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On the q-Charlier Multiple Orthogonal Polynomials*

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Abstract. We introduce a new family of special functions, namely q-Charlier multiple orthogonal polynomials. These polynomials are orthogonal with respect to q-analogues of Poisson distributions. We focus our attention on their structural properties. Raising and lowering operators as well as Rodrigues-type formulas are obtained. An explicit representation in terms of a q-analogue of the second of Appell's hypergeometric functions is given. A high-order linear q-difference equation with polynomial coefficients is deduced. Moreover, we show how to obtain the nearest neighbor recurrence relation from some difference operators involved in the Rodrigues-type formula.

Key words: multiple orthogonal polynomials; Hermite–Padé approximation; difference equations; classical orthogonal polynomials of a discrete variable; Charlier polynomials; q-polynomials

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1 Introduction

Let $\vec{\mu} = (\mu_1, \dots, \mu_r)$ be a vector of r positive Borel measures on $\mathbb{R}$, and let $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ be a multi-index. By $\mathbb{N}$ we denote the set of all nonnegative integers. A type II multiple orthogonal polynomial $P_{\vec{n}}$, corresponding to the multi-index $\vec{n}$, is a polynomial of degree $\leq |\vec{n}| = n_1 + \cdots + n_r$ which satisfies the orthogonality conditions [16]

$$\int_{\Omega_i} P_{\vec{n}}(x) x^k d\mu_i(x) = 0, \qquad k = 0, \dots, n_i - 1, \qquad i = 1, \dots, r, \tag{1.1}$$

where Ω_i is the smallest interval that contains $\operatorname{supp}(\mu_i)$. In this paper we will consider the situation when $P_{\vec{n}}$ is a monic multiple orthogonal polynomial and has exactly degree $|\vec{n}|$. If the measures in (1.1) are discrete

$$\mu_i = \sum_{k=0}^{N_i} \omega_{i,k} \delta_{x_{i,k}}, \qquad \omega_{i,k} > 0, \qquad x_{i,k} \in \mathbb{R}, \qquad N_i \in \mathbb{N} \cup \{+\infty\}, \qquad i = 1, 2, \dots, r, \tag{1.2}$$

where $\delta_{x_{i,k}}$ denotes the Dirac delta function and $x_{i_1,k} \neq x_{i_2,k}$, $k = 0, \dots, N_i$, whenever $i_1 \neq i_2$, the corresponding polynomial solution is called discrete multiple orthogonal polynomial.

In [4] was studied some type II discrete multiple orthogonal polynomials on the linear lattice $x(s) = s$. In particular, multiple Charlier polynomials were considered (when the component measures of $\vec{\mu}$ are different Poisson distributions). In this paper we will introduce a q-analogue

*This paper is a contribution to the Special Issue on Exact Solvability and Symmetry Avatars in honour of Luc Vinet. The full collection is available at http://www.emis.de/journals/SIGMA/ESSA2014.html of such multiple orthogonal polynomials (when the component measures of $\vec{\mu}$ are different q-Poisson distributions) and study their algebraic properties (structural properties). We are motivated by the recent applications that have been found for their predecessors: Multiple Charlier polynomials. In particular, these polynomials appear in remainder Padé approximation for the exponential function [18] and as common eigenstate of a set of r non-Hermitian oscillator Hamiltonians [13]. Furthermore, in [14] the authors pointed out a possible relationship of these polynomials to the orthogonal functions appearing in two speed totally asymmetric simple exclusion process (TASEP) [6]. Our new q-family of multiple orthogonal polynomials is likely to be of relevance in TASEP-like models as well as in q-extensions of the mathematical problems addressed in [13, 18].

Recently, two families of q-multiple orthogonal polynomials and some of their structural properties have been studied [3, 17]. Moreover, in [5, 12, 19] an $(r+1)$-order difference equation for some discrete multiple orthogonal polynomials was obtained. An interesting fact is that these polynomials are common eigenfunctions of two distinct linear difference operators of order $(r+1)$ since multiple orthogonal polynomials also satisfy $(r+2)$-term recurrence relations. The explicit expressions for the coefficients of this relation are the main ingredient for the study of some type of asymptotic behaviors for these polynomials. For instance, in [2] the weak asymptotics was studied for multiple Meixner polynomials of the first and second kind, respectively. The zero distribution of multiple Meixner polynomials was also studied. In [14] the recurrence relation is also used for studying the ratio asymptotic behavior of multiple Charlier polynomials (introduced in [4]) and from it the authors obtain the asymptotic distribution of the zeros after a suitable rescaling. Furthermore, this recurrence relation is a key-ingredient for attaining a Christoffel–Darboux kernel [7] among other applications, which plays important role in correlation kernel (for instance in the unitary random matrix model with external source).

In this paper, we will mainly focus on two structural properties satisfied by the multiple orthogonal polynomials introduced here, namely recurrence relations and the difference equation (with respect to the independent variable). The asymptotic analysis is out of the scope of this paper. Although we plan to address this question in a future publication.

The structure of the paper is as follows. Section 2 introduces the context and the background materials. In Section 3 we will define the q-Charlier multiple orthogonal polynomials (new in the literature) and obtain raising operators and then the Rodrigues-type formula for these special functions when r orthogonality conditions are considered. The explicit series representation in terms of a q-analogue of the second of Appell's hypergeometric functions is given for $\vec{n}=(n_1,n_2)$. In Section 4, an $(r+1)$-order q-difference equation on a non-uniform lattice $x(s)$ is obtained. Section 5 deals with the $(r+2)$-term recurrence relations. In particular, the nearest neighbor recurrence relation is obtained. Explicit expressions for the recurrence coefficients are given. Finally, in Section 6 some of our findings are summarized. A special limiting case when the parameter q involved in the components of the vector measure tends to one is pointed out. Under this limit the corresponding structural properties for the multiple Charlier polynomials are recovered.

2 Multiple Charlier polynomials

For the discrete measures (1.2) we have that $\operatorname{supp}(\mu_i)$ is the closure of $\{x_{i,k}\}_{k=0}^{N_i}$ and that Ω_i is the smallest closed interval on $\mathbb{R}$ which contains $\{x_{i,k}\}_{k=0}^{N_i}$. Moreover, the above orthogonality conditions (1.1) give a linear system of $|\vec{n}|$ homogeneous equations for the $|\vec{n}|+1$ unknown coefficients of $P_{\vec{n}}(x)$. This polynomial solution $P_{\vec{n}}$ always exists. We focus our attention on a unique solution (up to a multiplicative factor) with $\deg P_{\vec{n}}(x)=|\vec{n}|$. If this happen for every multi-index $\vec{n}$, we say that $\vec{n}$ is normal [16]. If the above system of measures forms an AT system [16] then every multi-index is normal. Indeed, we will deal with such system of discrete measures, where $\Omega_i=\Omega$ for each $i=1,2,\ldots,r$.

