^{\alpha_j - \alpha_i} \tau(i,j) A_{\alpha + \ell(\varepsilon_i - \varepsilon_j) - \beta} - \kappa \sum_{\alpha_i > \alpha_j} \sum_{\ell=0}^{\alpha_i - \alpha_j - 1} \tau(i,j) A_{\alpha + \ell(\varepsilon_j - \varepsilon_i) - \beta}$$
$$- \kappa \sum_{\beta_j > \beta_i} \sum_{\ell=1}^{\beta_j - \beta_i} A_{\alpha - \ell(\varepsilon_i - \varepsilon_j) - \beta} \tau(i,j)$$
$$+ \kappa \sum_{\beta_i > \beta_j} \sum_{\ell=0}^{\beta_i - \beta_j - 1} A_{\alpha - \ell(\varepsilon_j - \varepsilon_i) - \beta} \tau(i,j). \tag{5.1}$$

Proof. The statement follows from the equation

$$\big\langle x_i \mathcal{D}_i(x^\alpha \otimes T), x^\beta \otimes T' \big\rangle = \big\langle x^\alpha \otimes T, x_i \mathcal{D}_i\big(x^\beta \otimes T'\big)\big\rangle$$

and the lemma. ∎

The following is one of the main results of this section. Note that it is important to involve the multiplicity of the first part of γ.

Theorem 5.3. *Suppose $\gamma \in \boldsymbol{Z}_{N,n}$ such that γ^π is a partition and $\gamma_1^\pi = \gamma_m^\pi > \gamma_{m+1}$ then*

$$\left(\gamma_1 I + \kappa \sum_{\ell=m+1}^{N} \tau(1,\ell)\right) A_\gamma = -\kappa \sum_{j=m+1,\, \gamma_j \ge 0}^{N} \sum_{\ell=1}^{\gamma_1 - \gamma_j - 1} \tau(1,j) A_{\gamma + \ell(\varepsilon_j - \varepsilon_1)}$$
$$- \kappa \sum_{j=m+1,\, \gamma_j < 0}^{N} \left\{ \tau(1,j) \sum_{\ell=1}^{\gamma_1 - 1} A_{\gamma + \ell(\varepsilon_j - \varepsilon_1)} + \sum_{\ell=1}^{-\gamma_j} A_{\gamma - \ell(\varepsilon_1 - \varepsilon_j)} \tau(1,j) \right\}. \tag{5.2}$$

Each of the coefficients A_δ appearing on the second line of the equation satisfies $\delta \in \boldsymbol{Z}_{N,s}$ for some $s < n$ and for A_δ on the right-hand side of the first line $\delta = \delta^\pi - \gamma^\nu$ where $\delta^\pi \lhd \gamma^\pi$.

Proof. The formula follows from equation (5.1) by setting $i = 1$, $\beta_1 = 0$ and omitting the case $\alpha_j > \alpha_i$. Suppose $\gamma_j^\nu = 0$ for $j \le k$, and $\gamma_j^\pi = 0$ for $j > k$. The typical multi-index in the second line is

$$\big(\gamma_1 - \ell, \gamma_2^\pi, \dots, \gamma_k^\pi, -\gamma_{k+1}^\nu, \dots, -\gamma_j^\nu + \ell, \dots\big)$$

with $1 \le \ell \le \gamma_1 - 1$ or $1 \le \ell \le \gamma_j^\nu$. If $\gamma_1 \ge \ell$ then the sum of the nonnegative components is $n - \ell < n$; and if $\gamma_1 < \ell$ (possible if $\gamma_j^\nu > \gamma_1$) then the sum of the nonnegative components is $n - \gamma_1$. In both cases the multi-index is in $\bigcup_{s<n} \boldsymbol{Z}_{N,s}$. From [6, Lemma 10.1.3] $\gamma^\pi \succ (\gamma^\pi + \ell(\varepsilon_j - \varepsilon_1))^+$ for $1 \le \ell \le \gamma_1 - \gamma_j^\pi - 1$, thus the multi-indices on the right-hand side of the first line satisfy $\delta = \delta^\pi - \gamma^\nu$ with $(\delta^\pi)^+ \prec \gamma^\pi$, that is, $\delta^\pi \lhd \gamma^\pi$. ∎

If γ is $\rhd$-minimal with fixed γ^{ν} then there are no terms on the right side of the first line (that is, $\gamma_j \ge 0$ implies $\gamma_1 \ge \gamma_j \ge \gamma_1 - 1$).

Proposition 5.4. *Among $\gamma \in \boldsymbol{Z}_{N,n}$ such that $\gamma^{\nu} = \beta$ for some fixed β with $|\beta| = n$ and such that $\beta_j > 0$ exactly when $j > k$ the minimal multi-index for the order $\gamma^{(1)\pi} \rhd \gamma^{(2)\pi}$ is $\gamma^{(0)} = \big(p+1, \dots, \overset{(m)}{p+1}, p, \dots, \overset{(k)}{p}, -\beta_{k+1}, \dots, -\beta_N\big)$ where $p = \lfloor \frac{n}{k} \rfloor$ and $m = n - kp$ (so $0 \le m < k$). For this multi-index the right-hand side of (5.2) contains only A_δ with $\delta \in \bigcup_{s=0}^{n-1} \boldsymbol{Z}_{N,s}$.*

The proof is technical and is presented as Proposition A.5.