Definition 2.1. The system of positive discrete measures $\mu_1, \mu_2, \ldots, \mu_r$, given in (1.2), forms an AT system if there exist r continuous functions $\upsilon_1, \ldots, \upsilon_r$ on Ω with $\upsilon_i(x_k) = \omega_{i,k}$, $k = 0, \ldots, N_i$, $i = 1, 2, \ldots, r$, such that the $|\vec{n}|$ functions

$$\upsilon_1(x), x\upsilon_1(x), \ldots, x^{n_1-1}\upsilon_1(x), \ldots, \upsilon_r(x), x\upsilon_r(x), \ldots, x^{n_r-1}\upsilon_r(x),$$

form a Chebyshev system on Ω for each multi-index $\vec{n}$ with $|\vec{n}| < N + 1$, i.e., every linear combination $\sum_{i=1}^{r} Q_{n_i-1}(x)\upsilon_i(x)$, where $Q_{n_i-1} \in \mathbb{P}_{n_i-1} \setminus \{0\}$, has at most $|\vec{n}| - 1$ zeros on Ω.

The monic multiple Charlier polynomials [4] $C_{\vec{n}}^{\vec{\alpha}}(x)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$ satisfy the following orthogonality conditions with respect to r Poisson distributions with different positive parameters $\alpha_1, \ldots, \alpha_r$ (indexed by $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$)

$$\sum_{x=0}^{\infty} C_{\vec{n}}^{\vec{\alpha}}(x)(-x)_j \upsilon^{\alpha_i}(x) = 0, \qquad j = 0, \ldots, n_i - 1, \qquad i = 1, \ldots, r,$$

where

$$\upsilon^{\alpha_i}(x) = \begin{cases} \dfrac{\alpha_i^x}{\Gamma(x+1)}, & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Z}_-, \\ 0, & \text{otherwise,} \end{cases}$$

and $(x)_j = (x)(x+1)\cdots(x+j-1)$, $(x)_0 = 1$, $j \geq 1$, denotes the Pochhammer symbol.

In [4] the authors consider a normal multi-index $\vec{n} \in \mathbb{N}^r$, whenever $\alpha_i > 0$, $i = 1, 2, \ldots, r$, and with all the α_i different. Moreover, it was found the following raising operators

$$\mathcal{L}_{\vec{n}}^{\alpha_i}\left[C_{\vec{n}}^{\vec{\alpha}}(x)\right] = -C_{\vec{n}+\vec{e}_i}^{\vec{\alpha}}(x), \qquad i = 1, \ldots, r, \tag{2.1}$$

where

$$\mathcal{L}_{\vec{n}}^{\alpha_i} \stackrel{\text{def}}{=} \frac{\alpha_i}{\upsilon^{\alpha_i}(x)} \bigtriangledown \upsilon^{\alpha_i}(x),$$

and $\bigtriangledown f(x) = f(x) - f(x-1)$ denotes the backward difference operator. As a consequence of (2.1) there holds the Rodrigues-type formula

$$C_{\vec{n}}^{\vec{\alpha}}(x) = (-1)^{|\vec{n}|} \left(\prod_{j=1}^{r} \alpha_j^{n_j} \right) \Gamma(x+1) \mathcal{C}_{\vec{n}}^{\vec{\alpha}} \left(\frac{1}{\Gamma(x+1)} \right), \tag{2.2}$$

where

$$\mathcal{C}_{\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^{r} \left(\alpha_i^{-x} \bigtriangledown^{n_i} \alpha_i^x \right).$$

Two important structural properties are known for multiple Charlier polynomials [4], namely the $(r+1)$-order linear difference equation [12]

$$\prod_{i=1}^{r} \mathcal{L}_{\vec{n}}^{\alpha_i} \left[\bigtriangleup C_{\vec{n}}^{\vec{\alpha}}(x) \right] + \sum_{i=1}^{r} n_i \prod_{\substack{j=1 \\ j \neq i}}^{r} \mathcal{L}_{\vec{n}}^{\alpha_j} \left[C_{\vec{n}}^{\vec{\alpha}}(x) \right] = 0, \tag{2.3}$$

where $\bigtriangleup f(x) = f(x+1) - f(x)$, and the recurrence relation [4, 10]

$$xC_{\vec{n}}^{\vec{\alpha}}(x) = C_{\vec{n}+\vec{e}_k}^{\vec{\alpha}}(x) + (\alpha_k + |\vec{n}|)C_{\vec{n}}^{\vec{\alpha}}(x) + \sum_{i=1}^{r} \alpha_i n_i C_{\vec{n}-\vec{e}_i}^{\vec{\alpha}}(x), \tag{2.4}$$

where the multi-index $\vec{e}_i$ is the standard r dimensional unit vector with the i-th entry equals 1 and 0 otherwise.

Interestingly, the multiple Charlier polynomials $C_{\vec{n}}^{\vec{\alpha}}(x)$ are common eigenfunctions of the above two linear difference operators of order $(r+1)$, namely (2.3) and (2.4).

3 q-Charlier multiple orthogonal polynomials

Let us begin by recalling the definition of q-multiple orthogonal polynomials [3].

Definition 3.1. A polynomial $P_{\vec{n}}(x(s))$ on the lattice $x(s) = c_1 q^s + c_3$, $q \in \mathbb{R}^+ \setminus \{1\}$, $c_1, c_3 \in \mathbb{R}$, is said to be a q-multiple orthogonal polynomial of a multi-index $\vec{n} \in \mathbb{N}^r$ with respect to positive discrete measures $\mu_1, \mu_2, \ldots, \mu_r$ such that $\operatorname{supp}(\mu_i) \subset \Omega_i \subset \mathbb{R}$, $i = 1, 2, \ldots, r$, if there hold conditions

$$\begin{aligned} &(a) \quad \deg P_{\vec{n}}(x(s)) \leq |\vec{n}| = n_1 + n_2 + \cdots + n_r, \\ &(b) \quad \sum_{s=0}^{N_i} P_{\vec{n}}(x(s)) x(s)^k d\mu_i = 0, \qquad k = 0, \ldots, n_i - 1, \qquad N_i \in \mathbb{N} \cup \{+\infty\}. \end{aligned} \tag{3.1}$$