Theorem 5.5. *The coefficients A_α are rational functions of κ and are finite provided*

$$\kappa \notin \left\{-\frac{m}{c} \colon m, c \in \mathbb{N},\ 1 \le c \le \tau_1 - 1\right\} \cup \left\{\frac{m}{c} \colon m, c \in \mathbb{N},\ 1 \le c \le \ell(\tau) - 1\right\}.$$

Also $A_{-\alpha} = A_\alpha^*$ and $\tau(w)^* A_{w\alpha} \tau(w) = A_\alpha$ for all $\alpha \in \boldsymbol{Z}_N$, $w \in \mathcal{S}_N$ and permitted values of κ.

Proof. The NSJP $\zeta_{\alpha,T}$ is a rational function of κ with no poles in $-1/h_\tau < \kappa < 1/h_\tau$. The coefficients A_α are defined in terms of all the NSJP's and are also rational in κ. In equation (5.2) the operator on the left of A_γ is

$$\left(\gamma_1 I + \kappa \sum_{\ell=m+1}^{N} \tau(1,\ell)\right) = \tau(1,m)(\gamma_1 I + \kappa\tau(\omega_m))\tau(1,m),$$

where ω_m is the Jucys–Murphy element $\sum_{\ell=m+1}^{N} (m,\ell)$; the action $\tau(\omega_m)T = c(m,T)T$ for all $T \in \mathcal{Y}(\tau)$ shows that the eigenvalues of the operator are $\{\gamma_1 + \kappa c(m,T) \colon T \in \mathcal{Y}(\tau)\}$ and the operator is invertible provided $\kappa c(m,T) \notin \{-1,-2,-3,\dots\}$ for $1 \le m \le N$. The set of values of $c(m,T)$ is $\{j \in \mathbb{Z} \colon 1 - \ell(\tau) \le j \le \tau_1 - 1\}$. Thus an inductive argument based on n in $\boldsymbol{Z}_{N,n}$, the order in Proposition 5.4, and formula (5.2) shows there are unique solutions for $\{A_\alpha\}$ provided that the possible poles at $n + \kappa c(i,T) = 0$ are excluded. The relations $A_{-\alpha} = A_\alpha^*$ and $\tau(w)^* A_{w\alpha}\tau(w) = A_\alpha$ hold at least in an interval hence for all κ, excluding the poles. ∎

The largest interval around 0 without poles is $-\frac{1}{\tau_1 - 1} < \kappa < \frac{1}{\ell(\tau)-1}$. As illustration we describe A_γ for $\gamma \in \boldsymbol{Z}_{N,2}$. Above we showed

$$A_{\varepsilon_1 - \varepsilon_j} = -\kappa\, (I + \kappa\tau(\omega_1))^{-1}\tau(1,j), \qquad 2 \le j \le N.$$

Next for $\alpha = \varepsilon_1 + \varepsilon_2$ and $\beta = 2\varepsilon_j$ for $3 \le j \le N$ we find

$$\left(I + \kappa \sum_{i=3}^{N} \tau(1,i)\right) A_{\varepsilon_1+\varepsilon_2-2\varepsilon_j} = -\kappa(A_{\varepsilon_2-\varepsilon_1} + A_{\varepsilon_2-\varepsilon_j})\tau(1,j).$$

For $\alpha = \varepsilon_1 + \varepsilon_2$ and $\beta = \varepsilon_j + \varepsilon_{j+1}$ with $3 \le j \le N-1$

$$\left(I + \kappa \sum_{i=3}^{N} \tau(1,i)\right) A_{\varepsilon_1+\varepsilon_2-\varepsilon_j-\varepsilon_{j+1}} = -\kappa\big(A_{\varepsilon_2-\varepsilon_j}\tau(1,j+1) + A_{\varepsilon_2-\varepsilon_{j+1}}\tau(1,j)\big).$$

For $\alpha = 2\varepsilon_1$ and $\beta = 2\varepsilon_N$ we obtain

$$\begin{aligned}&(2I + \kappa\tau(\omega_1))A_{2\varepsilon_1-2\varepsilon_N}\\ &\qquad = -\kappa\left\{\sum_{\ell=2}^{N-1} \tau(1,\ell)A_{\varepsilon_1+\varepsilon_\ell-2\varepsilon_N} + \tau(1,N)A_{\varepsilon_1-\varepsilon_N} + (A_{\varepsilon_1-\varepsilon_N} + I)\tau(1,N)\right\}.\end{aligned}$$

The other coefficients for $n = 2$ are obtained using the relations

$$A_{-\alpha} = A_\alpha^* \qquad \text{and} \qquad \tau(w)^* A_{w\alpha}\tau(w) = A_\alpha.$$

6 The differential equation

We will show that μ satisfies a differential system in a distributional sense. Let $\mathbb{T}^N_{\mathrm{reg}} := \mathbb{T}^N \backslash \bigcup_{1\le i<j\le N} \{x\colon x_i = x_j\}$ (this avoids the singularities of the system) and $\partial_j := \frac{\partial}{\partial x_j}$ for $1 \le j \le N$. The system is

$$x_i \partial_i K(x) = \kappa \sum_{j\ne i} \frac{x_j}{x_i - x_j} \tau(i,j) K(x) + \kappa K(x) \sum_{j\ne i} \tau(i,j) \frac{x_i}{x_i - x_j}, \qquad 1 \le i \le N. \tag{6.1}$$

Any solution of this system satisfies $\sum_{i=1}^{N} x_i \partial_i K(x) = 0$ and thus is homogeneous of degree zero.

The relation $\langle x_i \mathcal{D}_i f, g\rangle_{\mathbb{T}} = \langle f, x_i \mathcal{D}_i g\rangle_{\mathbb{T}}$ extends to $C^{(1)}\big(\mathbb{T}^N; V_\tau\big)$, the continuously differentiable V_τ-valued functions, because Laurent polynomials are dense in this space. We have shown

$$\lim_{n\to\infty} \int_{\mathbb{T}^N} \big\{(x_i \mathcal{D}_i f(x))^* K_n(x) g(x) - f(x)^* K_n(x) x_i \mathcal{D}_i g(x)\big\} \mathrm{d}m(x) = 0.$$

Suppose $p, q \in C^{(1)}\big(\mathbb{T}^N\big)$ (scalar $\mathbb{C}$-valued) then by periodicity $\int_{\mathbb{T}^N} \frac{\partial}{\partial\theta_j}(pq)\mathrm{d}m = 0$ thus

$$\int_{\mathbb{T}^N} \left(\frac{\partial}{\partial\theta_j} p\right) q \mathrm{d}m = -\int_{\mathbb{T}^N} p \left(\frac{\partial}{\partial\theta_j} q\right) \mathrm{d}m.$$

Also from $\frac{\partial}{\partial\theta_j} f(x) = \mathrm{i}e^{\mathrm{i}\theta_j} \partial_j f(x) = \mathrm{i}x_j \partial_j f(x)$ and $\partial_j f^*(x) = -x_j^{-2} (\partial_j f)^*(x)$ we obtain

$$\int_{\mathbb{T}^N} (x_j \partial_j f(x))^* g(x) \mathrm{d}m = -\int_{\mathbb{T}^N} x_j^{-1} \big(-x_j^2\big) \partial_j f^*(x) g(x) \mathrm{d}m = \int_{\mathbb{T}^N} f^*(x) x_j \partial_j g(x) \mathrm{d}m.$$

That is, $x_j \partial_j$ is self-adjoint with respect to $\langle f, g\rangle := \int_{\mathbb{T}^N} f^*(x) g(x) \mathrm{d}m$. The result extends to $C^{(1)}\big(\mathbb{T}^N; V_\tau\big)$.

Specialize to a closed $\mathcal{S}_N$-invariant subset $E \subset \mathbb{T}^N_{\mathrm{reg}}$ which is the closure of its interior (for example $E_\varepsilon = \big\{x \in \mathbb{T}^N\colon \min_{i<j} |x_i - x_j| \ge \varepsilon\big\}$ for $\varepsilon > 0$), and let $f, g \in C^{(1)}\big(\mathbb{T}^N; V_\tau\big)$ have supports contained in E, that is, $f(x) = 0 = g(x)$ for $x \notin E$. For fixed f, g, i let

$$\begin{aligned}
I_n &= \int_{\mathbb{T}^N} \big\{(x_i \mathcal{D}_i f(x))^* K_n(x) g(x) - f(x)^* K_n(x) x_i \mathcal{D}_i g(x)\big\} \mathrm{d}m(x) \\
&= \int_{\mathbb{T}^N} \big\{(x_i \partial_i f(x))^* K_n(x) g(x) - f(x)^* K_n(x) x_i \partial_i g(x)\big\} \\
&\quad + \kappa \int_{\mathbb{T}^N} \sum_{j\ne i} \left(\tau(i,j) x_i \frac{f(x) - f(x(i,j))}{x_i - x_j}\right)^* K_n(x) g(x) \mathrm{d}m(x) \\
&\quad - \kappa \int_{\mathbb{T}^N} f(x)^* K_n(x) \sum_{j\ne i} \left(\tau(i,j) x_i \frac{g(x) - g(x(i,j))}{x_i - x_j}\right) \mathrm{d}m(x).
\end{aligned}$$

By using $\big(\frac{x_i}{x_i - x_j}\big)^* = -\frac{x_j}{x_i - x_j}$, $\tau(i,j)^* = \tau(i,j)$, and rearranging the sums we obtain

$$\begin{aligned}
I_n &= \int_{\mathbb{T}^N} \big\{(x_i \partial_i f(x))^* K_n(x) g(x) - f(x)^* K_n(x) x_i \partial_i g(x)\big\} \mathrm{d}m(x) \\
&\quad - \kappa \int_{\mathbb{T}^N} f(x)^* \sum_{j\ne i} \left\{\frac{x_j}{x_i - x_j} \tau(i,j) K_n(x) + K_n(x) \tau(i,j) \frac{x_i}{x_i - x_j}\right\} g(x) \mathrm{d}m(x) \\
&\quad + \kappa \sum_{j\ne i} \int_{\mathbb{T}^N} \big\{x_j f(x(i,j))^* \tau(i,j) K_n(x) g(x) + x_i f(x)^* K_n(x) \tau(i,j) g(x(i,j))\big\} \frac{\mathrm{d}m(x)}{x_i - x_j}.
\end{aligned}$$

Each integral in the third line is finite because f and g vanish on a neighborhood of $\bigcup_{1\le i<j\le N}\{x\colon x_i = x_j\}$. The terms inside $\{\cdot\}$ are invariant under the change of variable $x \mapsto x(i,j)$, because $\tau(i,j)K_n(x(i,j)) = K_n(x)\tau(i,j)$, but the denominator $(x_i - x_j)$ changes sign; thus the integrand is odd under (i,j) and the integral vanishes.