Let us consider the following r positive discrete measures on $\mathbb{R}^+$,

$$\mu_i = \sum_{s=0}^{\infty} \omega_i(k) \delta(k - s), \qquad \omega_i > 0, \qquad i = 1, 2, \ldots, r. \tag{3.2}$$

Here $\omega_i(s) = v_q^{\alpha_i}(s) \triangle x(s - 1/2)$, which involve the q-analogue of Poisson distributions

$$v_q^{\alpha_i}(s) = \begin{cases} \dfrac{\alpha_i^s}{\Gamma_q(s+1)}, & \text{if} \quad s \in \mathbb{R}^+ \cup \{0\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha_i > 0$, $i = 1, 2, \ldots, r$, with all the α_i different. Recall that the q-Gamma function is given by

$$\Gamma_q(s) = \begin{cases} f(s; q) = (1 - q)^{1-s} \dfrac{\prod\limits_{k \geq 0} (1 - q^{k+1})}{\prod\limits_{k \geq 0} (1 - q^{s+k})}, & 0 < q < 1, \\ q^{\frac{(s-1)(s-2)}{2}} f\left(s; q^{-1}\right), & q > 1. \end{cases} \tag{3.3}$$

See also [9, 15] for the above definition of the q-Gamma function.

Lemma 3.2. *The system of functions*

$$\alpha_1^s, x(s)\alpha_1^s, \ldots, x(s)^{n_1-1}\alpha_1^s, \ldots, \alpha_r^s, x(s)\alpha_r^s, \ldots, x(s)^{n_r-1}\alpha_r^s, \tag{3.4}$$

with $\alpha_i > 0$, $i = 1, 2, \ldots, r$, and $(\alpha_i/\alpha_j) \neq q^k$, $k \in \mathbb{Z}$, $i, j = 1, \ldots, r$, $i \neq j$, forms a Chebyshev system on $\mathbb{R}^+$ for every $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$.

Proof. This means that every linear combination $\sum_{i=1}^{r} Q_{n_i-1}(x(s))\alpha_i^s$ has at most $|\vec{n}| - 1$ zeros on $\mathbb{R}^+$ for every $Q_{n_i-1}(x(s)) \in \mathbb{P}_{n_i-1} \setminus \{0\}$. Since $x(s) = c_1 q^s + c_3$, where c_1, c_3 are constants, we consider $\sum_{i=1}^{r} Q_{n_i-1}(q^s)\alpha_i^s$, instead. Thus, the system (3.4) transforms into

$$a_{1,0}^s, a_{1,1}^s, \ldots, a_{1,n_1-1}^s, \ldots, a_{r,0}^s, a_{r,1}^s, \ldots, a_{r,n_r-1}^s,$$

where $a_{i,k} = (q^k \alpha_i)$, with $k = 0, \ldots, n_i - 1$, $i = 1, \ldots, r$. Observe that $a_{j,m} \neq a_{l,p}$ for $j \neq l$, $m \neq p$. Hence, identity $a_{i,k} = e^{\log a_{i,k}}$ yields the well-known Chebyshev system (see [16, p. 138])

$$e^{s \log a_{1,0}}, e^{s \log a_{1,1}}, \ldots, e^{s \log a_{1,n_1-1}}, \ldots, e^{s \log a_{r,0}}, e^{s \log a_{r,1}}, \ldots, e^{s \log a_{r,n_r-1}}.$$

Then, we conclude that the functions (3.4) form a Chebyshev system on $\mathbb{R}^+$. ■

As a consequence of Lemma 3.2 the system of measures $\mu_1, \mu_2, \ldots, \mu_r$ given in (3.2) forms an AT system on $\mathbb{R}^+$. Using this fact, we will rewrite Definition 3.1 for this system of measures. However, we first define the q-analogue of the Stirling polynomials denoted by $[s]_q^{(k)}$, which is a polynomial of degree k in the variable $x(s) = (q^s - 1)/(q - 1)$, as follows

$$[s]_q^{(k)} = \prod_{j=0}^{k-1} \frac{q^{s-j} - 1}{q - 1} = x(s)x(s-1)\cdots x(s-k+1) \qquad \text{for} \quad k > 0, \qquad \text{and} \qquad [s]_q^{(0)} = 1.$$

Definition 3.3. A polynomial $C_{q,\vec{n}}^{\vec{\alpha}}(s)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$ that verifies the orthogonality conditions

$$\sum_{s=0}^{\infty} C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(k)} \upsilon_q^{\alpha_i}(s) \triangle x(s - 1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \ldots, r, \tag{3.5}$$

(see (3.1) with respect to the measures (3.2)) is said to be the q-Charlier multiple orthogonal polynomial.

Let us point out some observations derived from the above definition. When $r = 1$ we recover the scalar q-Charlier polynomials [1]. The orthogonality conditions (3.1) have been written more conveniently as (3.5) since $[s]_q^{(k)} = q^{-\binom{k}{2}} x^k(s) + \text{lower terms} = \mathcal{O}(q^{ks})$. Here, the symbol $\mathcal{O}(\cdot)$ stands for big-O notation. Indeed, when q goes to 1, the symbol $[s]_q^{(k)}$ converges to $(-1)^k(-s)_k$, where $(s)_k$ denotes the Pochhammer symbol. Hence, one can recover the multiple Charlier polynomials given in [4] as a limiting case, provided that the lattice $x(s) = (q^s - 1)/(q - 1)$. Moreover, in the sequel we will only use this lattice. Finally, we have an AT-system of positive discrete measures, then the q-Charlier multiple orthogonal polynomial $C_{q,\vec{n}}^{\vec{\alpha}}(s)$ has exactly $|\vec{n}|$ different zeros on $\mathbb{R}^+$ (see [4, Theorem 2.1, pp. 26–27]).