Because terms like $\tau(i,j)\frac{x_i}{x_i-x_j}g(x)$ are in $C^{(1)}\big(\mathbb{T}^N;V_\tau\big)$ (assumption on the support of g) we can take the limit as $n\to\infty$ in the second line. By the adjoint property of $x_i\partial_i$ we find

$$\int_{\mathbb{T}^N}(x_i\partial_i f(x))^* K_n(x)g(x)\mathrm{d}m(x) = \int_{\mathbb{T}^N} f(x)^* x_i\partial_i\{K_n(x)g(x)\}\mathrm{d}m(x)$$
$$= \int_{\mathbb{T}^N} f(x)^*\big\{K_n(x)x_i\partial_i g(x) + (x_i\partial_i K_n(x))g(x)\big\}\mathrm{d}m(x).$$

Thus the fact that $\lim\limits_{n\to\infty} I_n = 0$ implies (recall the matrix-valued integral notation (4.1))

$$\lim_{n\to\infty}\int_{\mathbb{T}^N} f(x)^*(x_i\partial_i K_n(x))g(x)\mathrm{d}m(x)$$
$$= \kappa\int_{\mathbb{T}^N} f(x)^*\sum_{j\ne i}\frac{1}{x_i-x_j}\big\{x_j\tau(i,j)\mathrm{d}\mu(x) + \mathrm{d}\mu(x)\tau(i,j)x_i\big\}g(x). \tag{6.2}$$

This statement is valid for all $f,g\in C^{(1)}\big(\mathbb{T}^N;V_\tau\big)$ that vanish on a neighborhood of $\bigcup_{1\le i<j\le N}\{x\colon x_i = x_j\}$. The first line of the equation can be written as

$$\lim_{n\to\infty}\int_{\mathbb{T}^N}\big\{(x_i\partial_i f(x))^* K_n(x)g(x) - f(x)^* K_n(x)x_i\partial_i g(x)\big\}\mathrm{d}m(x)$$
$$= \int_{\mathbb{T}^N}\big\{(x_i\partial_i f(x))^*\mathrm{d}\mu(x)g(x) - f(x)^*\mathrm{d}\mu(x)x_i\partial_i g(x)\big\},$$

which expresses the distributional derivative of $\mathrm{d}\mu$. Thus the distribution-sense differential system is satisfied by $\mathrm{d}\mu$ on closed subsets of $\mathbb{T}^N_{\mathrm{reg}}$.

In the scalar case ($\tau = (N)$) the orthogonality weight is known to be (due to [1] for the symmetric Jack polynomials)

$$K(x) = \prod_{1\le i<j\le N}\big\{(x_i-x_j)\big(x_i^{-1}-x_j^{-1}\big)\big\}^\kappa$$

and the differential equation system reduces to (note $\tau(w) = 1$)

$$x_i\partial_i K(x) = \kappa K(x)\sum_{j\ne i}\frac{x_i+x_j}{x_i-x_j},\qquad 1\le i\le N.$$

The differential system (6.1) could be the subject for an article all by itself, but we can sketch a result about the absolutely continuous part of μ, namely that $\mathrm{d}\mu(x) = L(x)^* BL(x)\mathrm{d}m(x)$, $x\in\mathbb{T}^N_{\mathrm{reg}}$ where B is a locally constant positive matrix and $L(x)$ is a fundamental solution of

$$\partial_i L(x) = \kappa L(x)\left\{\sum_{j\ne i}\frac{1}{x_i-x_j}\tau(i,j) - \frac{\gamma}{x_i}I\right\},\qquad 1\le i\le N, \tag{6.3}$$
$$\gamma := \frac{1}{2N}\sum_{i=1}^{\ell(\tau)}\tau_i(\tau_i-2i+1) = \frac{1}{N}\sum_{j=1}^{N} c(j,T_0).$$

The effect of the term $\frac{\gamma}{x_i}I$ is to make $L(x)$ homogeneous of degree zero, that is, $\sum\limits_{i=1}^{N} x_i\partial_i L(x) = 0$, because

$$\sum_{1\le i<j\le N} \tau(i,j) = \left\{\sum_{j=1}^{N} c(j,T_0)\right\} I$$

(the sum of the contents in the diagram of τ).

The differential system is Frobenius integrable; this means that in the system $\partial_i L(x) = \kappa L(x)M_i(x)$, $1\le i\le N$, the two formal differentiations

$$\partial_j\partial_i L(x) = \kappa^2 L(x)M_j(x)M_i(x) + \kappa L(x)\partial_j M_i(x),$$
$$\partial_i\partial_j L(x) = \kappa^2 L(x)M_i(x)M_j(x) + \kappa L(x)\partial_i M_j(x),$$

are equal to each other for all i, j (see [2]). The system is analytic and thus any local solution can be continued analytically to any point in $\mathbb{C}^N_{\rm reg} := (\mathbb{C}\backslash\{0\})^N\backslash \bigcup\limits_{1\le i<j\le N}\{x\colon x_i = x_j\}$. When restricted to $\mathbb{T}^N_{\rm reg}$ there are solutions defined on each connected component. This is possible because $L(x)$ is constant on $\{ux\colon |u| = 1\}$ for fixed $x\in\mathbb{T}^N_{\rm reg}$ and each component is homotopic to a circle. Denote the component containing all the points $\big\{\big(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_N}\big)\colon -\pi<\theta_1<\theta_2<\cdots<\theta_N<\pi\big\}$ by $\mathcal{C}_0$. Since x and ux are in the same component for $|u| = 1$ we see that $x_0\in\mathcal{C}_0$ (recall $x_0 = \big(1,\omega,\dots,\omega^{N-1}\big)$, $\omega = e^{2\pi\mathrm{i}/N}$). The components are $\{\mathcal{C}_0 w\colon w\in\mathcal{S}_N,\ w(1) = 1\}$ (corresponding to the $(N-1)!$ circular permutations of $(1,2,\dots,N)$). Now fix the unique solution $L(x)$ such that $L(x_0 w) = I$ for each w with $w(1) = 1$. By differentiating $L(x)^{-1}L(x) = I$ obtain the system satisfied by L^{-1}:

$$\partial_i L(x)^{-1} = -\kappa\bigg\{\sum_{j\ne i}\frac{1}{x_i - x_j}\tau(i,j) - \frac{\gamma}{x_i}I\bigg\}L(x)^{-1}, \qquad 1\le i\le N.$$

The goal here is to replace f, g in formula (6.2) by $L^{-1}f$, $L^{-1}g$ and deduce the desired result. By use of $x_i\partial_i(L^{-1*}) = -\big(x_i\partial_i L^{-1}\big)^*$ we obtain

$$\begin{aligned}
&\big(x_i\partial_i\big(L(x)^{-1*}\big)\big)\mathrm{d}\mu(x)L(x)^{-1} + L(x)^{-1*}\mathrm{d}\mu(x)\big(x_i\partial_i L(x)^{-1}\big)\\
&\quad = \kappa L^{-1*}\bigg\{\sum_{j\ne i}\frac{-x_j}{x_i - x_j}\tau(i,j) - \gamma I\bigg\}\mathrm{d}\mu L^{-1} - \kappa L^{-1*}\mathrm{d}\mu\bigg\{\sum_{j\ne i}\frac{x_i}{x_i - x_j}\tau(i,j) - \gamma I\bigg\}L^{-1}\\
&\quad = -\kappa L^{-1*}\sum_{j\ne i}\left\{\frac{x_j}{x_i - x_j}\tau(i,j)\mathrm{d}\mu + \mathrm{d}\mu\tau(i,j)\frac{x_i}{x_i - x_j}\right\}L^{-1}.
\end{aligned}$$

Substitute this relation in formula (6.2)

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{T}^N} f(x)^* L(x)^{-1*}(x_i\partial_i K_n(x))L(x)^{-1}g(x)\mathrm{d}m(x)\\
&\quad + \int_{\mathbb{T}^N} f(x)^*(x_i\partial_i)\big(L(x)^{-1*}\big)\mathrm{d}\mu(x)L(x)^{-1}g(x)\\
&\quad + \int_{\mathbb{T}^N} f(x)^* L(x)^{-1*}\mathrm{d}\mu(x)x_i\partial_i L(x)^{-1}g(x) = 0.
\end{aligned}$$

The formula is valid because $L^{-1}f, L^{-1}g\in C^{(1)}\big(\mathbb{T}^N_{\rm reg};V_\tau\big)$ and vanish for $x\notin E$. The first line of the equation is the distributional derivative of μ so the equation is equivalent to

$$\int_{\mathbb{T}^N} f(x)^*\big[x_i\partial_i\big(L(x)^{-1*}\mathrm{d}\mu(x)L(x)^{-1}\big)\big]g(x) = 0, \qquad 1\le i\le N.$$

Thus all the partial derivatives of the distribution $L^{-1*}\mathrm{d}\mu L^{-1}$ vanish and $L^{-1*}\mathrm{d}\mu L^{-1} = B\mathrm{d}m$, where B is constant on each component of $\mathbb{T}^N_{\mathrm{reg}}$. We conclude $\mathrm{d}\mu(x) = L(x)^* B L(x)\mathrm{d}m(x)$ for $x \in \mathbb{T}^N_{\mathrm{reg}}$. Part (1) of Theorem 4.17 implies B is a positive matrix.

We have to point out that this result provides no information about the behavior of μ on the singular set $\bigcup\limits_{i<j}\{x\colon x_i = x_j\}$. We conjecture that μ does not have a singular part. This question seems a worthy topic for further investigations. Some of the difficulties in this problem come from the behavior of the solutions of system (6.3) in neighborhoods of the sets $\{x\colon x_i = x_j\}$; there are singularities of order $|x_i - x_j|^{\pm\kappa}$. Both signs appear because the eigenvalues of $\tau(i,j)$ are 1 and -1; a consequence of the assumption that τ is not one-dimensional.

A Appendix

A.1 The matrix Bochner theorem

For a matrix $A \in M_n(\mathbb{C})$ the operator norm is

$$\|A\| := \sup\{|Av|\colon |v| = 1\} = \sup\{|\langle u, Av\rangle|\colon |u| = 1 = |v|\}.$$

Proposition A.1. *Suppose F is a positive-definite $M_n(\mathbb{C})$-valued function on $\mathbb{Z}^N$ then*

(1) *$F(\mathbf{0})$ is positive-definite,*

(2) *$F(-\alpha) = F(\alpha)^*$ and $\|F(\alpha)\| \le \|F(\mathbf{0})\|$ for all $\alpha \in \mathbb{Z}^N$.*

Proof. Part (1) follows immediately from taking $f(\mathbf{0}) = u$, $f(\alpha) = 0$ for $\alpha \ne 0$. For part (2) fix $\alpha \ne 0$ and let $f(\mathbf{0}) = u$, $f(\alpha) = v$ and $f(\beta) = 0$ otherwise. By definition

$$u^* F(\mathbf{0})u + v^* F(\mathbf{0})v + v^* F(\alpha)u + u^* F(-\alpha)v \ge 0.$$