For monic q-Charlier multiple orthogonal polynomials we have r raising operators

$$\mathcal{D}_q^{\alpha_i} C_{q,\vec{n}}^{\vec{\alpha}}(s) = -q^{1/2} C_{q,\vec{n}+\vec{e}_i}^{\vec{\alpha}_{i,1/q}}(s), \qquad i = 1, \ldots, r, \tag{3.6}$$

where

$$\mathcal{D}_q^{\alpha_i} \stackrel{\text{def}}{=} \left(\frac{\alpha_i q^{|\vec{n}|}}{\upsilon_q^{\alpha_i/q}(s)} \nabla \upsilon_q^{\alpha_i}(s) \right), \qquad \nabla \stackrel{\text{def}}{=} \frac{\bigtriangledown}{\bigtriangledown x(s + 1/2)}, \qquad \vec{\alpha}_{i,1/q} = (\alpha_1, \ldots, \alpha_i/q, \ldots, \alpha_r).$$

Furthermore,

$$q^{-|\vec{n}|-1/2} \mathcal{D}_q^{\alpha_i} f(s) = [\alpha_i - x(s)] f(s) + x(s) \bigtriangledown f(s),$$

for any function $f(s)$ defined on the discrete variable s. Notice that we call $\mathcal{D}_q^{\alpha_i}$ a raising operator since the i-th component of the multi-index $\vec{n}$ in (3.6) is increased by 1. For finding this operator we have replace $[s]_q^{(k)}$ in (3.5) by the following finite-difference expression

$$[s]_q^{(k)} = \frac{q^{(k-1)/2}}{[k+1]_q^{(1)}} \nabla [s+1]_q^{(k+1)}, \tag{3.7}$$

and then used summation by parts along with conditions $\upsilon_q^{\alpha_i}(-1) = \upsilon_q^{\alpha_i}(\infty) = 0$.

In the sequel we will only consider monic q-Charlier multiple orthogonal polynomials. In addition, the following difference operators will appear regularly

$$\Delta \stackrel{\text{def}}{=} \frac{\triangle}{\triangle x(s - 1/2)}, \tag{3.8}$$

$$\nabla^{n_i} = \underbrace{\nabla \cdots \nabla}_{n_i \text{ times}}, \tag{3.9}$$

and $\bigtriangledown x_1(s) \stackrel{\text{def}}{=} \bigtriangledown x(s + 1/2) = \triangle x(s - 1/2) = q^{s-1/2}$.

Proposition 3.4. *There holds the following q-analogue of Rodrigues-type formula*

$$C_{q,\vec{n}}^{\vec{\alpha}}(s) = \mathcal{K}_q^{\vec{n},\vec{\alpha}}\Gamma_q(s+1)\mathcal{C}_{q,\vec{n}}^{\vec{\alpha}}\left(\frac{1}{\Gamma_q(s+1)}\right), \tag{3.10}$$

where

$$\mathcal{C}_{q,\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^{r}\mathcal{C}_{q,n_i}^{\alpha_i}, \qquad \mathcal{C}_{q,n_i}^{\alpha_i} = (\alpha_i)^{-s}\nabla^{n_i}(\alpha_i q^{n_i})^s, \tag{3.11}$$

and

$$\mathcal{K}_q^{\vec{n},\vec{\alpha}} = (-1)^{|\vec{n}|}q^{-\frac{|\vec{n}|}{2}}\left(\prod_{i=1}^{r}\alpha_i^{n_i}\right)\left(\prod_{i=1}^{r}q^{n_i\sum\limits_{j=i}^{r}n_j}\right).$$

Proof. Replacing $[s]_q^{(k)}$ in (3.5) by the right-hand side of expression (3.7) one has

$$\sum_{s=0}^{\infty}C_{q,\vec{n}}^{\vec{\alpha}}(s)\nabla[s+1]_q^{(k+1)}v_q^{\alpha_i}(s)\bigtriangledown x_1(s) = 0, \qquad 0\le k\le n_i-1, \qquad i=1,\dots,r.$$

Hence, using summation by parts and the raising operators (3.6) in a recursive way one obtains the Rodrigues-type formula (3.10). ■

Notice that equation (3.10) provides an explicit expression for the monic q-Charlier multiple orthogonal polynomials. Indeed, by using formula (3.2.29) from [15] as follows

$$\nabla^m f(s) = q^{\binom{m+1}{2}/2-ms}\sum_{k=0}^{m}\begin{bmatrix} m\\ k\end{bmatrix}(-1)^k q^{\binom{m-k}{2}}f(s-k),$$
$$\begin{bmatrix} m\\ k\end{bmatrix} = \frac{(q;q)_m}{(q;q)_k(q;q)_{m-k}}, \qquad m=1,2,\dots,$$

where

$$(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j) \qquad \text{for} \quad k>0, \qquad \text{and} \qquad (a;q)_0 = 1,$$

denotes the q-analogue of the Pochhammer symbol [9, 11, 15], one obtains the following relation for the q-Charlier multiple orthogonal polynomials (for multi-index $\vec{n}=(n_1,n_2)$)

$$\begin{aligned} C_{q,n_1,n_2}^{\alpha_1,\alpha_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2}q^{n_1^2+n_2^2+n_1n_2-(n_1+n_2)/2}\\ &\quad\times\left(\frac{\Gamma_q(s+1)}{\alpha_2^s}\nabla^{n_2}(\alpha_2q^{n_2})^s\right)\left(\frac{1}{\alpha_1^s}\nabla^{n_1}\frac{(\alpha_1q^{n_1})^s}{\Gamma_q(s+1)}\right)\\ &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2}q^{n_1^2+n_2^2+n_1n_2-(n_1+n_2-\binom{n_1+1}{2}-\binom{n_2+1}{2})/2}\\ &\quad\times\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}(-1)^{l+k}\begin{bmatrix} n_2\\ l\end{bmatrix}\begin{bmatrix} n_1\\ k\end{bmatrix}\frac{q^{\binom{n_2-l}{2}-ln_2+\binom{n_1-k}{2}-kn_1}}{\alpha_2^l\alpha_1^k}\frac{\Gamma_q(s+1)}{\Gamma_q(s-k-l+1)}. \end{aligned} \tag{3.12}$$

Observe that from (3.3) we have

$$\frac{\Gamma_q(s+1)}{\Gamma_q(s-l-k+1)} = [s]_q^{(k+l)} = \frac{(q^{-s};q)_{l+k}}{(q-1)^{l+k}}q^{s(k+l)-\binom{k+l}{2}}.$$

Now, using relation

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, \qquad k=0,1,\ldots,$$

(see [11, formula (1.8.18)]) the above expression (3.12) transforms into

$$\begin{aligned} C_{q,n_1,n_2}^{\alpha_1,\alpha_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\quad\times \sum_{k=0}^{n_1}\sum_{l=0}^{n_2} \frac{(q^{-s};q)_{k+l}(q^{-n_1};q)_k(q^{-n_2};q)_l}{q^{\binom{k+l}{2}}(q;q)_k(q;q)_l} \left(\frac{q^{s+1-n_1}}{\alpha_1(q-1)}\right)^k \left(\frac{q^{s+1-n_2}}{\alpha_2(q-1)}\right)^l. \end{aligned} \tag{3.13}$$

Finally, from (3.13) we have

$$\begin{aligned} C_{q,n_1,n_2}^{\alpha_1,\alpha_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\quad\times \lim_{\gamma\to+\infty} \Phi_2\left(q^{-s};q^{-n_1},q^{-n_2};\gamma,\gamma;\frac{\gamma q^{s+1-n_1}}{\alpha_1(1-q)},\frac{\gamma q^{s+1-n_2}}{\alpha_2(1-q)}\right), \end{aligned} \tag{3.14}$$

where

$$\Phi_2(\zeta;\beta,\beta';\gamma,\gamma';x,y) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{(\zeta;q)_{m+n}(\beta;q)_m(\beta';q)_n}{(\gamma;q)_m(\gamma';q)_n(q;q)_m(q;q)_n} q^{-mn}x^m y^n,$$

is a q-analogue of the second of Appell's hypergeometric functions of two variables (see [8, formula (23), p. 89]).