Thus $\operatorname{Im}(v^* F(\alpha)u + u^* F(-\alpha)v) = 0$ for all u, v. For $1 \le j,k \le n$ let $u = \varepsilon_j$, $v = c\varepsilon_k$, $c \in \mathbb{C}$, then

$$0 = \operatorname{Im}\left(\bar{c}F(\alpha)_{kj} + cF(-\alpha)_{jk}\right).$$

Set $c = 1$ and $c = \mathrm{i}$ to show $F(-\alpha)_{jk} = \overline{F(\alpha)_{kj}}$, that is, $F(-\alpha) = F(\alpha)^*$. Thus $\overline{u^* F(-\alpha)v} = v^* F(\alpha)u$ and

$$-2\operatorname{Re}(v^* F(\alpha)u) \le u^* F(\mathbf{0})u + v^* F(\mathbf{0})v.$$

Specialize to vectors u, v such that $|u| = 1 = |v|$ and $v^* F(\alpha)u = \langle v, F(\alpha)u\rangle = -\|F(\alpha)\|$. Since $F(\mathbf{0})$ is positive-definite it follows that $u^* F(\mathbf{0})u + v^* F(\mathbf{0})v \le 2\|F(\mathbf{0})\|$, and so $\|F(\alpha)\| \le \|F(\mathbf{0})\|$. ■

Proposition A.2. *Suppose $\alpha, \beta \in \mathbb{Z}^N$ and $\alpha, \beta \ne \mathbf{0}$ then*

$$\|F(\alpha) - F(\beta)\|^2 \le 2\|F(\mathbf{0})\|\{v^* F(\mathbf{0})v - \operatorname{Re}(v^* F(\alpha - \beta)v)\},$$

where $|v| = 1$ and $|(F(\alpha) - F(\beta))^* v| = \|F(\alpha) - F(\beta)\|$.

Proof. Assume $\alpha \ne \beta$ and let $f(\mathbf{0}) = u$, $f(\alpha) = w$, $f(\beta) = -w$, and $f(\gamma) = 0$ otherwise. By definition

$$\begin{aligned} &u^* F(\mathbf{0})u + 2w^* F(\mathbf{0})w + w^*(F(\alpha) - F(\beta))u + u^*(F(-\alpha) - F(-\beta))w \\ &\quad - w^* F(\alpha - \beta)w - w^* F(\beta - \alpha)w \ge 0. \end{aligned}$$

Let $u, v \in \mathbb{C}^n$ satisfy $|u| = 1 = |v|$ and $|(F(\alpha) - F(\beta))^* v| = v^*(F(\alpha) - F(\beta))u = \|F(\alpha) - F(\beta)\|$. Set $w = tv$ with $t \in \mathbb{R}$. The inequality becomes

$$u^* F(\mathbf{0})u + 2t^2 v^* F(\mathbf{0}t)v + 2t\|F(\alpha) - F(\beta)\| - 2t^2 \operatorname{Re}\left(v^* F(\alpha - \beta)v\right) \geq 0.$$

The discriminant of the quadratic polynomial in t must be nonpositive and this implies the stated inequality, since $u^* F(\mathbf{0})u \leq \|F(\mathbf{0})\|$. ∎

This is a sort of uniform continuity.

For any fixed $u \in \mathbb{C}^n$ the scalar function $g_u(\alpha) = u^* F(\alpha)u$ is positive-definite in the classical sense and thus by Bochner's theorem (see [13, pp. 17–21]) there exists a unique positive Baire measure μ_u on $\mathbb{T}^N$ such that

$$\int_{\mathbb{T}^N} x^{-\alpha} \mathrm{d}\mu_u(x) = u^* F(\alpha)u, \qquad \alpha \in \mathbb{Z}^N.$$

In particular, for $1 \leq j \leq n$ let $u = \varepsilon_j$ then $u^* F(\alpha)u = F(\alpha)_{jj}$ and set $\mu_{jj} = \mu_{\varepsilon_j}$. For $1 \leq j, k \leq n$ (with $j \neq k$) let $u = \varepsilon_j + \varepsilon_k$, $v = \varepsilon_j + \mathrm{i}\varepsilon_k$

$$\int_{\mathbb{T}^N} x^{-\alpha} \mathrm{d}\mu_u(x) = F(\alpha)_{jj} + F(\alpha)_{jk} + F(\alpha)_{kj} + F(\alpha)_{kk},$$
$$\int_{\mathbb{T}^N} x^{-\alpha} \mathrm{d}\mu_v(x) = F(\alpha)_{jj} + \mathrm{i}F(\alpha)_{jk} - \mathrm{i}F(\alpha)_{kj} + F(\alpha)_{kk}.$$

Thus

$$\int_{\mathbb{T}^N} x^{-\alpha} \mathrm{d}(\mu_u - \mathrm{i}\mu_v) = 2F(\alpha)_{jk} + (1 - \mathrm{i}) \int_{\mathbb{T}^N} x^{-\alpha} \mathrm{d}(\mu_{jj} + \mu_{kk}),$$

and define

$$\mu_{jk} = \frac{1}{2}(\mu_u - \mathrm{i}\mu_v) - \frac{1 - \mathrm{i}}{2}(\mu_{jj} + \mu_{kk}),$$

with the result

$$\int_{\mathbb{T}^N} x^{-\alpha} \mathrm{d}\mu_{jk}(x) = F(\alpha)_{jk}, \alpha \in \mathbb{Z}^N.$$

From the above equations we also obtain

$$\int_{\mathbb{T}^N} x^{-\alpha} \mathrm{d}(\mu_u + \mathrm{i}\mu_v) = 2F(\alpha)_{kj} + (1 + \mathrm{i}) \int_{\mathbb{T}^N} x^{-\alpha} \mathrm{d}(\mu_{jj} + \mu_{kk}).$$

The formula

$$\mu_{kj} = \frac{1}{2}(\mu_u + \mathrm{i}\mu_v) - \frac{1 + \mathrm{i}}{2}(\mu_{jj} + \mu_{kk}),$$

is consistent with the previous one and it can be directly verified that $\widehat{\mu_{kj}}(-\alpha) = \overline{\widehat{\mu_{jk}}(\alpha)}$ from the general equation $\widehat{\mu_u}(-\alpha) = \overline{\widehat{\mu_u}(\alpha)}$. This is a restatement of $F(-\alpha) = F(\alpha)^*$.

From $\|\mu_u\| = u^* F(\mathbf{0})u$ (measure/total variation norm) we obtain $\|\mu_{jj}\| = F(\mathbf{0})_{jj}$ and

$$\|\mu_{jk}\| \leq \frac{1}{\sqrt{2}}(F(\mathbf{0})_{jj} + F(\mathbf{0})_{kk}) + F(\mathbf{0})_{jj} + F(\mathbf{0})_{kk} + |\operatorname{Re} F(\mathbf{0})_{jk}| + |\operatorname{Im} F(\mathbf{0})_{jk}|.$$

In particular, if $F(\mathbf{0})=I$ then $\|\mu_{jk}\|\le 2+\sqrt{2}$. For the space $C(\mathbb{T}^N;\mathbb{C}^n)$ (continuous functions on $\mathbb{T}^N$, values in $\mathbb{C}^n$) define an inner product

$$\langle f,g\rangle_F := \sum_{i,j=1}^n \int_{\mathbb{T}^N} \overline{f(x)_i} g(x)_j \mathrm{d}\mu_{ij}(x),$$

then for $\alpha,\beta\in\mathbb{Z}^N$, and $1\le i,j\le n$ (with the standard unit basis vectors ε_i of $\mathbb{C}^n$)

$$\left\langle x^\alpha\varepsilon_i, x^\beta\varepsilon_j\right\rangle_F = \int_{\mathbb{T}^N} x^{-\alpha}x^\beta \mathrm{d}\mu_{ij}(x) = F(\alpha-\beta)_{ij}.$$

The following summarizes the above results.

Theorem A.3. *The Hermitian form $\langle\cdot,\cdot\rangle_F$ is bounded on $C(\mathbb{T}^N;\mathbb{C}^n)$, that is $|\langle f,g\rangle_F| \le B\|f\|_\infty\|g\|_\infty$ for some $B<\infty$, is positive-semidefinite, and if f, g are finitely supported functions on $\mathbb{Z}^N$ with values in $\mathbb{C}^n$ then for $\widehat{f}(x) := \sum_\alpha f(\alpha)x^\alpha$ and $\widehat{g}(x) := \sum_\alpha g(\alpha)x^\alpha$*

$$\left\langle \widehat{f},\widehat{g}\right\rangle_F = \sum_{\alpha,\beta} f(\alpha)^* F(\alpha-\beta) g(\beta).$$

Proof. The bound follows from the uniform bound on $\|\mu_{jk}\|$ for all j, k, depending only on $F(\mathbf{0})$. Suppose $\widehat{f}$, $\widehat{g}$ are trigonometric (Laurent) polynomials (equivalent to finite support on $\mathbb{Z}^N$), then

$$\begin{aligned}\left\langle \widehat{f},\widehat{g}\right\rangle_F &= \sum_{\alpha,\beta}\sum_{i,j=1}^n \int_{\mathbb{T}^N} \overline{f(\alpha)_i} x^{-\alpha} g(\beta)_j x^\beta \mathrm{d}\mu_{ij}(x) \\ &= \sum_{\alpha,\beta}\sum_{i,j=1}^n \overline{f(\alpha)_i} F(\alpha-\beta)_{ij} g(\beta)_j = \sum_{\alpha,\beta} f(\alpha)^* F(\alpha-\beta) g(\beta).\end{aligned}$$

By definition $\left\langle \widehat{f},\widehat{f}\right\rangle_F \ge 0$ and by the density of the trigonometric polynomials in $C(\mathbb{T}^N;\mathbb{C}^n)$ it follows that $\langle\cdot,\cdot\rangle_F$ is a positive semidefinite (it is possible that $\left\langle \widehat{f},\widehat{f}\right\rangle_F = 0$ for some $f\ne 0$) Hermitian form. ∎

The next result is used for the approximate identity arguments.

Theorem A.4. *Suppose $\sigma\in C(\mathbb{T}^N)$ and $\sigma(x)\ge 0$ for all $x\in\mathbb{T}^N$ then each $\sigma*\mu_{ij}\in C(\mathbb{T}^N)$ and $[\sigma*\mu_{ij}(x)]_{i,j=1}^n \in M_n(\mathbb{C})$ is positive semidefinite for each $x\in\mathbb{T}^N$.*

Proof. Suppose $(f_i)_{i=1}^n \in C(\mathbb{T}^N;\mathbb{C}^n)$. A convolution formula (similar to that in Lemma 4.13) shows that

$$\begin{aligned}\sum_{i,j=1}^n \int_{\mathbb{T}^N} \overline{f(x)_i} \sigma*\mu_{ij}(x) f(x)_j \mathrm{d}m(x) &= \sum_{i,j=1}^n \int_{\mathbb{T}^N}\int_{\mathbb{T}^N} \sigma\left(xy^{-1}\right) \overline{f(x)_i} f(x)_j \mathrm{d}\mu_{ij}(y)\mathrm{d}m(x) \\ &= \int_{\mathbb{T}^N} \sigma(u) \int_{\mathbb{T}^N} \sum_{i,j=1}^n \overline{f(yu)_i} f(yu)_j \mathrm{d}\mu_{ij}(y)\mathrm{d}m(u) \ge 0,\end{aligned}$$