Alternatively, in (3.13) the q-analogue of the Pochhammer symbol can be rewritten in terms of the q-falling factorials, which allows to express the q-Charlier multiple orthogonal polynomials in terms of the selected basis $[s]_q^{(k)}$, $k=0,1,\ldots$, i.e.

$$\begin{aligned} C_{q,n_1,n_2}^{\alpha_1,\alpha_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\quad\times \sum_{k=0}^{n_1}\sum_{l=0}^{n_2} \frac{[s]_q^{(k+l)}[n_1]_q^{(k)}[n_2]_q^{(l)}}{[k]_q![l]_q!} q^{\binom{k+1}{2}+\binom{l+1}{2}} \left(\frac{-1}{q^{2n_1}\alpha_1}\right)^k \left(\frac{-1}{q^{2n_2}\alpha_2}\right)^l. \end{aligned} \tag{3.15}$$

If we replace k by $j-l$ in (3.15), we obtain

$$\begin{aligned} C_{q,n_1,n_2}^{\alpha_1,\alpha_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\quad\times \sum_{l=0}^{n_2}\sum_{j=l}^{l+n_1} \frac{[s]_q^{(j)}[n_1]_q^{(j-l)}[n_2]_q^{(l)}}{[j-l]_q![l]_q!} \left(\frac{-1}{q^{2n_1}\alpha_1}\right)^{j-l} \left(\frac{-1}{q^{2n_2}\alpha_2}\right)^l q^{\binom{j-l+1}{2}+\binom{l+1}{2}} \\ &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\quad\times \sum_{j=0}^{n_1+n_2}\sum_{l=\max(0,j-n_1)}^{\min(j,n_2)} \frac{[s]_q^{(j)}[n_1]_q^{(j-l)}[n_2]_q^{(l)}}{[j-l]_q![l]_q!} \left(\frac{-1}{q^{2n_1}\alpha_1}\right)^{j-l} \left(\frac{-1}{q^{2n_2}\alpha_2}\right)^l q^{\binom{j-l+1}{2}+\binom{l+1}{2}} \\ &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\quad\times \sum_{j=0}^{n_1+n_2}\sum_{l=0}^{j} \frac{[s]_q^{(j)}[n_1]_q^{(j-l)}[n_2]_q^{(l)}}{[j-l]_q![l]_q!} \left(\frac{-1}{q^{2n_1}\alpha_1}\right)^{j-l} \left(\frac{-1}{q^{-2n_2}\alpha_2}\right)^l q^{\binom{j-l+1}{2}+\binom{l+1}{2}}. \end{aligned} \tag{3.16}$$

This expression is useful for computing some recurrence coefficients (see Section 5).

4 High-order q-difference equation

In this section we will find a lowering operator for the q-Charlier multiple orthogonal polynomials. Then we will combine it with the raising operators (3.6) to get an $(r+1)$-order q-difference equation (in the same fashion that [5] and [12]).

Lemma 4.1. *Let $\mathbb{V}$ be the linear subspace of polynomials $Q(s)$ on the lattice $x(s)$ of degree at most $|\vec{n}|-1$ defined by the following conditions*

$$\sum_{s=0}^{\infty} Q(s)[s]_q^{(k)} \upsilon_q^{q\alpha_j}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j - 2 \qquad \text{and} \qquad j = 1, \dots, r.$$

Then, the system $\{C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)\}_{i=1}^{r}$, where $\vec{\alpha}_{i,q} = (\alpha_1, \dots, q\alpha_i, \dots, \alpha_r)$, is a basis for $\mathbb{V}$.

Proof. From orthogonality relations

$$\sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)[s]_q^{(k)} \upsilon_q^{q\alpha_j}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j - 2, \qquad j = 1, \dots, r,$$

we have that polynomials $C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)$, $i = 1, \dots, r$, belong to $\mathbb{V}$.

Now, aimed to get a contradiction, let us assume that there exists constants λ_i, $i = 1, \dots, r$, such that

$$\sum_{i=1}^{r} \lambda_i C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s) = 0, \qquad \text{where} \qquad \sum_{i=1}^{r} |\lambda_i| > 0.$$

Then, multiplying the previous equation by $[s]_q^{(n_k-1)} \upsilon_q^{\alpha_k}(s) \bigtriangledown x_1(s)$ and then taking summation on s from 0 to ∞, one gets

$$\sum_{i=1}^{r} \lambda_i \sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)[s]_q^{(n_k-1)} \upsilon_q^{\alpha_k}(s) \bigtriangledown x_1(s) = 0.$$

Thus, taking into account relations

$$\sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)[s]_q^{(n_k-1)} \upsilon_q^{\alpha_k}(s) \bigtriangledown x_1(s) = c\delta_{i,k}, \qquad c \in \mathbb{R} \setminus \{0\}, \tag{4.1}$$

we deduce that $\lambda_k = 0$ for $k = 1, \dots, r$. Here $\delta_{i,k}$ represents the Kronecker delta symbol. Therefore, $\{C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)\}_{i=1}^{r}$ is linearly independent in $\mathbb{V}$. Furthermore, we know that any polynomial of $\mathbb{V}$ can be determined with $|\vec{n}|$ coefficients while $(|\vec{n}| - r)$ linear conditions are imposed on $\mathbb{V}$, consequently the dimension of $\mathbb{V}$ is at most r. Hence, the system $\{C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)\}_{i=1}^{r}$ spans $\mathbb{V}$, which completes the proof. ∎

Now we will prove that operator (3.8) is indeed a lowering operator for the sequence of q-Charlier multiple orthogonal polynomials $C_{q,\vec{n}}^{\vec{\alpha}}(s)$.

Lemma 4.2. *There holds the following relation*

$$\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s) = \sum_{i=1}^{r} q^{|\vec{n}|-n_i+1/2}[n_i]_q^{(1)} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s). \tag{4.2}$$

Proof. Using summation by parts we have

$$\begin{aligned}\sum_{s=0}^{\infty}\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(k)}\upsilon_q^{q\alpha_j}(s)\bigtriangledown x_1(s)&=-\sum_{s=0}^{\infty}C_{q,\vec{n}}^{\vec{\alpha}}(s)\nabla\big[[s]_q^{(k)}\upsilon_q^{q\alpha_j}(s)\big]\bigtriangledown x_1(s)\\&=-\sum_{s=0}^{\infty}C_{q,\vec{n}}^{\vec{\alpha}}(s)\varphi_{j,k}(s)\upsilon_q^{\alpha_j}(s)\bigtriangledown x_1(s),\end{aligned}\tag{4.3}$$

where

$$\varphi_{j,k}(s)=q^{1/2}[s]_q^{(k)}-q^{-1/2}\frac{x(s)}{\alpha_j}[s-1]_q^{(k)},$$

is a polynomial of degree $\leq k+1$ in the variable $x(s)$. Consequently, from the orthogonality conditions (3.5) we get

$$\sum_{s=0}^{\infty}\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(k)}\upsilon_q^{q\alpha_j}(s)\bigtriangledown x_1(s)=0,\qquad 0\leq k\leq n_j-2,\qquad j=1,\dots,r.$$

Hence, from Lemma 4.1, $\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)\in\mathbb{V}$. Moreover, $\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)$ can univocally be expressed as a linear combination of polynomials $\{C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)\}_{i=1}^{r}$, i.e.

$$\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)=\sum_{i=1}^{r}\beta_i C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s),\qquad \sum_{i=1}^{r}|\beta_i|>0.\tag{4.4}$$

Multiplying both sides of the equation (4.4) by $[s]_q^{(n_k-1)}\upsilon_q^{q\alpha_k}(s)\bigtriangledown x_1(s)$ and using relations (4.1) one has

$$\begin{aligned}\sum_{s=0}^{\infty}\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(n_k-1)}\upsilon_q^{q\alpha_k}(s)\bigtriangledown x_1(s)&=\sum_{i=1}^{r}\beta_i\sum_{s=0}^{\infty}C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)[s]_q^{(n_k-1)}\upsilon_q^{q\alpha_k}(s)\bigtriangledown x_1(s)\\&=\beta_k\sum_{s=0}^{\infty}C_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{i,q}}(s)[s]_q^{(n_k-1)}\upsilon_q^{q\alpha_k}(s)\bigtriangledown x_1(s).\end{aligned}\tag{4.5}$$

If we replace $[s]_q^{(k)}$ by $[s]_q^{(n_k-1)}$ in the left-hand side of equation (4.3), then left-hand side of equation (4.5) transforms into relation

$$\begin{aligned}\sum_{s=0}^{\infty}\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(n_k-1)}\upsilon_q^{q\alpha_k}(s)\bigtriangledown x_1(s)&=-\sum_{s=0}^{\infty}C_{q,\vec{n}}^{\vec{\alpha}}(s)\varphi_{k,n_k-1}(s)\upsilon_q^{\alpha_k}(s)\bigtriangledown x_1(s)\\&=\frac{q^{-1/2}}{\alpha_k}\sum_{s=0}^{\infty}C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(n_k)}\upsilon_q^{\alpha_k}(s)\bigtriangledown x_1(s).\end{aligned}\tag{4.6}$$

Here we have used that $x(s)[s-1]_q^{(n_k-1)}=[s]_q^{(n_k)}$ to get $\varphi_{k,n_k-1}(s)=-(q^{-1/2}/\alpha_k)[s]_q^{(n_k)}+$ lower terms.

On the other hand, from (3.6) one has that

$$\frac{1}{\alpha_k}\upsilon_q^{\alpha_k}(s)C_{q,\vec{n}}^{\vec{\alpha}}(s)=-q^{|\vec{n}|-1/2}\nabla\big[\upsilon_q^{q\alpha_k}(s)C_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{i,q}}(s)\big].\tag{4.7}$$

Then, by conveniently substituting (4.7) in the right-hand side of equation (4.6) and using once more summation by parts, we get

$$\sum_{s=0}^{\infty} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s) = -q^{|\vec{n}|-1} \sum_{s=0}^{\infty} [s]_q^{(n_k)} \nabla \left[\upsilon_q^{q\alpha_k}(s) C_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{i,q}}(s)\right] \bigtriangledown x_1(s)$$
$$= q^{|\vec{n}|-1} \sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{i,q}}(s) \Delta\left[[s]_q^{(n_k)}\right] \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s).$$

Since $\Delta[s]_q^{(n_k)} = q^{3/2-n_k}[n_k]_q^{(1)}[s]_q^{(n_k-1)}$ we finally have

$$\sum_{s=0}^{\infty} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s)$$
$$= q^{|\vec{n}|-n_k+1/2}[n_k]_q^{(1)} \sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{i,q}}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s).$$

Therefore, comparing this equation with (4.5) we obtain the coefficients in the expansion (4.4), i.e.

$$\beta_k = q^{|\vec{n}|-n_k+1/2}[n_k]_q^{(1)},$$

which proves relation (4.2). ■

Theorem 4.3. *The q-Charlier multiple orthogonal polynomial $C_{q,\vec{n}}^{\vec{\alpha}}(s)$ satisfies the following $(r+1)$-order q-difference equation*

$$\prod_{i=1}^{r} \mathcal{D}_q^{\alpha_i} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s) = -\sum_{i=1}^{r} q^{|\vec{n}|-n_i+1}[n_i]_q^{(1)} \prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{\alpha_j} C_{q,\vec{n}}^{\vec{\alpha}}(s). \tag{4.8}$$

Proof. Since operators (3.6) are commuting, we write

$$\prod_{i=1}^{r} \mathcal{D}_q^{\alpha_i} = \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{\alpha_j}\right) \mathcal{D}_q^{\alpha_i}, \tag{4.9}$$

and then using (3.6), by acting on equation (4.2) with the product of operators (4.9), we obtain the following relation

$$\prod_{i=1}^{r} \mathcal{D}_q^{\alpha_i} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s) = \sum_{i=1}^{r} q^{|\vec{n}|-n_i+1/2}[n_i]_q^{(1)} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{\alpha_j}\right) \mathcal{D}_q^{\alpha_i} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)$$
$$= -\sum_{i=1}^{r} q^{|\vec{n}|-n_i+1}[n_i]_q^{(1)} \prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{\alpha_j} C_{q,\vec{n}}^{\vec{\alpha}}(s),$$

which proves (4.8). ■

5 Recurrence relation

In this section we will study two types of recurrence relations, namely the nearest neighbor recurrence relation for any multi-index $\vec{n}$ and a step-line recurrence relation for $\vec{n}=(n_1,n_2)$.