(change-of-variable $x=yu$) because $y\mapsto f(yu)\in C(\mathbb{T}^N;\mathbb{C}^n)$ and the double sum is continuous in u and nonnegative by the above theorem. Now let $f(x)=g(x)v$ where $v\in\mathbb{C}^n$ and $g\in C(\mathbb{T}^N)$, $g\ge 0$, $\int_{\mathbb{T}^N} g^2\mathrm{d}m = 1$ and $g=0$ off an ε-neighborhood of a fixed $z\in\mathbb{T}^N$ (i.e., $|x-z|\ge\varepsilon$ implies $g(x)=0$) then

$$\sum_{i,j=1}^n \int_{\mathbb{T}^N} \overline{f(x)_i} \sigma*\mu_{ij}(x) f(x)_j \mathrm{d}m(x) = \int_{\mathbb{T}^N} g(x)^2 \sum_{i,j=1}^n \overline{v_i}(\sigma*\mu_{ij}(x)) v_j \mathrm{d}m(x) \ge 0.$$

Let $\varepsilon\to 0$ then the integral tends to $\sum\limits_{i,j=1}^n \overline{v_i}(\sigma*\mu_{ij}(z))v_j$, and this completes the proof. ∎

A.2 Some results for the index set

For $n = 1, 2, 3, \dots$

$$\#\boldsymbol{Z}_{N,n} = \sum_{j=1}^{N-1} \binom{N}{j} \binom{n-1}{j-1} \binom{N-j+n-1}{n}; \tag{A.1}$$

in each j-subset (there are $\binom{N}{j}$) of $[1, 2, \dots, N]$ take j-tuples $\alpha_{i_1}, \dots, \alpha_{i_j}$ with $\sum\limits_{\ell=1}^{j} \alpha_{i_\ell} = n$ and each $\alpha_{i_\ell} \geq 1$ ($\binom{n-1}{j-1}$ possibilities) and in the complement take $(N-j)$-tuples with $\sum\limits_{\ell=1}^{N-j} \alpha_{i_\ell} = -n$ and each $\alpha_{i_\ell} \leq 0$ ($\binom{N-j+n-1}{n}$ possibilities). For example when $n \geq 1$

$$\#\boldsymbol{Z}_{2,n} = 2, \quad \#\boldsymbol{Z}_{3,n} = 6n, \quad \#\boldsymbol{Z}_{4,n} = 10n^2 + 2, \quad \#\boldsymbol{Z}_{5,n} = \frac{5n}{3}\big(7n^2 + 5\big), \quad \dots .$$

Proposition A.5. *Among $\gamma \in \boldsymbol{Z}_{N,n}$ such that $\gamma^\nu = \beta$ for some fixed β with $|\beta| = n$ and such that $\beta_j > 0$ exactly when $j > k$ the minimal multi-index for the order $\gamma^{(1)\pi} \rhd \gamma^{(2)\pi}$ is $\gamma^{(0)} = \big(p+1, \dots, \overset{(m)}{p+1}, p, \dots, \overset{(k)}{p}, -\beta_{k+1}, \dots, -\beta_N\big)$ where $p = \lfloor \frac{n}{k} \rfloor$ and $m = n - kp$ (so $0 \leq m < k$).*

Proof. The claim is that $\big(\gamma_i^{(0)}\big)_{i=1}^{k}$ is $\prec$-minimal among partitions α of length $\leq k$ and $|\alpha| = n$. Argue by induction on the length. The statement is obviously true when $k = 1$. Suppose it is true for k and let α be a partition of length $\leq k+1$. Let

$$\begin{aligned} &n_1 = \sum_{i=1}^{k} \alpha_i, \qquad n_2 = n_1 + \alpha_{k+1}, \qquad p_1 = \left\lfloor \frac{n_1}{k} \right\rfloor, \qquad p_2 = \left\lfloor \frac{n_2}{k+1} \right\rfloor, \\ &m_1 = n_1 - kp_1, \qquad m_2 = n_2 - (k+1)p_2. \end{aligned}$$

Define $\gamma^{(1)}$ and $\gamma^{(2)}$ analogously to the above ($\gamma_i^{(s)} = p_s + 1$ for $1 \leq i \leq m_s$ and $\gamma_i^{(1)} = p_1$ for $m_1 < i \leq k$, $\gamma^{(2)} = p_2$ for $m_2 < i \leq k+1$). By the inductive hypothesis $\sum\limits_{j=1}^{i} \alpha_j \geq \sum\limits_{j=1}^{i} \gamma_j^{(1)}$. This implies $\alpha_{k+1} \leq \alpha_k \leq \gamma_k^{(1)}$. Thus $(k+1)p_2 \leq n_1 + \alpha_{k+1} \leq m_1 + kp_1 + \alpha_{k+1} \leq m_1 + (k+1)p_1$ and $p_2 \leq \frac{m_1}{k+1} + p_1$. Since p_1, p_2 are integers this implies $p_2 \leq p_1$. If $p_1 = p_2$ then $m_2 = m_1 - (p_1 - \alpha_{k+1})$, and clearly $\sum\limits_{j=1}^{i} \gamma_j^{(1)} \geq \sum\limits_{j=1}^{i} \gamma_j^{(2)}$ for $1 \leq i \leq k$. If $p_2 < p_1$ then $\gamma_j^{(2)} \leq p_2 + 1 \leq p_1 \leq \gamma_j^{(1)}$ for $1 \leq j \leq k$. Thus $\alpha \succeq \gamma^{(2)}$. ∎

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