For the main result of this section we will make use of the following lemma.

Lemma 5.1. *Let n_i be a positive integer and let $f(s)$ be a function defined on the discrete variable s. The following relation is valid*

$$\mathcal{C}_{q,n_i}^{\alpha_i}(x(s)f(s)) = q^{-n_i+1/2}x(n_i)(\alpha_i)^{-s}\nabla^{n_i-1}(\alpha_i q^{n_i})^s f(s) + q^{-n_i}[x(s)-x(n_i)]\mathcal{C}_{q,n_i}^{\alpha_i}f(s), \tag{5.1}$$

where difference operator $\mathcal{C}_{q,n_i}^{\alpha_i}$ is given in (3.11).

Proof. Let us act n_i-times with backward difference operators (3.9) on the product of functions $x(s)f(s)$. Assume momentarily that $n_i \geq N > 1$,

$$\begin{aligned}\nabla^{n_i}x(s)f(s) &= \nabla^{n_i-1}[\nabla x(s)f(s)] = \nabla^{n_i-1}\big[q^{-1/2}f(s)+x(s-1)\nabla f(s)\big]\\ &= q^{-1/2}\nabla^{n_i-1}f(s)+\nabla^{n_i-1}[x(s-1)\nabla f(s)]\\ &= q^{-1/2}\nabla^{n_i-1}f(s)+\nabla^{n_i-2}[\nabla x(s-1)\nabla f(s)].\end{aligned} \tag{5.2}$$

Repeating this process – but on the second term of the right-hand side of equation (5.2) – yields

$$\begin{aligned}\nabla^{n_i}x(s)f(s) &= \big[q^{1/2-n_i}+\cdots+q^{-5/2}+q^{-3/2}+q^{-1/2}\big]\nabla^{n_i-1}f(s)+x(s-n_i)\nabla^{n_i}f(s)\\ &= q^{1/2-n_i}x(n_i)\nabla^{n_i-1}f(s)+x(s-n_i)\nabla^{n_i}f(s),\end{aligned}$$

or equivalently,

$$\nabla^{n_i}x(s)f(s) = q^{-n_i+1/2}x(n_i)\nabla^{n_i-1}f(s)+q^{-n_i}[x(s)-x(n_i)]\nabla^{n_i}f(s), \qquad n_i\geq 1. \tag{5.3}$$

Now, aimed to involve difference operator $\mathcal{C}_{q,n_i}^{\alpha_i}$ in the above equation, we multiply equation (5.3) from the left side by $(\alpha_i)^{-s}$ and replace $f(s)$ by $(\alpha_i q^{n_i})^s f(s)$. Thus, equation (5.3) transforms into (5.1), which completes the proof. ■

Theorem 5.2. *The q-Charlier multiple orthogonal polynomials satisfy the following $(r+2)$-term recurrence relation*

$$\begin{aligned}x(s)C_{q,\vec{n}}^{\vec{\alpha}}(s) &= C_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha}}(s)+\left(\sum_{i=1}^{r}q^{|\vec{n}|_i}x(n_i)\left[(q-1)\left(\alpha_i q^{\sum_{j=i}^{r}n_j}\right)+1\right]+\alpha_k q^{|\vec{n}|+n_k+1}\right)C_{q,\vec{n}}^{\vec{\alpha}}(s)\\ &\quad+\sum_{i=1}^{r}q^{|\vec{n}|_i}x(n_i)\left[(q-1)\left(\alpha_i q^{\sum_{j=i}^{r}n_j}\right)+1\right]\alpha_i q^{|\vec{n}|+n_i-1}C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}}(s),\end{aligned} \tag{5.4}$$

where $|\vec{n}|_i = n_1+\cdots+n_{i-1}$, $|\vec{n}|_1=0$.

Proof. We will use Lemma 5.1. First, let us consider equation

$$\begin{aligned}(\alpha_k)^{-s}\nabla^{n_k+1}(\alpha_k q^{n_k+1})^s\frac{1}{\Gamma_q(s+1)} &= (\alpha_k)^{-s}\nabla^{n_k}\left[q^{-s+1/2}\bigtriangledown\frac{(\alpha_k q^{n_k+1})^s}{\Gamma_q(s+1)}\right]\\ &= q^{1/2}(\alpha_k)^{-s}\nabla^{n_k}(\alpha_k q^{n_k})^s\frac{1}{\Gamma_q(s+1)}\\ &\quad-(\alpha_k q^{n_k+1})^{-1}q^{1/2}(\alpha_k)^{-s}\nabla^{n_k}(\alpha_k q^{n_k})^s x(s)\frac{1}{\Gamma_q(s+1)},\end{aligned}$$

which can be rewritten in terms of difference operators (3.11) as follows

$$\mathcal{C}^{\alpha_k}_{q,n_k+1}\frac{1}{\Gamma_q(s+1)} = q^{1/2}\mathcal{C}^{\alpha_k}_{q,n_k}\frac{1}{\Gamma_q(s+1)} - (\alpha_k q^{n_k+1})^{-1}q^{1/2}\mathcal{C}^{\alpha_k}_{q,n_k}x(s)\frac{1}{\Gamma_q(s+1)}. \tag{5.5}$$

Since operators (3.11) are commuting the multiplication of equation (5.5) from the left-hand side by the product $\left(\prod\limits_{\substack{j=1\\ j\neq k}}^{r}\mathcal{C}^{\alpha_j}_{q,n_j}\right)$ yields

$$\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}x(s)\frac{1}{\Gamma_q(s+1)} = \alpha_k q^{n_k+1}\big(\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}} - q^{-1/2}\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}+\vec{e}_k}\big)\frac{1}{\Gamma_q(s+1)}. \tag{5.6}$$

Second, let us recursively use expression (5.1) involving the product of r difference operators $\mathcal{C}^{\alpha_1}_{q,n_1},\dots,\mathcal{C}^{\alpha_r}_{q,n_r}$ acting on the function $f(s)=1/\Gamma_q(s+1)$. Thus, we have

$$\begin{aligned} q^{|\vec{n}|}\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}x(s)\frac{1}{\Gamma_q(s+1)} &\\ = q^{1/2}\sum_{i=1}^{r}q^{|\vec{n}|_i}x(n_i)&\left[(q-1)\left(\alpha_i q^{\sum\limits_{j=i}^{r}n_j}\right)+1\right]\prod_{j=1}^{r}\mathcal{C}^{\alpha_j}_{q,n_j-\delta_{j,i}}\frac{1}{\Gamma_q(s+1)}\\ + \left(x(s)-\sum_{i=1}^{r}q^{|\vec{n}|_i}x(n_i)\left[(q-1)\left(\alpha_i q^{\sum\limits_{j=i}^{r}n_j}\right)+1\right]\right)&\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}\frac{1}{\Gamma_q(s+1)}. \end{aligned} \tag{5.7}$$

Hence, by using expressions (5.6), (5.7) one gets

$$\begin{aligned} x(s)\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}\frac{1}{\Gamma_q(s+1)} &= -\alpha_k q^{|\vec{n}|+n_k+1}q^{-1/2}\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}+\vec{e}_k}\frac{1}{\Gamma_q(s+1)}\\ &+ \left(\sum_{i=1}^{r}q^{|\vec{n}|_i}x(n_i)\left[(q-1)\left(\alpha_i q^{\sum\limits_{j=i}^{r}n_j}\right)+1\right]+\alpha_k q^{|\vec{n}|+n_k+1}\right)\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}\frac{1}{\Gamma_q(s+1)}\\ &- q^{1/2}\sum_{i=1}^{r}q^{|\vec{n}|_i}x(n_i)\left[(q-1)\left(\alpha_i q^{\sum\limits_{j=i}^{r}n_j}\right)+1\right]\prod_{j=1}^{r}\mathcal{C}^{\alpha_j}_{q,n_j-\delta_{j,i}}\frac{1}{\Gamma_q(s+1)}. \end{aligned}$$

Finally, multiplying from the left both sides of the previous expression by $\mathcal{K}^{\vec{n},\vec{\alpha}}_q\Gamma_q(s+1)$ and using Rodrigues-type formula (3.10) we obtain (5.4), which completes the proof. ∎

Observe that other recurrence relations different from the above nearest neighbor recurrence relation (5.4) can be obtained. Indeed, from (3.16) a 4-term recurrence relation for $\vec{n}=(n_1,n_2)$ can be obtained. In [4] it was given an approach for the recurrence relations of some discrete multiple orthogonal polynomials. This approach make use of the Rodrigues-type formulas along with some series representations (Kampé de Fériet series) for multiple orthogonal polynomials. Here we proceed in the same fashion.

Considering the expansion

$$C^{\alpha_1,\alpha_2}_{q,n_1,n_2}(s) = \sum_{j=0}^{n_1+n_2}c^{(j)}_{q,n_1,n_2}[s]^{(j)}_q,$$

the coefficients $c^{(j)}_{q,n_1,n_2}$ can be used to compute the recurrence coefficients in

$$\begin{aligned} x(s)P_{q,n_1,n_2}(s) &= q^{n_1+n_2}P_{q,n_1,n_2+1}(s) + b_{q,n_1,n_2}P_{q,n_1,n_2}(s) + c_{q,n_1,n_2}P_{q,n_1,n_2-1}(s)\\ &\quad + d_{q,n_1,n_2}P_{q,n_1-1,n_2-1}(s), \end{aligned}$$

where $P_{q,n_1,n_2}(s) = C^{\alpha_1,\alpha_2}_{q,n_1,n_2}(s)$. Indeed,

$$\begin{aligned}
b_{q,n_1,n_2} &= q^{n_1+n_2}\big(q^{-1}c^{(n_1+n_2-1)}_{q,n_1,n_2} - c^{(n_1+n_2)}_{q,n_1,n_2+1}\big) + x(n_1+n_2),\\
c_{q,n_1,n_2} &= q^{n_1+n_2}\big(q^{-2}c^{(n_1+n_2-2)}_{q,n_1,n_2} + q^{-(n_1+n_2)}c^{(n_1+n_2-1)}_{q,n_1,n_2}x(n_1+n_2-1)\\
&\quad - q^{-(n_1+n_2)}b_{q,n_1,n_2}c^{(n_1+n_2-1)}_{q,n_1,n_2} - c^{(n_1+n_2-1)}_{q,n_1,n_2+1}\big),\\
d_{q,n_1,n_2} &= q^{n_1+n_2}\big(q^{-3}c^{(n_1+n_2-3)}_{q,n_1,n_2} + q^{-(n_1+n_2)}c^{(n_1+n_2-2)}_{q,n_1,n_2}x(n_1+n_2-2)\\
&\quad - q^{-(n_1+n_2)}c_{q,n_1,n_2}c^{(n_1+n_2-2)}_{q,n_1,n_2-1} - q^{-(n_1+n_2)}b_{q,n_1,n_2}c^{(n_1+n_2-2)}_{q,n_1,n_2} - c^{(n_1+n_2-2)}_{q,n_1,n_2+1}\big).
\end{aligned}$$

From the explicit expression (3.16) we then get, after some calculations, that when $q \to 1$ we recover the recurrence coefficients given in [4].

6 Conclusions

In summary, let us recall some of our results. We have introduced a new family of special functions, i.e. q-Charlier multiple orthogonal polynomials. For these polynomials we have obtained a Rodrigues-type formula (3.10) as well as their explicit representation in terms of a q-analogue of the second of Appell's hypergeometric functions of two variables (3.14). Moreover, these polynomials are common eigenfunctions of two different $(r+1)$-order difference operators, namely (4.8) and (5.4). In the limiting situation $q \to 1$ we recover the corresponding structural relations for multiple Charlier polynomials [4]. Indeed, our relations (3.10), (4.8), and (5.4) transform into (2.2), (2.3), and (2.4), respectively.

Our algebraic approach for the nearest neighbor recurrence relation (5.4) does not require to introduce type I multiple orthogonality [20]. Indeed, the q-difference operators involved in the Rodrigues-type formula constitute the key-ingredient in our approach.

A description of the main term of the logarithm asymptotics of q-Charlier multiple orthogonal polynomials deserves our future attention. We expect to give it in terms of an algebraic function (see [2]). These results yield the Cauchy transform of the weak-star limit for scaling zero counting measure of the polynomials. In addition, the zero distribution of these type II multiple orthogonal polynomials will be studied.

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